



Representations of $*$ -Semigroups Associated to Invariant Kernels with Values Continuously Adjointable Operators

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Abstract. We consider positive semidefinite kernels valued in the $*$ -algebra of continuous and continuously adjointable operators on a VH-space (Vector Hilbert space in the sense of Loynes) and that are invariant under actions of $*$ -semigroups. For such a kernel we obtain two necessary and sufficient boundedness conditions in order for there to exist $*$ -representations of the underlying $*$ -semigroup on a VH-space linearisation, equivalently, on a reproducing kernel VH-space. We exhibit several situations when the latter boundedness condition is automatically fulfilled. For example, when specialising to the case of Hilbert modules over locally C^* -algebras, we show that both boundedness conditions are automatically fulfilled and, consequently, this general approach provides a rather direct proof of the general Stinespring–Kasparov type dilation theorem for completely positive maps on locally C^* -algebras and with values adjointable operators on Hilbert modules over locally C^* -algebras.

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Introduction

In 1965, Loynes published two articles [27] and [28] where he considered generalisations of the notions of inner product space and of Hilbert space, that he called VE-space (Vector Euclidean space) and, respectively, VH-space (Vector Hilbert space). These are vector spaces on which there are “inner products”

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with values in certain ordered $*$ -spaces, hence “vector valued inner products”, see Sects. 1.1–1.3 for precise definitions. His motivation was coming from stochastic processes [29] and the main results refer to a generalisation of B. Sz.-Nagy’ Dilation Theorem [47] for operator valued positive semidefinite maps on $*$ -semigroups [27], and to some other results on spectral theory of linear bounded operators on VH-spaces [28]. These ideas have been followed in prediction theory [7, 50, 51], in dilation theory [14–16], and a few others.

On the other hand, special cases of VH-spaces have been later considered independently of the Loynes’ articles. Thus, the concept of Hilbert module over a C^* -algebra was introduced in 1973 by Paschke [36], following Kaplansky [22], and independently by Rieffel 1 year later in [41], and these two articles triggered a whole domain of research, see e.g. [26] and [31] and the rich bibliography cited there. Hilbert modules over C^* -algebras are special cases of VH-spaces. Dilation theory plays a very important role in this theory and there are many dilation results of an impressive diversity, but the domain of Hilbert modules over C^* -algebras remained unrelated to that of VH-spaces. Another special case of a VH-space is that of Hilbert modules over H^* -algebras of Saworotnow [42]. Also, in 1985 Mallios [30] and later in 1988 Phillips [39] introduced and studied the concept of Hilbert module over locally C^* -algebra, which is yet another particular case of VH-space over an admissible space. The theory of Hilbert spaces over locally C^* -algebras is an active domain of research as well, e.g. see [21] and the rich bibliography cited there.

Taking into account the importance and the diversity of dilation theory, e.g. see [4], it is natural to ask for its unification under a general framework. Historically, the theory of positive semidefinite kernels, having values in operator spaces or ordered $*$ -spaces, e.g. see [12, 18, 32, 35], and [46], to cite just a few, turned out to provide, to a certain extent, such a unification framework, that can be made much more efficient when a certain “symmetry” is added, more precisely, the invariance under the action of a $*$ -semigroup, e.g. see [9]. Following [17], in this article we show that this unifying framework becomes significantly more successful when kernels with values linear operators on VH-spaces are employed. In [17] there is one extra assumption on the range of the kernels, namely that of boundedness in the sense of Loynes, which restricts the area of applicability to C^* -algebras and, in order to unify other dilation results, e.g. the dilation of completely positive maps on Hilbert modules over locally C^* -algebras, see [20], the boundedness condition should be relaxed.

This article is one step further in the programme, initiated in [17], of unifying dilation results under a setting comprising positive semidefinite kernels that are invariant under actions of $*$ -semigroups and with values continuous and continuously adjointable operators on VH-spaces, and a continuation of the work [5] in which we obtained a nontopological version of this kind of dilation theorem. From this point of view, the main result of this article is Theorem 2.10 that provides two necessary and sufficient conditions for the existence of $*$ -representations of the given $*$ -semigroup by continuous and continuously adjointable operators on VH-spaces. The boundedness condition

(b1) in Theorem 2.10 is the analog of the celebrated Sz.-Nagy's boundedness condition [47] (see [46] for a historical perspective of this issue) and is related to the continuity of linear operators in the range of the $*$ -representation, an obstruction caused by the gap between $*$ -semigroups and groups. The boundedness condition (b2) from Theorem 2.10 is new and refers to an obstruction related to the continuity of the adjoint operators which, in the case of VH-spaces, is not automatic.

Theorem 2.10 unifies most of the known dilation theorems for operator valued maps, in the chain of the two classical Naimark's theorems for operator valued positive semidefinite functions on commutative groups [34] and, respectively, for semispectral measures [33], that is, the Stinespring's Theorem [45] for operator valued completely positive maps on C^* -algebras, the Sz.-Nagy's Theorem [47] for operator valued positive semidefinite functions on $*$ -semigroups and its VH-space generalization of Loynes [27], as well as the dilation theorems for completely positive maps on C^* -algebras with values adjointable operators on Hilbert modules over C^* -algebras of Kasparov [23] and that for completely positive maps on locally C^* -algebras with values adjointable operators on Hilbert modules over locally C^* -algebras of Joița [20]. In this article, we explicitly show how the latter is obtained as a consequence of Theorem 3.2.

In the following we briefly describe the contents of this article. The first section is dedicated to notation and preliminary results on VH-spaces and their linear operators. Since we built on the fabric of dilation theory on VE-spaces over ordered $*$ -spaces, we first briefly review necessary concepts, results, and constructions from [5]. One of the main mathematical objects used in this research is that of Loynes' admissible space, that is, a complete topologically ordered $*$ -space. A list of nine examples, that we carefully present, indicates the unifying potential of this concept. VH-spaces and their linear operators are discussed in Sect. 1.3. Here, we draw attention to Lemma 1.3 that clarifies the locally convex topology on VH-spaces and to the six generic examples that illustrate the unifying potential of the concept of VH-space. Three of the main technical obstructions in this theory are related to the lack of a general Schwarz type inequality, to the existence of non-orthocomplemented VH-subspaces, and to the lack of a reliable substitute for the Riesz's Representation Theorem for continuous linear functionals. Consequently, many technical ingredients that are used in this article gravitate around finding sufficiently powerful surrogates of these missing tools. In this respect, in Lemma 1.6 we obtain a first surrogate of the Schwarz inequality related to positive operators and then refinements of this inequality are performed in Sect. 2.5.

The main section of this article refers to positive semidefinite kernels with values continuous and continuously adjointable linear operators on VH-spaces. Again, since we built on dilation results on VE-spaces over $*$ -ordered spaces investigated in [5], we first review the necessary terminology, results, and constructions corresponding to positive semidefinite kernels in the non-topological case. When working with kernels, there is a paradigmatic idea that the natural approach is through reproducing kernel spaces, e.g. see [3, 46] and

the rich bibliography cited there. For this reason, we first investigate basic properties of VH-space linearisations (Kolmogorov decompositions) and their interplay with reproducing kernel VH-spaces which, at this level of generality, require a careful treatment: most of the properties that we expect are true, but some of the proofs are rather different. We stress that our approach of the dilation constructions is through reproducing kernel spaces that has substantial advantages: the objects that are built preserve their concrete character to the largest possible extent, for example we always obtain (operator valued) function spaces and not abstract quotient spaces, in contrast to the GNS construction which is traditionally extensively used in dilation theory.

Theorem 2.10, the main result of this article, emphasises the boundedness condition (b1), the analog of the Sz.-Nagy's boundedness condition, and the boundedness condition (b2) that shows up due to topological obstructions of dealing with linear operators on locally convex spaces, especially in connection with the topological pathologies related to multiplication. Recently, a related phenomenon has been discussed by Żelazko [52] who introduced a class of continuous linear operators on locally convex spaces \mathcal{E} for which there is a certain control of the growth of their powers uniformly on \mathcal{E} , that he called *m-topologisable* (multiplicatively topologisable), see also [6]. In Sect. 2.5 we show that, when a positive semidefinite kernel has *m-topologisable* operators on its whole diagonal, a stronger condition than the boundedness condition (b2) is obtained by an iteration method, previously employed in spectral theory [11, 24, 25, 40], in particular, *m-topologisability* propagates throughout the kernel. However, the question whether condition (b2) holds at the level of generality of positive semidefinite kernels with values continuous and continuously adjointable operators on VH-spaces remains open.

The last section is dedicated to show the unifying coverage of our Theorem 2.10, by providing a direct proof of the dilation theorem from [20]. Theorem 3.2 is a remarkable consequence of Theorem 2.10 and of the previously obtained results for *m-topologisable* operators, which turns out to be the case in this context. This theorem shows that, for invariant positive semidefinite kernels with values adjointable operators on a Hilbert module over a locally C^* -algebra, the boundedness condition (b2) is automatic, hence the existence of $*$ -representations on a Hilbert locally C^* -module linearisation of the kernel, equivalently, on the reproducing kernel Hilbert locally C^* -module of the kernel, depends only on the boundedness condition (b1). Finally, we point out why the boundedness conditions (b1) discussed above is automatic as well in the special case of completely positive maps on locally C^* -algebras and with values adjointable operators on Hilbert modules over locally C^* -algebras, by an adaptation of the technique of Murphy [32] that solves the nonunital case by approximate identities in locally C^* -algebras.

1. Preliminaries

In this section we briefly review most of the definitions and some basic facts on ordered $*$ -spaces, VE-spaces over ordered $*$ -spaces, and their linear operators,

then review and get some facts on VH-spaces over admissible spaces and their linear operators.

1.1. VE-Spaces and Their Linear Operators

A complex vector space Z is called *ordered $*$ -space*, see [38], if:

- (a1) Z has an *involution* $*$, that is, a map $Z \ni z \mapsto z^* \in Z$ that is *conjugate linear* ($(sx + ty)^* = \bar{s}x^* + \bar{t}y^*$ for all $s, t \in \mathbb{C}$ and all $x, y \in Z$) and *involution* ($(z^*)^* = z$ for all $z \in Z$).
- (a2) In Z there is a *cone* Z^+ ($sx + ty \in Z^+$ for all numbers $s, t \geq 0$ and all $x, y \in Z^+$), that is *strict* ($Z^+ \cap -Z^+ = \{0\}$), and consisting of *selfadjoint elements* only ($z^* = z$ for all $z \in Z^+$). This cone is used to define a *partial order* on the real vector space of all selfadjoint elements in Z : $z_1 \geq z_2$ if $z_1 - z_2 \in Z^+$.

Recall that a $*$ -*algebra* \mathcal{A} is a complex algebra onto which there is defined an *involution* $\mathcal{A} \ni a \mapsto a^* \in \mathcal{A}$, that is, $(\lambda a + \mu b)^* = \bar{\lambda}a^* + \bar{\mu}b^*$, $(ab)^* = b^*a^*$, and $(a^*)^* = a$, for all $a, b \in \mathcal{A}$ and all $\lambda, \mu \in \mathbb{C}$.

An *ordered $*$ -algebra* \mathcal{A} is a $*$ -algebra such that it is an ordered $*$ -space, more precisely, it has the following property.

- (osa1) There exists a strict cone \mathcal{A}^+ in \mathcal{A} such that for any $a \in \mathcal{A}^+$ we have $a = a^*$.

Clearly, any ordered $*$ -algebra is an ordered $*$ -space. In particular, given $a \in \mathcal{A}$, we denote $a \geq 0$ if $a \in \mathcal{A}^+$ and, for $a = a^* \in \mathcal{A}$ and $b = b^* \in \mathcal{A}$, we denote $a \geq b$ if $a - b \geq 0$.

Given a complex linear space \mathcal{E} and an ordered $*$ -space space Z , a Z -*gramian*, also called a Z -*valued inner product*, is, by definition, a mapping $\mathcal{E} \times \mathcal{E} \ni (x, y) \mapsto [x, y] \in Z$ subject to the following properties:

- (ve1) $[x, x] \geq 0$ for all $x \in \mathcal{E}$, and $[x, x] = 0$ if and only if $x = 0$.
- (ve2) $[x, y] = [y, x]^*$ for all $x, y \in \mathcal{E}$.
- (ve3) $[x, \alpha y_1 + \beta y_2] = \alpha[x, y_1] + \beta[x, y_2]$ for all $\alpha, \beta \in \mathbb{C}$ and all $x_1, x_2 \in \mathcal{E}$.

A complex linear space \mathcal{E} onto which a Z -gramian $[\cdot, \cdot]$ is specified, for a certain ordered $*$ -space Z , is called a *VE-space* (Vector Euclidean space) over Z , cf. [27].

Given a pairing $[\cdot, \cdot]: \mathcal{E} \times \mathcal{E} \rightarrow Z$, where \mathcal{E} is some vector space and Z is an ordered $*$ -space, and assuming that $[\cdot, \cdot]$ satisfies only the axioms (ve2) and (ve3), then a *polarisation formula* holds

$$4[x, y] = \sum_{k=0}^3 i^k [x + i^k y, x + i^k y], \quad x, y \in \mathcal{E}. \quad (1.1)$$

In particular, this formula holds on a VE-space and it shows that the Z -gramian is perfectly defined by the Z -valued quadratic map $\mathcal{E} \ni x \mapsto [x, x] \in Z$.

A *VE-spaces isomorphism* is, by definition, a linear bijection $U: \mathcal{E} \rightarrow \mathcal{F}$, for two VE-spaces over the same ordered $*$ -space Z , which is *isometric*, that is, $[Ux, Uy]_{\mathcal{F}} = [x, y]_{\mathcal{E}}$ for all $x, y \in \mathcal{E}$.

In general VE-spaces, an analog of the Schwarz Inequality may not hold but some of its consequences can be proven using slightly different techniques,

cf. [27, 28]. Given two VE-spaces \mathcal{E} and \mathcal{F} , over the same ordered $*$ -space Z , one can consider the vector space $\mathcal{L}(\mathcal{E}, \mathcal{F})$ of all linear operators $T: \mathcal{E} \rightarrow \mathcal{F}$. A linear operator $T \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ is called *adjointable* if there exists $T^* \in \mathcal{L}(\mathcal{F}, \mathcal{E})$ such that

$$[Te, f]_{\mathcal{F}} = [e, T^*f]_{\mathcal{E}}, \quad e \in \mathcal{E}, f \in \mathcal{F}. \quad (1.2)$$

The operator T^* , if it exists, is uniquely determined by T and called its *adjoint*. Since an analog of the Riesz Representation Theorem for VE-spaces may not exist, in general, there may be not so many adjointable operators. Denote by $\mathcal{L}^*(\mathcal{E}, \mathcal{F})$ the vector space of all adjointable operators from $\mathcal{L}(\mathcal{E}, \mathcal{F})$. Note that $\mathcal{L}^*(\mathcal{E}) = \mathcal{L}^*(\mathcal{E}, \mathcal{E})$ is a $*$ -algebra with respect to the involution $*$ determined by the operation of taking the adjoint.

An operator $A \in \mathcal{L}(\mathcal{E})$ is called *selfadjoint* if $[Ae, f] = [e, Af]$, for all $e, f \in \mathcal{E}$. Any selfadjoint operator A is adjointable and $A = A^*$. By the polarisation formula (1.1), A is selfadjoint if and only if $[Ae, e] = [e, Ae]$, for all $e \in \mathcal{E}$. An operator $A \in \mathcal{L}(\mathcal{E})$ is *positive* if $[Ae, e] \geq 0$, for all $e \in \mathcal{E}$. Since the cone Z^+ consists of selfadjoint elements only, any positive operator is selfadjoint and hence adjointable. Note that any VE-space isomorphism U is adjointable, invertible, and $U^* = U^{-1}$, hence, equivalently, we can call it *unitary*.

A VE-module \mathcal{E} over an ordered $*$ -algebra \mathcal{A} is a right \mathcal{A} -module on which there exists an \mathcal{A} -gramian $[\cdot, \cdot]_{\mathcal{E}}: \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{A}$ with respect to which it is a VE-space, that is, (ve1)-(ve3) hold, and, in addition,

$$(\text{vem}) \quad [e, fa + gb]_{\mathcal{E}} = [e, f]_{\mathcal{E}}a + [e, g]_{\mathcal{E}}b \text{ for all } e, f, g \in \mathcal{E} \text{ and all } a, b \in \mathcal{A}.$$

Given an ordered $*$ -algebra \mathcal{A} and two VE-modules \mathcal{E} and \mathcal{F} over \mathcal{A} , an operator $T \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ is called a *module map* if

$$T(ea) = T(e)a, \quad e \in \mathcal{E}, a \in \mathcal{A}.$$

It is easy to see that any operator $T \in \mathcal{L}^*(\mathcal{E}, \mathcal{F})$ is a module map, e.g. see [5].

1.2. Admissible Spaces

The complex vector space Z is called *topologically ordered $*$ -space* if it is an ordered $*$ -space, that is, axioms (a1) and (a2) hold and, in addition,

- (a3) Z is a *Hausdorff locally convex space*.
- (a4) The topology of Z is *compatible* with the partial ordering in the sense that there exists a base of the topology, linearly generated by a family of neighbourhoods $\{C\}_{C \in \mathcal{C}_0}$ of the origin that are absolutely convex and *solid*, in the sense that, if $x \in C$ and $y \in Z$ are such that $0 \leq y \leq x$, then $y \in C$.

Remark 1.1. Axiom (a4) is equivalent with the following one:

- (a4') There exists a collection of seminorms $\{p_j\}_{j \in \mathcal{J}}$ defining the topology of Z that, for any $j \in \mathcal{J}$, p_j is *increasing*, in the sense that, $0 \leq x \leq y$ implies $p_j(x) \leq p_j(y)$.

To see this, e.g. see Lemma 1.1.1 and Remark 1.1.2 of [8], letting \mathcal{C}_0 be a family of open, absolutely convex and solid neighbourhoods of the origin

defining the topology of Z , for each $C \in \mathcal{C}_0$, consider the Minkowski seminorm p_C associated to C ,

$$p_C(x) = \inf\{\lambda \mid \lambda > 0, \quad x \in \lambda C\}, \quad x \in Z. \quad (1.3)$$

Clearly, $\{p_C \mid C \in \mathcal{C}_0\}$ define the topology of Z . Moreover, p_C is increasing. To see this, for any $\epsilon > 0$, there exists $p_C(x) \leq \lambda_\epsilon \leq p_C(x) + \epsilon$ such that $x \in \lambda_\epsilon C$. Since C is balanced, $\lambda_\epsilon C \subset (p_C(x) + \epsilon)C$, so $x \in (p_C(x) + \epsilon)C$. As C is also solid, if $0 \leq y \leq x$, then we have $y \in (p_C(x) + \epsilon)C$, from which we obtain $p_C(y) \leq p_C(x) + \epsilon$. Since $\epsilon > 0$ was arbitrary, we have that $p_C(y) \leq p_C(x)$.

Conversely, given any increasing continuous seminorm p on Z , the set

$$C_p := \{x \in Z \mid p(x) < 1\}$$

is absolutely convex. Moreover, it is solid since, if $x \in C_p$ with $0 \leq y \leq x$, then $p(y) \leq p(x) < 1$, so $y \in C_p$.

Given a family \mathcal{C}_0 of absolutely convex and solid neighbourhoods of the origin that generates the topology of Z , we denote by $S_{\mathcal{C}_0}(Z) = \{p_C \mid C \in \mathcal{C}_0\}$, where p_C is the Minkowski seminorm associated to C as in (1.3). The collection of all continuous increasing seminorms on Z is denoted by $S(Z)$. As a consequence of Remark 1.1, $S(Z)$ is in bijective correspondence with the family \mathcal{C} of all open, absolutely convex and solid neighbourhoods of the origin. Note that $S(Z)$ is a directed set: given $p, q \in S(Z)$, consider $r := p + q$. In fact, $S(Z)$ is a cone, i.e. it is closed under all finite linear combinations with positive coefficients.

Z is called an *admissible space*, cf. [27], if, in addition to the axioms (a1)–(a4),

- (a5) The cone Z_+ is *closed*, with respect to the specified topology of Z .
- (a6) The topology on Z is complete.

Finally, if, in addition to the axioms (a1)–(a6), the space Z satisfies also the following axiom:

- (a7) With respect to the specified partial ordering, any bounded monotone sequence is convergent.

then Z is called a *strongly admissible space* [27].

Examples 1.2. (1) Any C^* -algebra \mathcal{A} is an admissible space, as well as any closed \ast -subspace \mathcal{S} of a C^* -algebra \mathcal{A} , with the positive cone $\mathcal{S}^+ = \mathcal{A}^+ \cap \mathcal{S}$ and all other operations (addition, multiplication with scalars, and involution) inherited from \mathcal{A} .

(2) Any pre- C^* -algebra is a topologically ordered \ast -space. Any \ast -subspace \mathcal{S} of a pre- C^* -algebra \mathcal{A} is a topologically ordered \ast -space, with the positive cone $\mathcal{S}^+ = \mathcal{A}^+ \cap \mathcal{S}$ and all other operations inherited from \mathcal{A} .

(3) Any locally C^* -algebra, cf. [19, 39], (definition is recalled in Sect. 3.1) is an admissible space. In particular, any closed \ast -subspace \mathcal{S} of a locally C^* -algebra \mathcal{A} , with the cone $\mathcal{S}_+ = \mathcal{A}^+ \cap \mathcal{S}$ and all other operations inherited from \mathcal{A} , is an admissible space.

(4) Any locally pre- C^* -algebra is a topologically ordered $*$ -space. Any $*$ -subspace \mathcal{S} of a locally pre- C^* -algebra is a topologically ordered $*$ -space, with $\mathcal{S}^+ = \mathcal{A}^+ \cap \mathcal{S}$ and all other operations inherited from \mathcal{A} .

