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ONE-DIMENSIONAL LONG RANGE WIDOM–ROWLINSON MODEL WITH PERIODIC PARTICLE ACTIVITIES

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In this paper, we consider a one-dimensional long range Widom–Rowlinson model when particle activity parameters are periodic and biased. We show that if the interaction is sufficiently large versus particle activities then the model does not exhibit a phase transition at low temperatures.

Keywords: Gibbs state; ground state; partition function; contour; cluster.

1. Introduction and Formulation of Results

A gain in mixing entropy forces many multicomponent systems to a single phase. The system may pass to phases of prevailing particles of particular kind if some thermodynamical variables change. One of the basic models explaining this kind of phase separations lies in the relative strengths of repulsion between like and unlike particles. If the unlike particles experience a stronger repulsion than the like ones, at least at high density demixing phases are likely. The archetype for analogous systems is the Widom–Rowlinson model. The two particle Widom–Rowlinson model is a lattice gas model with two types of particles, allowed to share neighboring sites only if they are of the same type. The model was introduced (Ref. 1) as a continuum model of particles in space. The lattice variant was studied first in Ref. 2. The spin variables $\phi(x)$ belong to the spin space $\{-1, 0, +1\}$, where 0 corresponds to empty sites. The Hamiltonian of the model is defined as

$$H_0(\phi) = \sum_{x \in \mathbf{Z}^d} U_0(\phi(x)) + \sum_{x,y \in \mathbf{Z}^d} U_1(\phi(x), \phi(y)),$$

where the chemical potential is as follows:

$$U_0(\phi(x)) = \begin{cases} -\ln \lambda_- & \text{if } \phi(x) = -1 \\ 0 & \text{if } \phi(x) = 0 \\ -\ln \lambda_+ & \text{if } \phi(x) = +1 \end{cases},$$

and $\lambda_- > 0$ and $\lambda_+ > 0$ are the activity parameters of particles -1 and $+1$.

The hard-core pair interaction is given by

$$U_1(\phi(x), \phi(y)) = \begin{cases} \infty & \text{if } \phi(x)\phi(y) = -1 \text{ and } |x-y| = 1 \\ 0 & \text{otherwise} \end{cases}. \tag{1}$$

This hard-core model exhibits so-called hard constraints, i.e. their properties arise by forbidding certain configurations. For small values of $\beta\lambda_- = \beta\lambda_+$, there is a unique Gibbs state on which the overall densities of $+1$ and -1 particles are almost surely equal. At $d \geq 2$ and for sufficiently large values of $\beta\lambda_- = \beta\lambda_+$, the symmetry of -1 and $+1$ particles is broken: there are limiting Gibbs states with overwhelming densities of -1 and $+1$ particles. In nonsymmetric case $\lambda_- \neq \lambda_+$, most likely limiting Gibbs state is unique in $d \geq 2$, but rigorous proof is not known. The nonsymmetric case in $d = 1$ is considered in Ref. 3.

In this paper, the results of Ref. 3 is extended by considering the case when particle activities depend also on lattice sites.

Consider the one-dimensional long range Widom–Rowlinson model with the Hamiltonian

$$H(\phi) = \sum_{x \in \mathbf{Z}^1} U_0(\phi(x)) + \sum_{x,y \in \mathbf{Z}^1} U_1(\phi(x), \phi(y)) + \sum_{x,y \in \mathbf{Z}^1} U_2(\phi(x), \phi(y)), \tag{2}$$

where

$$U_0(\phi(x)) = \begin{cases} -\ln \lambda_-^x & \text{if } \phi(x) = -1 \\ 0 & \text{if } \phi(x) = 0 \\ -\ln \lambda_+^x & \text{if } \phi(x) = +1 \end{cases}.$$

$\lambda_-^x > 0$ and $\lambda_+^x > 0$ are the activity parameters of particles -1 and $+1$, that are periodic and depend on lattice sites $x \in \mathbf{Z}^1$: there is a positive integer p such that $\lambda_-^{x+p} = \lambda_-^x$ and $\lambda_+^{x+p} = \lambda_+^x$. $U(1)$ is defined as in (1) and

$$U_2(\phi(x), \phi(y)) = \begin{cases} -C|x-y|^{-\alpha} & \text{if } \phi(x)\phi(y) = 1 \\ 0 & \text{otherwise} \end{cases}.$$

We impose a condition $\alpha > 1$ for the existence of the thermodynamic limit.

