Technical Notes and Correspondence

Frequency-Domain Subspace Identification of Linear Time-Periodic (LTP) Systems

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***Abstract*—This paper proposes a new methodology for subspace-based state-space identification for linear time-periodic (LTP) systems. Since LTP systems can be lifted to equivalent linear time-invariant (LTI) systems, we first lift input–output data from an unknown LTP system as if they were collected from an equivalent LTI system. Then, we use frequency-domain subspace identification methods to find the LTI system estimate. Subsequently, we propose a novel method to obtain a time-periodic realization for the estimated lifted LTI system by exploiting the specific parametric structure of Fourier series coefficients of the frequency-domain lifting method. Our method can be used to obtain state-space estimates for unknown LTP systems as well as to obtain Floquet transforms for known LTP systems.**

***Index Terms*—Linear time-periodic (LTP) systems, subspace methods, system identification, time-varying systems.**

I. Introduction

In this paper, we introduce a frequency-domain subspace-based state-space identification method for linear time-periodic (LTP) systems. Many problems in engineering and biology, such as wind turbines [1], rotor bearing systems [2], aircraft models [3], locomotion [4], [5], and power distribution networks [6], require the consideration of time-periodic dynamics. As such, the analysis, identification, and control of LTP systems have received considerable attention [7]–[9].

A pioneering work by Wereley [7] introduced a frequency-domain analysis method for LTP systems. In this work, time-periodic system matrices in the LTP state-space formulation were expanded into their Fourier series coefficients. The principle of harmonic balance was used to obtain the concept of harmonic transfer functions (HTFs). Wereley's initial formulation for continuous-time LTP systems as infinite-dimensional operators was subsequently adapted to discrete time, which conveniently leads to finite-dimensional HTFs [10].

Most existing literature on LTP system identification [2], [11], including our own prior work on identification of legged locomotion [12]–[14], focus on using input–output HTF representations rather than state space. In addition, there are also contributions to state-space-based system identification for LTP systems [15], [16], analogous to subspace identification techniques commonly used for linear time-invariant (LTI) systems [17]. For instance, Verhaegen and Yu developed a subspace identification method for estimating successive state-transition matrices from time-domain data for linear time-varying (LTV) (including a special derivation for LTP) systems [15].

Critically, LTI subspace identification methods readily support both time-domain [17] and frequency-domain [18] data, whereas most subspace methods for LTP systems have focused on time-domain data [15], [16], and those state-space methods that do rely on frequency-domain data [19], [20] require that scheduling functions be known *a priori*. To the best of our knowledge, there are no general methods for frequency-domain subspace identification of LTP systems.

Here, we present a general subspace identification methodology for estimating state-space models from frequency-domain data for LTP systems. Our proposed methodology is based on the fact that LTP systems can be represented with equivalent LTI systems via lifting [10]. Based on this observation, we first lift the input–output data of an unknown LTP system as if they were collected from an equivalent LTI system, following previous methods [10]. We, then, estimate a discrete-time LTI state-space equivalent for the original LTP system by using an existing LTI frequency-domain subspace identification method [18]. A key property of the frequency-domain lifting method we utilize in this paper is the specific parametric structure of Fourier series coefficients associated with the original LTP system [10]. However, this structure is not, in general, preserved during the subspace identification process due to an inevitably unknown similarity transformation. In order to solve this issue, we identify a similarity transformation for the lifted LTI system that recovers the Fourier structure, although not the specific coefficients, because there is a subset of similarity transformations that preserves the Fourier structure but not its parameters. Our identification–realization algorithm also allows the realization of Floquet-transformed state-space models for LTP systems with arbitrary time-periodic system matrices (see Remark 3), whose analytic derivations are often very challenging and may even be impossible [21].

This paper is outlined as follows. We introduce the problem formulation in Section II. Then, in Section III, we show the existence of an equivalent discrete-time LTI system for a given LTP system via

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lifting and estimate its system matrices from frequency-domain data. In Section IV, we present a novel LTP realization algorithm for the estimated lifted LTI system. We provide an illustrative numerical example and a comparative analysis in Section V. Finally, we give our concluding remarks in Section VI.

II. Problem Formulation

In this paper, we consider single-input/single-output stable LTP systems represented by

$$\begin{aligned}\dot{\bar{x}}(t) &= \bar{A}(t)\bar{x}(t) + \bar{B}(t)u(t)\\ y(t) &= \bar{C}(t)\bar{x}(t) + \bar{D}(t)u(t)\end{aligned} \tag{1}$$

where $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, and $\bar{x}(t) \in \mathbb{R}^{n_p}$ represent the input, output, and state vectors, respectively. The system matrices are periodic with a fixed common period $T > 0$ (see Section III-B for the computation of T), with $\bar{A}(t) = \bar{A}(t + nT)$, $\bar{B}(t) = \bar{B}(t + nT)$, $\bar{C}(t) = \bar{C}(t + nT)$, and $\bar{D}(t) = \bar{D}(t + nT)$ $\forall n \in \mathbb{Z}$.

We formulate the identification problem as follows.

Given

- A single pair of input–output signals $u(t)$ and $y(t)$ in the form of a sum-of-cosines signal containing different frequency components that provide an LTP frequency response.

Estimate

- The four LTP system matrices that will be equivalent to (1) up to a similarity transform.

The remaining sections detail our solution methodology (see Appendix A for the procedure). Obviously, LTI subspace identification methods would result in oversimplified LTI systems due to ignorance of harmonic responses. On the other hand, one can use LTV subspace identification methods in the time domain to solve a discrete-time version of this problem [15], [16]. Our solution method is unique in that it solves the problem in the frequency domain and results in intuitive state-space estimates in Floquet-transformed forms.

III. Existence and Estimation of a Discrete-Time Lifted LTI System Representation

This section first introduces a system of transformations that needs to be used to prove the existence of a real-valued discrete-time LTI representation of (1). We, then, show how we estimate such an LTI system using input–output data of the original LTP system. Naturally, the original state-space form of (1) will not be available. Therefore, the transformations described in this section are not directly applied on the state-space form of (1); rather, the transformations map the input–output data into a form that makes it as if they were collected from the transformed (LTI) system.

Based on Floquet theory, there exists a transformation that converts (1) into the following form:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}(t)u(t)\\ y(t) &= \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)u(t)\end{aligned} \tag{2}$$

where $\mathbf{A}$, $\mathbf{B}(t)$, $\mathbf{C}(t)$, and $\mathbf{D}(t)$ can be obtained as real-valued (by doubling the system period, if necessary), as long as the system matrices in (1) are real-valued [21]. Note that deriving a Floquet transform is challenging even when the state-space is known. On the other hand, the Floquet transform is a similarity transformation and does not affect the input–output data. Hence, we assume, without loss of generality, that the LTP system to be identified has the state-space form given in (2). Note that Floquet-transformed forms are easier to work with since they have a time-invariant state matrix. Thus, we seek to find an LTP state-space estimate for (1) in a Floquet form such as (2).

A. Discretization via Bilinear (Tustin) Transform

In principle, we could directly lift (2) to a continuous-time LTI equivalent and utilize continuous-time LTI subspace identification methods. However, the Hankel (data) matrices used for continuous-time LTI systems may become ill-conditioned with increasing system dimension [22]. Therefore, we find it more convenient to work with discrete-time LTI systems. To this end, we transform (2) into an approximate discrete-time LTI system. This has two benefits. First, lifting discrete-time LTP systems yields finite-dimensional LTI representations, unlike infinite-dimensional ones in continuous-time models. Second, and more importantly, it generalizes the applicability of our solutions to both continuous-time and discrete-time LTP systems. To accomplish this, we utilize the time-varying bilinear (Tustin) transformation to obtain a discrete-time LTP state-space representation of (2). Note that (2) is a special case of LTV systems with time-periodic system matrices (and a time-invariant state matrix). Therefore, our special case reduces the transformations in [23] to the following:

$$\begin{aligned}\mathbf{x}_d[k+1] &= \mathbf{A}_d\mathbf{x}_d[k] + \mathbf{B}_d[k]\mathbf{u}_d[k]\\ \mathbf{y}_d[k] &= \mathbf{C}_d[k]\mathbf{x}_d[k] + \mathbf{D}_d[k]\mathbf{u}_d[k]\end{aligned} \tag{3}$$

where $\mathbf{x}_d[k]$ represents discrete-time states and

$$\begin{aligned}\mathbf{A}_d &= ((2/T_s)I + \mathbf{A})((2/T_s)I - \mathbf{A})^{-1}\\ \mathbf{B}_d[k] &= \left(2/\sqrt{T_s}\right)((2/T_s)I - \mathbf{A})^{-1}\mathbf{B}(kT_s)\\ \mathbf{C}_d[k] &= \left(2/\sqrt{T_s}\right)\mathbf{C}(kT_s)((2/T_s)I - \mathbf{A})^{-1}\\ \mathbf{D}_d[k] &= \mathbf{D}(kT_s) + \mathbf{C}(kT_s)((2/T_s)I - \mathbf{A})^{-1}\mathbf{B}(kT_s).\end{aligned} \tag{4}$$

Here, T_s is the sampling period yielding sampled input–output data as $\mathbf{u}_d[k] := u(kT_s)$ and $\mathbf{y}_d[k] := y(kT_s)$. Derivations for (3) can be found in [24]. Note that (3) is an LTP system, where $\mathbf{B}_d[k] = \mathbf{B}_d[k + nN]$ $\forall n \in \mathbb{Z}$ (also valid for $\mathbf{C}_d[k]$ and $\mathbf{D}_d[k]$) and N is the discrete-time system period defined as $N := T/T_s$. For the sake of simplicity, N is assumed to be even. The sampling period T_s determines N and, hence, the dimension of the lifted LTI equivalent. Using a higher sampling frequency allows capturing of high-frequency dynamics but also increases the complexity by increasing the lifted LTI system dimension. In addition, bilinear (Tustin) transformation causes frequency warping (distortions) at higher frequencies. To avoid this problem, we utilize the experimental design procedure in [25] by first prewarping the input frequencies that will be used while designing the sum-of-cosines input.

B. Lifting to a Time-Invariant Reformulation

One of the key properties of LTP systems is that a complex exponential input with frequency ω produces an output not only at the input frequency (which is the case for LTI systems) but also at different harmonics $\omega \pm k\omega_p$, $k \in \mathbb{Z}$ separated by the system frequency $\omega_p = 2\pi/T$, with possibly different magnitudes and phases in the steady state (this also allows estimating T from input–output data). In this context, the concept of HTFs was developed to represent each harmonic response of the LTP system with a distinct transfer function $G_k(w + k\omega_p)$ for $k \in \mathbb{Z}$ [7]. This approach represents an LTP system as the superposition of multiple modulated LTI systems. As such, HTFs can be used as a lifting technique to transform an LTP system into an LTI equivalent [10]. This motivates our use of HTFs as the frequency-domain lifting method to obtain an LTI equivalent state-space model for (3). Among possible alternatives (see [10] for a survey), we use frequency lifting in state space due to the convenient structure of Fourier series coefficients for periodic system matrices in lifted (semi)Toeplitz matrices.