(5) Let \mathcal{H} be an infinite dimensional separable Hilbert space and let \mathcal{C}_1 be the trace-class ideal, that is, the collection of all linear bounded operators A on \mathcal{H} such that $\text{tr}(|A|) < \infty$. \mathcal{C}_1 is a $*$ -ideal of $\mathcal{B}(\mathcal{H})$ and complete under the norm $\|A\|_1 = \text{tr}(|A|)$. Positive elements in \mathcal{C}_1 are defined in the sense of positivity in $\mathcal{B}(\mathcal{H})$. In addition, the norm $\|\cdot\|_1$ is increasing, since $0 \leq A \leq B$ implies $\text{tr}(A) \leq \text{tr}(B)$, hence \mathcal{C}_1 is a normed admissible space.

(6) Let V be a complex Banach space and let V' be its conjugate dual space. On the vector space $\mathcal{B}(V, V')$ of all bounded linear operators $T: V \rightarrow V'$, a natural notion of positive operator can be defined: T is *positive* if $(Tv)(v) \geq 0$ for all $v \in V$. Let $\mathcal{B}(V, V')^+$ be the collection of all positive operators and note that it is a strict cone that is closed with respect to the weak operator topology. The involution $*$ in $\mathcal{B}(V, V')$ is defined in the following way: for any $T \in \mathcal{B}(V, V')$, $T^* = T'|V$, that is, the restriction to V of the dual operator $T': V'' \rightarrow V'$. With respect to the weak operator topology, the cone $\mathcal{B}(V, V')^+$, and the involution $*$ just defined, $\mathcal{B}(V, V')$ becomes an admissible space. See Weron [50], as well as Gaspar and Gaspar [14].

(7) Let X be a nonempty set and denote by $\mathcal{K}(X)$ the collection of all complex valued kernels on X , that is, $\mathcal{K}(X) = \{k \mid k: X \times X \rightarrow \mathbb{C}\}$, considered as a complex vector space with the operations of addition and multiplication of scalars defined elementwise. An involution $*$ can be defined on $\mathcal{K}(X)$ as follows: $k^*(x, y) = \overline{k(y, x)}$, for all $x, y \in X$ and all $k \in \mathcal{K}(X)$. The cone $\mathcal{K}(X)^+$ consists of all *positive semidefinite* kernels, that is, those kernels $k \in \mathcal{K}(X)$ with the property that, for any $n \in \mathbb{N}$ and any $x_1, \dots, x_n \in X$, the complex matrix $[k(x_i, x_j)]_{i,j=1}^n$ is positive semidefinite. Then $\mathcal{K}(X)$ is an ordered $*$ -space.

Further, consider the set $\mathcal{P}_0(X)$ of all finite subsets of X . For each $A \in \mathcal{P}_0(X)$, let $A = \{x_1, \dots, x_n\}$ and define the seminorm $p_A: \mathcal{K}(X) \rightarrow \mathbb{R}$ by

$$p_A(k) = \|[k(x_i, x_j)]_{i,j=1}^n\|, \quad k \in \mathcal{K}(X),$$

the norm being the operator norm of the $n \times n$ matrix $[k(x_i, x_j)]_{i,j=1}^n$. Since a reordering of the elements x_1, \dots, x_n produces a unitary equivalent matrix, the definition of p_A does not depend on which order of the elements of the set A is considered. It is easy to see that each seminorm p_A is increasing and that, with the locally convex topology defined by $\{p_A\}_{A \in \mathcal{P}_0(X)}$, $\mathcal{K}(X)$ is an admissible space.

(8) Let \mathcal{A} and \mathcal{B} be two C^* -algebras. Recall that, in this case, the specified strict cone \mathcal{A}^+ linearly generates \mathcal{A} . On $\mathcal{L}(\mathcal{A}, \mathcal{B})$, the vector space of all linear maps $\varphi: \mathcal{A} \rightarrow \mathcal{B}$, we define an involution: $\varphi^*(a) = \varphi(a^*)^*$, for all $a \in \mathcal{A}$. A linear map $\varphi \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ is called *positive* if $\varphi(\mathcal{A}^+) \subseteq \mathcal{B}^+$. It is easy to see that $\mathcal{L}(\mathcal{A}, \mathcal{B})^+$, the collection of all positive maps from $\mathcal{L}(\mathcal{A}, \mathcal{B})$, is a cone, and that it is strict because \mathcal{A}^+ linearly generates \mathcal{A} . In addition, any

$\varphi \in \mathcal{L}(\mathcal{A}, \mathcal{B})^+$ is selfadjoint, again due to the fact that \mathcal{A}^+ linearly generates \mathcal{A} . Consequently, $\mathcal{L}(\mathcal{A}, \mathcal{B})$ has a natural structure of ordered \ast -space.

On $\mathcal{L}(\mathcal{A}, \mathcal{B})$ we consider the collection of seminorms $\{p_a\}_{a \in \mathcal{A}^+}$ defined by $p_a(\varphi) = \|\varphi(a)\|$, for all $\varphi \in \mathcal{L}(\mathcal{A}, \mathcal{B})$. All these seminorms are increasing and the topology generated by $\{p_a\}_{a \in \mathcal{A}^+}$ is Hausdorff and complete. Consequently, $\mathcal{L}(\mathcal{A}, \mathcal{B})$ is an admissible space.

With a slightly more involved topology, it can be shown that the same conclusion holds for the case when \mathcal{A} and \mathcal{B} are locally C^\ast -algebras.

(9) Let $\{Z_\alpha\}_{\alpha \in A}$ be a family of admissible spaces such that, for each $\alpha \in A$, Z_α^+ is the specified strict cone of positive elements in Z_α , and the topology of Z_α is generated by the family of increasing seminorms $\{p_{\alpha,j}\}_{j \in \mathcal{J}_\alpha}$. On the product space $Z = \prod_{\alpha \in A} Z_\alpha$ let $Z^+ = \prod_{\alpha \in A} Z_\alpha^+$ and observe that Z^+ is a strict cone. Letting the involution \ast on Z be defined elementwise, it follows that Z^+ consists on selfadjoint elements only. In this way, Z is an ordered \ast -space.

For each $\beta \in A$ and each $j \in \mathcal{J}_\beta$, let

$$q_j^{(\beta)}((z_\alpha)_{\alpha \in A}) = p_j^{(\beta)}(z_\beta), \quad (z_\alpha)_{\alpha \in A} \in Z. \quad (1.4)$$

It is easy to show that $q_j^{(\beta)}$ is an increasing seminorm on Z and that, with the topology generated by the family of increasing seminorms $\{q_j^{(\beta)}\}_{\substack{\beta \in A \\ j \in \mathcal{J}_\beta}}$, Z becomes an admissible space.

1.3. Vector Hilbert Spaces and Their Linear Operators

If Z is a topologically ordered \ast -space, any VE-space \mathcal{E} over Z can be made in a natural way into a Hausdorff locally convex space by considering the topology $\tau_\mathcal{E}$, the weakest topology on \mathcal{E} that makes the quadratic map $Q : E \ni h \mapsto [h, h] \in Z$ continuous. More precisely, letting \mathcal{C}_0 be a collection of open, absolutely convex and solid neighbourhoods of the origin in Z , that generates the topology of Z as in axiom (a5), the collection of sets

$$D_C = \{x \in \mathcal{E} \mid [x, x] \in C\}, \quad C \in \mathcal{C}_0, \quad (1.5)$$

is a topological base of open and absolutely convex neighbourhoods of the origin of \mathcal{E} that linearly generates $\tau_\mathcal{E}$, cf. [27]. We are interested in explicitly defining the topology $\tau_\mathcal{E}$ in terms of seminorms.

Lemma 1.3. *Let Z be a topologically ordered \ast -space and \mathcal{E} a VE-space over Z .*

- (1) $(\mathcal{E}; \tau_\mathcal{E})$ is a Hausdorff locally convex space.
- (2) For every continuous increasing seminorm p on Z

$$\tilde{p}(h) = p([h, h])^{1/2}, \quad h \in \mathcal{E}, \quad (1.6)$$

is a continuous seminorm on $(\mathcal{E}; \tau_\mathcal{E})$.

- (3) Let $\{p_j\}_{j \in \mathcal{J}}$ be a family of increasing seminorms defining the topology of Z as in axiom (a4'). Then, with the definition (1.6), the family of seminorms $\{\tilde{p}_j\}_{j \in \mathcal{J}}$ generates $\tau_\mathcal{E}$.
- (4) The gramian $[\cdot, \cdot] : \mathcal{E} \times \mathcal{E} \rightarrow Z$ is jointly continuous.

Statements (1) and (4) are proven in Theorem 1 in [27]. Statement (2) is claimed in Proposition 1.1.1 in [8] but, unfortunately, the proof provided there is irremediably flawed, so we provide full details.

Proof of Lemma 1.3. We first prove that, if p is a continuous and increasing seminorm on Z , \tilde{p} is a quasi seminorm on \mathcal{E} . Indeed, for any $\lambda \in \mathbb{C}$ and any $h \in \mathcal{E}$

$$\tilde{p}(\lambda h) = p([\lambda h, \lambda h])^{1/2} = |\lambda|p([h, h])^{1/2} = |\lambda|\tilde{p}(h),$$

hence \tilde{p} is positively homogeneous.

For arbitrary $h, k \in \mathcal{E}$ we have

$$[h \pm k, h \pm k] = [h, h] + [k, k] \pm [h, k] \pm [k, h] \geq 0,$$

in particular,

$$[h, k] + [k, h] \leq [h, h] + [k, k]. \quad (1.7)$$

and

$$0 \leq [h \pm k, h \pm k] \leq [h - k, h - k] + [h + k, h + k] = 2([h, h] + [k, k]). \quad (1.8)$$

Since p is increasing, it follows that

$$\begin{aligned} \tilde{p}(h + k) &= (p([h + k, h + k]))^{1/2} \leq \sqrt{2}(p([h, h]) + p([k, k]))^{1/2} \\ &\leq \sqrt{2}(p([h, h])^{1/2} + p([k, k])^{1/2}) = \sqrt{2}(\tilde{p}(h) + \tilde{p}(k)). \end{aligned}$$

This concludes the proof that \tilde{p} is a quasi seminorm.

Also, since \tilde{p} is the composition of the square root function $\sqrt{\cdot}$, a homeomorphism of \mathbb{R}_+ onto itself, with p and the quadratic map $\mathcal{E} \ni x \mapsto [x, x] \in Z$, clearly \tilde{p} is continuous with respect to the topology $\tau_{\mathcal{E}}$. This observation shows that, if $\{p_j\}_{j \in \mathcal{J}}$ is a family of increasing seminorms generating the topology of Z , then $\{\tilde{p}_j\}_{j \in \mathcal{J}}$ is a family of quasi seminorms generating $\tau_{\mathcal{E}}$. In particular, $(\mathcal{E}; \tau_{\mathcal{E}})$ is a topological vector space.

We prove now that \tilde{p} satisfies the triangle inequality, hence it is a seminorm. To see this, consider the unit quasi ball

$$U_{\tilde{p}} = \{h \in \mathcal{E} \mid \tilde{p}(h) < 1\}.$$

Since \tilde{p} is continuous, $U_{\tilde{p}}$ is open, hence absorbing for each of its points. Since \tilde{p} is positively homogeneous, $U_{\tilde{p}}$ is balanced. We prove that $U_{\tilde{p}}$ is convex as well. Let $h, k \in U_{\tilde{p}}$ and $0 \leq t \leq 1$ arbitrary. Then,

$$\begin{aligned} 0 &\leq [th + (1 - t)k, th + (1 - t)k] \\ &= t^2[h, h] + (1 - t)^2[k, k] + t(1 - t)([h, k] + [k, h]) \end{aligned}$$

and then using (1.7),

$$\begin{aligned} &\leq t^2[h, h] + (1 - t)^2[k, k] + t(1 - t)([h, h] + [k, k]) \\ &= t[h, h] + (1 - t)[k, k], \end{aligned}$$

hence, since p is increasing, it follows

$$\begin{aligned}\tilde{p}(th + (1-t)k) &= p([th + (1-t)k, th + (1-t)k])^{1/2} \\ &\leq (tp([h, h]) + (1-t)p([k, k]))^{1/2} < 1,\end{aligned}$$

hence $th + (1-t)k \in U_{\tilde{p}}$.

It is a routine exercise to show that \tilde{p} is the gauge of $U_{\tilde{p}}$

$$\tilde{p}(h) = \inf\{t > 0 \mid h \in tU_{\tilde{p}}\},$$

hence, by Proposition IV.1.14 in [10], it follows that \tilde{p} is a seminorm.

Statement (4) is a consequence of the polarisation formula (1.1). \square

From now on, any time we have a VE-space \mathcal{E} over a topologically ordered \ast -space Z , we consider on \mathcal{E} the topology $\tau_{\mathcal{E}}$ defined as in Lemma 1.3. With respect to this topology, we call \mathcal{E} a *topological VE-space* over Z . Denote

$$S(\mathcal{E}) := S_{\mathcal{C}}(\mathcal{E}) = \{\tilde{p}_C \mid C \in \mathcal{C}\}, \quad (1.9)$$

where \mathcal{C} is the collection of all open, absolutely convex and solid neighbourhoods of the origin of Z as in (1.5). Note that $S(\mathcal{E})$ is directed, more precisely, given $\tilde{p}_C, \tilde{p}_D \in S(\mathcal{E})$ consider $S(Z) \ni q := p_C + p_D$ and define $\tilde{q}(h) := q([h, h]_{\mathcal{E}})^{1/2}$. Also note that $S(\mathcal{E})$ is closed under positive scalar multiplication.

If Z is an admissible space and \mathcal{E} is a topological VE-space whose locally convex topology is complete, then \mathcal{E} is called a *VH-space* (Vector Hilbert space). Any topological VE-space \mathcal{E} on an admissible space Z can be embedded as a dense subspace of a VH-space \mathcal{H} over Z , uniquely determined up to an isomorphism, cf. Theorem 2 in [27].

Examples 1.4. (1) Any Hilbert module \mathcal{H} over a C^* -algebra \mathcal{A} , e.g. see [26, 31], can be viewed as a VH-space \mathcal{H} over the admissible space \mathcal{A} , see Example 1.2.(1). In particular, any closed subspace \mathcal{S} of \mathcal{H} is a VH-space over the admissible space \mathcal{A} .

(2) Any Hilbert module \mathcal{H} over a locally C^* -algebra \mathcal{A} , e.g. see [19, 39], can be viewed as a VH-space \mathcal{H} over the admissible space \mathcal{A} , see Example 1.2.(2). In particular, any closed subspace \mathcal{S} of \mathcal{H} is a VH-space over the admissible space \mathcal{A} .

(3) With notation as in Example 1.2.(5), consider \mathcal{C}_2 the ideal of Hilbert-Schmidt operators on \mathcal{H} . Then $[A, B] = A^*B$, for all $A, B \in \mathcal{C}_2$, is a gramian with values in the admissible space \mathcal{C}_1 with respect to which \mathcal{C}_2 becomes a VH-space. Observe that, since \mathcal{C}_1 is a normed admissible space, by Lemma 1.3 it follows that \mathcal{C}_2 is a normed VH-space, with norm $\|A\|_2 = \text{tr}(|A|^2)^{1/2}$, for all $A \in \mathcal{C}_2$. More abstract versions of this example have been considered by Saworotnow in [42].

(4) Let $\{\mathcal{E}_{\alpha}\}_{\alpha \in A}$ be a family of VH-spaces such that, for each $\alpha \in A$, \mathcal{E}_{α} is a VH-space over the admissible space Z_{α} . As in Example 1.2, consider the admissible space $Z = \prod_{\alpha \in A} Z_{\alpha}$ and the vector space $\mathcal{E} = \prod_{\alpha \in A} \mathcal{E}_{\alpha}$ on which we define

$$[(e_{\alpha})_{\alpha \in A}, (f_{\alpha})_{\alpha \in A}] = ([e_{\alpha}, f_{\alpha}])_{\alpha \in A} \in Z, \quad (e_{\alpha})_{\alpha \in A}, (f_{\alpha})_{\alpha \in A} \in \mathcal{E}.$$

Then \mathcal{E} is a VE-space over Z . On Z consider the topology generated by the family of increasing seminorms $\{q_j^{(\beta)}\}_{\substack{\beta \in A \\ j \in \mathcal{J}_\beta}}$ defined at (1.4), with respect to which Z becomes an admissible space. For each $\beta \in A$ and each $j \in \mathcal{J}_\beta$, in view of Lemma 1.3, consider the seminorm

$$\tilde{q}_j^{(\beta)}((e_\alpha)_{\alpha \in A}) = p_j^{(\beta)}([e_\alpha, e_\alpha])^{1/2}, \quad (e_\alpha)_{\alpha \in A} \in \mathcal{E}.$$

The family of seminorms $\{\tilde{q}_j^{(\beta)}\}_{\substack{\beta \in A \\ j \in \mathcal{J}_\beta}}$ generates on \mathcal{E} the topology with respect to which it is a VH-space over Z .

(5) Let Z be an admissible space and $\mathcal{E}_1, \dots, \mathcal{E}_n$ VH-spaces over Z . On $\mathcal{E} = \prod_{j=1}^n \mathcal{E}_j$ define

$$[(e_j)_{j=1}^n, (f_j)_{j=1}^n]_{\mathcal{E}} = \sum_{j=1}^n [e_j, f_j]_{\mathcal{E}_j}, \quad (e_j)_{j=1}^n, (f_j)_{j=1}^n \in \mathcal{E}, \quad (1.10)$$

and observe that $(\mathcal{E}; [\cdot, \cdot]_{\mathcal{E}})$ is a VE-space over Z . In addition, for any $p \in S(Z)$ letting $\tilde{p}: \mathcal{E} \rightarrow \mathbb{R}_+$ be defined as in (1.6), $\tilde{p}(e) = p([e, e]_{\mathcal{E}})^{1/2}$, for all $e \in \mathcal{E}$, it is easy to see that \mathcal{E} is a VH-space over Z . It is clear that we can denote this VH-space by $\bigoplus_{j=1}^n \mathcal{E}_j$ and call it the *direct sum VH-space* of the VH-spaces $\mathcal{E}_1, \dots, \mathcal{E}_n$.

(6) Let \mathcal{H} be a Hilbert space and \mathcal{E} a VH-space over the admissible space Z . On the algebraic tensor product $\mathcal{H} \otimes \mathcal{E}$ define a gramian by

$$[h \otimes e, l \otimes f]_{\mathcal{H} \otimes \mathcal{E}} = \langle h, l \rangle_{\mathcal{H}} [e, f]_{\mathcal{E}} \in Z, \quad h, l \in \mathcal{H}, e, f \in \mathcal{E},$$

and then extend it to $\mathcal{H} \otimes \mathcal{E}$ by linearity. It can be proven that, in this way, $\mathcal{H} \otimes \mathcal{E}$ is a VE-space over Z . Since Z is an admissible space, $\mathcal{H} \otimes \mathcal{E}$ can be topologised as in Lemma 1.3 and then completed to a VH-space $\mathcal{H} \widehat{\otimes} \mathcal{E}$ over Z .

If $\mathcal{H} = \mathbb{C}^n$ for some $n \in \mathbb{N}$ then, with notation as in item (5), it is clear that $\mathbb{C}^n \otimes \mathcal{E}$ is isomorphic with $\bigoplus_{j=1}^n \mathcal{E}_j$, with $\mathcal{E}_j = \mathcal{E}$ for all $j = 1, \dots, n$.

Remark 1.5. If \mathcal{E} and \mathcal{F} are two VH-spaces over the same admissible space Z , by $\mathcal{L}_c(\mathcal{E}, \mathcal{F})$ we denote the space of all continuous operators from \mathcal{E} to \mathcal{F} . Let \mathcal{C}_0 be a system of open and absolutely convex neighbourhoods of the origin defining the topology of Z . Since $S(\mathcal{E})$ is directed and it is closed under positive scalar multiplication, the continuity of a linear operator $T \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ is equivalent with: for any $p \in S_{\mathcal{C}_0}(\mathcal{F})$, there exists $q \in S(\mathcal{E})$ and a constant $c \geq 0$ such that $p(Th) \leq cq(h)$ for all $h \in \mathcal{E}$. We will use this fact frequently in this article.

For \mathcal{E} and \mathcal{F} two VH-spaces over the same admissible space Z , we denote by $\mathcal{L}_c^*(\mathcal{E}, \mathcal{F})$ the subspace of $\mathcal{L}^*(\mathcal{E}, \mathcal{F})$ consisting of all continuous and continuously adjointable operators. Note that $\mathcal{L}_c^*(\mathcal{E}) = \mathcal{L}_c^*(\mathcal{E}, \mathcal{E})$ is an ordered *-subalgebra of $\mathcal{L}^*(\mathcal{E})$.