Let V_N be an interval with the center at the origin and with the length of $2N$, and $\Phi(N)$ denote the set of all configurations $\phi(V_N)$. We denote the concatenation of the configurations $\phi(V_N)$ and $\phi^i(\mathbf{Z^1} - V_N)$ by χ i.e. $\chi(x) = \phi(x)$, if $x \in V_N$ and $\chi(x) = \phi^i(x)$, if $x \in \mathbf{Z^1} - V_N$. Define

$$H_N(\phi|\phi^i) = \sum_{\substack{x \in \mathbf{Z}^1 \\ x \in V_N}} U_0(\chi(x)) + \sum_{\substack{x,y \in \mathbf{Z}^1 \\ x > y \\ \{x,y\} \cap V_N \neq \emptyset}} (U_1(\chi(x), \chi(y)) + U_2(\chi(x), \chi(y))).$$

The finite-volume Gibbs distribution corresponding to the boundary conditions ϕ^i is

$$\mathbf{P}^{\mathbf{i}}_{\mathbf{N}}(\phi|\phi^i) = \frac{\exp(-\beta H_N(\phi|\phi^i))}{\Xi(N, \phi^i)},$$

where β is the inverse temperature and the partition function $\Xi(N, \phi^i) = \sum_{\phi \in V_N} \exp(-\beta H_N(\phi|\phi^i))$. We say that a probability measure $\mathbf{P}$ on the configuration space $\{-1, 0, 1\}^{\mathbf{Z}^1}$ is an infinite-volume Gibbs state if for each N and for $\mathbf{P}$ almost all ϕ^i in $\{-1, 0, 1\}^{\mathbf{Z}^1}$, we have

$$\mathbf{P}(\phi(V_N) = \varphi(V_N)|\phi(\mathbf{Z}^1 - V_N) = \phi^i(\mathbf{Z}^1 - V_N)) = \mathbf{P}^{\mathbf{i}}_{\mathbf{N}}(\varphi|\phi^i).$$

In this paper, we investigate the problem of uniqueness of Gibbs states of the model (2). The case $\alpha > 2$ is well known: since the interactions between distant spins decrease rapidly, the total interaction of complementary half-lines is finite and the phase transition is absent (Refs. 4–6). The case $2 > \alpha > 1$ is open for different possibilities. In the homogeneous and symmetric case $\lambda^x_- = \lambda^x_+,\ x \in \mathbf{Z}^1$, most likely the model exhibits a phase transition at sufficiently low temperatures as in ferromagnetic Ising model with long range interaction (Refs. 7 and 8).

We will treat the model (2) by a special method (Refs. 9 and 10) developed for the case when the interactions between distant spins does not decrease rapidly. This low temperature regime method mixes two independent realizations of Gibbs fields and reduces the problem of phase transition to percolation type problems of special clusters connecting fixed segments with the boundary. The procedure of mixing of two independent realizations in other words "coupling" have had successful effects in numerous different cases (Refs. 11–15). In pursuance of Refs. 9 and 10 for investigation of Gibbs states of model (2), we explore stability properties of ground states and by applying of uniqueness criterion from Ref. 10 (see Theorem 1 below), and we prove the following.

Theorem 1. *Let* $\sum_{x=0}^{p}(\ln \lambda^x_- - \ln \lambda^x_+) \neq 0$ *and the interaction constant* C *is sufficiently large. Then the inverse temperature* β_{cr} *exists such that if* $\beta > \beta_{cr}$ *then the model* (2) *has at most one limiting Gibbs state.*

As it was mentioned above for weak interaction potentials $U_2(\phi(x), \phi(y))$, the model (2) does not exhibit phase transition. Nevertheless, the condition on constant C is necessary in order to avoid cases when in some part of the period local clusters of similar particles may withstand the influence of remaining particles leading to possible phase coexistence (Ref. 16). The structure of ground states of one-dimensional Ising model with long range interaction and additional nonconstant external field was investigated in Ref. 17.

2. Proofs

We say that a configuration ϕ^{gr} is a ground state, if for any finite perturbation ϕ' of the configuration ϕ^{gr}, the expression $H(\phi') - H(\phi^{gr})$ is non-negative.

$\phi(B)$ denotes the restriction of the configuration ϕ to the set B. We say that the ground state ϕ^{gr} of the model (2) satisfies the Peierls stability condition with positive constant t, if $H(\phi') - H(\phi^{gr}) \geq t|A|$ for any finite set $A \subset \mathbf{Z^1}$ ($|A|$ denotes the number of sites of A and ϕ' is a perturbation of ϕ^{gr} on the set A).

Without loss of generality, we suppose that $\sum_{x=1}^{p}(\ln \lambda_-^x - \ln \lambda_+^x) = \Delta > 0$.