The lifting method starts by analyzing the input–output spectra of the LTP systems by referring the concept of exponentially modulated periodic (EMP) signals. An input signal $\mathbf{u}_d[k]$ is a discrete-time EMP signal if there is a nonzero complex number z such that

$$\mathbf{u}_d[k+pN] = \mathbf{u}_d[k]z^{pN} \tag{5}$$

over one period for $p \in \mathbb{Z}$ and can be written as follows:

$$\mathbf{u}_d[k] := z^k \sum_{n=-N/2}^{N/2-1} U_n e^{j2\pi\frac{nk}{N}} \tag{6}$$

where U_n are called modulated Fourier series coefficients for EMP signals and are defined as follows:

$$U_n := \frac{1}{N}\sum_{k=0}^{N-1}\left(\mathbf{u}_d[k]z^{-k}\right)e^{-j2\pi\frac{nk}{N}}. \tag{7}$$

It has been shown that when an LTP system is given an EMP input, the state and output signals are also EMP in the steady state [7]. Therefore, similar to (6), we obtain the state vector as follows:

$$\mathbf{x}_d[k] = z^k \sum_{n\in\mathbf{I_N}} X_n e^{j2\pi\frac{nk}{N}} \tag{8}$$

and a similar expression for $\mathbf{y}_d[k]$, where $\mathbf{I_N}$ defines the interval $\mathbf{I_N} = [-N/2,\ N/2-1]$. In addition, the discrete-time Fourier synthesis equation for $\mathbf{B}_d[k]$ is computed as follows:

$$\mathbf{B}_d[k] = \sum_{n\in\mathbf{I_N}} B_n e^{j2\pi\frac{nk}{N}}. \tag{9}$$

Similar expressions are also valid for $\mathbf{C}_d[k]$ and $\mathbf{D}_d[k]$. Substituting Fourier synthesis equations into (3) yields

$$0 = z^k \sum_{n\in\mathbf{I_N}}\left(zX_n e^{j2\pi\frac{n}{N}} - \mathbf{A}_d X_n - \sum_{m\in\mathbf{I_N}} B_{n-m}U_m\right)e^{j2\pi\frac{nk}{N}}. \tag{10}$$

The exponentials $\{e^{j2\pi\frac{nk}{N}}|n\in\mathbf{I_N}\}$ constitute an orthonormal basis. Thus, by the principle of harmonic balance, each term enclosed by the brackets must be zero to ensure that the overall sum is zero. Therefore, for all $n\in\mathbf{I_N}$, we have

$$ze^{j2\pi\frac{n}{N}}X_n = \mathbf{A}_d X_n + \sum_{m\in\mathbf{I_N}} B_{n-m}U_m. \tag{11}$$

Note that the above equation is valid since Fourier coefficients B_m are also periodic with N. For the output, we also have

$$Y_n = \sum_{m\in\mathbf{I_N}} C_{n-m}X_m + \sum_{m\in\mathbf{I_N}} D_{n-m}U_m \tag{12}$$

for all $n\in\mathbf{I_N}$. Similar to continuous-time systems, (11) and (12) can be represented with (semi) Toeplitz matrices to obtain an LTI state-space model. To this end, we first define the N-block state $\mathcal{X}_d$, input $\mathcal{U}_d$, and output $\mathcal{Y}_d$ vectors, whose ith block for $i=1,2,\ldots,N$ is given by

$$\mathcal{X}_d(i) = X_{i-1-\frac{N}{2}},\quad \mathcal{U}_d(i) = U_{i-1-\frac{N}{2}},\quad \mathcal{Y}_d(i) = Y_{i-1-\frac{N}{2}}. \tag{13}$$

In addition, the time-invariant reformulation of the unlifted N-periodic output matrix can be obtained as follows:

$$\mathcal{C}_d := \begin{bmatrix} C_0 & C_{-1} & \ldots & C_{-\frac{N}{2}} & C_{\frac{N}{2}-1} & C_{\frac{N}{2}-2} & \ldots & C_1 \\ C_1 & C_0 & \ldots & C_{-\frac{N}{2}+1} & C_{-\frac{N}{2}} & C_{\frac{N}{2}-1} & \ldots & C_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ C_{-1} & C_{-2} & \ldots & C_{\frac{N}{2}-1} & C_{\frac{N}{2}-2} & C_{\frac{N}{2}-3} & \ldots & C_0 \end{bmatrix}. \tag{14}$$

Similarly, (semi)Toeplitz forms for $\mathcal{B}_d$ and $\mathcal{D}_d$ matrices can be obtained in terms of their Fourier series coefficients $\{B_n|n\in\mathbf{I_N}\}$ and $\{D_n|n\in\mathbf{I_N}\}$, respectively. Note that, since $\mathbf{A}_d$ is time-invariant, its Toeplitz form $\mathcal{A}_d$ includes only $\mathbf{A}_d$ in its diagonals as follows:

$$\mathcal{A}_d := \text{blkdiag}\{\mathbf{A}_d\} \mid \mathcal{A}_d \in \mathbb{R}^{Nn_p\times Nn_p} \tag{15}$$

where blkdiag represents a block-diagonal matrix and $\mathbf{A}_d$ is repeated blockwise on diagonals. Finally, we define a modulation matrix $\mathcal{N}_d$ to capture the exponential terms in (11) as follows:

$$\mathcal{N}_d := \text{blkdiag}\left\{e^{j2\pi\frac{n}{N}}I_{n_p} \mid \forall n\in\mathbf{I_N}\right\}. \tag{16}$$

We also define

$$\mathcal{A}_{dN} := \mathcal{N}_d^{-1}\mathcal{A}_d,\quad \mathcal{B}_{dN} := \mathcal{N}_d^{-1}\mathcal{B}_d. \tag{17}$$

Now, (11) and (12) can be represented as follows:

$$\begin{aligned} z\mathcal{X}_d &= \mathcal{A}_{dN}\mathcal{X}_d + \mathcal{B}_{dN}\mathcal{U}_d \\ \mathcal{Y}_d &= \mathcal{C}_d\mathcal{X}_d + \mathcal{D}_d\mathcal{U}_d. \end{aligned} \tag{18}$$

This is called the harmonic state-space (HSS) model, and it represents a lifted LTI equivalent of (1) for a general class of input–output signals. Following sections explain how we transform this HSS model into a more intuitional single-input multioutput (SIMO) LTI equivalent by limiting the space of EMP inputs.

C. SIMO LTI Equivalent

The input to the original LTP system (1) is a sum-of-cosines signal in the form $u(t) = \sum_{m=1}^{M} 2K\cos(\omega_m t)$. As stated earlier, each cosine input at ω_m produces an output spectra at $\pm\omega_m \pm k\omega_p$ for $k\in\mathbb{Z}$, since cosine triggers both $\pm\omega_m$. Hence, the input frequencies should be carefully selected to avoid any coincidence of harmonic responses (see [26] for illustrative explanations). Once this is satisfied, we can separate the input–output response of each individual cosine signal in the frequency domain. At this point, we write each single cosine input as follows:

$$u_c(t) = 2K\cos(\omega_m t) = \underbrace{Ke^{j\omega_m t}}_{u_c^+(t)} + \underbrace{Ke^{-j\omega_m t}}_{u_c^-(t)}. \tag{19}$$

Let the output of (1) to inputs $u_c^+(t)$, $u_c^-(t)$, and $u_c(t)$ be $y_c^+(t)$, $y_c^-(t)$, and $y_c(t)$, respectively, where $y_c(t) = y_c^+(t) + y_c^-(t)$. Ensuring that $\omega_m \neq 0.5k\omega_p$ for $k\in\mathbb{Z}$, one can also guarantee that there will be no coincidence in harmonic responses of the single-cosine input [26]. Thus, we can simulate (1) with $u_c(t)$ and only use $y_c^+(t)$ as the output, assuming that our input was $u_c^+(t)$. We choose distinct exponential modulation $z = e^{j\omega_m}$ in (5) for each individual input signal. Hence, the modulated Fourier series coefficient vector in (18) becomes $\mathcal{U}_d = [0\ \ldots\ 0\ K\ 0\ \ldots\ 0]^T$ with K on row $(N/2+1)$ for each input. More importantly, with its current form, $\mathcal{U}_d$ selects only column $(N/2+1)$

in (14) for $\mathcal{B}_d$ and $\mathcal{D}_d$, yielding

$$\begin{aligned} z\mathcal{X}_d &= \mathcal{A}_{dN}\mathcal{X}_d + \bar{\mathcal{B}}_{dN}\bar{\mathcal{U}}_d \\ \mathcal{Y}_d &= \mathcal{C}_d\mathcal{X}_d + \bar{\mathcal{D}}_d\bar{\mathcal{U}}_d \end{aligned} \tag{20}$$

where $\bar{\mathcal{U}}_d = K$, $z = e^{j\omega_m}$, and

$$\begin{aligned} \bar{\mathcal{B}}_{dN} &:= \mathcal{N}_d^{-1}\begin{bmatrix} B_{-N/2} & \ldots & B_0 & \ldots & B_{N/2-1}\end{bmatrix}^T \\ \bar{\mathcal{D}}_d &:= \begin{bmatrix} D_{-N/2} & \ldots & D_0 & \ldots & D_{N/2-1}\end{bmatrix}^T. \end{aligned} \tag{21}$$

D. Transforming to a Real-Valued State-Space Model

One problem with LTI subspace identification methods is that they rely on real-valued input–output data in the time domain to estimate real-valued system matrices [27]–[31]. Hence, we need to transform (20) into a system that, if it were converted to the time-domain, would produce real-valued states and outputs, given real-valued inputs. Note that this system would not correspond to our original time-domain system. Rather, the time-domain equivalent of (20) is a fictitious system, useful only for the purpose of analysis. This SIMO LTI system has N states and outputs. Thus, we have

$$\begin{aligned} \mathcal{X}_d[k] &:= \begin{bmatrix} \bar{\mathcal{X}}_{-N/2}[k] & \ldots & \bar{\mathcal{X}}_0[k] & \ldots & \bar{\mathcal{X}}_{N/2-1}[k]\end{bmatrix}^T \\ \mathcal{Y}_d[k] &:= \begin{bmatrix} \bar{\mathcal{Y}}_{-N/2}[k] & \ldots & \bar{\mathcal{Y}}_0[k] & \ldots & \bar{\mathcal{Y}}_{N/2-1}[k]\end{bmatrix}^T. \end{aligned} \tag{22}$$

Considering (20) as an LTI system in the z-domain and by utilizing the block-diagonal structure of $\mathcal{A}_{dN}$ (noting that $\mathbf{A}_d$ is stable), one can simply solve for each state equation in the steady state as follows:

$$\bar{\mathcal{X}}_m[k] = \sum_{i=0}^{k-1}\left(e^{-j2\pi\frac{m}{N}}I_{n_p}\mathbf{A}_d\right)^{k-i-1}\left(e^{-j2\pi\frac{m}{N}}I_{n_p}B_m\right)u[i] \tag{23}$$

where $u[k] = 2K\cos(\omega_m kT_s)$. This follows since $\bar{\mathcal{U}}_d = K$ in the z-domain corresponds to a single cosine input signal for the time-domain signal. (We write the input as in (19) and ignore the negative frequency component for the sake of our analysis.) Also, note that $B_m = B^*_{-m}$ since $\mathbf{B}_d[k]$ is real-valued by definition. Hence, we can state that $\bar{\mathcal{X}}_m[k] = \bar{\mathcal{X}}^*_{-m}[k]$, except for $\bar{\mathcal{X}}_{-N/2}[k]$ and $\bar{\mathcal{X}}_0[k]$, which are both real-valued as seen in (23). A similar analysis can be done for $\mathcal{Y}_d[k]$ by using (23). However, solutions for each LTI output signal $\bar{\mathcal{Y}}_m[k]$ are more challenging since complex-conjugate state solutions are now multiplied with shifted versions of Fourier series coefficients as illustrated in (14). To achieve our goal, we first write the steady-state solutions for each output signal as follows:

$$\bar{\mathcal{Y}}_m[k] = \sum_{n\in\mathbf{I_N}} C_{m-n}\bar{\mathcal{X}}_n[k]. \tag{24}$$

By using lengthy but straightforward calculations, one can show that $\bar{\mathcal{Y}}_m[k] = \bar{\mathcal{Y}}^*_{-m}[k]$ in the steady state. Having shown the complex-conjugate nature of the time-domain state and output signals, we define two complex-valued transformation matrices $\mathcal{T}_x$ and $\mathcal{T}_y$ as follows:

$$\underline{\mathcal{X}}_d[k] := \mathcal{T}_x\mathcal{X}_d[k], \quad \underline{\mathcal{Y}}_d[k] := \mathcal{T}_y\mathcal{Y}_d[k] \tag{25}$$

where $\mathcal{T}_x$ can be defined as follows:

$$\mathcal{T}_x := 0.5\begin{bmatrix} 2I_{n_p} & 0 & 0 & 0 \\ 0 & I_{(N/2-1)n_p} & 0 & J_{(N/2-1)n_p} \\ 0 & 0 & 2I_{n_p} & 0 \\ 0 & -jJ_{(N/2-1)n_p} & 0 & jI_{(N/2-1)n_p} \end{bmatrix} \tag{26}$$

with a similar expression for $\mathcal{T}_y$, where $I_{\bar{n}}$ is the usual $\bar{n}\times\bar{n}$ identity and $J_{\bar{n}}$ is an antidiagonal $\bar{n}\times\bar{n}$ matrix (i.e., 1 for the entries where $i = \bar{n} - j + 1$, 0 otherwise) with associated sizes. Equation (25) transforms (20) into the following:

$$\begin{aligned} z\mathcal{X} &= \mathcal{T}_x\mathcal{A}_{dN}\mathcal{T}_x^{-1}\mathcal{X} + \mathcal{T}_x\bar{\mathcal{B}}_{dN}\bar{\mathcal{U}}_d \\ \mathcal{Y} &= \mathcal{T}_y\mathcal{C}_d\mathcal{T}_x^{-1}\mathcal{X} \quad + \mathcal{T}_y\bar{\mathcal{D}}_d\bar{\mathcal{U}}_d \end{aligned} \tag{27}$$

where $\mathcal{X} := \mathcal{T}_x\mathcal{X}_d$ and $\mathcal{Y} := \mathcal{T}_y\mathcal{Y}_d$. Note that $\mathcal{T}_x$ and $\mathcal{T}_y$ also transform the system matrices to real-valued equivalents.