Lemma 1.6. *Let \mathcal{H} be a topological VE-space over the topologically ordered *-space Z . Let $T \in \mathcal{L}_c^*(\mathcal{H})$ be a positive operator and $p \in S(Z)$. Then there exist $q \in S(Z)$ and $c(T, p) \geq 0$ such that*

$$p([Th, h]_{\mathcal{H}}) \leq c(T, p) q([h, h]_{\mathcal{H}}), \quad h \in \mathcal{H}.$$

Proof. To a certain extent, we use an argument in [27]. From

$$[Th - h, Th - h]_{\mathcal{H}} = [Th, Th]_{\mathcal{H}} - 2[Th, h]_{\mathcal{H}} + [h, h]_{\mathcal{H}} \geq 0,$$

and taking into account that T is positive, we obtain

$$0 \leq 2[Th, h]_{\mathcal{H}} \leq [Th, Th]_{\mathcal{H}} + [h, h]_{\mathcal{H}}.$$

From here, for any seminorm $p \in S(Z)$, using that p is increasing, T is continuous, and Remark 1.5, it follows that there exist $q \in S(Z)$ and a constant $c(T, p) \geq 0$ such that, for all $h \in \mathcal{H}$ we have

$$p([Th, h]_{\mathcal{H}}) \leq \frac{1}{2}(p([Th, Th]_{\mathcal{H}}) + p([h, h]_{\mathcal{H}})) \leq c(T, p) q([h, h]_{\mathcal{H}}). \quad \square$$

Let \mathcal{H}_1 and \mathcal{H}_2 be two VH spaces over the same admissible space Z , with their family of seminorms $S(\mathcal{H}_1) = \{\tilde{p}_{\mathcal{H}_1} \mid p \in S(Z)\}$ and, respectively, $S(\mathcal{H}_2) = \{\tilde{p}_{\mathcal{H}_2} \mid p \in S(Z)\}$. Then the *strict topology* on $\mathcal{L}^*(\mathcal{H}_1, \mathcal{H}_2)$ is defined by the seminorms $T \mapsto \tilde{p}_{\mathcal{H}_2}(T\xi)$ for $\tilde{p}_{\mathcal{H}_2} \in S(\mathcal{H}_2)$, $\xi \in \mathcal{H}_1$ and $T \mapsto \tilde{p}_{\mathcal{H}_1}(T^*\eta)$ for $\tilde{p}_{\mathcal{H}_1} \in S(\mathcal{H}_1)$, $\eta \in \mathcal{H}_2$, for all $p \in S(Z)$ with the seminorms $\tilde{p}_{\mathcal{H}_1}$ on \mathcal{H}_1 and $\tilde{p}_{\mathcal{H}_2}$ on \mathcal{H}_2 defined at (1.6). Equivalently, we can use all $p \in S_{\mathcal{C}_0}(Z)$, where \mathcal{C}_0 is a collection of open, absolutely convex, and solid neighbourhoods of 0 and that generates the topology of Z , as in Sect. 1.2.

Lemma 1.7. *Let \mathcal{H}_1 and \mathcal{H}_2 be two VH-spaces over the same admissible space Z . Then $\mathcal{L}^*(\mathcal{H}_1, \mathcal{H}_2)$ with the strict topology is complete.*

Proof. Let $(T_i)_i$ be a Cauchy net in $\mathcal{L}^*(\mathcal{H}_1, \mathcal{H}_2)$ with respect to the strict topology. Then, $(T_i\xi)_i$ is a Cauchy net in \mathcal{H}_2 for all $\xi \in \mathcal{H}_1$ and $(T_i^*\eta)_i$ is a Cauchy net in \mathcal{H}_1 for all $\eta \in \mathcal{H}_2$, since they are Cauchy with respect to all seminorms in $S(\mathcal{H}_2)$ and $S(\mathcal{H}_1)$, respectively. Since \mathcal{H}_1 and \mathcal{H}_2 are complete, we have that $T_i\xi \xrightarrow{i} x_\xi$ and $T_i^*\eta \xrightarrow{i} y_\eta$ for some $x_\xi \in \mathcal{H}_2$ and $y_\eta \in \mathcal{H}_1$.

Define the linear operators $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ by $T\xi = x_\xi$ and $R : \mathcal{H}_2 \rightarrow \mathcal{H}_1$ by $R\eta = y_\eta$. Then, by the continuity of the gramians, see Lemma 1.3, we have

$$[T\xi, \eta]_{\mathcal{H}_2} = \lim_i [T_i\xi, \eta]_{\mathcal{H}_2} = \lim_i [\xi, T_i^*\eta]_{\mathcal{H}_1} = [\xi, R\eta]_{\mathcal{H}_1}.$$

Therefore, T is adjointable with $T^* = R$, and $T_i \xrightarrow{i} T$ in the strict topology of $\mathcal{L}^*(\mathcal{H}_1, \mathcal{H}_2)$. \square

A subspace \mathcal{M} of a VH-space \mathcal{H} is *orthocomplemented*, or *accessible* [27], if every element $h \in \mathcal{H}$ can be written as $h = g + k$ where g is in \mathcal{M} and k is such that $[l, k] = 0$ for all $l \in \mathcal{M}$, that is, k is in the *orthogonal companion* \mathcal{M}^\perp of \mathcal{M} . Observe that if such a decomposition exists it is unique and hence the *orthogonal projection* $P_{\mathcal{M}}$ onto \mathcal{M} can be defined by $P_{\mathcal{M}}h = g$. Any orthogonal projection P is selfadjoint and idempotent, in particular we have $[Ph, k] = [Ph, Pk]$ for all $j, k \in \mathcal{H}$, hence P is positive and contractive, in the sense $[Ph, Ph] \leq [h, h]$ for all $h \in \mathcal{H}$, hence P is continuous. Conversely, any selfadjoint idempotent operator is an orthogonal projection onto its range subspace. Any orthocomplemented subspace is closed.

2. Positive Semidefinite Kernels with Values Adjointable Operators

Our main result is Theorem 2.10 that provides necessary and sufficient conditions for a positive semidefinite kernel with values adjointable operators and invariant under an action of a $*$ -semigroup to give rise to a $*$ -representation of the given $*$ -semigroup on a VH-space. We first provide some preliminary results on positive semidefinite kernels with values adjointable operators in a VE-space, cf. [5].

2.1. Kernels with Values Adjointable Operators

Let X be a nonempty set and let \mathcal{H} be a VE-space over the ordered $*$ -space Z . A map $\mathbf{k}: X \times X \rightarrow \mathcal{L}(\mathcal{H})$ is called a *kernel* on X and valued in $\mathcal{L}(\mathcal{H})$. In case the kernel \mathbf{k} has all its values in $\mathcal{L}^*(\mathcal{H})$, an *adjoint* kernel $\mathbf{k}^*: X \times X \rightarrow \mathcal{L}^*(\mathcal{H})$ can be associated by $\mathbf{k}^*(x, y) = \mathbf{k}(y, x)^*$ for all $x, y \in X$. The kernel \mathbf{k} is called *Hermitian* if $\mathbf{k}^* = \mathbf{k}$.

Let $\mathcal{F} = \mathcal{F}(X; \mathcal{H})$ denote the complex vector space of all functions $f: X \rightarrow \mathcal{H}$ and let $\mathcal{F}_0 = \mathcal{F}_0(X; \mathcal{H})$ be its subspace of those functions having finite support. A pairing $[\cdot, \cdot]_{\mathcal{F}_0}: \mathcal{F}_0 \times \mathcal{F}_0 \rightarrow Z$ can be defined by

$$[g, h]_{\mathcal{F}_0} = \sum_{y \in X} [g(y), h(y)]_{\mathcal{H}}, \quad g, h \in \mathcal{F}_0. \quad (2.1)$$

This pairing is clearly a Z -gramian on \mathcal{F}_0 , hence $(\mathcal{F}_0; [\cdot, \cdot]_{\mathcal{F}_0})$ is a VE-space.

Another pairing $[\cdot, \cdot]_{\mathbf{k}}$ can be defined on \mathcal{F}_0 by

$$[g, h]_{\mathbf{k}} = \sum_{x, y \in X} [\mathbf{k}(y, x)g(x), h(y)]_{\mathcal{H}}, \quad g, h \in \mathcal{F}_0. \quad (2.2)$$

In general, the pairing $[\cdot, \cdot]_{\mathbf{k}}$ is linear in the second variable and conjugate linear in the first variable. If, in addition, $\mathbf{k} = \mathbf{k}^*$ then the pairing $[\cdot, \cdot]_{\mathbf{k}}$ is Hermitian as well, that is,

$$[g, h]_{\mathbf{k}} = [h, g]_{\mathbf{k}}^*, \quad g, h \in \mathcal{F}_0.$$

A *convolution operator* $K: \mathcal{F}_0 \rightarrow \mathcal{F}$ can be associated to the kernel \mathbf{k} by

$$(Kg)(y) = \sum_{x \in X} \mathbf{k}(y, x)g(x), \quad g \in \mathcal{F}_0, \quad (2.3)$$

and it is easy to see that K is a linear operator. There is a natural relation between the pairing $[\cdot, \cdot]_{\mathbf{k}}$ and the convolution operator K given by

$$[g, h]_{\mathbf{k}} = [Kg, h]_{\mathcal{F}_0}, \quad g, h \in \mathcal{F}_0.$$

Given $n \in \mathbb{N}$, the kernel k is called *n-positive* if for any $x_1, x_2, \dots, x_n \in X$ and any $h_1, h_2, \dots, h_n \in \mathcal{H}$ we have

$$\sum_{i, j=1}^n [\mathbf{k}(x_i, x_j)h_j, h_i]_{\mathcal{H}} \geq 0. \quad (2.4)$$

The kernel k is called *positive semidefinite* (or *of positive type*) if it is n -positive for all natural numbers n .

Lemma 2.1 (Lemma 3.1 from [17]). *Assume that the kernel $\mathbf{k}: X \times X \rightarrow \mathcal{L}^*(\mathcal{H})$ is 2-positive. Then:*

- (1) \mathbf{k} is Hermitian.
- (2) If, for some $x \in X$, we have $\mathbf{k}(x, x) = 0$, then $\mathbf{k}(x, y) = 0$ for all $y \in X$.
- (3) There exists a unique decomposition $X = X_0 \cup X_1$, such that $X_0 \cap X_1 = \emptyset$, $\mathbf{k}(x, y) = 0$ for all $x, y \in X_0$ and $\mathbf{k}(x, x) \neq 0$ for all $x \in X_1$.

Given an $\mathcal{L}^*(\mathcal{H})$ -valued kernel \mathbf{k} on a nonempty set X , for some VE-space \mathcal{H} on an ordered \ast -space Z , a *VE-space linearisation* or, equivalently, a *VE-space Kolmogorov decomposition* of \mathbf{k} is, by definition, a pair $(\mathcal{K}; V)$, subject to the following conditions:

- (vel1) \mathcal{K} is a VE-space over the same ordered \ast -space Z .
- (vel2) $V: X \rightarrow \mathcal{L}^*(\mathcal{H}, \mathcal{K})$ satisfies $\mathbf{k}(x, y) = V(x) \ast V(y)$ for all $x, y \in X$.

The VE-space linearisation $(\mathcal{K}; V)$ is called *minimal* if

- (vel3) $\text{Lin } V(X)\mathcal{H} = \mathcal{K}$.

Two VE-space linearisations $(V; \mathcal{K})$ and $(V'; \mathcal{K}')$ of the same kernel \mathbf{k} are called *unitary equivalent* if there exists a VE-space isomorphism $U: \mathcal{K} \rightarrow \mathcal{K}'$ such that $UV(x) = V'(x)$ for all $x \in X$.

The uniqueness of a minimal VE-space linearisation $(\mathcal{K}; V)$ of a positive semidefinite kernel \mathbf{k} , modulo unitary equivalence, follows in the usual way, see [5].

Let \mathcal{H} be a VE-space over the ordered \ast -space Z , and let X be a nonempty set. A VE-space \mathcal{R} , over the same ordered \ast -space Z , is called an *\mathcal{H} -reproducing kernel VE-space on X* if there exists a Hermitian kernel $\mathbf{k}: X \times X \rightarrow \mathcal{L}^*(\mathcal{H})$ such that the following axioms are satisfied:

- (rk1) \mathcal{R} is a subspace of $\mathcal{F}(X; \mathcal{H})$, with all algebraic operations.
- (rk2) For all $x \in X$ and all $h \in \mathcal{H}$, the \mathcal{H} -valued function $\mathbf{k}_x h = \mathbf{k}(\cdot, x)h \in \mathcal{R}$.
- (rk3) For all $f \in \mathcal{R}$ we have $[f(x), h]_{\mathcal{H}} = [f, \mathbf{k}_x h]_{\mathcal{R}}$, for all $x \in X$ and $h \in \mathcal{H}$.

As a consequence of (rk2), $\text{Lin}\{\mathbf{k}_x h \mid x \in X, h \in \mathcal{H}\} \subseteq \mathcal{R}$. The reproducing kernel VE-space \mathcal{R} is called *minimal* if the following property holds as well:

- (rk4) $\text{Lin}\{\mathbf{k}_x h \mid x \in X, h \in \mathcal{H}\} = \mathcal{R}$.

Observe that if \mathcal{R} is an \mathcal{H} -reproducing kernel VE-space on X with kernel \mathbf{k} , then \mathbf{k} is positive semidefinite and uniquely determined by \mathcal{R} hence, we can talk about *the \mathcal{H} -reproducing kernel \mathbf{k} corresponding to \mathcal{R}* . On the other hand, a minimal reproducing kernel VE-space \mathcal{R} is uniquely determined by its reproducing kernel \mathbf{k} .

Letting \mathcal{H} be a VE-space over an ordered \ast -space Z , for X a nonempty set, an *evaluation operator* $E_x: \mathcal{F}(X; \mathcal{H}) \rightarrow \mathcal{H}$ can be defined for each $x \in X$ by letting $E_x f = f(x)$ for all $f \in \mathcal{F}(X; \mathcal{H})$. Clearly, E_x is linear. If $\mathcal{R} \subseteq \mathcal{F}(X; \mathcal{H})$, with all algebraic operations, is a VE-space over Z , then \mathcal{R} is an \mathcal{H} -reproducing kernel VE-space if and only if, for all $x \in X$, the restriction of the evaluation operator E_x to \mathcal{R} is adjointable as a linear operator $\mathcal{R} \rightarrow \mathcal{H}$, e.g. see [5].

Proposition 2.2 (Proposition 2.4 in [5]). *Let X be a nonempty set, \mathcal{H} a VE-space over an ordered $*$ -space Z , and let $\mathbf{k}: X \times X \rightarrow \mathcal{L}^*(\mathcal{H})$ be a Hermitian kernel.*

(1) *Any \mathcal{H} -reproducing kernel VE-space \mathcal{R} with kernel \mathbf{k} is a VE-space linearisation $(\mathcal{R}; V)$ of \mathbf{k} , with $V(x) = \mathbf{k}_x$ for all $x \in X$.*

(2) *For any minimal VE-space linearisation $(\mathcal{K}; V)$ of \mathbf{k} , letting*

$$\mathcal{R} = \{V(\cdot)^* f \mid f \in \mathcal{K}\}, \quad (2.5)$$

we obtain an \mathcal{H} -reproducing kernel VE-space with reproducing kernel \mathbf{k} .

Let a (multiplicative) semigroup Γ act on X , denoted by $\xi \cdot x$, for all $\xi \in \Gamma$ and all $x \in X$. By definition, we have

$$\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x \text{ for all } \alpha, \beta \in \Gamma \text{ and all } x \in X. \quad (2.6)$$

Equivalently, this means that we have a semigroup morphism $\Gamma \ni \xi \mapsto \xi \cdot \in G(X)$, where $G(X)$ denotes the semigroup, with respect to composition, of all maps $X \rightarrow X$. In case the semigroup Γ has a unit ϵ , the action is called *unital* if $\epsilon \cdot x = x$ for all $x \in X$, equivalently, $\epsilon \cdot = \text{Id}_X$.

Assume that Γ is a $*$ -semigroup, that is, there is an *involution* $*$ on Γ : $(\xi\eta)^* = \eta^*\xi^*$ and $(\xi^*)^* = \xi$ for all $\xi, \eta \in \Gamma$. Note that, in case Γ has a unit ϵ then $\epsilon^* = \epsilon$.

Given a VE-space \mathcal{H} we consider those Hermitian kernels $\mathbf{k}: X \times X \rightarrow \mathcal{L}^*(\mathcal{H})$ that are *invariant* under the action of Γ on X , that is,

$$\mathbf{k}(y, \xi \cdot x) = \mathbf{k}(\xi^* \cdot y, x) \text{ for all } x, y \in X \text{ and all } \xi \in \Gamma. \quad (2.7)$$

A triple $(\mathcal{K}; \pi; V)$ is called an *invariant VE-space linearisation* of the kernel \mathbf{k} and the action of Γ on X , shortly a Γ -*invariant VE-space linearisation* of \mathbf{k} , if:

(ikd1) $(\mathcal{K}; V)$ is a VE-space linearisation of the kernel \mathbf{k} .

(ikd2) $\pi: \Gamma \rightarrow \mathcal{L}^*(\mathcal{K})$ is a $*$ -representation, that is, a multiplicative $*$ -morphism.

(ikd3) V and π are related by the formula: $V(\xi \cdot x) = \pi(\xi)V(x)$, for all $x \in X$, $\xi \in \Gamma$.

If $(\mathcal{K}; \pi; V)$ is a Γ -invariant VE-space linearisation of the kernel \mathbf{k} then \mathbf{k} is invariant under the action of Γ on X .

If, in addition to the axioms (ikd1)–(ikd3), the triple $(\mathcal{K}; \pi; V)$ has the property

$$(ikd4) \quad \text{Lin } V(X)\mathcal{H} = \mathcal{K},$$

that is, the VE-space linearisation $(\mathcal{K}; V)$ is minimal, then $(\mathcal{K}; \pi; V)$ is called a *minimal Γ -invariant VE-space linearisation* of \mathbf{k} and the action of Γ on X .

Theorem 2.3 (Theorem 2.8 in [5]). *Let Γ be a $*$ -semigroup that acts on the nonempty set X and let $\mathbf{k}: X \times X \rightarrow \mathcal{L}^*(\mathcal{H})$ be a kernel, for some VE-space \mathcal{H} over an ordered $*$ -space Z . The following assertions are equivalent:*

- (1) \mathbf{k} is positive semidefinite, in the sense of (2.4), and invariant under the action of Γ on X , that is, (2.7) holds.
- (2) \mathbf{k} has a Γ -invariant VE-space linearisation $(\mathcal{K}; \pi; V)$.

- (3) \mathbf{k} admits an \mathcal{H} -reproducing kernel VE-space \mathcal{R} and there exists a $*$ -representation $\rho: \Gamma \rightarrow \mathcal{L}^*(\mathcal{R})$ such that $\rho(\xi)\mathbf{k}_x h = \mathbf{k}_{\xi \cdot x} h$ for all $\xi \in \Gamma$, $x \in X$, $h \in \mathcal{H}$.

In addition, in case any of the assertions (1), (2), or (3) holds, then a minimal Γ -invariant VE-space linearisation can be constructed, any minimal Γ -invariant VE-space linearisation is unique up to unitary equivalence, a pair $(\mathcal{R}; \rho)$ as in assertion (3) with \mathcal{R} minimal can be always obtained and, in this case, it is uniquely determined by \mathbf{k} as well.

Because we will use some of the constructions provided by the proof of Theorem 2.3 we recall those needed. Assuming that \mathbf{k} is positive semidefinite, by Lemma 2.1.(1) it follows that \mathbf{k} is Hermitian, that is, $\mathbf{k}(x, y)^* = \mathbf{k}(y, x)$ for all $x, y \in X$. We consider the convolution operator K defined at (2.3) and let $\mathcal{G} = \mathcal{G}(X; \mathcal{H})$ be its range, more precisely,

$$\begin{aligned} \mathcal{G} &= \{f \in \mathcal{F} \mid f = Kg \text{ for some } g \in \mathcal{F}_0\} \\ &= \left\{ f \in \mathcal{F} \mid f(y) = \sum_{x \in X} \mathbf{k}(y, x)g(x) \text{ for some } g \in \mathcal{F}_0 \text{ and all } x \in X \right\}. \end{aligned} \quad (2.8)$$

A pairing $[\cdot, \cdot]_{\mathcal{G}}: \mathcal{G} \times \mathcal{G} \rightarrow Z$ can be defined by

$$[e, f]_{\mathcal{G}} = [Kg, h]_{\mathcal{F}_0} = \sum_{y \in X} [e(y), h(y)]_{\mathcal{H}} = \sum_{x, y \in X} [\mathbf{k}(y, x)g(x), h(y)]_{\mathcal{H}}, \quad (2.9)$$

where $f = Kh$ and $e = Kg$ for some $g, h \in \mathcal{F}_0$. The pairing $[\cdot, \cdot]_{\mathcal{G}}$ is a Z -valued gramian, that is, it satisfies all the requirements (ve1)–(ve3). $(\mathcal{G}; [\cdot, \cdot]_{\mathcal{G}})$ is a VE-space that we denote by \mathcal{K} . For each $x \in X$ define $V(x): \mathcal{H} \rightarrow \mathcal{G}$ by

$$V(x)h = Kh_x, \quad h \in \mathcal{H}, \quad (2.10)$$

where $h_x = \delta_x h \in \mathcal{F}_0$ is the function that takes the value h at x and is null elsewhere. Equivalently,

$$(V(x)h)(y) = (Kh_x)(y) = \sum_{z \in X} \mathbf{k}(y, z)(h_x)(z) = \mathbf{k}(y, x)h, \quad y \in X. \quad (2.11)$$

Note that $V(x)$ is an operator from the VE-space \mathcal{H} to the VE-space $\mathcal{G} = \mathcal{K}$ and it can be shown that $V(x)$ is adjointable for all $x \in X$.

On the other hand, for any $x, y \in X$, by (2.11), we have

$$V(y)^*V(x)h = (V(x)h)(y) = \mathbf{k}(y, x)h, \quad h \in \mathcal{H},$$

hence $(V; \mathcal{K})$ is a VE-space linearisation of \mathbf{k} and it is minimal as well, more precisely, \mathcal{G} is the range of the convolution operator K defined at (2.3).

For each $\xi \in \Gamma$ let $\pi(\xi): \mathcal{F} \rightarrow \mathcal{F}$ be defined by

$$(\pi(\xi)f)(y) = f(\xi^* \cdot y), \quad f \in \mathcal{F}, \quad y \in X, \quad \xi \in \Gamma. \quad (2.12)$$

$\pi(\xi)$ leaves \mathcal{G} invariant. Denote by the same symbol $\pi(\xi)$ the map $\pi(\xi): \mathcal{G} \rightarrow \mathcal{G}$.