Lemma 1. *The model* (2) *has a unique ground state* $\phi^{gr} \equiv 1$.

Proof. Let ϕ' be a perturbation of $\phi^{gr} \equiv 1$ on a set A such that $\phi'(x)\phi'(y) \neq -1$ for all adjacent spins. Let $I_k = [1 + kp, p + kp] \cap \mathbf{Z^1}$, then readily $\mathbf{Z^1} = \cup_{-\infty}^{\infty} I_k$. Suppose that all indices for which $I_{l_i} \cap A \neq \emptyset$ are $\{l_1, \ldots, l_s\}$, then

$$\begin{aligned} H(\phi') - H(\phi^{gr}) &= \sum_{i=1}^{s}(H(\phi'(I_{l_i})) - H(\phi^{gr}(I_{l_i}))) \\ &\quad + \sum_{*}(U_1(\phi'(x), \phi'(y)) - U_1(\phi(x), \phi(y)))\,, \end{aligned}$$

where the summation in $\sum_*$ is taken over all pairs (x, y) not belonging to the same I_k. Since the long range interaction is ferromagnetic, we readily get the following:

$$H(\phi') - H(\phi^{gr}) \geq \sum_{i=1}^{s}(H(\phi'(I_{l_i})) - H(\phi^{gr}(I_{l_i})))\,. \tag{3}$$

Consider $H(\phi'(I_{l_j})) - H(\phi^{gr}(I_{l_j}))$ for some l_j. If $\phi'(I_{l_j})$ consists of only -1 particles then $H(\phi'(I_{l_j})) - H(\phi^{gr}(I_{l_j})) \geq \Delta > 0$. If not then $\phi'(I_{l_j})$ is a union of spin blocks -1, 0 and $+1$ particles and since in each merger between distinct blocks we lose at least $U(1)$, we readily get $H(\phi'(I_{l_j})) - H(\phi^{gr}(I_{l_j})) \geq (C \cdot U(1) - \sum_{i=1}^{p} \max(\ln \lambda_-^x, \ln \lambda_+^x)) > 0$ for sufficiently large values of C. Thus, in both cases $H(\phi'(I_{l_j})) - H(\phi^{gr}(I_{l_j})) > 0$. □

Lemma 2. *The unique ground state ϕ^{gr} of the model (2) satisfies the Peierls stability condition.*

Proof. Let ϕ' be a perturbation of $\phi^{gr} \equiv 1$ on a set A. Let us choose the constant C such that $C \cdot U(1) - \sum_{i=1}^{p} \max(\ln \lambda_-^x, \ln \lambda_+^x)) > \Delta$. Then by (3)

$$H(\phi') - H(\phi^{gr}) \geq \sum_{i=1}^{s}(H(\phi'(I_{l_i})) - H(\phi^{gr}(I_{l_i}))) \geq \Delta \cdot s$$

and for $t = \frac{\Delta}{p}$, we readily get the required inequality

$$H(\phi') - H(\phi^{gr}) \geq t \cdot |A|\,.$$ □

The following theorem installs a strong relationship between stable ground states and Gibbs states at low temperatures:

Theorem 2. (see Ref. 10) *Suppose that a one-dimensional model has a unique ground state satisfying Peierls stability condition and a constant $\gamma < 1$ exists such* *that for any number L and any interval $I = [a, b]$ with length n and for any configuration $\phi(I)$*

$$\sum_{\substack{B \subset Z^1 \\ B \cap I \neq \emptyset \\ B \cap (Z^1 - [a-L, b+L]) \neq \emptyset}} |U(B)| \leq (\text{const.})\, n^{\gamma} L^{\gamma-1} \,. \tag{4}$$

A value of the inverse temperature β_{cr} exists such that if $\beta > \beta_{cr}$ then the model has at most one limiting Gibbs state.

Now Theorem 1 follows from Theorem 2. Indeed, by Lemmas 1 and 2, the ground state ϕ^{gr} is unique and stable, and the condition $|U_2(\phi(x), \phi(y)| \leq C|x-y|^{-\alpha}$ with $\alpha > 1$ readily implies (4).

3. Final Notes

Theorem 1 shows that if parameters of particle activities are periodic and biased in the Widom–Rowlinson model, the ferromagnetic influence of the boundary particles on like particles inside the volume vanishes when volume infinitely grows: in spite of strong long range attraction potential between similar particles, the phase in sufficiently large volume is almost independent on the configuration outside the volume.

We think that Theorem 1 is held at all values of the temperature. Since the main method (Ref. 10) used in this paper stands on low temperature estimations of configurations differing on ground states, we are stick to low temperature region.

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