E. Estimating an LTI Equivalent via Subspace Identification

At this point, we could utilize a variety of LTI subspace identification methods [17], [18], [27], [32], [33]. Although we could not find a general benchmarking study on these algorithms, it has been shown that CVA [18] performs better than N4SID [17] and MOESP [32] in terms of prediction error and computational complexity [34]. Moreover, CVA [18] is MATLAB's (The MathWorks Inc., Natick, MA, USA) built-in frequency-domain subspace identification method. Hence, we use CVA for estimating the equivalent LTI system by carefully selecting the estimated system dimension (see Remark 1). Q2

Remark 1: In classical LTI subspace identification, the estimated system order $\hat{n}$ is chosen based on large drops in singular values of Hankel matrices [17]. However, one needs to be aware of the specific parametric structure of LTP systems while selecting $\hat{n}$. Let the eigenvalues of $\mathbf{A}_d$ be $S_d = \{\lambda_i^d\}_{i=1}^{n_p}$. Lifting to (17) results in $\mathcal{A}_{dN}$ with the following eigenvalues:

$$S = \left\{\left\{\lambda_i^d e^{-j2\pi\frac{k}{N}}\right\}_{i=1}^{n_p} \middle| \forall k\in\mathbf{I_N}\right\}. \tag{28}$$

Once $\hat{n}$ is chosen based on the singular values (not the eigenvalues), the user should check the eigenvalues of the estimated state matrix for the phase structure defined in (28). This phase structure will both reveal the underlying LTP system's dimension n_p as well as the number of harmonics that will appear in the state vector N_h. The user might need to use expert knowledge to decide on $\hat{n}$ to maintain the phase structure of (28). The correct choice of $\hat{n}$ will yield eigenvalues as follows:

$$\hat{S} = \left\{\left\{\lambda_i^d e^{-j2\pi\frac{k}{N}}\right\}_{i=1}^{n_p} \middle| \forall k\in[-N_h, N_h]\right\}. \tag{29}$$

Note that, under these constraints, $\hat{n}$ would be equal to the cardinality of $\hat{S}$, i.e., $\hat{n} = |\hat{S}| = (2N_h+1)n_p$, and this will limit the dimensions of $\hat{\mathcal{X}}$ (and associated system matrices) in (30). It is quite possible that the user could also limit the output harmonics in (13) based on the LTP frequency response. This choice will be independent of $\hat{n}$ and it will limit the dimensions of $\hat{\mathcal{Y}}$ (and associated system matrices) in (30). ∎

The CVA method estimates a quadruple of real-valued LTI system matrices as $[\hat{\bar{A}}, \hat{\bar{B}}, \hat{\bar{C}}, \hat{\bar{D}}]$, which is equivalent to (27) up to a similarity transformation. However, we need to backsubstitute the transformations in (17) to find an equivalent lifted LTI system for the unknown LTP system. To this end, we use $\hat{A} = \hat{\bar{A}}$, $\hat{B} = \hat{\bar{B}}$, $\hat{C} = \mathcal{T}_y^{-1}\hat{\bar{C}}$, and $\hat{D} = \mathcal{T}_y^{-1}\hat{\bar{D}}$ and obtain the equivalent lifted LTI system as follows:

$$\begin{aligned} z\hat{\mathcal{X}} &= \hat{A}\hat{\mathcal{X}} + \hat{B}\bar{\mathcal{U}}_d \\ \hat{\mathcal{Y}} &= \hat{C}\hat{\mathcal{X}} + \hat{D}\bar{\mathcal{U}}_d \end{aligned} \tag{30}$$

where $\hat{A}\in\mathbb{R}^{\hat{n}\times\hat{n}}$, $\hat{B}\in\mathbb{R}^{\hat{n}\times 1}$, $\hat{C}\in\mathbb{C}^{N\times\hat{n}}$, and $\hat{D}\in\mathbb{C}^{N\times 1}$. Note that we do not substitute $\mathcal{T}_x$ back since it is already in the form of a similarity transformation.

At this point, our method provides a parametric system representation, which is equivalent to the lifted LTI form (27) of the original LTP system. However, the main drawback of this representation—lifted

LTI—is that it is unintuitive and requires additional processes (unlifting the signals) to predict the output of the original LTP system. In Section IV, we introduce an LTP realization method that collapses the lifted LTI system to an LTP system in Floquet form.

IV. Time-Periodic Realization for the Estimated Lifted LTI Equivalent

The specific parametric structure of Fourier series coefficients is not generally preserved during subspace identification. Finding a computationally effective solution to this problem remains an open issue [35]. Motivated by this, we propose a time-periodic realization method for lifted LTI systems. Unlike the previous work that considers unforced LTP systems [36], we provide a framework for a general class of LTP systems with inputs. Our goal can be defined as finding a similarity transformation matrix $\mathcal{T}$ such that

$$\begin{bmatrix} \mathcal{T}^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \begin{bmatrix} \mathcal{T} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_{\mathcal{S}} & B_{\mathcal{S}} \\ C_{\mathcal{S}} & D_{\mathcal{S}} \end{bmatrix} \tag{31}$$

where $[A_{\mathcal{S}}, B_{\mathcal{S}}, C_{\mathcal{S}}, D_{\mathcal{S}}]$ represents the parametric structure of Fourier series coefficients, as defined in (14), (15), and (21).

Assumption 1: $\hat{A}$ has nonrepeated eigenvalues, and hence, it is diagonalizable via a similarity transformation matrix $\mathcal{T}_D$. Note that this also constrains $\mathbf{A}_d$ in (3) to be diagonalizable.

Assumption 1 is reasonable since even small perturbations eliminate repeated eigenvalues. We find such a $\mathcal{T}_D$ using an eigenvalue decomposition of $\hat{A}$ and transform the system as follows:

$$\begin{aligned} \hat{A}_D &= \mathcal{T}_D^{-1} \hat{A} \mathcal{T}_D, \quad & \hat{B}_D &= \mathcal{T}_D^{-1} \hat{B} \\ \hat{C}_D &= \hat{C} \mathcal{T}_D, & \hat{D}_D &= \hat{D}. \end{aligned} \tag{32}$$

Recall that there is a freedom in performing the eigenvalue decomposition; hence, when doing this, we ensure that $\mathcal{T}_D$ is selected such that eigenvalues of $\hat{A}_D$ enjoy the same parametric phase structure and ordering as (28). Finally, because of the SIMO structure of the lifted system, there are additional constraints (14) on the output matrix $\hat{C}_D$ but not on $\hat{B}_D$ (the input matrix is a column vector). These constraints can be satisfied with a similarity transformation $\mathcal{T}_C$. Note that such a transformation should maintain the parametric form of $\hat{A}_D$.

Proposition 1: Given Assumption 1, the similarity transformation matrix $\mathcal{T}_C \in \mathbb{C}^{\hat{n} \times \hat{n}}$ that satisfies

$$\mathcal{T}_C^{-1} \hat{A}_D \mathcal{T}_C = \hat{A}_D \tag{33}$$

has a diagonal structure as $\mathcal{T}_C = \mathrm{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_{\hat{n}}\}$.

Given Assumption 1, the proof is trivial. Hence, we use $\mathcal{T}_C$ to put $\hat{C}_D$ into the desired parametric form $C_{\mathcal{S}}$ as follows:

$$\hat{C}_D \mathcal{T}_C = C_{\mathcal{S}} \tag{34}$$

where $C_{\mathcal{S}}$ is the $N \times \hat{n}$ center columns of $\mathcal{C}_d$ in (14). Note that $C_{\mathcal{S}}$ is still in a parametric representation. However, we know that (34) projects $\hat{C}_D$ onto $C_{\mathcal{S}}$ such that there will be multiple equality constraints due to the same complex Fourier series coefficients in main diagonals and subdiagonals, as shown in (14). However, these terms will not be numerically equal due to inevitable noise, and hence, we first find the optimal Fourier series coefficient candidates. For simplicity, we will show the computations as if each Fourier series coefficient in $C_{\mathcal{S}}$ is a complex-valued scalar term, although they are vectors ($\mathbb{C}^{1 \times n_p}$). However, each variable in these vectors individually satisfies the form of $C_{\mathcal{S}}$ and is multiplied with a different element of the diagonal similarity transformation matrix. Hence, we can process them separately and combine the results. With this in mind, we choose the candidate solutions as the mean of their occurrences in $C_{\mathcal{S}}$ as follows:

For $-N_h - 1 \le m \le N_h$, we have

$$\bar{C}_m = \sum_{i=1}^{2N_h+1} \frac{\hat{C}_D(N/2 - N_h + m + i, i)\gamma_i}{2N_h + 1}. \tag{35}$$

For $m > N_h$, we have

$$\bar{C}_m = \sum_{i=1}^{3N_h+1-m} \frac{\hat{C}_D(N/2 - N_h + m + i, i)\gamma_i}{3N_h + 1 - m}. \tag{36}$$

For $m < -N_h - 1$, we have

$$\bar{C}_m = \sum_{i=1}^{3N_h+1+m} \frac{\hat{C}_D(i, i - N_h - 1 - m)\gamma_{(i-N_h-1-m)}}{3N_h + 1 + m}. \tag{37}$$

Now, we can generate $C_{\mathcal{S}}$ in terms of the estimated (and transformed) output matrix $\hat{C}_D$ and the diagonal similarity transformation matrix $\mathcal{T}_C$. We equate each variable on the left-hand side of (34) to their corresponding value in $C_{\mathcal{S}}$ using the complex Fourier series coefficients defined by (35)–(37). Note that these equalities will constrain the similarity transformation matrix $\mathcal{T}_C$. However, we expect to have infinitely many solutions that satisfy (34). Therefore, we formulate the set of all possible solutions and select one towards an LTP realization of the estimated system. For instance, for the fundamental harmonic, the first equality can be written by using (35) as follows:

$$\hat{C}_D(N/2 + 1 - N_h, 1)\gamma_1 = \sum_{i=1}^{2N_h+1} \frac{\hat{C}_D(N/2 - N_h + i, i)\gamma_i}{2N_h + 1}. \tag{38}$$

Organizing terms and multiplying both sides by $2N_h + 1$ yield

$$2N_h \hat{C}_D(N/2 + 1 - N_h, 1)\gamma_1 = \sum_{i=2}^{2N_h+1} \hat{C}_D(N/2 - N_h + i, i)\gamma_i. \tag{39}$$

We utilize a vector form for (39) as $\nu_0^1 \Gamma = 0$, where

$$\begin{aligned} \nu_0^1 &:= \left[2N_h \hat{C}_D(N/2 + 1 - N_h, 1), -\hat{C}_D(N/2 + 2 - N_h, 2), \ldots\right] \\ \Gamma &:= \left[\gamma_1, \gamma_2, \ldots, \gamma_{(2N_h+1)}\right]^T. \end{aligned} \tag{40}$$

Here, ν_0^1 represents the coefficients of the first constraint for the 0th Fourier series coefficient. Similarly, the ith constraint on the 0th Fourier series coefficient can be written as follows:

$$\nu_0^i := \left[\ldots, -\hat{C}_D(.,.), 2N_h \hat{C}_D(.,.), -\hat{C}_D(.,.), \ldots\right]. \tag{41}$$

Once we derive all constraint equations for all Fourier series coefficients, we combine in matrix multiplication form as follows:

$$\mathcal{V}\Gamma = 0 \tag{42}$$

where $\mathcal{V}$ includes all coefficient vectors for all complex Fourier series coefficients. We expect a complex-valued similarity transformation matrix and write (42) in real-valued form as follows:

$$\underbrace{\begin{bmatrix} \mathrm{Re}\{\mathcal{V}\} & -\mathrm{Im}\{\mathcal{V}\} \\ \mathrm{Im}\{\mathcal{V}\} & \mathrm{Re}\{\mathcal{V}\} \end{bmatrix}}_{\bar{\mathcal{V}}} \underbrace{\begin{bmatrix} \mathrm{Re}\{\Gamma\} \\ \mathrm{Im}\{\Gamma\} \end{bmatrix}}_{\bar{\Gamma}} = 0. \tag{43}$$

Note that $\bar{\mathcal{V}} \in \mathbb{R}^{(2M) \times (4N_h+2)}$, where $M > 2N_h + 1$.