π is a $*$ -representation of the semigroup Γ on the complex vector space \mathcal{G} and, taking into account that \mathbf{k} is invariant under the action of Γ on X , for all $\xi \in \Gamma$, $x, y \in X$, $h \in \mathcal{H}$, we have

$$(V(\xi \cdot x)h)(y) = \mathbf{k}(y, \xi \cdot x)h = \mathbf{k}(\xi^* \cdot y, x)h = (V(x)h)(\xi^* \cdot y) = (\pi(\xi)V(x)h)(y), \quad (2.13)$$

which proves (ikd3). Thus, $(\mathcal{K}; \pi; V)$, here constructed, is a Γ -invariant VE-space linearisation of the Hermitian kernel \mathbf{k} . Note that $(\mathcal{K}; \pi; V)$ is minimal, that is, the axiom (ikd4) holds, since the VE-space linearisation $(\mathcal{K}; V)$ is minimal.

The construction of $(\mathcal{K}; \pi; V)$ just presented is essentially a minimal \mathcal{H} -reproducing kernel VE-space one. In particular, it proves the statement (3) as well. On the other hand, Proposition 2.2 provides an explicit connection between the collection of all minimal Γ -invariant VE-space linearisations $(\mathcal{K}; \pi; V)$ of \mathbf{k} , identified by unitary equivalence, and the unique minimal \mathcal{H} -reproducing kernel VE-space \mathcal{R} of \mathbf{k} . On \mathcal{R} a Z -valued gramian is defined by

$$[V(\cdot)^* f, V(\cdot)^* g]_{\mathcal{R}} = [f, g]_{\mathcal{K}}, \quad f, g \in \mathcal{K}. \quad (2.14)$$

2.2. VE-Module Linearisations

Given an ordered $*$ -algebra \mathcal{A} and a VE-module \mathcal{E} over \mathcal{A} , an \mathcal{E} -reproducing kernel VE-module over \mathcal{A} is just an \mathcal{E} -reproducing kernel VE-space over \mathcal{A} , with definition as in Sect. 2.1, which is also a VE-module over \mathcal{A} .

Proposition 2.4. *Let Γ be a $*$ -semigroup that acts on the nonempty set X and let $\mathbf{k}: X \times X \rightarrow \mathcal{L}^*(\mathcal{H})$ be a kernel, for some VE-module \mathcal{H} over an ordered $*$ -algebra \mathcal{A} . The following assertions are equivalent:*

- (1) \mathbf{k} is positive semidefinite, in the sense of (2.4), and invariant under the action of Γ on X , that is, (2.7) holds.
- (2) \mathbf{k} has a Γ -invariant VE-module (over \mathcal{A}) linearisation $(\mathcal{K}; \pi; V)$.
- (3) \mathbf{k} admits an \mathcal{H} -reproducing kernel VE-module \mathcal{R} and there exists a $*$ -representation $\rho: \Gamma \rightarrow \mathcal{L}^*(\mathcal{R})$ such that $\rho(\xi)\mathbf{k}_x h = \mathbf{k}_{\xi \cdot x} h$ for all $\xi \in \Gamma$, $x \in X$, $h \in \mathcal{H}$.

In addition, in case any of the assertions (1), (2), or (3) holds, then a minimal Γ -invariant VE-module linearisation can be constructed, any minimal Γ -invariant VE-module linearisation is unique up to unitary equivalence, a pair $(\mathcal{R}; \rho)$ as in assertion (3) with \mathcal{R} minimal can be always obtained and, in this case, it is uniquely determined by \mathbf{k} as well.

We briefly recall the construction made in the implication (1) \Rightarrow (2), for later use. We first observe that, since \mathcal{H} is a module over \mathcal{A} , the space $\mathcal{F}(X; \mathcal{H})$ has a natural structure of right module over \mathcal{A} , more precisely, for any $f \in \mathcal{F}(X; \mathcal{H})$ and $a \in \mathcal{A}$

$$(fa)(x) = f(x)a, \quad x \in X.$$

In particular, the space $\mathcal{F}_0(X; \mathcal{H})$ is a submodule of $\mathcal{F}(X; \mathcal{H})$. On the other hand, by assumption, for each $x, y \in X$, $\mathbf{k}(x, y) \in \mathcal{L}^*(\mathcal{H})$, hence $\mathbf{k}(x, y)$ is a module map. These imply that the convolution operator $K: \mathcal{F}_0(X; \mathcal{H}) \rightarrow$

$\mathcal{F}(X; \mathcal{H})$ defined as in (2.3) is a module map. Indeed, for any $f \in \mathcal{F}_0(X; \mathcal{H})$, $a \in \mathcal{A}$, and $y \in X$,

$$((Kf)a)(x) = \sum_{x \in X} \mathbf{k}(y, x) f(x) a = K(fa)(x).$$

Then, the space $\mathcal{G}(X; \mathcal{H})$ which, with the definition as in (2.8), is the range of the convolution operator K , is a module over \mathcal{A} as well.

When endowed with the \mathcal{A} valued gramian $[\cdot, \cdot]_{\mathcal{G}}$ defined as in (2.9), we have

$$[e, fa]_{\mathcal{G}} = [e, f]_{\mathcal{G}} a, \quad e, f \in \mathcal{G}(X; \mathcal{H}), \quad a \in \mathcal{A}. \quad (2.15)$$

Indeed, let $e = Kg$ and $f = Kh$ for some $g, h \in \mathcal{F}_0(X; \mathcal{H})$. Then,

$$\begin{aligned} [e, fa]_{\mathcal{G}} &= [Kg, ha]_{\mathcal{F}_0} = \sum_{y \in X} [e(y), h(y)a]_{\mathcal{H}} \\ &= \sum_{y \in X} [e(y), h(y)]_{\mathcal{H}} a = [Kg, h]_{\mathcal{F}_0} a = [e, f]_{\mathcal{G}} a. \end{aligned}$$

From (2.15) and the proof of the implication (1) \Rightarrow (2) in Theorem 2.3, it follows that $\mathcal{K} = \mathcal{G}(X; \mathcal{H})$ is a VE-module over the ordered \ast -algebra \mathcal{A} and hence, the triple $(\mathcal{K}; \pi; V)$ is a minimal Γ -invariant VE-module linearisation of \mathbf{k} .

2.3. VH-Space Linearisations and Reproducing Kernels

Let \mathcal{H} be a VH-space over the admissible space Z , and consider a kernel $\mathbf{k}: X \times X \rightarrow \mathcal{L}_c^*(\mathcal{H})$. A *VH-space linearisation* of \mathbf{k} , or *VH-space Kolmogorov decomposition* of \mathbf{k} , is a pair $(\mathcal{K}; V)$, subject to the following conditions:

- (vhl1) \mathcal{K} is a VH-space over the same ordered \ast -space Z .
- (vhl2) $V: X \rightarrow \mathcal{L}_c^*(\mathcal{H}, \mathcal{K})$ satisfies $\mathbf{k}(x, y) = V(x) \ast V(y)$ for all $x, y \in X$.

The VH-space linearisation $(\mathcal{K}; V)$ is called *minimal* if

- (vhl3) $\text{Lin } V(X)\mathcal{H}$ is dense in \mathcal{K} .

It is useful to observe that any VH-space linearisation is a VE-space linearisation with some differences between them: the former requires that both the kernel \mathbf{k} and all the operators $V(x)$, $x \in X$, are all continuous and continuously adjointable operators. As concerning minimality, the two concepts are significantly different.

Two VH-space linearisations $(V; \mathcal{K})$ and $(V'; \mathcal{K}')$ of the same kernel \mathbf{k} are called *unitary equivalent* if there exists a unitary operator $U: \mathcal{K} \rightarrow \mathcal{K}'$ such that $UV(x) = V'(x)$ for all $x \in X$.

The uniqueness of a minimal VH-space linearisation $(\mathcal{K}; V)$ of a positive semidefinite kernel \mathbf{k} , modulo unitary equivalence, follows in the usual way, taking into account that unitary operators are continuous, e.g. see [17].

A VH-space \mathcal{R} over the ordered \ast -space Z is called an *\mathcal{H} -reproducing kernel VH-space on X* if there exists a Hermitian kernel $\mathbf{k}: X \times X \rightarrow \mathcal{L}_c^*(\mathcal{H})$ such that the following axioms are satisfied:

- (rk1) \mathcal{R} is a subspace of $\mathcal{F}(X; \mathcal{H})$, with all algebraic operations.
- (rk2) For all $x \in X$ and all $h \in \mathcal{H}$, the \mathcal{H} -valued function $\mathbf{k}_x h = \mathbf{k}(\cdot, x)h \in \mathcal{R}$.
- (rk3) For all $f \in \mathcal{R}$ we have $[f(x), h]_{\mathcal{H}} = [f, \mathbf{k}_x h]_{\mathcal{R}}$, for all $x \in X$ and $h \in \mathcal{H}$.

(rk4) For all $x \in X$ the evaluation operator $\mathcal{R} \ni f \mapsto f(x) \in \mathcal{H}$ is continuous.

Note that, when comparing a reproducing kernel VH-space with a reproducing kernel VE-space, for the same kernel \mathbf{k} , there are at least two differences. First, in the former, we have a VH-space and the values of the kernel are all continuous and continuously adjointable operators. Second, the axiom (rk4) is new even when compared to the classical case of reproducing kernel Hilbert spaces, when this is actually a consequence of the other axioms. As the following result shows, these differences have consequences that differentiate the reproducing kernel VH-space from the reproducing kernel VE-space and from the reproducing kernel Hilbert space.

Lemma 2.5. *Let \mathcal{R} be an \mathcal{H} -reproducing kernel VH-space with reproducing kernel \mathbf{k} .*

- (1) *For any $x \in X$, $\mathbf{k}_x \in \mathcal{L}_c^*(\mathcal{H}, \mathcal{R})$.*
- (2) *For any $x, y \in X$, $\mathbf{k}(x, y) = \mathbf{k}_x^* \mathbf{k}_y$.*
- (3) *\mathbf{k} is positive semidefinite.*
- (4) *The orthogonal space of $\text{Lin}\{\mathbf{k}_x h \mid x \in X, h \in \mathcal{H}\} \subseteq \mathcal{R}$ is the null space.*
- (5) *\mathbf{k} is uniquely determined by \mathcal{R} .*

Proof. Clearly, for arbitrary $x \in X$, the map $\mathbf{k}_x : \mathcal{H} \rightarrow \mathcal{R}$ is a linear operator. From (rk3) it follows that \mathbf{k}_x is adjointable and its adjoint \mathbf{k}_x^* is $E_x : \mathcal{R} \rightarrow \mathcal{H}$, the evaluation operator $E_x(f) = f(x)$, for $f \in \mathcal{R}$ which, by (rk4), is assumed to be continuous. On the other hand, by (rk3), for arbitrary $x, y \in X$, we have

$$[\mathbf{k}_x h, \mathbf{k}_y g]_{\mathcal{R}} = [(\mathbf{k}_x h)(y), g]_{\mathcal{H}} = [\mathbf{k}(y, x)h, g]_{\mathcal{H}}, \quad h, g \in \mathcal{H},$$

hence the assertion (2) is proven. In particular, $\mathbf{k}(x, x) = \mathbf{k}_x^* \mathbf{k}_x$ is a positive operator. Since, by assumption, $\mathbf{k}(x, x) \in \mathcal{L}_c^*(\mathcal{H})$, we can apply Lemma 1.6 and obtain that, for every seminorm $p \in S(Z)$ there exist a seminorm $q \in S(Z)$ and a constant $c \geq 0$ such that

$$p([\mathbf{k}_x h, \mathbf{k}_x h]_{\mathcal{R}}) = p([\mathbf{k}(x, x)h, h]_{\mathcal{H}}) \leq c q([h, h]), \quad h \in \mathcal{H},$$

hence \mathbf{k}_x is continuous. This concludes the proof of assertion (1).

Let $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$, and $h_1, \dots, h_n \in \mathcal{H}$ be arbitrary. Then

$$\sum_{j,k=1}^n [\mathbf{k}(x_j, x_k)h_k, h_j]_{\mathcal{H}} = \sum_{j,k=1}^n [\mathbf{k}_{x_j}^* \mathbf{k}_{x_k} h_k, h_j]_{\mathcal{H}} = \left[\sum_{k=1}^n \mathbf{k}_{x_k} h_k, \sum_{j=1}^n \mathbf{k}_{x_j} h_j \right]_{\mathcal{R}} \geq 0,$$

hence assertion (3) is proven.

Let $f \in \mathcal{R}$ be an \mathcal{H} -valued function orthogonal to all \mathcal{H} -valued functions $\mathbf{k}_x h$, with $x \in X$ and $h \in \mathcal{H}$. By (rk3), for each $x \in X$,

$$0 = [f, \mathbf{k}_x h]_{\mathcal{R}} = [f(x), h]_{\mathcal{H}}, \quad h \in \mathcal{H},$$

and hence, since the gramian $[\cdot, \cdot]_{\mathcal{H}}$ is nondegenerate it follows that $f(x) = 0$. Therefore, $f = 0$ and assertion (4) is proven as well.

In order to see that assertion (5) is true, observe that once the \mathcal{H} -reproducing kernel VH-space \mathcal{R} on the set X is given, all the evaluation

operators E_x are uniquely determined by \mathcal{R} . Since $E_x = \mathbf{k}_x^*$, from (2) it follows

$$\mathbf{k}(y, x) = \mathbf{k}_y^* \mathbf{k}_x = E_y E_x^*, \quad x, y \in X,$$

hence the kernel \mathbf{k} is uniquely determined by \mathcal{R} . □

Assertion (4) in the previous lemma says that reproducing kernel VH-spaces have a built-in minimality property but, due to the fact that not any closed subspace of a VH-space is orthocomplemented, the following definition makes sense. An \mathcal{H} -reproducing kernel VH-space \mathcal{R} on X is called *minimal* if

(5) $\text{Lin}\{\mathbf{k}_x h \mid x \in X, h \in \mathcal{H}\}$ is dense in \mathcal{R} .

Proposition 2.6. *Let \mathcal{H} be a VH-space over some admissible space Z and \mathbf{k} an $\mathcal{L}_c^*(\mathcal{H})$ -valued kernel on X and assume that there exists an \mathcal{H} -reproducing kernel VH-space \mathcal{K} on X with reproducing kernel \mathbf{k} .*

(1) *The closure of $\text{Lin}\{\mathbf{k}_x h \mid x \in X, h \in \mathcal{H}\}$ in \mathcal{K} is a minimal \mathcal{H} -reproducing kernel VH-space on X with kernel \mathbf{k} .*

(2) *If \mathcal{R} is another minimal \mathcal{H} -reproducing kernel VH-space on X with the same reproducing kernel \mathbf{k} , then $\mathcal{R} \subseteq \mathcal{K}$. In particular, the minimal \mathcal{H} -reproducing kernel VH-space on X with reproducing kernel \mathbf{k} is unique.*

Proof. (1) This statement is clear from the axioms (rk1)–(rk4).

(2) Clearly, $\mathcal{L} = \text{Lin}\{\mathbf{k}_x h \mid x \in X, h \in \mathcal{H}\}$ is contained in both \mathcal{R} and \mathcal{K} . In addition,

$$[f, g]_{\mathcal{R}} = [f, g]_{\mathcal{K}}, \quad f, g \in \mathcal{L},$$

and, with notation as in Lemma 1.3, we have $\tilde{p}_{\mathcal{R}}|_{\mathcal{L}} = \tilde{p}_{\mathcal{K}}|_{\mathcal{L}}$ and hence $\tau_{\mathcal{R}}|_{\mathcal{L}} = \tau_{\mathcal{K}}|_{\mathcal{L}}$. By the minimality of \mathcal{R} , for any $f \in \mathcal{R}$ there exists a net $(f_i)_i$ in \mathcal{L} such that $f_i \xrightarrow{i} f$ and

$$[f(x), h]_{\mathcal{H}} = [f, \mathbf{k}_x h]_{\mathcal{R}} = \lim_i [f_i, \mathbf{k}_x h]_{\mathcal{R}}, \quad x \in X, h \in \mathcal{H}.$$

But, $(f_i)_i$ is a Cauchy net in $(\mathcal{L}; \tau_{\mathcal{R}}|_{\mathcal{L}}) = (\mathcal{L}; \tau_{\mathcal{K}}|_{\mathcal{L}})$ and hence, there exists $g \in \mathcal{K}$ such that $f_i \xrightarrow{i} g$, which implies

$$[g(x), h]_{\mathcal{H}} = [g, \mathbf{k}_x h]_{\mathcal{K}} = \lim_i [f_i, \mathbf{k}_x h]_{\mathcal{K}}.$$

Since, for arbitrary fixed $x \in X$ and $h \in \mathcal{H}$ we have

$$[f_i, \mathbf{k}_x h]_{\mathcal{K}} = [f_i, \mathbf{k}_x h]_{\mathcal{R}}, \quad \text{for any } i,$$

taking into account that Z is separated, it follows

$$[f(x), h]_{\mathcal{H}} = [g(x), h]_{\mathcal{H}}, \quad x \in X, h \in \mathcal{H},$$

hence $f = g \in \mathcal{R}$. This proves $\mathcal{R} \subseteq \mathcal{K}$. □

Observe that, given X a nonempty set and \mathcal{H} a VH-space, for any $x \in X$ one can define a general *evaluation operator* $E_x: \mathcal{F}(X; \mathcal{H}) \rightarrow \mathcal{H}$ by $E_x(f) = f(x)$, for all $f \in \mathcal{F}(X; \mathcal{H})$. In particular, evaluation operators can be defined if instead of $\mathcal{F}(X; \mathcal{H})$ we can consider any vector subspace \mathcal{S} of $\mathcal{F}(X; \mathcal{H})$.

Proposition 2.7. *Let X be a nonempty set, \mathcal{H} a VH-space over an admissible space Z , and let $\mathcal{R} \subseteq \mathcal{F}(X; \mathcal{H})$, with all algebraic operations, be a VH-space over Z . Then \mathcal{R} is an \mathcal{H} -reproducing kernel VH-space if and only if, for all $x \in X$, the evaluation operator $E_x \in \mathcal{L}_c^*(\mathcal{R}, \mathcal{H})$, that is, E_x is continuous and continuously adjointable.*

Proof. Assume first that \mathcal{R} is an \mathcal{H} -reproducing kernel VH-space on X and let \mathbf{k} be its reproducing kernel. For any $h \in \mathcal{H}$ and any $f \in \mathcal{R}$ we have

$$[E_x f, h]_{\mathcal{H}} = [f(x), h]_{\mathcal{H}} = [f, \mathbf{k}_x h]_{\mathcal{R}}. \quad (2.16)$$

Since $\mathbf{k}_x \in \mathcal{L}(\mathcal{H}, \mathcal{R})$, it follows that E_x is adjointable and, in addition, $E_x^* = \mathbf{k}_x$, for all $x \in X$. It was proven in Lemma 2.5 that $\mathbf{k}_x \in \mathcal{L}_c^*(\mathcal{H}, \mathcal{R})$, hence $E_x \in \mathcal{L}_c^*(\mathcal{R}, \mathcal{H})$.

Conversely, assume that, for all $x \in X$, the evaluation operator $E_x \in \mathcal{L}_c^*(\mathcal{R}, \mathcal{H})$. Equation (2.16) suggests to define the kernel \mathbf{k} in the following way:

$$\mathbf{k}(y, x)h = (E_x^* h)(y), \quad x, y \in X, \quad h \in \mathcal{H}. \quad (2.17)$$

Then $k(y, x): \mathcal{H} \rightarrow \mathcal{H}$ is a linear operator and, letting $\mathbf{k}_x = \mathbf{k}(\cdot, x)$ for all $x \in X$, we have $\mathbf{k}_x h = E_x^* h$ for all $x \in X$ and all $h \in \mathcal{H}$. The reproducing property (rk3) holds:

$$[f(x), h]_{\mathcal{H}} = [E_x f, h]_{\mathcal{H}} = [f, E_x^* h]_{\mathcal{R}} = [f, \mathbf{k}_x h]_{\mathcal{R}}, \quad f \in \mathcal{R}, \quad h \in \mathcal{H}, \quad x \in X.$$

The axioms (rk1), (rk2), and (rk3) are clearly satisfied. We prove that \mathbf{k} is a Hermitian kernel. To see this, fix $x, y \in X$ and $h, l \in \mathcal{H}$. Then

$$\begin{aligned} [\mathbf{k}(y, x)h, l]_{\mathcal{H}} &= [(\mathbf{k}_x h)(y), l]_{\mathcal{H}} = [\mathbf{k}_x h, \mathbf{k}_y l]_{\mathcal{R}} \\ &= [\mathbf{k}_y l, \mathbf{k}_x h]_{\mathcal{R}}^* = [\mathbf{k}(x, y)l, h]_{\mathcal{R}}^* = [h, \mathbf{k}(x, y)l]_{\mathcal{R}}. \end{aligned}$$

Therefore, $\mathbf{k}(y, x)$ is adjointable and $\mathbf{k}(y, x)^* = \mathbf{k}(x, y)$, hence \mathbf{k} is a Hermitian kernel. We have proven that \mathbf{k} is the reproducing kernel of \mathcal{R} . \square

There is a very close connection between VH-space linearisations and reproducing kernel VH-spaces, similar, to a certain extent, to the connection between VE-space linearisations and reproducing kernel VE-spaces, as in Proposition 2.2.

Proposition 2.8. *Let X be a nonempty set, \mathcal{H} a VH-space over an admissible space Z , and let $\mathbf{k}: X \times X \rightarrow \mathcal{L}_c^*(\mathcal{H})$ be a Hermitian kernel.*

(1) *For any VH-space linearisation $(\mathcal{K}; V)$ of \mathbf{k} , letting \mathcal{K}_0 denote the closure of the linear span of $V(X)\mathcal{H}$ in \mathcal{K} and $V_0(x)h := V(x)h \in \mathcal{K}_0$, for all $x \in X$ and all $h \in \mathcal{H}$, we obtain a minimal VH-space linearisation $(\mathcal{K}_0; V_0)$ of \mathbf{k} .*

(2) *For any minimal VH-space linearisation $(\mathcal{K}; V)$ of \mathbf{k} , letting*

$$\mathcal{R} = \{V(\cdot)^* f \mid f \in \mathcal{K}\}, \quad (2.18)$$

we obtain the minimal \mathcal{H} -reproducing kernel VH-space with reproducing kernel \mathbf{k} .

(3) *Any \mathcal{H} -reproducing kernel VH-space \mathcal{R} with kernel \mathbf{k} is a VH-space linearisation $(\mathcal{R}; V)$ of \mathbf{k} , with $V(x) = \mathbf{k}_x$ for all $x \in X$. In addition, the*

\mathcal{H} -reproducing kernel VH-space \mathcal{R} is minimal if and only if the VH-space linearisation $(\mathcal{R}; V)$ is minimal.

Proof. (1) Clearly \mathcal{K}_0 is a VH-subspace of \mathcal{K} . By its very definition, $V_0(x) \in \mathcal{L}_c(\mathcal{H}, \mathcal{K}_0)$, for all $x \in X$. Fixing $x \in X$, we consider the linear operator $W(x) = V(x)^*|_{\mathcal{K}_0} : \mathcal{K}_0 \rightarrow \mathcal{H}$ and observe that $W(x) \in \mathcal{L}_c(\mathcal{K}_0, \mathcal{H})$. Then,

$$\begin{aligned} [W(x)k, h]_{\mathcal{H}} &= [V(x)^*k, h]_{\mathcal{H}} = [k, V(x)h]_{\mathcal{K}} \\ &= [k, V(x)h]_{\mathcal{K}_0} = [k, V_0(x)h]_{\mathcal{K}_0}, \quad h \in \mathcal{H}, \quad k \in \mathcal{K}_0, \end{aligned}$$

hence $W(x)$ is the adjoint operator of $V_0(x)$, hence $V_0(x) \in \mathcal{L}_c^*(\mathcal{H}, \mathcal{K}_0)$. In addition,

$$\begin{aligned} [V_0(x)^*V_0(y)h, g]_{\mathcal{H}} &= [V_0(y)h, V_0(x)g]_{\mathcal{K}_0} = [V(y)h, V(x)g]_{\mathcal{K}} \\ &= [V(x)^*V(y)h, g]_{\mathcal{H}} = [\mathbf{k}(x, y)h, g]_{\mathcal{H}}, \quad h, g \in \mathcal{H}, \end{aligned}$$

hence $(\mathcal{K}_0; V_0)$ is a VH-space linearisation of \mathbf{k} . Since, by definition, \mathcal{K}_0 coincides with the closure of the linear span of $V_0(X)\mathcal{H}$, it follows that it is minimal as well.

(2) Let $(\mathcal{K}; \pi; V)$ be a minimal VH-space linearisation of the kernel \mathbf{k} . Define \mathcal{R} as in (2.18) that is, \mathcal{R} consists of all functions $X \ni x \mapsto V(x)^*f \in \mathcal{H}$, in particular $\mathcal{R} \subseteq \mathcal{F}(X; \mathcal{H})$, and we endow \mathcal{R} with the algebraic operations inherited from the complex vector space $\mathcal{F}(X; \mathcal{H})$. We consider the correspondence

$$\mathcal{K} \ni f \mapsto Uf = V(\cdot)^*f \in \mathcal{R}. \quad (2.19)$$

From Proposition 2.2, we know that $(\mathcal{R}; [\cdot, \cdot]_{\mathcal{R}})$ with the Z -gramian $[Uf, Ug]_{\mathcal{R}} = [f, g]_{\mathcal{K}}$ is a VE-space, that $U : \mathcal{K} \rightarrow \mathcal{R}$ is a unitary operator of VE-spaces \mathcal{K} and \mathcal{R} , and that $(\mathcal{R}; [\cdot, \cdot]_{\mathcal{R}})$ is an \mathcal{H} -reproducing kernel VE-space with reproducing kernel \mathbf{k} . In addition, by (2.19) and the definition of the natural topology of a VH-space, see Lemma 1.3, it follows that U is a homeomorphism, hence \mathcal{R} is a VH-space. Therefore, the axioms (rk1)–(rk3) hold and the minimality of \mathcal{R} follows from the minimality of \mathcal{K} . It only remains to show that the axiom (rk4) holds as well.

We show that $\mathbf{k}_x \in \mathcal{L}_c^*(\mathcal{H}, \mathcal{R})$ for all $x \in X$. First recall that $\mathbf{k}_x \in \mathcal{L}^*(\mathcal{H}, \mathcal{R})$ for all $x \in X$ by the reproducing kernel axiom. We first prove that \mathbf{k}_x is continuous. By the continuity of $V(x)$ for arbitrary $x \in X$, for any $p \in S(Z)$ there exist $q \in S(Z)$ and $c_p(x) \geq 0$ such that, for all $h \in \mathcal{H}$, $p([\mathbf{k}_x(h), \mathbf{k}_x(h)]_{\mathcal{R}}) = p([\mathbf{k}_x h, \mathbf{k}_x h]_{\mathcal{R}}) = p([V(x)h, V(x)h]_{\mathcal{K}}) \leq c_p(x) q([h, h]_{\mathcal{H}})$, hence \mathbf{k}_x is continuous.

Finally we show that \mathbf{k}_x^* is continuous. Let $p \in S(Z)$. Then, by the continuity of $V(x)^*$ for arbitrary $x \in X$, for some $q \in S(Z)$ and $c_p(x) \geq 0$, we have

$$\begin{aligned} p([\mathbf{k}_x^* f, \mathbf{k}_x^* f]_{\mathcal{H}}) &= p([f(x), f(x)]_{\mathcal{H}}) = p([V(x)^*g, V(x)^*g]_{\mathcal{H}}) \\ &\leq c_p(x) q([g, g]_{\mathcal{K}}) = c_p(x) q([Ug, Ug]_{\mathcal{R}}) \\ &= c_p(x) q([f, f]_{\mathcal{R}}), \quad f \in \mathcal{R}, \end{aligned}$$

where $g \in \mathcal{K}$ is the unique vector such that $Ug = V(\cdot)^*g = f$. Hence the continuity of \mathbf{k}_x^* is proven.

(3) Assume that $(\mathcal{R}; [\cdot, \cdot]_{\mathcal{R}})$ is an \mathcal{H} -reproducing kernel VH-space on X , with reproducing kernel \mathbf{k} . We let $\mathcal{K} = \mathcal{R}$ and define $V(x): \mathcal{H} \rightarrow \mathcal{K}$ by

$$V(x)h = \mathbf{k}_x h, \quad x \in X, \quad h \in \mathcal{H}. \quad (2.20)$$

Then, $V(x) \in \mathcal{L}^*(\mathcal{H}, \mathcal{K})$, with $V(x)^*: f \ni \mathcal{K} = \mathcal{R} \mapsto f(x) \in \mathcal{H}$ for all $x \in X$. From Lemma 2.5, we see that, actually, $V(x) \in \mathcal{L}_c^*(\mathcal{H}, \mathcal{K})$ and that $V(y)^*V(x) = \mathbf{k}(y, x)$ for all $x, y \in X$. Thus, $(\mathcal{K}; V)$ is a VH-space linearisation of \mathbf{k} . \square

Let us observe that, until now, we did not say anything about the existence of reproducing kernel VH-spaces or, equivalently, of VH-space linearisations, associated to a given positive semidefinite \mathcal{H} -kernel. This question is considered in the next subsection and answered in Corollary 2.12, as a consequence of Theorem 2.10, by providing a necessary and sufficient condition (b2). We present some cases when this boundedness condition is automatically fulfilled, for example, in Sect. 2.5 for a class of positive semidefinite kernels having a certain property of m-topologisability, or the case when \mathcal{H} is a Hilbert module over a locally C^* -algebra, see in Sect. 3.2.

2.4. Dilation in VH-Spaces

Let \mathcal{H} be a VH-space over an admissible space Z , let $\mathbf{k}: X \times X \rightarrow \mathcal{L}_c^*(\mathcal{H})$ be a kernel on some nonempty set X , and let Γ be a $*$ -semigroup that acts at left on X . As in the case of VE-space operator valued kernels, we call \mathbf{k} Γ -invariant, if (2.7) holds. A triple $(\mathcal{K}; \pi; V)$ is called a Γ -invariant VH-space linearisation for \mathbf{k} if

- (ihl1) $(\mathcal{K}; V)$ is a VH-space linearisation of \mathbf{k} .
- (ihl2) $\pi: \Gamma \rightarrow \mathcal{L}_c^*(\mathcal{K})$ is a $*$ -representation.
- (ihl3) $V(\xi \cdot x) = \pi(\xi)V(x)$ for all $\xi \in \Gamma$ and all $x \in X$.

Also, $(\mathcal{K}; \pi; V)$ is *minimal* if the VH-space linearisation $(\mathcal{K}; V)$ is minimal, that is, \mathcal{K} is the closure of the linear span of $V(X)\mathcal{H}$.

Remark 2.9. Let $(\mathcal{K}; \pi; V)$ be a Γ -invariant VH-space linearisation for the positive semidefinite kernel $\mathbf{k}: X \times X \rightarrow \mathcal{L}_c^*(\mathcal{H})$ and consider the minimal VH-space linearisation $(\mathcal{K}_0; V_0)$ as in Proposition 2.8, that is, \mathcal{K}_0 is the closure of the linear span of $V(X)\mathcal{H}$ and $V_0: X \rightarrow \mathcal{L}_c^*(\mathcal{H}, \mathcal{K}_0)$ is defined by $V_0(x)h = V(x)h$, for all $x \in X$ and all $h \in \mathcal{H}$. We observe that for every $\xi \in \Gamma$, the operator $\pi(\xi)$ leaves \mathcal{K}_0 invariant: for any $x \in X$ and any $h \in \mathcal{H}$, by (ihl3) we have $\pi(\xi)V(x)h = V(\xi \cdot x)h \in \mathcal{K}_0$, and then use linearity and continuity of $\pi(\xi)$. Thus, we can define $\pi_0: \Gamma \rightarrow \mathcal{L}_c^*(\mathcal{K}_0)$ by $\pi_0(\xi)k = \pi(\xi)k \in \mathcal{K}_0$ for any $\xi \in \Gamma$ and any $k \in \mathcal{K}_0$. Then, it is easy to see that $(\mathcal{K}_0; \pi_0; V_0)$ is a minimal Γ -invariant linearisation of \mathbf{k} .

The following is a topological version of Theorem 2.3.

Theorem 2.10. *Let Γ be a $*$ -semigroup that acts on the nonempty set X and let $\mathbf{k}: X \times X \rightarrow \mathcal{L}_c^*(\mathcal{H})$ be a kernel, for some VH-space \mathcal{H} over an admissible space Z . Then the following assertions are equivalent:*

(1) \mathbf{k} is positive semidefinite, in the sense of (2.4), and invariant under the action of Γ on X , that is, (2.7) holds, and, in addition, the following conditions hold:

(b1) For any $\xi \in \Gamma$ and any seminorm $p \in S(Z)$, there exists a seminorm $q \in S(Z)$ and a constant $c_p(\xi) \geq 0$ such that for all $n \in \mathbb{N}$, $\{h_i\}_{i=1}^n \in \mathcal{H}$, $\{x_i\}_{i=1}^n \in X$ we have

$$p\left(\sum_{i,j=1}^n [\mathbf{k}(\xi \cdot x_i, \xi \cdot x_j)h_j, h_i]_{\mathcal{H}}\right) \leq c_p(\xi) q\left(\sum_{i,j=1}^n [\mathbf{k}(x_i, x_j)h_j, h_i]_{\mathcal{H}}\right).$$

(b2) For any $x \in X$ and any seminorm $p \in S(Z)$, there exists a seminorm $q \in S(Z)$ and a constant $c_p(x) \geq 0$ such that for all $n \in \mathbb{N}$, $\{y_i\}_{i=1}^n \in X$, $\{h_i\}_{i=1}^n \in \mathcal{H}$ we have

$$p\left(\sum_{i,j=1}^n [\mathbf{k}(x, y_i)h_i, \mathbf{k}(x, y_j)h_j]_{\mathcal{H}}\right) \leq c_p(x) q\left(\sum_{i,j=1}^n [\mathbf{k}(y_j, y_i)h_i, h_j]_{\mathcal{H}}\right).$$

(2) \mathbf{k} has a Γ -invariant VH-space linearisation $(\mathcal{K}; \pi; V)$.

(3) \mathbf{k} admits an \mathcal{H} -reproducing kernel VH-space \mathcal{R} and there exists a \ast -representation $\rho: \Gamma \rightarrow \mathcal{L}_c^*(\mathcal{R})$ such that $\rho(\xi)\mathbf{k}_x h = \mathbf{k}_{\xi \cdot x} h$ for all $\xi \in \Gamma$, $x \in X$, $h \in \mathcal{H}$.

In addition, in case any of the assertions (1), (2), or (3) holds, then a minimal Γ -invariant VH-space linearisation of \mathbf{k} can be constructed, any minimal Γ -invariant VH-space linearisation of \mathbf{k} is unique up to unitary equivalence, and the pair $(\mathcal{R}; \rho)$ as in assertion (3) is uniquely determined by \mathbf{k} as well.

Proof. (1) \Rightarrow (2). We consider the notation and the minimal Γ -invariant VE-space linearisation $(\mathcal{G}; V; \pi)$ defined as in (2.8)–(2.12). Consider the VE-space $(\mathcal{G}; [\cdot, \cdot]_{\mathcal{G}})$ with its natural topology defined as in Sect. 1.3. We show that, for all $\xi \in \Gamma$, $\pi(\xi)$ is continuous as a linear operator on the locally convex space \mathcal{G} . By the boundedness condition (b1), for any $p \in S(Z)$ there exists $q \in S(Z)$ and $c_p(\xi) \geq 0$ such that, for all $f \in \mathcal{G}$, with $f = Kg$ for some $g \in \mathcal{F}_0$, we have

$$\begin{aligned} p([\pi(\xi)f, \pi(\xi)f]_{\mathcal{G}}) &= p([\pi(\xi^*)\pi(\xi)f, f]_{\mathcal{G}}) = p([\pi(\xi^*\xi)f, f]_{\mathcal{G}}) \\ &= p\left(\sum_{x,y \in X} [\mathbf{k}(\xi^*\xi \cdot y, x)g(x), g(y)]_{\mathcal{H}}\right) \\ &= p\left(\sum_{x,y \in X} [\mathbf{k}(\xi \cdot y, \xi \cdot x)g(x), g(y)]_{\mathcal{H}}\right) \\ &\leq c_p(\xi) q\left(\sum_{x,y \in X} [\mathbf{k}(y, x)g(x), g(y)]_{\mathcal{H}}\right) \\ &= c_p(\xi) q([f, f]_{\mathcal{G}}). \end{aligned}$$

Hence the continuity of $\pi(\xi)$ is proven.

Let \mathcal{K} be the VH-space completion of the VE-space \mathcal{G} . It follows that $\pi(\xi)$ extends uniquely to a continuous operator on \mathcal{K} and that π is a \ast -representation of Γ in $\mathcal{L}_c^*(\mathcal{K})$.

We now show that all the operators $V(x)$ defined as in (2.10) are continuous as linear operators defined on \mathcal{H} and with values in \mathcal{G} . Fix $x \in X$ and $p \in S(Z)$, but arbitrary. By Lemma 1.6, for some $q \in S(Z)$ and $c_p(x) \geq 0$, for all $h \in \mathcal{H}$ we have

$$\begin{aligned} p([V(x)h, V(x)h]_{\mathcal{G}}) &= p([V(x)^*V(x)h, h]_{\mathcal{H}}) \\ &= p([\mathbf{k}(x, x)h, h]_{\mathcal{H}}) \leq c_p(x) q([h, h]_{\mathcal{H}}). \end{aligned}$$

This proves the continuity of $V(x)$.

On the other hand, the operators $V(x)^*$ obtained as in Theorem 2.3 are continuous on \mathcal{G} for all $x \in X$. To see this, using the boundedness condition (b2), for any $p \in S(Z)$ there exist $q \in S(Z)$ and $c_p(x) \geq 0$ such that, for all $f \in \mathcal{G}$ we have

$$\begin{aligned} p([V(x)^*f, V(x)^*f]_{\mathcal{H}}) &= p([f(x), f(x)]_{\mathcal{H}}) \\ &= p\left(\sum_{y, z \in X} [\mathbf{k}(x, y)g(y), \mathbf{k}(x, z)g(z)]_{\mathcal{H}}\right) \\ &\leq c_p(x) q\left(\sum_{y, z \in X} [\mathbf{k}(z, y)g(y), g(z)]_{\mathcal{H}}\right) = c_p(x) q([f, f]_{\mathcal{G}}), \end{aligned}$$

where $Kg = f$ for some $g \in \mathcal{F}_0$. Hence $V(x)^*$ is continuous and, consequently, it extends uniquely to a continuous operator $V(x)^*: \mathcal{K} \rightarrow \mathcal{H}$. A continuity argument establishes the fact that $V(x): \mathcal{H} \rightarrow \mathcal{K}$ is adjointable with adjoint $V(x)^*: \mathcal{K} \rightarrow \mathcal{H}$. Hence $V(x) \in \mathcal{L}_c^*(\mathcal{H}, \mathcal{K})$. By (2.11) we obtain $V(y)^*V(x) = \mathbf{k}(y, x)$ for all $x, y \in X$, and by (2.13) $\pi(\xi)V(x) = V(\xi \cdot x)$ for all $\xi \in \Gamma$ and $x \in X$. Therefore $(\mathcal{K}; \pi; V)$ is a Γ -invariant VH-space linearisation of \mathbf{k} . Clearly, it is minimal. The uniqueness of the minimal invariant VH-space linearisation follows as usually.

(2) \Rightarrow (1). Let $(\mathcal{K}; \pi; V)$ be a Γ -invariant VH-space linearisation of \mathbf{k} . We already know from Theorem 2.3 that \mathbf{k} is positive semidefinite and that \mathbf{k} is invariant under the action of Γ on X . To show that (b1) holds, letting $p \in S(Z)$ be a seminorm and $\xi \in \Gamma$, since the operator $\pi(\xi)$ is continuous, there exist $q \in S(Z)$ and $c_p(\xi) \geq 0$, such that, for all $n \in \mathbb{N}$, $\{h_i\}_{i=1}^n \in \mathcal{H}$, $\{x_i\}_{i=1}^n \in X$, we have

$$\begin{aligned} p\left(\sum_{i,j=1}^n [\mathbf{k}(\xi \cdot x_i, \xi \cdot x_j)h_j, h_i]_{\mathcal{H}}\right) &= p\left(\sum_{i,j=1}^n [V(\xi \cdot x_i)^*V(\xi \cdot x_j)h_j, h_i]_{\mathcal{H}}\right) \\ &= p\left(\sum_{i,j=1}^n [V(\xi \cdot x_j)h_j, V(\xi \cdot x_i)h_i]_{\mathcal{K}}\right) \\ &= p\left(\left[\pi(\xi)\left(\sum_{j=1}^n V(x_j)h_j\right), \pi(\xi)\left(\sum_{i=1}^n V(x_i)h_i\right)\right]_{\mathcal{K}}\right) \\ &\leq c_p(\xi) q\left(\left[\sum_{j=1}^n V(x_j)h_j, \sum_{i=1}^n V(x_i)h_i\right]_{\mathcal{K}}\right) \\ &= c_p(\xi) q\left(\sum_{i,j=1}^n [\mathbf{k}(x_i, x_j)h_j, h_i]_{\mathcal{H}}\right). \end{aligned}$$

We show that (b2) holds. Let $x \in X$ and $p \in S(Z)$ be fixed. Since the operator $V(x)^* \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ is continuous, for some $q \in S(Z)$ and $c_p(x) \geq 0$, and arbitrary $n \in \mathbb{N}$, $\{y_i\}_{i=1}^n \in X$, $\{h_i\}_{i=1}^n \in \mathcal{H}$, we have

$$\begin{aligned} p\left(\sum_{i,j=1}^n [\mathbf{k}(x, y_i)h_i, \mathbf{k}(x, y_j)h_j]_{\mathcal{H}}\right) &= p\left(\left[V(x)^*\left(\sum_{i=1}^n V(y_i)h_i\right), V(x)^*\left(\sum_{j=1}^n V(y_j)h_j\right)\right]_{\mathcal{H}}\right) \\ &\leq c_p(x) q\left(\left[\sum_{i=1}^n V(y_i)h_i, \sum_{j=1}^n V(y_j)h_j\right]_{\mathcal{H}}\right) \\ &= c_p(x) q\left(\sum_{i,j=1}^n [\mathbf{k}(y_j, y_i)h_i, h_j]_{\mathcal{H}}\right). \end{aligned}$$

(2) \Rightarrow (3). Basically, this is a consequence of Proposition 2.2. Here are the details. Let $(\mathcal{K}; \pi; V)$ be a minimal Γ -invariant VH-space linearisation of the kernel \mathbf{k} and the action of Γ on X . Defining \mathcal{R} as in (2.18), from Proposition 2.2 it follows that \mathcal{R} has a natural structure of minimal \mathcal{H} -reproducing kernel VH-space with reproducing kernel \mathbf{k} . Letting $\rho(\xi) = U\pi(\xi)U^{-1}$, where $U: \mathcal{K} \rightarrow \mathcal{R}$ is the unitary operator defined as in (2.19), we obtain a \ast -representation of Γ on the VH-space \mathcal{R} such that $\mathbf{k}_{\xi \cdot x} = \rho(\xi)\mathbf{k}_x$ for all $\xi \in \Gamma$ and $x \in X$. By continuity of $\pi(\xi)$ for any $\xi \in \Gamma$, $\rho(\xi)$ is continuous for any $\xi \in \Gamma$ as well.