Proposition 2: $\bar{\mathcal{V}}$ is rank deficient, and hence, the nullspace of $\bar{\mathcal{V}}$ (with dimension 2) defines the subspace of similarity transformation matrices that satisfy (34). ∎

Proof: We start by replacing the right-hand side of (38) with $\bar{C}_m$ to show that each $\hat{C}_D(.,.)$ in $\bar{\mathcal{V}}$ can be written in terms of $\bar{C}_m$ (see Remark 2). Hence, we can rewrite ν_0^1 as follows:

$$\nu_0^1 := \left[2N_h\bar{C}_0/\gamma_1, -\bar{C}_0/\gamma_2, \ldots, -\bar{C}_0/\gamma_{(2N_h+1)}\right]. \tag{44}$$

At this point, we can expand ν_0^1 as follows:

$$\nu_0^1 := \underbrace{\left[2N_h\bar{C}_0, -\bar{C}_0, \ldots, -\bar{C}_0\right]}_{\underline{\nu}_0^1} \underbrace{\begin{bmatrix} 1/\gamma_1 & & \\ & \ddots & \\ & & 1/\gamma_{(2N_h+1)} \end{bmatrix}}_{\underline{\gamma}} \tag{45}$$

where the summation of the elements of $\underline{\nu}_0^1$ is 0. We can also apply the same expansion on $\bar{\mathcal{V}}$ as follows:

$$\bar{\mathcal{V}} = \underbrace{\begin{bmatrix} \mathrm{Re}\{\underline{\mathcal{V}}\} & -\mathrm{Im}\{\underline{\mathcal{V}}\} \\ \mathrm{Im}\{\underline{\mathcal{V}}\} & \mathrm{Re}\{\underline{\mathcal{V}}\} \end{bmatrix}}_{\bar{\mathcal{V}}^\dagger} \underbrace{\begin{bmatrix} \mathrm{Re}\{\underline{\gamma}\} & -\mathrm{Im}\{\underline{\gamma}\} \\ \mathrm{Im}\{\underline{\gamma}\} & \mathrm{Re}\{\underline{\gamma}\} \end{bmatrix}}_{\underline{\gamma}^\dagger} \tag{46}$$

such that the summation of columns in $\bar{\mathcal{V}}^\dagger$ would be 0, based on (45). Thus, one of the columns can always be written in terms of the others, proving that $\bar{\mathcal{V}}$ is rank deficient.

Note that the rank of $\bar{\mathcal{V}}$ equals the rank of $\bar{\mathcal{V}}^\dagger$ since $\underline{\gamma}^\dagger$ is full rank by definition. Further, the way we define $\bar{\mathcal{V}}^\dagger$ ensures that the column space of its left and right halves is orthogonal to each other. Hence, we simply add the dimensions of nullspaces of the left and right halves to obtain the overall dimension of the nullspace of $\bar{\mathcal{V}}$. We know that the left and right halves are rank deficient by (45). In order to find the dimension of the nullspace of the left half, we consider the constraint equations $\nu_0^i\ \forall i = \{1, 2, \ldots, 2N_h + 1\}$ for $\bar{C}_0$ only, which will generate the coefficient vectors (also valid for $\mathrm{Im}\{\underline{\gamma}\}$) as follows:

$$\begin{bmatrix} 2N_h\mathrm{Re}\{\bar{C}_0\} & -\mathrm{Re}\{\bar{C}_0\} & \ldots & -\mathrm{Re}\{\bar{C}_0\} \\ -\mathrm{Re}\{\bar{C}_0\} & 2N_h\mathrm{Re}\{\bar{C}_0\} & \ddots & -\mathrm{Re}\{\bar{C}_0\} \\ \vdots & \ddots & \ddots & \vdots \\ -\mathrm{Re}\{\bar{C}_0\} & \ldots & \ldots & 2N_h\mathrm{Re}\{\bar{C}_0\} \end{bmatrix}. \tag{47}$$

Putting (47) to row echelon form, one can simply show that $\mathrm{Re}\{\underline{\mathcal{V}}\}$ is only rank-1 deficient (as $\mathrm{Im}\{\underline{\gamma}\}$). Considering the same derivations for the right half, one can find the dimension of the nullspace of $\bar{\mathcal{V}}$ as 2. ■

Remark 2: Note that (35)–(37) define the "optimal" Fourier series coefficients as the mean of their occurrences. In the proof of Proposition 2, we assume that we can write each $\hat{C}_D(.,.)$ in $\bar{\mathcal{V}}$ in terms of $\bar{C}_m$. However, this equivalence is only valid for the noise-free case. Noise will perturb the constraint equations, and numerically, $\bar{\mathcal{V}}$ will be full rank. The reason we computed the dimension of the nullspace is that we can use this dimension to choose the number of least significant eigenvectors of $\bar{\mathcal{V}}$ when generating the solution space for $\bar{\Gamma}$. ■

Now, we know that the dimension of the nullspace for $\bar{\mathcal{V}}$ should be 2. Therefore, we use SVD to find the eigenvectors for $\bar{\mathcal{V}}$. Then, we choose two eigenvectors v_1 and v_2 corresponding to least significant singular values as the basis vectors of the nullspace of $\bar{\mathcal{V}}$. Hence, the solution set for $\bar{\Gamma}$ can be written as $\bar{\Gamma} = \alpha_1 v_1 + \alpha_2 v_2$, and any choice of ($\alpha_1$, α_2) pair yields a valid solution for $\bar{\Gamma}$ that will construct the similarity transformation matrix $\mathcal{T}_C$, which transforms (32) into the following:

$$\begin{aligned} \hat{A}_{\mathcal{S}} &= \mathcal{T}_C^{-1}\hat{A}_D\mathcal{T}_C, \quad \hat{B}_{\mathcal{S}} = \mathcal{T}_C^{-1}\hat{B}_D \\ \hat{C}_{\mathcal{S}} &= \hat{C}_D\mathcal{T}_C, \qquad \hat{D}_{\mathcal{S}} = \hat{D}_D. \end{aligned} \tag{48}$$

Given (48), we identify the Fourier series coefficients for system matrices and construct the LTP state-space realization as follows:

$$\begin{aligned} \hat{\mathbf{x}}_d[k+1] &= \hat{\mathbf{A}}_d\hat{\mathbf{x}}_d[k] + \hat{\mathbf{B}}_d[k]\mathbf{u}_d[k] \\ \hat{\mathbf{y}}_d[k] &= \hat{\mathbf{C}}_d[k]\hat{\mathbf{x}}_d[k] + \hat{\mathbf{D}}_d[k]\mathbf{u}_d[k] \end{aligned} \tag{49}$$

by using the Fourier synthesis equations, such as (9). Finally, an inverse bilinear (Tustin) transformation on (49) yields

$$\begin{aligned} \dot{\hat{\mathbf{x}}}(t) &= \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}(t)u(t) \\ \hat{y}(t) &= \hat{\mathbf{C}}(t)\hat{\mathbf{x}}(t) + \hat{\mathbf{D}}(t)u(t) \end{aligned} \tag{50}$$

where

$$\begin{aligned} \hat{\mathbf{A}} &= (2/T_s)\left(\hat{\mathbf{A}}_d + I\right)^{-1}(\hat{\mathbf{A}}_d - I) \\ \hat{\mathbf{B}}(kT_s) &= \left(2/\sqrt{T_s}\right)\left(\hat{\mathbf{A}}_d + I\right)^{-1}\hat{\mathbf{B}}_d[k] \\ \hat{\mathbf{C}}(kT_s) &= \left(2/\sqrt{T_s}\right)\hat{\mathbf{C}}_d[k]\left(\hat{\mathbf{A}}_d + I\right)^{-1} \\ \hat{\mathbf{D}}(kT_s) &= \hat{\mathbf{D}}_d[k] - \hat{\mathbf{C}}_d[k]\left(\hat{\mathbf{A}}_d + I\right)^{-1}\hat{\mathbf{B}}_d[k] \end{aligned} \tag{51}$$

and the intersample behavior is obtained via linear interpolation.

Remark 3: Note that one can use this methodology to obtain Floquet transforms for known LTP systems. To accomplish this, one can simply equate the system matrices in (30) to those in (27) by skipping the LTI subspace identification part.

V. Numerical Example

In this section, we provide a numerical example to illustrate the practicality of the proposed method as well as to present a comparative analysis with one of the time-domain LTP subspace identification methods in the literature [15].

The numerical example we consider is in the following form:

$$\begin{aligned} \dot{\bar{x}}(t) &= \bar{A}(t)\bar{x}(t) + \bar{B}(t)u(t) \\ y(t) &= \bar{C}(t)\bar{x}(t) \end{aligned} \tag{52}$$

with the following system matrices:

$$\begin{aligned} \bar{A}(t) &= \begin{bmatrix} -2\mathbf{s}^2(t) + 0.5\mathbf{s}(2t) & \mathbf{s}(t) + \mathbf{s}(2t) \\ -\mathbf{c}^2(t) + \mathbf{s}(2t) & -2\mathbf{c}^2(t) - 0.5\mathbf{s}(2t) \end{bmatrix} \\ \bar{B}(t) &= \begin{bmatrix} -\mathbf{s}(t)(1 + \beta_b\mathbf{c}(t)) \\ \mathbf{c}(t)(1 + \beta_b\mathbf{c}(t)) \end{bmatrix} \\ \bar{C}(t) &= \left[\,\mathbf{c}(\mathbf{t})(\mathbf{1} + \beta_c\mathbf{c}(\mathbf{t}))\ \ \mathbf{s}(t)(1 + \beta_c\mathbf{c}(t))\,\right] \end{aligned} \tag{53}$$

where $\mathbf{s}(t) = \sin(4\pi t)$, $\mathbf{c}(t) = \cos(4\pi t)$, $\mathbf{s}(2t) = \sin(8\pi t)$, $\beta_b = 0.5$, and $\beta_c = 0.3$.

We simulate the LTP system with a sinusoidal input signal as the sum of different frequency cosine inputs. In order to design our input signal, we first choose the sampling frequency as $f_s = 1$ kHz. We plan to use the summation of 400 different frequency cosine signals in the range of 0.1–250 Hz for 200 s. Instead of choosing equidistant frequency values in continuous-time, we transform our limits into discrete-time frequency equivalents using the technique presented in [25], and then,

TABLE I
NRMSE OF IDENTIFICATION DATA WITH DIFFERENT NOISE REALIZATIONS

SNR	∞	40	30	20
Our Method	10^{-8}	0.1921	0.6417	2.4961
Verhaegen [15]	10^{-13}	0.9355	1.7326	3.3741

TABLE II
NRMSE FOR TEST SIGNALS

	Sinusoid	Noise	Step	Squarewave
Our Method	0.00002	0.00001	0.00003	0.00002
Verhaegen [15]	0.02020	0.00760	0.02790	0.02010

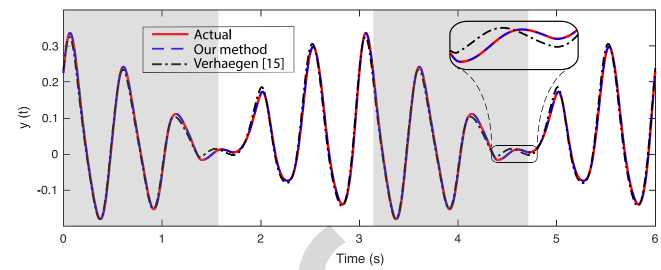

Fig. 1. Comparison of the proposed method and the method proposed in [15] for predicting the output of a square wave input signal with period π. Shaded and white regions represent the $+0.5$ and -0.5 regions of the square wave, respectively.

choose 400 equidistant frequency values in discrete-time to avoid distortion (warping) at high frequencies. Then, we transform the discrete-time frequency values back into continuous-time. This process is called *prewarping* [25].

Once we obtain the input–output data from the unknown system, we apply the proposed subspace identification method to estimate an LTP realization for the original system (see Appendix A). We estimate an equivalent representation for (52) in the form of (50) with the system matrices as follows:

$$\hat{\mathbf{A}} = \begin{bmatrix} 0 & 1 \\ -170.4848 & -2.0001 \end{bmatrix}$$
$$\hat{\mathbf{B}}(t) = \begin{bmatrix} 0 \\ 12.5671 + 6.2836\mathbf{c}(t) \end{bmatrix} \; \hat{\mathbf{C}}(t) \quad = \begin{bmatrix} 1 + 0.3\mathbf{c}(t) & 0 \end{bmatrix}. \tag{54}$$

Note that we neglected the sine terms with magnitude less than 10^{-8} for clarity. Since it is challenging to derive the Floquet transform for $\bar{A}(t)$ given in (53), we numerically computed a similarity transformation matrix that will give us the Floquet multipliers ($\{e^{\lambda_i T}\}|_{i=1}^{n_p}$, where $\{\lambda_i\}|_{i=1}^{n_p}$ are the eigenvalues of $\mathbf{A}$ [21] as $\mu_{1,2} = 0.5903 \pm 0.1419j$. On the other hand, Floquet multipliers of $\hat{\mathbf{A}}$, which is computed through our subspace identification method, are $\hat{\mu}_{1,2} = 0.5911 \pm 0.1360j$, which are very close to the numerical solution. In order to evaluate the prediction performance, we compute the normalized root-mean-squared error (nrmse) on identification data (see Table I). We also contaminate the output data $y(t)$ with zero mean white Gaussian noise to quantify the prediction performance with different signal-to-noise (SNR) conditions. As seen in Table I, the proposed method generates accurate output predictions for the noise-free case. For the noisy cases, we performed 100 independent noise realizations and report mean nrmse errors.

To provide a comparative analysis, we implemented the time-domain subspace identification method proposed in [15] for the same example defined in (52). We simulated (52) with a white noise sequence and collected sampled input–output data. Note that the method proposed in [15] works with discrete-time LTP systems. However, since we are working with sampled data, it is fair to compare the input–output data of the two methods. Note that the nrmse results presented in Table I for [15] are based on the prediction performance of its own identification signal (noise sequence). Our method works slightly better than that presented in [15] for predicting the identification signals under different noise realizations. In addition, we tested both methods with different test signals, such as a sinusoidal noise sequence and step and square wave input signals (see Table II). Again, the proposed method works slightly better for the prediction of different test signals as compared with the method proposed in [15]. To illustrate, we show a comparison plot for the square wave test signal prediction performance of the two methods in Fig. 1. The minor difference in prediction performance can be spotted in this comparison plot.

The comparison of our method with that proposed in [15] reveals that both methods are accurate in predicting identification and test signals. However, we emphasize certain points for a complete discussion. First, the LTP state-space model generated by our method is more intuitive than the model obtained using [15], which seeks to find a time-invariant state-space quadruple for each discrete-time step. Therefore, for an N-periodic discrete-time LTP system, the model obtained using [15] generates N different state-space quadruples, which are much more difficult to interpret than the form in (54) generated by our method. Moreover, the Floquet form in (54) is more preferable due to the time-invariant state matrix. Nevertheless, even though both methods work with a single input–output data pair, the model obtained using [15] finds and works with the smallest data length. Therefore, the method proposed in [15] is more advantageous in terms of using less data.

VI. CONCLUSION

In this note, we propose a new method for subspace-based state-space identification of LTP systems using frequency response data. Our solution is based on the fact that LTP systems can be transformed into equivalent discrete-time LTI systems. To accomplish this, we utilize a bilinear (Tustin) transformation and a frequency-domain lifting method available in the literature. Then, we estimate an LTI system representation that can predict the input–output data of the original system.

We, then, introduce a novel method to obtain a time-periodic realization for the estimated equivalent lifted LTI system. Note that the proposed LTP realization method works with the complexity of a standard subspace identification procedure. Finally, the estimated LTP system has a time-invariant state matrix. Therefore, our method allows finding Floquet transforms for known LTP systems via system identification.

APPENDIX A

Following is a summary of implementation details.

1) Simulate (1) with a sum-of-cosines input, selecting the frequencies as defined in [25].
2) Obtain the sampled data $\mathbf{u}_d[k]$ and $\mathbf{y}_d[k]$ for (3).
3) Use (13) and (7) to obtain $\mathcal{U}_d$ and $\mathcal{Y}_d$.
4) Process each frequency separately; choose $\bar{\mathcal{U}}_d = K$ and use (25) to obtain $\mathcal{Y}$.
5) Combine $\bar{\mathcal{U}}_d$ and $\mathcal{Y}$ for each frequency in vectors, and use the CVA [18] to obtain (30) (backsubstitute $\mathcal{T}_x$ and $\mathcal{T}_y$).
6) Perform eigenvalue decomposition on $\hat{A}$, and perform the similarity transformation in (32).

7) Construct the constraint equation in (42), and use the SVD to find the nullspace vectors.
8) Choose a solution from the nullspace, and do the similarity transformation in (48) to obtain (49).
9) Use (50) as the inverse bilinear (Tustin) transform.

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Technical Notes and Correspondence

Frequency-Domain Subspace Identification of Linear Time-Periodic (LTP) Systems

İsmail Uyanık, Uluç Saranlı, Mustafa Mert Ankaralı, Noah J. Cowan, and Ömer Morgül, *Member, IEEE*

***Abstract*—This paper proposes a new methodology for subspace-based state-space identification for linear time-periodic (LTP) systems. Since LTP systems can be lifted to equivalent linear time-invariant (LTI) systems, we first lift input–output data from an unknown LTP system as if they were collected from an equivalent LTI system. Then, we use frequency-domain subspace identification methods to find the LTI system estimate. Subsequently, we propose a novel method to obtain a time-periodic realization for the estimated lifted LTI system by exploiting the specific parametric structure of Fourier series coefficients of the frequency-domain lifting method. Our method can be used to obtain state-space estimates for unknown LTP systems as well as to obtain Floquet transforms for known LTP systems.**

***Index Terms*—Linear time-periodic (LTP) systems, subspace methods, system identification, time-varying systems.**

I. Introduction

In this paper, we introduce a frequency-domain subspace-based state-space identification method for linear time-periodic (LTP) systems. Many problems in engineering and biology, such as wind turbines [1], rotor bearing systems [2], aircraft models [3], locomotion [4], [5], and power distribution networks [6], require the consideration of time-periodic dynamics. As such, the analysis, identification, and control of LTP systems have received considerable attention [7]–[9].

A pioneering work by Wereley [7] introduced a frequency-domain analysis method for LTP systems. In this work, time-periodic system matrices in the LTP state-space formulation were expanded into their Fourier series coefficients. The principle of harmonic balance was used to obtain the concept of harmonic transfer functions (HTFs). Wereley's initial formulation for continuous-time LTP systems as infinite-dimensional operators was subsequently adapted to discrete time, which conveniently leads to finite-dimensional HTFs [10].

Most existing literature on LTP system identification [2], [11], including our own prior work on identification of legged locomotion [12]–[14], focus on using input–output HTF representations rather than state space. In addition, there are also contributions to state-space-based system identification for LTP systems [15], [16], analogous to subspace identification techniques commonly used for linear time-invariant (LTI) systems [17]. For instance, Verhaegen and Yu developed a subspace identification method for estimating successive state-transition matrices from time-domain data for linear time-varying (LTV) (including a special derivation for LTP) systems [15].

Critically, LTI subspace identification methods readily support both time-domain [17] and frequency-domain [18] data, whereas most subspace methods for LTP systems have focused on time-domain data [15], [16], and those state-space methods that do rely on frequency-domain data [19], [20] require that scheduling functions be known *a priori*. To the best of our knowledge, there are no general methods for frequency-domain subspace identification of LTP systems.

Here, we present a general subspace identification methodology for estimating state-space models from frequency-domain data for LTP systems. Our proposed methodology is based on the fact that LTP systems can be represented with equivalent LTI systems via lifting [10]. Based on this observation, we first lift the input–output data of an unknown LTP system as if they were collected from an equivalent LTI system, following previous methods [10]. We, then, estimate a discrete-time LTI state-space equivalent for the original LTP system by using an existing LTI frequency-domain subspace identification method [18]. A key property of the frequency-domain lifting method we utilize in this paper is the specific parametric structure of Fourier series coefficients associated with the original LTP system [10]. However, this structure is not, in general, preserved during the subspace identification process due to an inevitably unknown similarity transformation. In order to solve this issue, we identify a similarity transformation for the lifted LTI system that recovers the Fourier structure, although not the specific coefficients, because there is a subset of similarity transformations that preserves the Fourier structure but not its parameters. Our identification–realization algorithm also allows the realization of Floquet-transformed state-space models for LTP systems with arbitrary time-periodic system matrices (see Remark 3), whose analytic derivations are often very challenging and may even be impossible [21].

This paper is outlined as follows. We introduce the problem formulation in Section II. Then, in Section III, we show the existence of an equivalent discrete-time LTI system for a given LTP system via

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lifting and estimate its system matrices from frequency-domain data. In Section IV, we present a novel LTP realization algorithm for the estimated lifted LTI system. We provide an illustrative numerical example and a comparative analysis in Section V. Finally, we give our concluding remarks in Section VI.

II. Problem Formulation

In this paper, we consider single-input/single-output stable LTP systems represented by

$$\begin{aligned}\dot{\bar{x}}(t) &= \bar{A}(t)\bar{x}(t) + \bar{B}(t)u(t)\\ y(t) &= \bar{C}(t)\bar{x}(t) + \bar{D}(t)u(t)\end{aligned} \tag{1}$$

where $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, and $\bar{x}(t) \in \mathbb{R}^{n_p}$ represent the input, output, and state vectors, respectively. The system matrices are periodic with a fixed common period $T > 0$ (see Section III-B for the computation of T), with $\bar{A}(t) = \bar{A}(t+nT)$, $\bar{B}(t) = \bar{B}(t+nT)$, $\bar{C}(t) = \bar{C}(t+nT)$, and $\bar{D}(t) = \bar{D}(t+nT)\ \forall n \in \mathbb{Z}$.

We formulate the identification problem as follows.

Given

- A single pair of input–output signals $u(t)$ and $y(t)$ in the form of a sum-of-cosines signal containing different frequency components that provide an LTP frequency response.

Estimate

- The four LTP system matrices that will be equivalent to (1) up to a similarity transform.

The remaining sections detail our solution methodology (see Appendix A for the procedure). Obviously, LTI subspace identification methods would result in oversimplified LTI systems due to ignorance of harmonic responses. On the other hand, one can use LTV subspace identification methods in the time domain to solve a discrete-time version of this problem [15], [16]. Our solution method is unique in that it solves the problem in the frequency domain and results in intuitive state-space estimates in Floquet-transformed forms.

III. Existence and Estimation of a Discrete-Time Lifted LTI System Representation

This section first introduces a system of transformations that needs to be used to prove the existence of a real-valued discrete-time LTI representation of (1). We, then, show how we estimate such an LTI system using input–output data of the original LTP system. Naturally, the original state-space form of (1) will not be available. Therefore, the transformations described in this section are not directly applied on the state-space form of (1); rather, the transformations map the input–output data into a form that makes it as if they were collected from the transformed (LTI) system.

Based on Floquet theory, there exists a transformation that converts (1) into the following form:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}(t)u(t)\\ y(t) &= \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)u(t)\end{aligned} \tag{2}$$

where $\mathbf{A}$, $\mathbf{B}(t)$, $\mathbf{C}(t)$, and $\mathbf{D}(t)$ can be obtained as real-valued (by doubling the system period, if necessary), as long as the system matrices in (1) are real-valued [21]. Note that deriving a Floquet transform is challenging even when the state-space is known. On the other hand, the Floquet transform is a similarity transformation and does not affect the input–output data. Hence, we assume, without loss of generality, that the LTP system to be identified has the state-space form given in (2). Note that Floquet-transformed forms are easier to work with since they have a time-invariant state matrix. Thus, we seek to find an LTP state-space estimate for (1) in a Floquet form such as (2).

A. Discretization via Bilinear (Tustin) Transform

In principle, we could directly lift (2) to a continuous-time LTI equivalent and utilize continuous-time LTI subspace identification methods. However, the Hankel (data) matrices used for continuous-time LTI systems may become ill-conditioned with increasing system dimension [22]. Therefore, we find it more convenient to work with discrete-time LTI systems. To this end, we transform (2) into an approximate discrete-time LTI system. This has two benefits. First, lifting discrete-time LTP systems yields finite-dimensional LTI representations, unlike infinite-dimensional ones in continuous-time models. Second, and more importantly, it generalizes the applicability of our solutions to both continuous-time and discrete-time LTP systems. To accomplish this, we utilize the time-varying bilinear (Tustin) transformation to obtain a discrete-time LTP state-space representation of (2). Note that (2) is a special case of LTV systems with time-periodic system matrices (and a time-invariant state matrix). Therefore, our special case reduces the transformations in [23] to the following:

$$\begin{aligned}\mathbf{x}_d[k+1] &= \mathbf{A}_d\mathbf{x}_d[k] + \mathbf{B}_d[k]\mathbf{u}_d[k]\\ \mathbf{y}_d[k] &= \mathbf{C}_d[k]\mathbf{x}_d[k] + \mathbf{D}_d[k]\mathbf{u}_d[k]\end{aligned} \tag{3}$$

where $\mathbf{x}_d[k]$ represents discrete-time states and

$$\begin{aligned}\mathbf{A}_d &= ((2/T_s)I + \mathbf{A})((2/T_s)I - \mathbf{A})^{-1}\\ \mathbf{B}_d[k] &= \left(2/\sqrt{T_s}\right)((2/T_s)I - \mathbf{A})^{-1}\mathbf{B}(kT_s)\\ \mathbf{C}_d[k] &= \left(2/\sqrt{T_s}\right)\mathbf{C}(kT_s)((2/T_s)I - \mathbf{A})^{-1}\\ \mathbf{D}_d[k] &= \mathbf{D}(kT_s) + \mathbf{C}(kT_s)((2/T_s)I - \mathbf{A})^{-1}\mathbf{B}(kT_s).\end{aligned} \tag{4}$$

Here, T_s is the sampling period yielding sampled input–output data as $\mathbf{u}_d[k] := u(kT_s)$ and $\mathbf{y}_d[k] := y(kT_s)$. Derivations for (3) can be found in [24]. Note that (3) is an LTP system, where $\mathbf{B}_d[k] = \mathbf{B}_d[k+nN]\ \forall n \in \mathbb{Z}$ (also valid for $\mathbf{C}_d[k]$ and $\mathbf{D}_d[k]$) and N is the discrete-time system period defined as $N := T/T_s$. For the sake of simplicity, N is assumed to be even. The sampling period T_s determines N and, hence, the dimension of the lifted LTI equivalent. Using a higher sampling frequency allows capturing of high-frequency dynamics but also increases the complexity by increasing the lifted LTI system dimension. In addition, bilinear (Tustin) transformation causes frequency warping (distortions) at higher frequencies. To avoid this problem, we utilize the experimental design procedure in [25] by first prewarping the input frequencies that will be used while designing the sum-of-cosines input.