(3) \Rightarrow (2). Assume that $(\mathcal{R}; [\cdot, \cdot]_{\mathcal{R}})$ is an \mathcal{H} -reproducing kernel VH-space on X , with reproducing kernel \mathbf{k} and $\rho: \Gamma \rightarrow \mathcal{L}_c^*(\mathcal{R})$ is a \ast -representation such that $\mathbf{k}_{\xi \cdot x} = \rho(\xi)\mathbf{k}_x$ for all $\xi \in \Gamma$ and $x \in X$. We let $\mathcal{K} = \mathcal{R}$ and define $V(x): \mathcal{H} \rightarrow \mathcal{K}$ by

$$V(x)h = \mathbf{k}_x h, \quad x \in X, \quad h \in \mathcal{H}.$$

By Proposition 2.2, it follows that $(\mathcal{K}; V)$ is a VH-space linearisation of \mathbf{k} . Then, letting $\pi = \rho$, $(\mathcal{K}; \pi; V)$ is a minimal Γ -invariant VH-space linearisation of \mathbf{k} . \square

Remarks 2.11. (1) With notation as in Theorem 2.10, let \mathcal{C}_0 be a family of open, absolutely convex and solid neighbourhoods of the origin defining the topology of Z , and let $S_{\mathcal{C}_0}(Z) = \{p_C \mid C \in \mathcal{C}_0\}$ be defined as in Section 1.2. Then, the boundedness conditions (b1) and (b2) in the assertion (1) of Theorem 2.10 can, equivalently, be stated only for all $p \in S_{\mathcal{C}_0}(Z)$.

(2) In the particular case when Γ is a group and $\xi^* = \xi^{-1}$ for all $\xi \in \Gamma$, the boundedness condition (i) in assertion (1) is always fulfilled, due to the Γ -invariance of the kernel \mathbf{k} .

As a consequence of Theorem 2.10, for a given positive semidefinite \mathcal{H} -kernel \mathbf{k} , we can show that the boundedness condition (b2) is necessary and sufficient for the existence of a VH-space linearisation and, equivalently, for the existence of an \mathcal{H} -reproducing kernel VH-space associated to \mathbf{k} .

Corollary 2.12. *Let \mathbf{k} be a positive semidefinite \mathcal{H} -kernel on X , for some VH-space \mathcal{H} over an admissible space Z . Then, the following assertions are equivalent:*

- (1) *The following condition holds:*
 (b2) *For any $x \in X$ and any seminorm $p \in S(Z)$, there exists a seminorm $q \in S(Z)$ and a constant $c_p(x) \geq 0$ such that for all $n \in \mathbb{N}$, $\{y_i\}_{i=1}^n \in X$, $\{h_i\}_{i=1}^n \in \mathcal{H}$ we have*

$$p\left(\sum_{i,j=1}^n [\mathbf{k}(x, y_i)h_i, \mathbf{k}(x, y_j)h_j]_{\mathcal{H}}\right) \leq c_p(x) q\left(\sum_{i,j=1}^n [\mathbf{k}(y_j, y_i)h_i, h_j]_{\mathcal{H}}\right).$$

- (2) \mathbf{k} has a VH-space linearisation $(\mathcal{K}; V)$.
 (3) \mathbf{k} admits an \mathcal{H} -reproducing kernel VH-space \mathcal{R} .

2.5. Condition (b2) of Theorem 2.10

Condition (b2) of Theorem 2.10 for a positive semidefinite kernel can be considered as a weaker version of an inequality for positive semidefinite kernels taking values in $\mathcal{B}^*(\mathcal{H})$, obtained in Proposition 3.2. of [17], where \mathcal{H} is a VH-space and $\mathcal{B}^*(\mathcal{H})$ is the C^* -algebra of all adjointable and bounded, in Loynes sense, operators on \mathcal{H} , cf. [27]. Consequently, it is natural to ask to which extent of generality condition (b2) is automatically satisfied or not. Here, we show a rather general class of $\mathcal{L}_c^*(\mathcal{H})$ valued kernels, for an arbitrary topological VE-space \mathcal{H} , that guarantees the validity of condition (b2).

We first prove a Schwarz type inequality for positive operators.

Lemma 2.13. *Let $T \in \mathcal{L}^*(\mathcal{H})$ be a positive operator on a topological VE-space \mathcal{H} over the topologically ordered $*$ -space Z . Let $p \in S(Z)$. Then*

$$p([Th, h]) \leq p([Th, Th])^{\frac{1}{2}} p([h, h])^{\frac{1}{2}}, \quad h \in \mathcal{H}.$$

Proof. For any $h_1, h_2 \in \mathcal{H}$ and any number $\lambda > 0$ we have

$$\begin{aligned} 0 &\leq [h_1 - \lambda h_2, h_1 - \lambda h_2] \\ &= [h_1, h_1] + \lambda^2 [h_2, h_2] - \lambda [h_2, h_1] - \lambda [h_1, h_2]. \end{aligned}$$

By definition of the partial ordering on Z and dividing by λ we obtain

$$[h_2, h_1] + [h_1, h_2] \leq \frac{1}{\lambda} [h_1, h_1] + \lambda [h_2, h_2]. \quad (2.21)$$

Letting $h_1 := h$, $h_2 := Th$ in (2.21), since T is positive, applying p on both sides of (2.21), and taking into account that p is increasing, we obtain

$$2p([Th, h]) \leq \frac{1}{\lambda} p([h, h]) + \lambda p([Th, Th]), \quad h \in \mathcal{H}. \quad (2.22)$$

Since the left-hand side in (2.22) does not depend on $\lambda > 0$, it follows that, for any $h \in \mathcal{H}$, we have

$$2p([Th, h]) \leq \inf_{\lambda > 0} \frac{1}{\lambda} p([h, h]) + \lambda p([Th, Th]) = p([Th, Th])^{\frac{1}{2}} p([h, h])^{\frac{1}{2}}. \quad \square$$

We now reformulate Lemma 2.13 in case of a positive semidefinite kernel.

Lemma 2.14. *Let $\mathbf{k}: X \times X \rightarrow \mathcal{L}^*(\mathcal{H})$ be a positive semidefinite kernel, where \mathcal{H} is a topological VE-space over a topologically ordered $*$ -space Z . Then for every $p \in S(Z)$, $n \in \mathbb{N}$, $x, \{y_i\}_{i=1}^n \in X$ and $\{h_i\}_{i=1}^n \in \mathcal{H}$ we have the inequality*

$$\begin{aligned} & p\left(\sum_{i,j=1}^n [\mathbf{k}(x, y_i)h_i, \mathbf{k}(x, y_j)h_j]\right) \\ & \leq p\left(\sum_{i,j=1}^n [\mathbf{k}(x, x)\mathbf{k}(x, y_i)h_i, \mathbf{k}(x, y_j)h_j]\right)^{\frac{1}{2}} p\left(\sum_{i,j=1}^n [\mathbf{k}(y_i, y_j)h_j, h_i]\right)^{\frac{1}{2}}. \end{aligned}$$

Proof. By Theorem 2.3, there is a minimal VE-space linearisation $(\mathcal{K}; \pi; V)$ of \mathbf{k} , where $\Gamma = \{\epsilon\}$ is the trivial $*$ -group and the unital action of Γ on X . For each fixed $x \in X$, consider the positive operator $T := V(x)V(x)^*: \mathcal{K} \rightarrow \mathcal{K}$ and for arbitrary $\{y_i\}_{i=1}^n \in X$ and all $\{h_i\}_{i=1}^n \in \mathcal{H}$ the corresponding element $h := \sum_{i=1}^n V(y_i)h_i \in \mathcal{K}$. Given any $p \in S(Z)$, by applying Lemma 2.13 for these T and h , and taking into account that $V(z)^*V(t) = \mathbf{k}(z, t)$ for any $z, t \in X$, we obtain the required inequality. \square

For a topological VE-space \mathcal{H} over the topologically ordered $*$ -space Z , following [52], an operator $T \in \mathcal{L}(\mathcal{H})$ is called *m-topologisable* if for every $p \in S(Z)$, there exists a constant $D_p \geq 0$ and a continuous seminorm r on \mathcal{H} such that, for every $n \in \mathbb{N}$ and every $h \in \mathcal{H}$,

$$\tilde{p}(T^n h) = p([T^n h, T^n h])^{\frac{1}{2}} \leq D_p^n r(h). \quad (2.23)$$

Observe that, m-topologisable operators are those continuous linear operators $T: \mathcal{H} \rightarrow \mathcal{H}$ for which there is a certain control of the growth of their powers uniformly on \mathcal{H} .

The inequality in the following lemma, which can be viewed as a stronger version of the inequality from Lemma 1.6 for the special case of an m-topologisable positive operator, is a generalisation of the celebrated Krein-Reid-Lax-Dieudonné inequality; the iteration method through which we obtain it is following a similar idea as that in [24, 25, 40], and [11]. This iteration method is used also in Proposition 2.6 in [36] in order to prove that any adjointable everywhere defined operator on a Hilbert C^* -module is bounded in the sense of Loynes [28].

Lemma 2.15. *Let $T \in \mathcal{L}^*(\mathcal{H})$ be an m-topologisable positive operator on a topological VE-space \mathcal{H} over the topologically ordered $*$ -space Z . Let $p \in S(Z)$. Then there is a constant C , depending only on T and p , such that*

$$p([Th, h]) \leq C p([h, h]), \quad h \in \mathcal{H}.$$

Proof. Using the fact $[T^{2^n}h, T^{2^n}h] = [T^{2^{n+1}}h, h]$ for any $h \in \mathcal{H}$ and any $n \in \mathbb{N}$, as well as successively applying Lemma 2.13, we obtain

$$\begin{aligned} p([Th, h]) &\leq p([Th, Th])^{\frac{1}{2}} p([h, h])^{\frac{1}{2}} \\ &\leq p([T^2h, T^2h])^{\frac{1}{4}} p([h, h])^{\frac{1}{2} + \frac{1}{4}} \\ &\vdots \\ &\leq p([T^{2^n}h, T^{2^n}h])^{\frac{1}{2^{n+1}}} p([h, h])^{\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{n+1}}} \\ &\leq D_p^{\frac{2^n}{2^n}} r(h)^{\frac{1}{2^n}} p([h, h])^{\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{n+1}}}, \end{aligned}$$

where the last inequality follows from the m -topologisability of T , with some constant D_p . Taking limits as $n \rightarrow \infty$, we obtain the required inequality with $C = D_p$. \square

Remark 2.16. The conclusion of Lemma 2.15 can be obtained for a class of positive operators T larger than that of m -topologisable ones, namely, in (2.23) it is sufficient that $r: \mathcal{H} \rightarrow [0, +\infty)$ is an arbitrary function.

It now follows that if an m -topologisability condition is imposed on the kernel \mathbf{k} , a stronger inequality than that in condition (b2) of Theorem 2.10 is obtained. In particular, this kind of kernels always have VH-space linearisation, equivalently, their reproducing kernel VH-spaces always exist.

Proposition 2.17. *Let $\mathbf{k}: X \times X \rightarrow \mathcal{L}^*(\mathcal{H})$ be a positive semidefinite kernel, where \mathcal{H} is a topological VE-space over a topologically ordered $*$ -space Z . Assume that for every $x \in X$, the operator $\mathbf{k}(x, x)$ is m -topologisable. Then, for any $x \in X$ and any seminorm $p \in S(Z)$, there exists a constant $c_p(x) \geq 0$ such that for all $n \in \mathbb{N}$, $\{y_i\}_{i=1}^n \in X$, $\{h_i\}_{i=1}^n \in \mathcal{H}$ we have*

$$p\left(\sum_{i,j=1}^n [\mathbf{k}(x, y_i)h_i, \mathbf{k}(x, y_j)h_j]\right) \leq c_p(x) p\left(\sum_{i,j=1}^n [\mathbf{k}(y_j, y_i)h_i, h_j]\right).$$

Proof. Since $\mathbf{k}(x, x)$ is an m -topologisable positive operator, by taking $T := \mathbf{k}(x, x)$ and $h := \sum_{i=1}^n \mathbf{k}(x, y_i)h_i$ in Lemma 2.15, for some constant $c_p(x) \geq 0$ we have

$$p\left(\sum_{i,j=1}^n [\mathbf{k}(x, x)\mathbf{k}(x, y_i)h_i, \mathbf{k}(x, y_j)h_j]\right) \leq c_p(x) p\left(\sum_{i,j=1}^n [\mathbf{k}(x, y_i)h_i, \mathbf{k}(x, y_j)h_j]\right). \quad (2.24)$$

Then, by Lemma 2.14, we have

$$\begin{aligned} p\left(\sum_{i,j=1}^n [\mathbf{k}(x, y_i)h_i, \mathbf{k}(x, y_j)h_j]\right) &\leq p\left(\sum_{i,j=1}^n [\mathbf{k}(x, x)\mathbf{k}(x, y_i)h_i, \mathbf{k}(x, y_j)h_j]\right)^{\frac{1}{2}} \\ &\quad \times p\left(\sum_{i,j=1}^n [\mathbf{k}(y_i, y_j)h_j, h_i]\right)^{\frac{1}{2}} \end{aligned}$$

whence, by (2.24),

$$\begin{aligned} &\leq c_p(x)^{1/2} p\left(\sum_{i,j=1}^n [\mathbf{k}(x, y_i)h_i, \mathbf{k}(x, y_j)h_j]\right)^{1/2} \\ &\quad \times p\left(\sum_{i,j=1}^n [\mathbf{k}(y_i, y_j)h_j, h_i]\right)^{\frac{1}{2}}. \end{aligned} \quad (2.25)$$

A standard argument implies now the required inequality. \square

Remark 2.18. The inequality in Proposition 2.17 is stronger than condition (b2) in Theorem 2.10 and one can ask whether the inequality obtained in Lemma 1.6, which does not require any extra condition on the positive operator T , may be used instead of Lemma 2.15, in order to obtain the validity of the inequality in the condition (b2), in general. Unfortunately, an inspection of the proof of Proposition 2.17, more precisely (2.25), shows that this is not the case and, if condition (b2) has to be proven in general, this way does not work and probably a completely new idea is needed. On the other hand, we do not have a counter-example of positive semidefinite kernels for which condition (b2) does not hold: in view of [6], such a counter-example should be very pathological, if exists. The question formulated at the beginning of this subsection remains open.

The next proposition shows that under very general assumptions of positivity, an m -topologisable diagonal of the kernel propagates an even stronger continuity property throughout the kernel.

Proposition 2.19. *Let $\mathbf{k}: X \times X \rightarrow \mathcal{L}_c^*(\mathcal{H})$ be a 2-positive kernel, for some topological VE -space space \mathcal{H} over a topologically ordered $*$ -space Z . If $\mathbf{k}(x, x)$ is m -topologisable for all $x \in X$ then, for any $x, y \in X$ and any $p \in S(Z)$, there exists $C \geq 0$ such that, for all $h \in \mathcal{H}$ the following inequality holds*

$$p([\mathbf{k}(y, x)h, \mathbf{k}(y, x)h]) \leq C p([h, h]). \quad (2.26)$$

In particular, the linear operator $\mathbf{k}(y, x)$ is m -topologisable for all $x, y \in X$.

Proof. Let us fix $x, y \in X$ and $p \in S(Z)$, and let $h, g \in \mathcal{H}$ vary. By the 2-positivity assumption, we have

$$[\mathbf{k}(x, y)g, h] + [\mathbf{k}(y, x)h, g] \leq [\mathbf{k}(x, x)h, h] + [\mathbf{k}(y, y)g, g],$$

and take $g = C_y^{-1}\mathbf{k}[y, x]h$, where $C_y > 0$ is a constant as in Lemma 2.15 when applied to the m -topologisable and positive operator $T = \mathbf{k}(y, y)$. Then, taking into account that, by Lemma 2.1, $\mathbf{k}(x, y) = \mathbf{k}(y, x)^*$, it follows

$$2C_y^{-1}[\mathbf{k}(y, x)h, \mathbf{k}(y, x)h] \leq [\mathbf{k}(x, x)h, h] + C_y^{-2}[\mathbf{k}(y, y)\mathbf{k}(y, x)h, \mathbf{k}(y, x)h],$$

and, since the left side is in Z^+ and p is increasing, we obtain

$$\begin{aligned}
2C_y^{-1}p([\mathbf{k}(y, x)h, \mathbf{k}(y, x)h]) &\leq p([\mathbf{k}(x, x)h, h]) + C_y^{-2}p([\mathbf{k}(y, y)\mathbf{k}(y, x)h, \mathbf{k}(y, x)h]) \\
&\leq p([\mathbf{k}(x, x)h, h]) + C_y^{-1}p([\mathbf{k}(y, x)h, \mathbf{k}(y, x)h]) \\
&\leq C_x p([h, h]) + C_y^{-1}p([\mathbf{k}(y, x)h, \mathbf{k}(y, x)h]),
\end{aligned}$$

which provides the required inequality (2.26), with $C = C_x C_y$.

Finally, given arbitrary $x, y \in X$, for any $p \in S(Z)$ and any natural number $n \geq 1$, by iterating the inequality (2.26) n times we get

$$p([\mathbf{k}(y, x)^n h, \mathbf{k}(y, x)^n h]) \leq C^n p([h, h]), \quad h \in \mathcal{H},$$

hence $\mathbf{k}(y, x)$ is m-topologisable. \square

Remark 2.20. The conclusion of Proposition 2.19 can be obtained as a consequence of Proposition 2.17 if the assumption of 2-positivity of the kernel is replaced by its positive semidefiniteness.

2.6. Kernels with Values Adjointable Operators on VH-Modules

In the following we point out an application of Theorem 2.10 to linear maps with values adjointable operators on VH-modules over admissible $*$ -algebras. By definition, an *admissible $*$ -algebra* \mathcal{A} is a $*$ -algebra that is, in the same time, an admissible space. A *VH-module* \mathcal{E} over an admissible $*$ -algebra \mathcal{A} is, by definition, a VH-space over \mathcal{A} , viewed as an admissible space, which is a VE-module, as well. Given a VH-module \mathcal{E} over an admissible $*$ -algebra \mathcal{A} , an \mathcal{H} -reproducing kernel VH-module over \mathcal{A} is just an \mathcal{E} -reproducing kernel VH-space over \mathcal{A} , with definition as in Sect. 2.3, which is also a VH-module over \mathcal{A} . We have the following consequence of Theorem 2.10 and Proposition 2.4.

Proposition 2.21. *Let Γ be a $*$ -semigroup that acts on the nonempty set X and let $\mathbf{k}: X \times X \rightarrow \mathcal{L}_c^*(\mathcal{H})$ be a kernel, for some VH-module \mathcal{H} over an admissible $*$ -algebra \mathcal{A} . Then, assertion (1) in Theorem 2.10 is equivalent with each of the following assertions:*

- (2) \mathbf{k} has a Γ -invariant VH-module (over \mathcal{A}) linearisation $(\mathcal{K}; \pi; V)$.
- (3) \mathbf{k} admits an \mathcal{H} -reproducing kernel VH-module \mathcal{R} and there exists a $*$ -representation $\rho: \Gamma \rightarrow \mathcal{L}_c^*(\mathcal{R})$ such that $\rho(\xi)\mathbf{k}_x h = \mathbf{k}_{\xi \cdot x} h$ for all $\xi \in \Gamma$, $x \in X$, $h \in \mathcal{H}$.

In addition, in case any of the assertions (1), (2), or (3) holds, then a minimal Γ -invariant VH-module linearisation can be constructed, any minimal Γ -invariant VH-module linearisation is unique up to unitary equivalence, a pair $(\mathcal{R}; \rho)$ as in assertion (3) with \mathcal{R} minimal can be always obtained and, in this case, it is uniquely determined by \mathbf{k} as well.

If $\varphi: \mathcal{B} \rightarrow \mathcal{L}_c^*(\mathcal{H})$ is a linear map, for some $*$ -algebra \mathcal{B} and some VH-module \mathcal{H} over an admissible $*$ -algebra \mathcal{A} , one can define a kernel $\mathbf{k}: \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{L}_c^*(\mathcal{H})$ by

$$\mathbf{k}(a, b) = \varphi(a^* b), \quad a, b \in \mathcal{B}. \quad (2.27)$$

It is immediate to verify that, letting the $*$ -semigroup \mathcal{B} act on itself by multiplication, \mathbf{k} is \mathcal{B} -invariant, in the sense of (2.7). Consequently, the following holds.

Corollary 2.22. *Let $\varphi: \mathcal{B} \rightarrow \mathcal{L}_c^*(\mathcal{H})$ be a linear map, for some $*$ -algebra \mathcal{B} and some VH -module \mathcal{H} over an admissible $*$ -algebra \mathcal{A} . The following assertions are equivalent:*

- (1) *The map φ is positive semidefinite, in the sense that the kernel \mathbf{k} defined at (2.27) is positive semidefinite, and*
- (b1) *For any $b \in \mathcal{B}$ and any seminorm $p \in S(\mathcal{A})$, there exist a seminorm $q \in S(\mathcal{A})$ and a constant $c_p(b) \geq 0$ such that, for all $n \in \mathbb{N}$, $\{h_i\}_{i=1}^n \in \mathcal{H}$, $\{a_i\}_{i=1}^n \in \mathcal{B}$, we have*

$$p\left(\sum_{i,j=1}^n [\varphi(a_i^* b^* b a_j) h_j, h_i]_{\mathcal{H}}\right) \leq c_p(b) q\left(\sum_{i,j=1}^n [\varphi(a_i^* a_j) h_j, h_i]_{\mathcal{H}}\right).$$

- (b2) *For any $b \in \mathcal{B}$ and any seminorm $p \in S(\mathcal{A})$, there exist a seminorm $q \in S(\mathcal{A})$ and a constant $c_p(b) \geq 0$ such that, for all $n \in \mathbb{N}$, $\{a_i\}_{i=1}^n \in \mathcal{B}$, $\{h_i\}_{i=1}^n \in \mathcal{H}$, we have*

$$p\left(\sum_{i,j=1}^n [\varphi(b^* a_i) h_i, \varphi(b^* a_j) h_j]_{\mathcal{H}}\right) \leq c_p(b) q\left(\sum_{i,j=1}^n [\varphi(a_j^* a_i) h_i, h_j]_{\mathcal{H}}\right).$$

- (2) *There exist a VH -module \mathcal{K} over the admissible $*$ -algebra \mathcal{A} , a linear map $V: \mathcal{B} \rightarrow \mathcal{L}_c^*(\mathcal{H}, \mathcal{K})$, and a $*$ -representation $\pi: \mathcal{B} \rightarrow \mathcal{L}_c^*(\mathcal{K})$, such that:*

- (i) $\varphi(a^* b) = V(a)^* V(b)$ for all $a, b \in \mathcal{B}$.
- (ii) $V(ab) = \pi(a) V(b)$ for all $a, b \in \mathcal{B}$.

In addition, if this happens, then the triple $(\mathcal{K}; \pi; V)$ can always be chosen minimal, in the sense that \mathcal{K} is the closed linear span of the set $V(\mathcal{B})\mathcal{H}$, and any two minimal triples as before are unique, modulo unitary equivalence.

- (3) *There exist an \mathcal{H} -reproducing kernel VH -module \mathcal{R} on \mathcal{A} and a $*$ -representation $\rho: \mathcal{B} \rightarrow \mathcal{L}_c^*(\mathcal{R})$ such that:*

- (i) \mathcal{R} has the reproducing kernel $\mathcal{B} \times \mathcal{B} \ni (a, b) \mapsto \varphi(a^* b) \in \mathcal{L}^*(\mathcal{H})$.
- (ii) $\rho(a)\varphi(\cdot)b = \varphi(\cdot ab)$ for all $a, b \in \mathcal{B}$ and $h \in \mathcal{H}$.

In addition, the reproducing kernel VH -module \mathcal{R} as in (3) can always be constructed minimal and in this case it is uniquely determined by φ .

In case the $*$ -algebra \mathcal{B} is unital, Corollary 2.22 takes a form that is closer to a topological version of Kasparov's Theorem [23] and its generalisation [20].

Corollary 2.23. *Let \mathcal{B} be a unital $*$ -algebra and $\varphi: \mathcal{A} \rightarrow \mathcal{L}_c^*(\mathcal{H})$ a linear map, for some VH -module \mathcal{H} over an ordered $*$ -algebra \mathcal{A} . Then, assertion (1) in Corollary 2.22 is equivalent with*

- (2)' *There exist a VH -module \mathcal{K} over \mathcal{A} , a $*$ -representation $\pi: \mathcal{B} \rightarrow \mathcal{L}_c^*(\mathcal{K})$, and $W \in \mathcal{L}_c^*(\mathcal{H}, \mathcal{K})$ such that*

$$\varphi(b) = W^* \pi(b) W, \quad b \in \mathcal{B}. \quad (2.28)$$

In addition, if this happens, then the triple $(\mathcal{K}; \pi; W)$ can always be chosen minimal, in the sense that \mathcal{K} is the closed linear span of the set $\pi(\mathcal{A})W\mathcal{H}$, and any two minimal triples as before are unique, modulo unitary equivalence.

3. *-Representations on Hilbert Modules over Locally C^* -Algebras

In the following we specialise to the case when \mathcal{H} is a Hilbert module over a locally C^* -algebra. After a review of preliminary material on locally C^* -algebras and Hilbert modules over locally C^* -algebras, we show, by an application of Proposition 2.17, that the boundedness condition (b2) in Theorem 2.10 is automatic in this case. Then, as an application, we show how the Kasparov type dilation theorem in [20] can be proven from here in a rather direct way.

3.1. Hilbert Modules Over Locally C^* -Algebras

A $*$ -algebra \mathcal{A} that has a complete Hausdorff topology induced by a family of C^* -seminorms, that is, seminorms p on \mathcal{A} that satisfy the C^* -condition $p(a^*a) = p(a)^2$ for all $a \in \mathcal{A}$, is called a *locally C^* -algebra* [19] (equivalent names are *(Locally Multiplicatively Convex) LMC*-algebras* [30, 43], or *b^* -algebra* [1, 2], or *pro C^* -algebra* [48]), [39]. Note that, any C^* -seminorm is *submultiplicative*, $p(ab) \leq p(a)p(b)$ for all $a, b \in \mathcal{A}$, cf. [44], and *$*$ -invariant*, $p(a^*) = p(a)$ for all $a \in \mathcal{A}$. Denote the collection of all continuous C^* -seminorms by $S_*(\mathcal{A})$. Then $S_*(\mathcal{A})$ is a directed set under pointwise maximum seminorm, namely, given $p, q \in S_*(\mathcal{A})$, letting $r(a) := \max\{p(a), q(a)\}$ for all $a \in \mathcal{A}$, then r is a continuous C^* -seminorm and $p, q \leq r$. Locally C^* -algebras were studied in [1, 2, 19, 39, 43], and [53], to cite a few.

It follows from Corollary 2.8 in [19] that any locally C^* -algebra is, in particular, an admissible space, more precisely, a directed family of increasing seminorms generating the topology in axiom (a5') in Sect. 1.2 is $S_*(\mathcal{A})$. Note that $S_*(\mathcal{A}) \subset S(\mathcal{A})$ and, although they generate the same topology on \mathcal{A} , these two sets are quite different. For instance, while $S(\mathcal{A})$ is a cone, $S_*(\mathcal{A})$ is not even stable under positive scalar multiplication.

By $b(\mathcal{A})$ we denote the C^* -algebra of all bounded elements in \mathcal{A} , i.e. all $a \in \mathcal{A}$ such that $\|a\|_\infty := \sup\{p(a) \mid p \in S_*(\mathcal{A})\} < \infty$. Then $\|a\|_\infty$ defines a C^* -norm on $b(\mathcal{A})$. Also, $b(\mathcal{A})$ is dense in \mathcal{A} , see [39] or [13].

An *approximate unit* of \mathcal{A} is an increasing net $(e_j)_{j \in \mathcal{J}}$ of positive elements in \mathcal{A} with $p(e_j) \leq 1$ for any $p \in S_*(\mathcal{A})$ and any $j \in \mathcal{J}$, satisfying $p(x - xe_j) \xrightarrow{j} 0$ and $p(x - e_jx) \xrightarrow{j} 0$ for all $p \in S_*(\mathcal{A})$ and all $x \in \mathcal{A}$. For any locally C^* -algebra, there exists an approximate unit, cf. [19, 39].

A *pre-Hilbert module over a locally C^* -algebra \mathcal{A}* , or a *pre-Hilbert \mathcal{A} -module* is a topological VE-module \mathcal{H} over \mathcal{A} . Note that the topology on the pre-Hilbert \mathcal{A} -module \mathcal{H} is given by the family of seminorms $\{\tilde{p}\}_{p \in S_*(\mathcal{A})}$, where $\tilde{p}(h) = p([h, h])^{1/2}$ for all $p \in S_*(\mathcal{A})$ and all $h \in \mathcal{H}$. A pre-Hilbert \mathcal{A} -module \mathcal{H} is called a *Hilbert \mathcal{A} -module* if it is complete, e.g. see [39].

Let \mathcal{H} be a pre-Hilbert \mathcal{A} -module, let $p \in S_*(\mathcal{A})$ and let $x, y \in \mathcal{H}$. Then a Schwarz type inequality holds, e.g. see [53], as follows

$$p([h, k]_{\mathcal{H}}) \leq p([h, h]_{\mathcal{H}})^{1/2} p([k, k]_{\mathcal{H}})^{1/2}, \quad h, k \in \mathcal{H}. \quad (3.1)$$

For a Hilbert \mathcal{A} -module \mathcal{H} and $p \in S_*(\mathcal{A})$, denote $\mathcal{I}_p^{\mathcal{A}} := \{a \in \mathcal{A} \mid p(a) = 0\}$, or simply \mathcal{I}_p when there will be no danger of confusion on the

ambient locally C^* -algebra, and $\tilde{\mathcal{I}}_p^{\mathcal{H}} := \{x \in \mathcal{H} \mid [x, x] \in \mathcal{I}_p\}$, or simply $\tilde{\mathcal{I}}_p$. Then \mathcal{I}_p is a closed $*$ -ideal in \mathcal{A} and it is known, cf. [2], that the quotient $\mathcal{A}_p := \mathcal{A}/\mathcal{I}_p$ is a C^* -algebra with C^* -norm $\|a + \mathcal{I}_p\|_{\mathcal{A}_p} := p(a)$ for $a \in \mathcal{A}$. Also, $\tilde{\mathcal{I}}_p$ is a closed \mathcal{A} -submodule in \mathcal{H} and the quotient module $\mathcal{H}_p := \mathcal{H}/\tilde{\mathcal{I}}_p$ is a Hilbert module over the C^* -algebra \mathcal{A}_p , with module action given by

$$(h + \tilde{\mathcal{I}}_p)(a + \mathcal{I}_p) := ha + \tilde{\mathcal{I}}_p, \quad h \in \mathcal{H}, \quad a \in \mathcal{A},$$

and gramian given by

$$[h + \tilde{\mathcal{I}}_p, k + \tilde{\mathcal{I}}_p]_{\mathcal{H}_p} := [h, k]_{\mathcal{H}} + \mathcal{I}_p, \quad h, k \in \mathcal{H}, \quad a \in \mathcal{A}.$$

On the other hand, when \mathcal{H} and \mathcal{K} are Hilbert modules over the same locally C^* -algebra \mathcal{A} , the space of all adjointable linear operators $T: \mathcal{H} \rightarrow \mathcal{K}$, denoted by $\mathcal{L}^*(\mathcal{H}, \mathcal{K})$, has some additional properties, when compared to VH-spaces. Any operator $T \in \mathcal{L}^*(\mathcal{H}, \mathcal{K})$ is automatically a module map and continuous, cf. [49] or Lemma 3.2 in [53], in particular, $T(h \cdot a) = T(h) \cdot a$ for all $h \in \mathcal{H}$, $a \in \mathcal{A}$ and $\mathcal{L}^*(\mathcal{H}, \mathcal{K}) = \mathcal{L}_c^*(\mathcal{H}, \mathcal{K})$, see Sect. 1.3 for notation.

For fixed $p \in S_*(\mathcal{A})$, any operator $T \in \mathcal{L}^*(\mathcal{H}, \mathcal{K})$ induces an adjointable, hence a continuous module map operator T_p from the Hilbert \mathcal{A}_p -module \mathcal{H}_p to the Hilbert \mathcal{A}_p -module \mathcal{K}_p , via

$$T_p(h + \tilde{\mathcal{I}}_p^{\mathcal{H}}) := Th + \tilde{\mathcal{I}}_p^{\mathcal{K}}, \quad h \in \mathcal{H}, \quad (3.2)$$

with adjoint

$$T_p^*(k + \tilde{\mathcal{I}}_p^{\mathcal{K}}) := T^*k + \tilde{\mathcal{I}}_p^{\mathcal{H}}, \quad k \in \mathcal{K}. \quad (3.3)$$

Moreover, there is a constant $C \geq 0$ such that

$$\tilde{p}_{\mathcal{K}}(Th) \leq C \tilde{p}_{\mathcal{H}}(h), \quad h \in \mathcal{H}, \quad (3.4)$$

see [39] and [53].

A topology on $\mathcal{L}^*(\mathcal{H}, \mathcal{K})$ can be defined via the collection of seminorms $\{\bar{p}_{\mathcal{H}, \mathcal{K}}\}_{p \in S_*(\mathcal{A})}$: for arbitrary $p \in S_*(\mathcal{A})$,

$$\bar{p}_{\mathcal{H}, \mathcal{K}}(T) := \|T_p\|, \quad T \in \mathcal{L}^*(\mathcal{H}, \mathcal{K}), \quad (3.5)$$

where $\|\cdot\|$ denotes the operator norm in $\mathcal{L}^*(\mathcal{H}_p, \mathcal{K}_p)$, equivalently, $\|T_p\|$ is the infimum of all $C \geq 0$ satisfying inequality (3.4). For the case $\mathcal{H} = \mathcal{K}$, these seminorms become C^* -seminorms and they turn $\mathcal{L}^*(\mathcal{H})$ into a locally C^* -algebra, c.f. [39] and [53].

For a locally C^* -algebra \mathcal{A} , let $M_n(\mathcal{A})$ denote the $*$ -algebra of all $n \times n$ matrices over \mathcal{A} . $M_n(\mathcal{A})$ becomes a locally C^* -algebra considered with the topology generated by the C^* -seminorms

$$p_n([a_{ij}]_{i,j=1}^n) := \|[a_{ij} + \mathcal{I}_p]_{i,j=1}^n\|_{M_n(\mathcal{A}_p)}, \quad [a_{ij}]_{i,j=1}^n \in M_n(\mathcal{A}),$$

where $\|\cdot\|_{M_n(\mathcal{A}_p)}$ is the C^* -norm on the C^* -algebra $M_n(\mathcal{A}_p)$.

3.2. Kernels with Values Adjointable Operators in Hilbert Locally C^* -Modules

Let \mathcal{H} be a Hilbert module over a locally C^* -algebra \mathcal{A} and $\mathbf{k}: X \times X \rightarrow \mathcal{L}^*(\mathcal{H})$ a positive semidefinite kernel. Then, for each seminorm $p \in S_*(\mathcal{A})$, a kernel

$$\mathbf{k}_p: X \times X \rightarrow \mathcal{L}^*(\mathcal{H}_p), \quad \mathbf{k}_p(x, y) := \mathbf{k}(x, y)_p \quad \text{for all } x, y \in X \quad (3.6)$$

is defined, where $\mathbf{k}(x, y)_p$ is defined as in (3.2). It is easy to see that \mathbf{k}_p is positive semidefinite.

An \mathcal{H} -reproducing kernel Hilbert \mathcal{A} -module \mathcal{R} is a Hilbert \mathcal{A} -module, which satisfies, along with (vhrk2) and (vhrk3),

(vhrk1)' \mathcal{R} is a submodule of the \mathcal{A} -module $\mathcal{F}(X; \mathcal{H})$, with all algebraic operations.

Note that, in this case, the axiom (vhrk4) is automatically satisfied due to the fact that, in the case of Hilbert modules over locally C^* -algebras, any adjointable, hence continuous, linear operator has continuous adjoint.

The following lemma shows that, in this special case of kernels with values adjointable operators on Hilbert modules over locally C^* -algebras, the boundedness condition (b2) in Theorem 2.10 is automatic, using Proposition 2.17.

Lemma 3.1. *Let \mathcal{A} be a locally C^* -algebra, let \mathcal{H} be a Hilbert \mathcal{A} -module, let X be a nonempty set and let $\mathbf{k}: X \times X \rightarrow \mathcal{L}^*(\mathcal{H})$ be a positive semidefinite kernel. Then for any seminorm $p \in S_*(\mathcal{A})$ and any $x \in X$ there exists a constant $c_p(x) \geq 0$ such that for all $\{y_i\}_{i=1}^n \in X$, $\{h_i\}_{i=1}^n \in \mathcal{H}$ we have*

$$p\left(\sum_{i,j=1}^n [\mathbf{k}(x, y_i)h_i, \mathbf{k}(x, y_j)h_j]_{\mathcal{H}}\right) \leq c_p(x) p\left(\sum_{i,j=1}^n [\mathbf{k}(y_j, y_i)h_i, h_j]_{\mathcal{H}}\right).$$

Proof. By Proposition 2.17, it is enough to show that $\mathbf{k}(x, x)$ is an m -topologisable operator for every $x \in X$. Letting $T := \mathbf{k}(x, x)$ in inequality (3.4), we get a constant $c_p(x) \geq 0$ such that

$$\tilde{p}(\mathbf{k}(x, x)h) \leq c_p(x)\tilde{p}(h), \quad h \in \mathcal{H}.$$

This gives, for all $n \in \mathbb{N}$ and $h \in \mathcal{H}$,

$$\begin{aligned} \tilde{p}(\mathbf{k}(x, x)^n h) &\leq c_p(x)\tilde{p}(\mathbf{k}(x, x)^{n-1}h) \\ &\leq \cdots \leq c_p(x)^{n-1}\tilde{p}(\mathbf{k}(x, x)h) \leq c_p(x)^n\tilde{p}(h), \end{aligned}$$

hence the operator $\mathbf{k}(x, x)$ is m -topologisable. \square

As a consequence of the previous lemma and Proposition 2.21, we have

Theorem 3.2. *Let Γ be a $*$ -semigroup that acts on the nonempty set X and let $\mathbf{k}: X \times X \rightarrow \mathcal{L}^*(\mathcal{H})$ be a kernel, for some Hilbert module \mathcal{H} over a locally C^* -algebra \mathcal{A} . Then the following assertions are equivalent:*

- (1) \mathbf{k} is positive semidefinite and invariant under the action of Γ on X and, additionally, the following condition hold:

- (b1) For any $\xi \in \Gamma$ and any seminorm $p \in S(Z)$, there exists a seminorm $q \in S_\ast(A)$ and a constant $c_p(\xi) \geq 0$ such that for all $n \in \mathbb{N}$, $\{h_i\}_{i=1}^n \in \mathcal{H}$, $\{x_i\}_{i=1}^n \in X$ we have

$$p\left(\sum_{i,j=1}^n [\mathbf{k}(\xi \cdot x_i, \xi \cdot x_j)h_j, h_i]_{\mathcal{H}}\right) \leq c_p(\xi) q\left(\sum_{i,j=1}^n [\mathbf{k}(x_i, x_j)h_j, h_i]_{\mathcal{H}}\right).$$

- (2) \mathbf{k} has a Γ -invariant Hilbert \mathcal{A} -module linearisation $(\mathcal{K}; \pi; V)$, that is,
 (ihl1) $(\mathcal{K}; V)$ is a Hilbert \mathcal{A} -module linearisation of \mathbf{k} .
 (ihl2) $\pi: \Gamma \rightarrow \mathcal{L}^\ast(\mathcal{H})$ is a \ast -representation.
 (ihl3) $V(\xi \cdot x) = \pi(\xi)V(x)$ for all $\xi \in \Gamma$ and all $x \in X$.
 (3) \mathbf{k} admits an \mathcal{H} -reproducing kernel Hilbert \mathcal{A} -module \mathcal{R} and there exists a \ast -representation $\rho: \Gamma \rightarrow \mathcal{L}^\ast(\mathcal{R})$ such that $\rho(\xi)\mathbf{k}_x h = \mathbf{k}_{\xi \cdot x} h$ for all $\xi \in \Gamma$, $x \in X$, $h \in \mathcal{H}$.

As a consequence of the previous theorem, it follows that positive semidefinite kernels with values adjointable operators on Hilbert modules over locally C^\ast -algebras always have Hilbert modules linearisations, equivalently, they admit reproducing kernel Hilbert modules.

Corollary 3.3. Let $\mathbf{k}: X \times X \rightarrow \mathcal{L}^\ast(\mathcal{H})$ be a kernel on a nonempty set X , for some Hilbert module \mathcal{H} over a locally C^\ast -algebra \mathcal{A} . Then the following assertions are equivalent:

- (1) \mathbf{k} is positive semidefinite.
- (2) \mathbf{k} has a Hilbert \mathcal{A} -module linearisation $(\mathcal{K}; V)$.
- (3) \mathbf{k} admits an \mathcal{H} -reproducing kernel Hilbert \mathcal{A} -module \mathcal{R} .

3.3. Completely Positive Maps

Let \mathcal{H} be a Hilbert \mathcal{A} -module for some locally C^\ast -algebra \mathcal{A} . Let \mathcal{B} be another locally C^\ast -algebra, let $\varphi: \mathcal{B} \rightarrow \mathcal{L}^\ast(\mathcal{H})$ be a linear map, and consider the kernel \mathbf{k} associated to φ as in (2.27), that is, $\mathbf{k}(a, b) = \varphi(a^\ast b)$ for all $a, b \in \mathcal{B}$. Then \mathbf{k} is invariant under the (multiplicative) action of \mathcal{B} on itself. Keeping in mind that, any \ast -algebra is, in particular, a (multiplicative) \ast -semigroup, note that a \mathcal{B} -invariant Hilbert module linearisation of \mathbf{k} simply is a \mathcal{B} -invariant VH-space linearisation $(\mathcal{K}; \pi; V)$ of \mathbf{k} , such that \mathcal{K} is a Hilbert \mathcal{A} -module.

Remark 3.4. Let \mathcal{C} and \mathcal{D} be locally C^\ast -algebras. For a linear map $\varphi: \mathcal{C} \rightarrow \mathcal{D}$, recall that φ is *completely positive* if for every $n \in \mathbb{N}$, the map $\varphi^{(n)}: M_n(\mathcal{C}) \rightarrow M_n(\mathcal{D})$, $[a_{ij}]_{i,j=1}^n \mapsto [\varphi(a_{ij})]_{i,j=1}^n$ is positive. Let \mathbf{k} be the kernel associated to φ as in (2.27). Then \mathbf{k} is positive semidefinite if and only if φ is completely positive. This follows from the fact that any positive matrix in $M_n(\mathcal{C})$ can be written as the sum of positive matrices of form $[x_i^\ast x_j]_{i,j=1}^n$, e.g. see [37].

Recall the definition of the strict topology on $\mathcal{L}^\ast(\mathcal{H}, \mathcal{K})$ for two VH-spaces \mathcal{H} and \mathcal{K} over the same admissible space Z in Sect. 1.2. Given a linear completely positive map $\varphi: \mathcal{B} \rightarrow \mathcal{L}^\ast(\mathcal{H})$ for \mathcal{B} a locally C^\ast -algebra, \mathcal{H} a VH-space over a topologically admissible space Z , we say that φ is *strict* if $(\varphi(e_i))_i$ is a Cauchy net in the strict topology for some approximate unit $(e_i)_i$ in \mathcal{B} .