B. Lifting to a Time-Invariant Reformulation

One of the key properties of LTP systems is that a complex exponential input with frequency ω produces an output not only at the input frequency (which is the case for LTI systems) but also at different harmonics $\omega \pm k\omega_p$, $k \in \mathbb{Z}$ separated by the system frequency $\omega_p = 2\pi/T$, with possibly different magnitudes and phases in the steady state (this also allows estimating T from input–output data). In this context, the concept of HTFs was developed to represent each harmonic response of the LTP system with a distinct transfer function $G_k(w + k\omega_p)$ for $k \in \mathbb{Z}$ [7]. This approach represents an LTP system as the superposition of multiple modulated LTI systems. As such, HTFs can be used as a lifting technique to transform an LTI system into an LTI equivalent [10]. This motivates our use of HTFs as the frequency-domain lifting method to obtain an LTI equivalent state-space model for (3). Among possible alternatives (see [10] for a survey), we use frequency lifting in state space due to the convenient structure of Fourier series coefficients for periodic system matrices in lifted (semi)Toeplitz matrices.

The lifting method starts by analyzing the input–output spectra of the LTP systems by referring the concept of exponentially modulated periodic (EMP) signals. An input signal $\mathbf{u}_d[k]$ is a discrete-time EMP signal if there is a nonzero complex number z such that

$$\mathbf{u}_d[k+pN] = \mathbf{u}_d[k]z^{pN} \tag{5}$$

over one period for $p \in \mathbb{Z}$ and can be written as follows:

$$\mathbf{u}_d[k] := z^k \sum_{n=-N/2}^{N/2-1} U_n e^{j2\pi\frac{nk}{N}} \tag{6}$$

where U_n are called modulated Fourier series coefficients for EMP signals and are defined as follows:

$$U_n := \frac{1}{N}\sum_{k=0}^{N-1}\left(\mathbf{u}_d[k]z^{-k}\right)e^{-j2\pi\frac{nk}{N}}. \tag{7}$$

It has been shown that when an LTP system is given an EMP input, the state and output signals are also EMP in the steady state [7]. Therefore, similar to (6), we obtain the state vector as follows:

$$\mathbf{x}_d[k] = z^k \sum_{n\in\mathbf{I_N}} X_n e^{j2\pi\frac{nk}{N}} \tag{8}$$

and a similar expression for $\mathbf{y}_d[k]$, where $\mathbf{I_N}$ defines the interval $\mathbf{I_N} = [-N/2,\ N/2-1]$. In addition, the discrete-time Fourier synthesis equation for $\mathbf{B}_d[k]$ is computed as follows:

$$\mathbf{B}_d[k] = \sum_{n\in\mathbf{I_N}} B_n e^{j2\pi\frac{nk}{N}}. \tag{9}$$

Similar expressions are also valid for $\mathbf{C}_d[k]$ and $\mathbf{D}_d[k]$. Substituting Fourier synthesis equations into (3) yields

$$0 = z^k \sum_{n\in\mathbf{I_N}}\left(zX_n e^{j2\pi\frac{n}{N}} - \mathbf{A}_d X_n - \sum_{m\in\mathbf{I_N}} B_{n-m}U_m\right)e^{j2\pi\frac{nk}{N}}. \tag{10}$$

The exponentials $\{e^{j2\pi\frac{nk}{N}} | n \in \mathbf{I_N}\}$ constitute an orthonormal basis. Thus, by the principle of harmonic balance, each term enclosed by the brackets must be zero to ensure that the overall sum is zero. Therefore, for all $n \in \mathbf{I_N}$, we have

$$ze^{j2\pi\frac{n}{N}}X_n = \mathbf{A}_d X_n + \sum_{m\in\mathbf{I_N}} B_{n-m}U_m. \tag{11}$$

Note that the above equation is valid since Fourier coefficients B_m are also periodic with N. For the output, we also have

$$Y_n = \sum_{m\in\mathbf{I_N}} C_{n-m}X_m + \sum_{m\in\mathbf{I_N}} D_{n-m}U_m \tag{12}$$

for all $n \in \mathbf{I_N}$. Similar to continuous-time systems, (11) and (12) can be represented with (semi) Toeplitz matrices to obtain an LTI state-space model. To this end, we first define the N-block state $\mathcal{X}_d$, input $\mathcal{U}_d$, and output $\mathcal{Y}_d$ vectors, whose ith block for $i = 1, 2, \ldots, N$ is given by

$$\mathcal{X}_d(i) = X_{i-1-\frac{N}{2}},\quad \mathcal{U}_d(i) = U_{i-1-\frac{N}{2}},\quad \mathcal{Y}_d(i) = Y_{i-1-\frac{N}{2}}. \tag{13}$$

In addition, the time-invariant reformulation of the unlifted N-periodic output matrix can be obtained as follows:

$$\mathcal{C}_d := \begin{bmatrix} C_0 & C_{-1} & \ldots & C_{-\frac{N}{2}} & C_{\frac{N}{2}-1} & C_{\frac{N}{2}-2} & \ldots & C_1 \\ C_1 & C_0 & \ldots & C_{-\frac{N}{2}+1} & C_{-\frac{N}{2}} & C_{\frac{N}{2}-1} & \ldots & C_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ C_{-1} & C_{-2} & \ldots & C_{\frac{N}{2}-1} & C_{\frac{N}{2}-2} & C_{\frac{N}{2}-3} & \ldots & C_0 \end{bmatrix}. \tag{14}$$

Similarly, (semi)Toeplitz forms for $\mathcal{B}_d$ and $\mathcal{D}_d$ matrices can be obtained in terms of their Fourier series coefficients $\{B_n | n \in \mathbf{I_N}\}$ and $\{D_n | n \in \mathbf{I_N}\}$, respectively. Note that, since $\mathbf{A}_d$ is time-invariant, its Toeplitz form $\mathcal{A}_d$ includes only $\mathbf{A}_d$ in its diagonals as follows:

$$\mathcal{A}_d := \text{blkdiag}\{\mathbf{A}_d\} \mid \mathcal{A}_d \in \mathbb{R}^{Nn_p \times Nn_p} \tag{15}$$

where blkdiag represents a block-diagonal matrix and $\mathbf{A}_d$ is repeated blockwise on diagonals. Finally, we define a modulation matrix $\mathcal{N}_d$ to capture the exponential terms in (11) as follows:

$$\mathcal{N}_d := \text{blkdiag}\left\{e^{j2\pi\frac{n}{N}} I_{n_p} \mid \forall n \in \mathbf{I_N}\right\}. \tag{16}$$

We also define

$$\mathcal{A}_{dN} := \mathcal{N}_d^{-1}\mathcal{A}_d,\quad \mathcal{B}_{dN} := \mathcal{N}_d^{-1}\mathcal{B}_d. \tag{17}$$

Now, (11) and (12) can be represented as follows:

$$\begin{aligned} z\mathcal{X}_d &= \mathcal{A}_{dN}\mathcal{X}_d + \mathcal{B}_{dN}\mathcal{U}_d \\ \mathcal{Y}_d &= \mathcal{C}_d\mathcal{X}_d + \mathcal{D}_d\mathcal{U}_d. \end{aligned} \tag{18}$$

This is called the harmonic state-space (HSS) model, and it represents a lifted LTI equivalent of (1) for a general class of input–output signals. Following sections explain how we transform this HSS model into a more intuitional single-input multioutput (SIMO) LTI equivalent by limiting the space of EMP inputs.

C. SIMO LTI Equivalent

The input to the original LTP system (1) is a sum-of-cosines signal in the form $u(t) = \sum_{m=1}^{M} 2K\cos(\omega_m t)$. As stated earlier, each cosine input at ω_m produces an output spectra at $\pm\omega_m \pm k\omega_p$ for $k \in \mathbb{Z}$, since cosine triggers both $\pm\omega_m$. Hence, the input frequencies should be carefully selected to avoid any coincidence of harmonic responses (see [26] for illustrative explanations). Once this is satisfied, we can separate the input–output response of each individual cosine signal in the frequency domain. At this point, we write each single cosine input as follows:

$$u_c(t) = 2K\cos(\omega_m t) = \underbrace{Ke^{j\omega_m t}}_{u_c^+(t)} + \underbrace{Ke^{-j\omega_m t}}_{u_c^-(t)}. \tag{19}$$

Let the output of (1) to inputs $u_c^+(t)$, $u_c^-(t)$, and $u_c(t)$ be $y_c^+(t)$, $y_c^-(t)$, and $y_c(t)$, respectively, where $y_c(t) = y_c^+(t) + y_c^-(t)$. Ensuring that $\omega_m \neq 0.5k\omega_p$ for $k \in \mathbb{Z}$, one can also guarantee that there will be no coincidence in harmonic responses of the single-cosine input [26]. Thus, we can simulate (1) with $u_c(t)$ and only use $y_c^+(t)$ as the output, assuming that our input was $u_c^+(t)$. We choose distinct exponential modulation $z = e^{j\omega_m}$ in (5) for each individual input signal. Hence, the modulated Fourier series coefficient vector in (18) becomes $\mathcal{U}_d = [0\ \ldots\ 0\ K\ 0\ \ldots\ 0]^T$ with K on row $(N/2+1)$ for each input. More importantly, with its current form, $\mathcal{U}_d$ selects only column $(N/2+1)$

in (14) for $\mathcal{B}_d$ and $\mathcal{D}_d$, yielding

$$\begin{aligned} z\mathcal{X}_d &= \mathcal{A}_{dN}\mathcal{X}_d + \bar{\mathcal{B}}_{dN}\bar{\mathcal{U}}_d \\ \mathcal{Y}_d &= \mathcal{C}_d\mathcal{X}_d + \bar{\mathcal{D}}_d\bar{\mathcal{U}}_d \end{aligned} \tag{20}$$

where $\bar{\mathcal{U}}_d = K$, $z = e^{j\omega_m}$, and

$$\begin{aligned} \bar{\mathcal{B}}_{dN} &:= \mathcal{N}_d^{-1}\left[\, B_{-N/2} \quad \ldots \quad B_0 \quad \ldots \quad B_{N/2-1} \,\right]^T \\ \bar{\mathcal{D}}_d &:= \left[\, D_{-N/2} \quad \ldots \quad D_0 \quad \ldots \quad D_{N/2-1} \,\right]^T. \end{aligned} \tag{21}$$

D. Transforming to a Real-Valued State-Space Model

One problem with LTI subspace identification methods is that they rely on real-valued input–output data in the time domain to estimate real-valued system matrices [27]–[31]. Hence, we need to transform (20) into a system that, if it were converted to the time-domain, would produce real-valued states and outputs, given real-valued inputs. Note that this system would not correspond to our original time-domain system. Rather, the time-domain equivalent of (20) is a fictitious system, useful only for the purpose of analysis. This SIMO LTI system has N states and outputs. Thus, we have

$$\begin{aligned} \mathcal{X}_d[k] &:= \left[\, \bar{\mathcal{X}}_{-N/2}[k] \quad \ldots \quad \bar{\mathcal{X}}_0[k] \quad \ldots \quad \bar{\mathcal{X}}_{N/2-1}[k] \,\right]^T \\ \mathcal{Y}_d[k] &:= \left[\, \bar{\mathcal{Y}}_{-N/2}[k] \quad \ldots \quad \bar{\mathcal{Y}}_0[k] \quad \ldots \quad \bar{\mathcal{Y}}_{N/2-1}[k] \,\right]^T. \end{aligned} \tag{22}$$

Considering (20) as an LTI system in the *z*-domain and by utilizing the block-diagonal structure of $\mathcal{A}_{dN}$ (noting that $\mathbf{A}_d$ is stable), one can simply solve for each state equation in the steady state as follows:

$$\bar{\mathcal{X}}_m[k] = \sum_{i=0}^{k-1} \left(e^{-j2\pi\frac{m}{N}} I_{n_p}\mathbf{A}_d\right)^{k-i-1} \left(e^{-j2\pi\frac{m}{N}} I_{n_p} B_m\right) u[i] \tag{23}$$

where $u[k] = 2K\cos(\omega_m kT_s)$. This follows since $\bar{\mathcal{U}}_d = K$ in the *z*-domain corresponds to a single cosine input signal for the time-domain signal. (We write the input as in (19) and ignore the negative frequency component for the sake of our analysis.) Also, note that $B_m = B_{-m}^*$ since $\mathbf{B}_d[k]$ is real-valued by definition. Hence, we can state that $\bar{\mathcal{X}}_m[k] = \bar{\mathcal{X}}_{-m}^*[k]$, except for $\bar{\mathcal{X}}_{-N/2}[k]$ and $\bar{\mathcal{X}}_0[k]$, which are both real-valued as seen in (23). A similar analysis can be done for $\mathcal{Y}_d[k]$ by using (23). However, solutions for each LTI output signal $\bar{\mathcal{Y}}_m[k]$ are more challenging since complex-conjugate state solutions are now multiplied with shifted versions of Fourier series coefficients as illustrated in (14). To achieve our goal, we first write the steady-state solutions for each output signal as follows:

$$\bar{\mathcal{Y}}_m[k] = \sum_{n\in\mathbf{I_N}} C_{m-n}\bar{\mathcal{X}}_n[k]. \tag{24}$$

By using lengthy but straightforward calculations, one can show that $\bar{\mathcal{Y}}_m[k] = \bar{\mathcal{Y}}_{-m}^*[k]$ in the steady state. Having shown the complex-conjugate nature of the time-domain state and output signals, we define two complex-valued transformation matrices $\mathcal{T}_x$ and $\mathcal{T}_y$ as follows:

$$\underline{\mathcal{X}}_d[k] := \mathcal{T}_x\mathcal{X}_d[k], \quad \underline{\mathcal{Y}}_d[k] := \mathcal{T}_y\mathcal{Y}_d[k] \tag{25}$$

where $\mathcal{T}_x$ can be defined as follows:

$$\mathcal{T}_x := 0.5\begin{bmatrix} 2I_{n_p} & 0 & 0 & 0 \\ 0 & I_{(N/2-1)n_p} & 0 & J_{(N/2-1)n_p} \\ 0 & 0 & 2I_{n_p} & 0 \\ 0 & -jJ_{(N/2-1)n_p} & 0 & jI_{(N/2-1)n_p} \end{bmatrix} \tag{26}$$

with a similar expression for $\mathcal{T}_y$, where $I_{\bar{n}}$ is the usual $\bar{n}\times\bar{n}$ identity and $J_{\bar{n}}$ is an antidiagonal $\bar{n}\times\bar{n}$ matrix (i.e., 1 for the entries where $i = \bar{n} - j + 1$, 0 otherwise) with associated sizes. Equation (25) transforms (20) into the following:

$$\begin{aligned} z\mathcal{X} &= \mathcal{T}_x\mathcal{A}_{dN}\mathcal{T}_x^{-1}\mathcal{X} + \mathcal{T}_x\bar{\mathcal{B}}_{dN}\bar{\mathcal{U}}_d \\ \mathcal{Y} &= \mathcal{T}_y\mathcal{C}_d\mathcal{T}_x^{-1}\mathcal{X} \quad + \mathcal{T}_y\bar{\mathcal{D}}_d\bar{\mathcal{U}}_d \end{aligned} \tag{27}$$

where $\mathcal{X} := \mathcal{T}_x\mathcal{X}_d$ and $\mathcal{Y} := \mathcal{T}_y\mathcal{Y}_d$. Note that $\mathcal{T}_x$ and $\mathcal{T}_y$ also transform the system matrices to real-valued equivalents.

E. Estimating an LTI Equivalent via Subspace Identification

At this point, we could utilize a variety of LTI subspace identification methods [17], [18], [27], [32], [33]. Although we could not find a general benchmarking study on these algorithms, it has been shown that CVA [18] performs better than N4SID [17] and MOESP [32] in terms of prediction error and computational complexity [34]. Moreover, CVA [18] is MATLAB's (The MathWorks Inc., Natick, MA, USA) built-in frequency-domain subspace identification method. Hence, we use CVA for estimating the equivalent LTI system by carefully selecting the estimated system dimension (see Remark 1).

Remark 1: In classical LTI subspace identification, the estimated system order $\hat{n}$ is chosen based on large drops in singular values of Hankel matrices [17]. However, one needs to be aware of the specific parametric structure of LTP systems while selecting $\hat{n}$. Let the eigenvalues of $\mathbf{A}_d$ be $S_d = \{\lambda_i^d\}_{i=1}^{n_p}$. Lifting to (17) results in $\mathcal{A}_{dN}$ with the following eigenvalues:

$$S = \left\{\left\{\lambda_i^d e^{-j2\pi\frac{k}{N}}\right\}_{i=1}^{n_p} \middle| \forall k\in\mathbf{I_N}\right\}. \tag{28}$$

Once $\hat{n}$ is chosen based on the singular values (not the eigenvalues), the user should check the eigenvalues of the estimated state matrix for the phase structure defined in (28). This phase structure will both reveal the underlying LTP system's dimension n_p as well as the number of harmonics that will appear in the state vector N_h. The user might need to use expert knowledge to decide on $\hat{n}$ to maintain the phase structure of (28). The correct choice of $\hat{n}$ will yield eigenvalues as follows:

$$\hat{S} = \left\{\left\{\lambda_i^d e^{-j2\pi\frac{k}{N}}\right\}_{i=1}^{n_p} \middle| \forall k\in[-N_h, N_h]\right\}. \tag{29}$$

Note that, under these constraints, $\hat{n}$ would be equal to the cardinality of $\hat{S}$, i.e., $\hat{n} = |\hat{S}| = (2N_h+1)n_p$, and this will limit the dimensions of $\hat{\mathcal{X}}$ (and associated system matrices) in (30). It is quite possible that the user could also limit the output harmonics in (13) based on the LTP frequency response. This choice will be independent of $\hat{n}$ and it will limit the dimensions of $\hat{\mathcal{Y}}$ (and associated system matrices) in (30). ■

The CVA method estimates a quadruple of real-valued LTI system matrices as $[\hat{\bar{A}}, \hat{\bar{B}}, \hat{\bar{C}}, \hat{\bar{D}}]$, which is equivalent to (27) up to a similarity transformation. However, we need to backsubstitute the transformations in (17) to find an equivalent lifted LTI system for the unknown LTP system. To this end, we use $\hat{A} = \hat{\bar{A}}$, $\hat{B} = \hat{\bar{B}}$, $\hat{C} = \mathcal{T}_y^{-1}\hat{\bar{C}}$, and $\hat{D} = \mathcal{T}_y^{-1}\hat{\bar{D}}$ and obtain the equivalent lifted LTI system as follows:

$$\begin{aligned} z\hat{\mathcal{X}} &= \hat{A}\hat{\mathcal{X}} + \hat{B}\bar{\mathcal{U}}_d \\ \hat{\mathcal{Y}} &= \hat{C}\hat{\mathcal{X}} + \hat{D}\bar{\mathcal{U}}_d \end{aligned} \tag{30}$$

where $\hat{A}\in\mathbb{R}^{\hat{n}\times\hat{n}}$, $\hat{B}\in\mathbb{R}^{\hat{n}\times 1}$, $\hat{C}\in\mathbb{C}^{N\times\hat{n}}$, and $\hat{D}\in\mathbb{C}^{N\times 1}$. Note that we do not substitute $\mathcal{T}_x$ back since it is already in the form of a similarity transformation.

At this point, our method provides a parametric system representation, which is equivalent to the lifted LTI form (27) of the original LTP system. However, the main drawback of this representation—lifted

LTI—is that it is unintuitive and requires additional processes (unlifting the signals) to predict the output of the original LTP system. In Section IV, we introduce an LTP realization method that collapses the lifted LTI system to an LTP system in Floquet form.

IV. Time-Periodic Realization for the Estimated Lifted LTI Equivalent

The specific parametric structure of Fourier series coefficients is not generally preserved during subspace identification. Finding a computationally effective solution to this problem remains an open issue [35]. Motivated by this, we propose a time-periodic realization method for lifted LTI systems. Unlike the previous work that considers unforced LTP systems [36], we provide a framework for a general class of LTP systems with inputs. Our goal can be defined as finding a similarity transformation matrix $\mathcal{T}$ such that

$$\begin{bmatrix} \mathcal{T}^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \begin{bmatrix} \mathcal{T} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_{\mathcal{S}} & B_{\mathcal{S}} \\ C_{\mathcal{S}} & D_{\mathcal{S}} \end{bmatrix} \tag{31}$$

where $[A_{\mathcal{S}}, B_{\mathcal{S}}, C_{\mathcal{S}}, D_{\mathcal{S}}]$ represents the parametric structure of Fourier series coefficients, as defined in (14), (15), and (21).

Assumption 1: $\hat{A}$ has nonrepeated eigenvalues, and hence, it is diagonalizable via a similarity transformation matrix $\mathcal{T}_D$. Note that this also constrains $\mathbf{A}_d$ in (3) to be diagonalizable.

Assumption 1 is reasonable since even small perturbations eliminate repeated eigenvalues. We find such a $\mathcal{T}_D$ using an eigenvalue decomposition of $\hat{A}$ and transform the system as follows:

$$\begin{aligned} \hat{A}_D &= \mathcal{T}_D^{-1} \hat{A} \mathcal{T}_D, \quad & \hat{B}_D &= \mathcal{T}_D^{-1} \hat{B} \\ \hat{C}_D &= \hat{C} \mathcal{T}_D, & \hat{D}_D &= \hat{D}. \end{aligned} \tag{32}$$

Recall that there is a freedom in performing the eigenvalue decomposition; hence, when doing this, we ensure that $\mathcal{T}_D$ is selected such that eigenvalues of $\hat{A}_D$ enjoy the same parametric phase structure and ordering as (28). Finally, because of the SIMO structure of the lifted system, there are additional constraints (14) on the output matrix $\hat{C}_D$ but not on $\hat{B}_D$ (the input matrix is a column vector). These constraints can be satisfied with a similarity transformation $\mathcal{T}_C$. Note that such a transformation should maintain the parametric form of $\hat{A}_D$.

Proposition 1: Given Assumption 1, the similarity transformation matrix $\mathcal{T}_C \in \mathbb{C}^{\hat{n} \times \hat{n}}$ that satisfies

$$\mathcal{T}_C^{-1} \hat{A}_D \mathcal{T}_C = \hat{A}_D \tag{33}$$

has a diagonal structure as $\mathcal{T}_C = \text{diag}\{\gamma_1, \gamma_2, \dots, \gamma_{\hat{n}}\}$.