Theorem 3.5 (Theorem 4.6 in [20]). *Let \mathcal{A} and \mathcal{B} be locally C^* -algebras and \mathcal{H} be a Hilbert module over \mathcal{A} . Let $\varphi : \mathcal{B} \rightarrow \mathcal{L}^*(\mathcal{H})$ be a linear map. Then the following are equivalent:*

- (1) *φ is a completely positive, strict, continuous map.*
- (2) *There exists \mathcal{K} a Hilbert module over \mathcal{A} , a continuous $*$ -representation $\pi : \mathcal{B} \rightarrow \mathcal{L}^*(\mathcal{K})$ and $W \in \mathcal{L}^*(\mathcal{H}, \mathcal{K})$ such that $\varphi(a) = W^* \pi(a) W$ for all $a \in \mathcal{B}$.*

Moreover, in case any of assertions (1), (2) holds, the space \mathcal{K} in (2) can be constructed minimal, in the sense that \mathcal{K} is the closure of $\text{Lin}\{\pi(b)Wh \mid b \in \mathcal{B}, h \in \mathcal{H}\}$, and any such minimal Hilbert module is unique up to unitary equivalence.

We show that this theorem can be obtained as a consequence of our Theorem 3.2 which tells us that, basically, we have to take care of two technical obstructions: the boundedness condition (b1) and the lack of unit of the algebra \mathcal{B} . We first prove two technical results that will be needed for solving the obstruction with the boundedness condition (b1).

The following lemma uses an idea from the proof of Theorem 2.4 in [32].

Lemma 3.6. *Let \mathcal{A} be a C^* -algebra and \mathcal{H} be a Hilbert C^* -module over \mathcal{A} . Let \mathcal{B} be a C^* -algebra and $\varphi : \mathcal{B} \rightarrow \mathcal{L}^*(\mathcal{H})$ be a completely positive map. Then, for any $b \in \mathcal{B}$ there exists a constant $c(b) \geq 0$ such that, for all $n \in \mathbb{N}$, $\{h_i\}_{i=1}^n \in \mathcal{H}$, $\{x_i\}_{i=1}^n \in \mathcal{B}$ we have*

$$\left\| \sum_{i,j=1}^n [\varphi(x_i^* b^* b x_j) h_j, h_i]_{\mathcal{H}} \right\|_{\mathcal{A}} \leq c(b) \left\| \sum_{i,j=1}^n [\varphi(x_i^* x_j) h_j, h_i]_{\mathcal{H}} \right\|_{\mathcal{A}}.$$

Proof. We use Theorem 2.3 with $\Gamma = X = \mathcal{B}$ and \mathcal{H} in order to obtain a \mathcal{B} -invariant VE-space linearisation $(\mathcal{K}; \pi; V)$, where \mathcal{K} is a VE-space over \mathcal{A} , $\pi : \mathcal{B} \rightarrow \mathcal{L}^*(\mathcal{K})$ is a $*$ -representation and $V : \mathcal{B} \rightarrow \mathcal{L}^*(\mathcal{H}, \mathcal{K})$. As in Proposition 2.4, \mathcal{K} is a VE-module over \mathcal{A} and, via (1.6), it is a pre-Hilbert \mathcal{A} -module.

Consider $\tilde{\mathcal{B}} = \mathcal{B} \oplus \mathbb{C}$, the unitization of the C^* -algebra \mathcal{B} . Then π extends uniquely to a $*$ -representation $\tilde{\pi} : \tilde{\mathcal{B}} \rightarrow \mathcal{L}^*(\mathcal{K})$, where $\tilde{\pi}((a, \lambda)) = \pi(a) + \lambda I_{\mathcal{K}}$. Let $u \in \tilde{\mathcal{B}}$ be a unitary element. It is straightforward to check that $\tilde{\pi}(u)$ is a unitary operator, hence continuous.

Now, consider arbitrary $b \in \mathcal{B}$ and let $u_i \in \tilde{\mathcal{B}}$ be unitary elements and $\lambda_i \in \mathbb{C}$ be scalars such that $b = \sum_{i=1}^m \lambda_i u_i$. Then

$$\pi(b) = \tilde{\pi}(b) = \tilde{\pi} \left(\sum_{i=1}^m \lambda_i u_i \right) = \sum_{i=1}^m \lambda_i \tilde{\pi}(u_i),$$

therefore $\pi(b) : \mathcal{K} \rightarrow \mathcal{K}$ is continuous. Taking into account that \mathcal{K} is topologised by the norm $\mathcal{K} \ni k \mapsto \|[k, k]_{\mathcal{K}}\|_{\mathcal{A}}$, this means that, there exists a constant $c(b) \geq 0$ such that

$$\|[\pi(b)k, \pi(b)k]_{\mathcal{K}}\| \leq c(b) \|[k, k]_{\mathcal{K}}\|_{\mathcal{A}}, \quad k \in \mathcal{K},$$

whence, in view of (2.8)–(2.13), we obtain the required inequality. \square

Lemma 3.7. *Let \mathcal{A} be a locally C^* -algebra and \mathcal{H} be a Hilbert module over \mathcal{A} . Let \mathcal{B} be a locally C^* -algebra and let $\varphi: \mathcal{B} \rightarrow \mathcal{L}^*(\mathcal{H})$ be a continuous and completely positive map. Then, for any $b \in \mathcal{B}$ and any $p \in S_*(\mathcal{A})$, there exists a constant $c_p(b) \geq 0$ such that, for all $n \in \mathbb{N}$, $\{h_i\}_{i=1}^n \in \mathcal{H}$, $\{x_i\}_{i=1}^n \in \mathcal{B}$, we have*

$$p\left(\sum_{i,j=1}^n [\varphi(x_i^* b^* b x_j) h_j, h_i]_{\mathcal{H}}\right) \leq c_p(b) p\left(\sum_{i,j=1}^n [\varphi(x_i^* x_j) h_j, h_i]_{\mathcal{H}}\right).$$

Proof. Throughout the proof, we fix $p \in S_*(\mathcal{A})$. Since $S_*(\mathcal{B})$ is directed and φ is continuous, we can find $r \in S_*(\mathcal{B})$ and $d_p \geq 0$ such that

$$\bar{p}(\varphi(x)) \leq d_p r(x), \text{ for all } x \in \mathcal{B}, \quad (3.7)$$

where the seminorm \bar{p} is defined as in (3.5). If $r(x) = 0$, for some $x \in \mathcal{B}$, by (3.7)

$$\bar{p}(\varphi(x)) \leq d_p r(x) = 0,$$

therefore $\bar{p}(\varphi(x)) = 0$, and hence $\varphi(x)_p = 0$ on \mathcal{H}_p . It follows that the map $\varphi_p: \mathcal{B}_r \rightarrow \mathcal{L}^*(\mathcal{H}_p)$ defined by

$$\varphi_p(b + \mathcal{I}_r^{\mathcal{B}}) := \varphi(b)_p, \quad b \in \mathcal{B}, \quad (3.8)$$

where $\mathcal{B}_r = \mathcal{B}/\mathcal{I}_r^{\mathcal{B}}$, is a well defined linear map. Moreover, φ_p is completely positive: this can be checked directly by considering the associated kernel and proving that it is positive semidefinite.

Finally, applying Lemma 3.6 for the map φ_p , we get that for any $b \in \mathcal{B}$, considering its coset $b + \mathcal{I}_r^{\mathcal{B}} \in \mathcal{B}_r$, there exists a constant $c_p(b + \mathcal{I}_r^{\mathcal{B}}) \geq 0$ such that for all $n \in \mathbb{N}$, all $h_1, \dots, h_n \in \mathcal{H}$, and all $x_1, \dots, x_n \in \mathcal{B}$, considering their cosets $\{h_i + \tilde{\mathcal{I}}_p\}_{i=1}^n \in \mathcal{H}_p$ and $\{x_i + \mathcal{I}_r^{\mathcal{B}}\}_{i=1}^n \in \mathcal{B}_r$, we have

$$\begin{aligned} p\left(\sum_{i,j=1}^n [\varphi(x_i^* b^* b x_j) h_j, h_i]_{\mathcal{H}}\right) &= \left\| \sum_{i,j=1}^n [\varphi_p((x_i^* b^* b x_j) + \mathcal{I}_r^{\mathcal{B}})(h_j + \tilde{\mathcal{I}}_p), (h_i + \tilde{\mathcal{I}}_p)]_{\mathcal{H}_p} \right\|_{\mathcal{A}_p} \\ &= \left\| \sum_{i,j=1}^n [\varphi_p((x_i + \mathcal{I}_r^{\mathcal{B}})^*(b + \mathcal{I}_r^{\mathcal{B}})^*(b + \mathcal{I}_r^{\mathcal{B}})(x_j + \mathcal{I}_r^{\mathcal{B}}))(h_j + \tilde{\mathcal{I}}_p), (h_i + \tilde{\mathcal{I}}_p)]_{\mathcal{H}_p} \right\|_{\mathcal{A}_p} \\ &\leq c_p(b + \mathcal{I}_r^{\mathcal{B}}) \left\| \sum_{i,j=1}^n [\varphi_p((x_i + \mathcal{I}_r^{\mathcal{B}})^*(x_j + \mathcal{I}_r^{\mathcal{B}}))(h_j + \tilde{\mathcal{I}}_p), (h_i + \tilde{\mathcal{I}}_p)]_{\mathcal{H}_p} \right\|_{\mathcal{A}_p} \\ &= c_p(b + \mathcal{I}_r^{\mathcal{B}}) p\left(\sum_{i,j=1}^n [\varphi(x_i^* x_j) h_j, h_i]_{\mathcal{H}}\right). \end{aligned}$$

Since once p is fixed r is also fixed, it is clear that we can write $c_p(b + \mathcal{I}_r^{\mathcal{B}}) = c_p(b)$, and the lemma is proven. \square

Proof of Theorem 3.5. (1) \Rightarrow (2). Consider the kernel $\mathbf{k}(a, b) = \varphi(a^* b)$, $a, b \in \mathcal{B}$. By Theorem 3.2, Lemma 3.7 and Lemma 3.1, we get a minimal \mathcal{B} -invariant Hilbert \mathcal{A} -module linearisation $(\mathcal{K}; \pi; V)$ of \mathbf{k} .

We check that V is linear. For $b_1, b_2, c \in \mathcal{B}$ and $\lambda \in \mathbb{C}$ we have

$$\begin{aligned} V(b_1 + \lambda b_2)^* V(c) &= \varphi((b_1 + \lambda b_2)^* c) = \varphi(b_1^* c) + \bar{\lambda} \varphi(b_2^* c) \\ &= V(b_1)^* V(c) + \bar{\lambda} V(b_2)^* V(c) = (V(b_1) + \lambda V(b_2))^* V(c) \end{aligned}$$

and, by the minimality of \mathcal{K} , it follows that $V(b_1 + \lambda b_2) = V(b_1) + \lambda V(b_2)$.

We show that $V : \mathcal{B} \rightarrow \mathcal{L}^*(\mathcal{H}, \mathcal{K})$ is continuous. By the continuity of φ , for any seminorm $p \in S_*(\mathcal{A})$, there exist $r \in S_*(\mathcal{B})$ and $c_p \geq 0$ such that

$$\bar{p}_{\mathcal{H}}(\varphi(b^* b)) \leq c_p r(b)^2, \quad b \in \mathcal{B}, \quad (3.9)$$

hence, for all $b \in \mathcal{B}$

$$\bar{p}_{\mathcal{H}, \mathcal{K}}(V(b))^2 = \|V(b)_p\|_{\mathcal{L}(\mathcal{H}_p, \mathcal{K}_p)}^2 = \|V(b)_p^* V(b)_p\|_{\mathcal{L}(\mathcal{H}_p)} = \bar{p}_{\mathcal{H}}(\varphi(b^* b)) \leq c_p r(b)^2.$$

This shows that V is continuous and hence the mapping $\mathcal{B} \ni b \mapsto V(b)^* \in \mathcal{L}^*(\mathcal{K}, \mathcal{H})$ is also continuous, since $\bar{p}_{\mathcal{H}, \mathcal{K}}(V(b)) = \bar{p}_{\mathcal{K}, \mathcal{H}}(V(b)^*)$ for all $p \in S_*(\mathcal{A})$.

Now let $(e_j)_{j \in \mathcal{J}}$ be an approximate unit of \mathcal{B} with respect to which φ is strict. Since $V(e_j)^* V(b) = \varphi(e_j b)$ and $e_j b \xrightarrow{j} b$ for any $b \in \mathcal{B}$, it follows that, for any $p \in S_*(\mathcal{A})$, $b \in \mathcal{B}$, $h \in \mathcal{H}$, we have

$$\tilde{p}_{\mathcal{H}}(V(e_j)^* V(b)h - \varphi(b)h) = \tilde{p}_{\mathcal{H}}(\varphi(e_j b - b)h) \leq \bar{p}_{\mathcal{H}}(\varphi(e_j b - b)) \tilde{p}_{\mathcal{H}}(h) \rightarrow 0.$$

It follows that $V(e_j)^* y$ converges to $\sum_{l=1}^n \varphi(b_l)h_l$ whenever $y = \sum_{l=1}^n V(b_l)h_l$, i.e. for all $y \in \mathcal{K}_0$. Let $p \in S_*(\mathcal{A})$. Since, $\mathcal{B} \ni b \mapsto V(b)^* \in \mathcal{L}^*(\mathcal{K}, \mathcal{H})$ is continuous, there exists $r \in S_*(\mathcal{B})$ such that

$$\bar{p}_{\mathcal{K}, \mathcal{H}}(V(e_i - e_j)^*) \leq d r(e_i - e_j) \leq c, \quad i, j \in \mathcal{J},$$

with $d \geq 0$ and $c > 0$ some constant numbers independent of $i, j \in \mathcal{J}$. Given $\epsilon > 0$, choose $y_0 \in \mathcal{K}_0$ such that

$$\tilde{p}_{\mathcal{K}}(y - y_0) \leq \frac{\epsilon}{2c}$$

and $i, j \in \mathcal{J}$ such that

$$\tilde{p}_{\mathcal{H}}(V(e_i - e_j)^* y_0) \leq \frac{\epsilon}{2}.$$

Using these inequalities we have

$$\begin{aligned} \tilde{p}_{\mathcal{H}}(V(e_i)^* y - V(e_j)^* y) &= p([V(e_i)^* y - V(e_j)^* y, V(e_i)^* y - V(e_j)^* y]_{\mathcal{H}})^{\frac{1}{2}} \\ &= p([V(e_i - e_j)^* y_0 + V(e_i - e_j)^*(y - y_0), V(e_i - e_j)^* y_0 + V(e_i - e_j)^*(y - y_0)]_{\mathcal{H}})^{\frac{1}{2}} \\ &= \tilde{p}_{\mathcal{H}}(V(e_i - e_j)^* y_0 + V(e_i - e_j)^*(y - y_0)) \\ &\leq \tilde{p}_{\mathcal{H}}(V(e_i - e_j)^* y_0) + \tilde{p}_{\mathcal{H}}(V(e_i - e_j)^*(y - y_0)) \\ &\leq \frac{\epsilon}{2} + \bar{p}_{\mathcal{K}, \mathcal{H}}(V(e_i - e_j)^*) \tilde{p}_{\mathcal{K}}(y - y_0) \\ &\leq \frac{\epsilon}{2} + c \frac{\epsilon}{2c} = \epsilon. \end{aligned}$$

Hence $(V(e_j)^* y)_{j \in \mathcal{J}}$ is a Cauchy net in \mathcal{H} for all $y \in \mathcal{K}$, hence convergent.

Now let $p \in S_*(\mathcal{B})$ and assume that $i, j \in \mathcal{J}$ are such that $j \leq i$, hence $0 \leq e_j \leq e_i$. Since $e_i, e_j \in b(\mathcal{B})$, the C^* -algebra of all bounded elements of \mathcal{B} , we have $(e_i - e_j) \leq (e_i - e_j)^2$, hence

$$\begin{aligned} \widetilde{p}_{\mathcal{K}}(V(e_i)h - V(e_j)h)^2 &= p([h, (V(e_i) - V(e_j))^*(V(e_i - e_j))h]_{\mathcal{H}}) \\ &= p([h, \varphi((e_i - e_j)^2)h]_{\mathcal{H}}) \\ &\leq p([h, \varphi(e_i - e_j)h]_{\mathcal{H}}). \end{aligned}$$

Since $(\varphi(e_j))_j$ is a Cauchy net for the strict topology of $\mathcal{L}^*(\mathcal{H})$, by Lemma 2.13 with $T := \varphi(e_i - e_j)$ and a standard argument, it follows that the net $(V(e_j)h)_j$ is Cauchy in \mathcal{K} . Hence we have that $(V(e_j))_j$ is a Cauchy net in $\mathcal{L}^*(\mathcal{H}, \mathcal{K})$. By Lemma 1.7, $\mathcal{L}^*(\mathcal{H}, \mathcal{K})$ with the strict topology is complete, hence there is $W \in \mathcal{L}^*(\mathcal{H}, \mathcal{K})$ such that $V(e_j)$ converges to W , with respect to the strict topology.

We prove now that the $*$ -representation $\pi: \mathcal{B} \rightarrow \mathcal{L}^*(\mathcal{K})$ is continuous. Let $p \in S_*(\mathcal{A})$ arbitrary. Since $\varphi: \mathcal{B} \rightarrow \mathcal{L}^*(\mathcal{H})$ is continuous, there exist $r \in S_*(\mathcal{B})$ and a constant $c_p \geq 0$ such that (3.7) holds. Define $\pi_p: \mathcal{B}_r \rightarrow \mathcal{L}^*(\mathcal{K}_p)$ by

$$\pi_p(b + \mathcal{I}_r^{\mathcal{B}}) := \pi(b)_p, \quad b \in \mathcal{B}. \quad (3.10)$$

In order for the definition in (3.10) to be correct, we have to show that, if $b \in \mathcal{B}$ is such that $r(b) = 0$ then $\pi(b)_p = 0$. Indeed, first observe that, since r is submultiplicative, from (3.9) it follows that, for any $x, y \in \mathcal{B}$, we have $\widetilde{p}_{\mathcal{H}}(\varphi(y^*bx)) = 0$, that is, $\varphi(y^*bx)h \in \widetilde{\mathcal{I}}_p$ for all $h \in \mathcal{H}$. Then, for arbitrary $h, g \in \mathcal{H}$ and $x, y \in \mathcal{B}$, we have

$$\begin{aligned} [\pi(b)_p(V(x)h + \widetilde{\mathcal{I}}_p), (V(y)g + \widetilde{\mathcal{I}}_p)]_{\mathcal{K}_p} &= [V(y)^*\pi(b)V(x)h, g]_{\mathcal{H}} + \mathcal{I}_p \\ &= [\varphi(y^*bx)h, g]_{\mathcal{H}} + \mathcal{I}_p = \mathcal{I}_p. \end{aligned}$$

Since \mathcal{K}_0 , the span of $V(\mathcal{B})\mathcal{H}$, is dense in \mathcal{K} , it follows that $\pi_p(b + \mathcal{I}_r^{\mathcal{B}}) = 0$ hence, π_p in (3.10) is correctly defined. It is easy to see that π_p is a $*$ -morphism of the C^* -algebra \mathcal{B}_r with values in the C^* -algebra $\mathcal{L}^*(\mathcal{K}_p)$, hence bounded. Letting $d_p = \|\pi_p\| \geq 0$, where $\|\pi_p\|$ denotes the operator norm of this $*$ -morphism π_p , it follows that

$$\widetilde{p}_{\mathcal{K}}(\pi(b)) \leq d_p r(b), \quad b \in \mathcal{B},$$

which proves the continuity of the $*$ -representation π .

For any $b \in \mathcal{B}$ and $h \in \mathcal{H}$, by the continuity of V and of $\pi(b)$, we have

$$\pi(b)Wh = \lim_j \pi(b)V(e_j)h = \lim_j V(be_j)h = V(b)h,$$

hence $\pi(b)W = V(b)$. Since the span of $V(\mathcal{B})\mathcal{H}$ is dense in \mathcal{K} , it follows that the span of $\pi(\mathcal{B})W\mathcal{H}$ is dense in \mathcal{K} . Finally, for any $h \in \mathcal{H}$ and $b \in \mathcal{B}$ we have

$$W^*\pi(b)Wh = W^*V(b)h = \lim_i V(e_i)^*V(b)h = \lim_i \varphi(e_ib)h = \varphi(b)h,$$

hence $W^*\pi(b)W = \varphi(b)$. Uniqueness up to unitary equivalence follows as usually.

(2) \Rightarrow (1). It can be shown, as in the proof of (2) \Rightarrow (1) of Theorem 2.3, that the associated kernel \mathbf{k} to φ is positive semidefinite hence, as in Remark 3.4, we have that φ is completely positive.

Since the span of $\pi(\mathcal{B})W\mathcal{H}$ is dense in \mathcal{K} and π is continuous, it follows that $\pi(e_i) \xrightarrow{i} I_{\mathcal{K}}$ strictly for any approximate unit $(e_i)_i$ of \mathcal{B} , where $I_{\mathcal{K}}$ is the identity operator of \mathcal{K} . From this we obtain that $\varphi(e_i) \xrightarrow{i} W^*W$ strictly.

On the other hand, since $\varphi(b) = W^*\pi(b)W$ for all $b \in \mathcal{B}$, and the maps W^* , W , $\pi(b)$ and π are continuous, it follows that φ is continuous. \square

Remark 3.8. During the proof of the implication (1) \Rightarrow (2) from Theorem 3.5, while proving that $(V(e_i)^*y)_i$ is a Cauchy net for any $y \in \mathcal{K}$, one can also use the Schwarz inequality (3.1) instead of subadditivity of the seminorm $\tilde{p}_{\mathcal{H}}$. An even simpler approach is to use inequality (1.8) in Sect. 1.3 to get

$$p([h_1 + h_2, h_1 + h_2]_{\mathcal{H}}) \leq 2(p([h_1, h_1]_{\mathcal{H}}) + p([h_2, h_2]_{\mathcal{H}}))$$

for any $h_1, h_2 \in \mathcal{H}$. Using this inequality with $h_1 = V(e_i - e_j)^*y_0$ and $h_2 = V(e_i - e_j)^*(y - y_0)$ provides a valid proof as well.

Similarly, while proving that the net $(V(e_j)h)_j$ is Cauchy in \mathcal{K} , one can use the Schwarz inequality (3.1) instead of Lemma 2.13. The details are left to the reader.

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