Given Assumption 1, the proof is trivial. Hence, we use $\mathcal{T}_C$ to put $\hat{C}_D$ into the desired parametric form $C_{\mathcal{S}}$ as follows:

$$\hat{C}_D \mathcal{T}_C = C_{\mathcal{S}} \tag{34}$$

where $C_{\mathcal{S}}$ is the $N \times \hat{n}$ center columns of $\mathcal{C}_d$ in (14). Note that $C_{\mathcal{S}}$ is still in a parametric representation. However, we know that (34) projects $\hat{C}_D$ onto $C_{\mathcal{S}}$ such that there will be multiple equality constraints due to the same complex Fourier series coefficients in main diagonals and subdiagonals, as shown in (14). However, these terms will not be numerically equal due to inevitable noise, and hence, we first find the optimal Fourier series coefficient candidates. For simplicity, we will show the computations as if each Fourier series coefficient in $C_{\mathcal{S}}$ is a complex-valued scalar term, although they are vectors ($\mathbb{C}^{1 \times n_p}$). However, each variable in these vectors individually satisfies the form of $C_{\mathcal{S}}$ and is multiplied with a different element of the diagonal similarity transformation matrix. Hence, we can process them separately and combine the results. With this in mind, we choose the candidate solutions as the mean of their occurrences in $C_{\mathcal{S}}$ as follows:

For $-N_h - 1 \leq m \leq N_h$, we have

$$\bar{C}_m = \sum_{i=1}^{2N_h+1} \frac{\hat{C}_D(N/2 - N_h + m + i, i)\gamma_i}{2N_h + 1}. \tag{35}$$

For $m > N_h$, we have

$$\bar{C}_m = \sum_{i=1}^{3N_h+1-m} \frac{\hat{C}_D(N/2 - N_h + m + i, i)\gamma_i}{3N_h + 1 - m}. \tag{36}$$

For $m < -N_h - 1$, we have

$$\bar{C}_m = \sum_{i=1}^{3N_h+1+m} \frac{\hat{C}_D(i, i - N_h - 1 - m)\gamma_{(i-N_h-1-m)}}{3N_h + 1 + m}. \tag{37}$$

Now, we can generate $C_{\mathcal{S}}$ in terms of the estimated (and transformed) output matrix $\hat{C}_D$ and the diagonal similarity transformation matrix $\mathcal{T}_C$. We equate each variable on the left-hand side of (34) to their corresponding value in $C_{\mathcal{S}}$ using the complex Fourier series coefficients defined by (35)–(37). Note that these equalities will constrain the similarity transformation matrix $\mathcal{T}_C$. However, we expect to have infinitely many solutions that satisfy (34). Therefore, we formulate the set of all possible solutions and select one towards an LTP realization of the estimated system. For instance, for the fundamental harmonic, the first equality can be written by using (35) as follows:

$$\hat{C}_D(N/2 + 1 - N_h, 1)\gamma_1 = \sum_{i=1}^{2N_h+1} \frac{\hat{C}_D(N/2 - N_h + i, i)\gamma_i}{2N_h + 1}. \tag{38}$$

Organizing terms and multiplying both sides by $2N_h + 1$ yield

$$2N_h \hat{C}_D(N/2 + 1 - N_h, 1)\gamma_1 = \sum_{i=2}^{2N_h+1} \hat{C}_D(N/2 - N_h + i, i)\gamma_i. \tag{39}$$

We utilize a vector form for (39) as $\nu_0^1 \Gamma = 0$, where

$$\begin{aligned} \nu_0^1 &:= \left[2N_h \hat{C}_D(N/2 + 1 - N_h, 1), -\hat{C}_D(N/2 + 2 - N_h, 2), \dots\right] \\ \Gamma &:= \left[\gamma_1, \gamma_2, \dots, \gamma_{(2N_h+1)}\right]^T. \end{aligned} \tag{40}$$

Here, ν_0^1 represents the coefficients of the first constraint for the 0th Fourier series coefficient. Similarly, the ith constraint on the 0th Fourier series coefficient can be written as follows:

$$\nu_0^i := \left[\dots, -\hat{C}_D(.,.), 2N_h \hat{C}_D(.,.), -\hat{C}_D(.,.), \dots\right]. \tag{41}$$

Once we derive all constraint equations for all Fourier series coefficients, we combine in matrix multiplication form as follows:

$$\mathcal{V}\Gamma = 0 \tag{42}$$

where $\mathcal{V}$ includes all coefficient vectors for all complex Fourier series coefficients. We expect a complex-valued similarity transformation matrix and write (42) in real-valued form as follows:

$$\underbrace{\begin{bmatrix} \text{Re}\{\mathcal{V}\} & -\text{Im}\{\mathcal{V}\} \\ \text{Im}\{\mathcal{V}\} & \text{Re}\{\mathcal{V}\} \end{bmatrix}}_{\bar{\mathcal{V}}} \underbrace{\begin{bmatrix} \text{Re}\{\Gamma\} \\ \text{Im}\{\Gamma\} \end{bmatrix}}_{\bar{\Gamma}} = 0. \tag{43}$$

Note that $\bar{\mathcal{V}} \in \mathbb{R}^{(2M) \times (4N_h+2)}$, where $M > 2N_h + 1$.

Proposition 2: $\bar{\mathcal{V}}$ is rank deficient, and hence, the nullspace of $\bar{\mathcal{V}}$ (with dimension 2) defines the subspace of similarity transformation matrices that satisfy (34). ■

Proof: We start by replacing the right-hand side of (38) with $\bar{C}_m$ to show that each $\hat{C}_D(.,.)$ in $\bar{\mathcal{V}}$ can be written in terms of $\bar{C}_m$ (see Remark 2). Hence, we can rewrite ν_0^1 as follows:

$$\nu_0^1 := \left[2N_h\bar{C}_0/\gamma_1, -\bar{C}_0/\gamma_2, \ldots, -\bar{C}_0/\gamma_{(2N_h+1)}\right]. \tag{44}$$

At this point, we can expand ν_0^1 as follows:

$$\nu_0^1 := \underbrace{[2N_h\bar{C}_0, -\bar{C}_0, \ldots, -\bar{C}_0]}_{\underline{\nu}_0^1} \underbrace{\begin{bmatrix} 1/\gamma_1 & & \\ & \ddots & \\ & & 1/\gamma_{(2N_h+1)} \end{bmatrix}}_{\underline{\gamma}} \tag{45}$$

where the summation of the elements of $\underline{\nu}_0^1$ is 0. We can also apply the same expansion on $\bar{\mathcal{V}}$ as follows:

$$\bar{\mathcal{V}} = \underbrace{\begin{bmatrix} \mathrm{Re}\{\underline{\mathcal{V}}\} & -\mathrm{Im}\{\underline{\mathcal{V}}\} \\ \mathrm{Im}\{\underline{\mathcal{V}}\} & \mathrm{Re}\{\underline{\mathcal{V}}\} \end{bmatrix}}_{\bar{\mathcal{V}}^\dagger} \underbrace{\begin{bmatrix} \mathrm{Re}\{\underline{\gamma}\} & -\mathrm{Im}\{\underline{\gamma}\} \\ \mathrm{Im}\{\underline{\gamma}\} & \mathrm{Re}\{\underline{\gamma}\} \end{bmatrix}}_{\underline{\gamma}^\dagger} \tag{46}$$

such that the summation of columns in $\bar{\mathcal{V}}^\dagger$ would be 0, based on (45). Thus, one of the columns can always be written in terms of the others, proving that $\bar{\mathcal{V}}$ is rank deficient.

Note that the rank of $\bar{\mathcal{V}}$ equals the rank of $\bar{\mathcal{V}}^\dagger$ since $\underline{\gamma}^\dagger$ is full rank by definition. Further, the way we define $\bar{\mathcal{V}}^\dagger$ ensures that the column space of its left and right halves is orthogonal to each other. Hence, we simply add the dimensions of nullspaces of the left and right halves to obtain the overall dimension of the nullspace of $\bar{\mathcal{V}}$. We know that the left and right halves are rank deficient by (45). In order to find the dimension of the nullspace of the left half, we consider the constraint equations $\nu_0^i\ \forall i = \{1, 2, \ldots, 2N_h+1\}$ for $\bar{C}_0$ only, which will generate the coefficient vectors (also valid for $\mathrm{Im}\{\underline{\gamma}\}$) as follows:

$$\begin{bmatrix} 2N_h\mathrm{Re}\{\bar{C}_0\} & -\mathrm{Re}\{\bar{C}_0\} & \ldots & -\mathrm{Re}\{\bar{C}_0\} \\ -\mathrm{Re}\{\bar{C}_0\} & 2N_h\mathrm{Re}\{\bar{C}_0\} & \ddots & -\mathrm{Re}\{\bar{C}_0\} \\ \vdots & \ddots & \ddots & \vdots \\ -\mathrm{Re}\{\bar{C}_0\} & \ldots & \ldots & 2N_h\mathrm{Re}\{\bar{C}_0\} \end{bmatrix}. \tag{47}$$

Putting (47) to row echelon form, one can simply show that $\mathrm{Re}\{\underline{\mathcal{V}}\}$ is only rank-1 deficient (as $\mathrm{Im}\{\underline{\gamma}\}$). Considering the same derivations for the right half, one can find the dimension of the nullspace of $\bar{\mathcal{V}}$ as 2. ■

Remark 2: Note that (35)–(37) define the "optimal" Fourier series coefficients as the mean of their occurrences. In the proof of Proposition 2, we assume that we can write each $\hat{C}_D(.,.)$ in $\bar{\mathcal{V}}$ in terms of $\bar{C}_m$. However, this equivalence is only valid for the noise-free case. Noise will perturb the constraint equations, and numerically, $\bar{\mathcal{V}}$ will be full rank. The reason we computed the dimension of the nullspace is that we can use this dimension to choose the number of least significant eigenvectors of $\bar{\mathcal{V}}$ when generating the solution space for $\bar{\Gamma}$. ■

Now, we know that the dimension of the nullspace for $\bar{\mathcal{V}}$ should be 2. Therefore, we use SVD to find the eigenvectors for $\bar{\mathcal{V}}$. Then, we choose two eigenvectors v_1 and v_2 corresponding to least significant singular values as the basis vectors of the nullspace of $\bar{\mathcal{V}}$. Hence, the solution set for $\bar{\Gamma}$ can be written as $\bar{\Gamma} = \alpha_1 v_1 + \alpha_2 v_2$, and any choice of ($\alpha_1$, α_2) pair yields a valid solution for $\bar{\Gamma}$ that will construct the similarity transformation matrix $\mathcal{T}_C$, which transforms (32) into the following:

$$\begin{aligned} \hat{A}_{\mathcal{S}} &= \mathcal{T}_C^{-1}\hat{A}_D\mathcal{T}_C, \quad & \hat{B}_{\mathcal{S}} &= \mathcal{T}_C^{-1}\hat{B}_D \\ \hat{C}_{\mathcal{S}} &= \hat{C}_D\mathcal{T}_C, & \hat{D}_{\mathcal{S}} &= \hat{D}_D. \end{aligned} \tag{48}$$

Given (48), we identify the Fourier series coefficients for system matrices and construct the LTP state-space realization as follows:

$$\begin{aligned} \hat{\mathbf{x}}_d[k+1] &= \hat{\mathbf{A}}_d\hat{\mathbf{x}}_d[k] + \hat{\mathbf{B}}_d[k]\mathbf{u}_d[k] \\ \hat{\mathbf{y}}_d[k] &= \hat{\mathbf{C}}_d[k]\hat{\mathbf{x}}_d[k] + \hat{\mathbf{D}}_d[k]\mathbf{u}_d[k] \end{aligned} \tag{49}$$

by using the Fourier synthesis equations, such as (9). Finally, an inverse bilinear (Tustin) transformation on (49) yields

$$\begin{aligned} \dot{\hat{\mathbf{x}}}(t) &= \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}(t)u(t) \\ \hat{y}(t) &= \hat{\mathbf{C}}(t)\hat{\mathbf{x}}(t) + \hat{\mathbf{D}}(t)u(t) \end{aligned} \tag{50}$$

where

$$\begin{aligned} \hat{\mathbf{A}} &= (2/T_s)\left(\hat{\mathbf{A}}_d + I\right)^{-1}(\hat{\mathbf{A}}_d - I) \\ \hat{\mathbf{B}}(kT_s) &= \left(2/\sqrt{T_s}\right)\left(\hat{\mathbf{A}}_d + I\right)^{-1}\hat{\mathbf{B}}_d[k] \\ \hat{\mathbf{C}}(kT_s) &= \left(2/\sqrt{T_s}\right)\hat{\mathbf{C}}_d[k]\left(\hat{\mathbf{A}}_d + I\right)^{-1} \\ \hat{\mathbf{D}}(kT_s) &= \hat{\mathbf{D}}_d[k] - \hat{\mathbf{C}}_d[k]\left(\hat{\mathbf{A}}_d + I\right)^{-1}\hat{\mathbf{B}}_d[k] \end{aligned} \tag{51}$$

and the intersample behavior is obtained via linear interpolation.

Remark 3: Note that one can use this methodology to obtain Floquet transforms for known LTP systems. To accomplish this, one can simply equate the system matrices in (30) to those in (27) by skipping the LTI subspace identification part.

V. Numerical Example

In this section, we provide a numerical example to illustrate the practicality of the proposed method as well as to present a comparative analysis with one of the time-domain LTP subspace identification methods in the literature [15].

The numerical example we consider is in the following form:

$$\begin{aligned} \dot{\bar{x}}(t) &= \bar{A}(t)\bar{x}(t) + \bar{B}(t)u(t) \\ y(t) &= \bar{C}(t)\bar{x}(t) \end{aligned} \tag{52}$$

with the following system matrices:

$$\begin{aligned} \bar{A}(t) &= \begin{bmatrix} -2\mathbf{s}^2(t) + 0.5\mathbf{s}(2t) & \mathbf{s}(t) + \mathbf{s}(2t) \\ -\mathbf{c}^2(t) + \mathbf{s}(2t) & -2\mathbf{c}^2(t) - 0.5\mathbf{s}(2t) \end{bmatrix} \\ \bar{B}(t) &= \begin{bmatrix} -\mathbf{s}(t)(1 + \beta_b\mathbf{c}(t)) \\ \mathbf{c}(t)(1 + \beta_b\mathbf{c}(t)) \end{bmatrix} \\ \bar{C}(t) &= \left[\,\mathbf{c}(\mathbf{t})(\mathbf{1} + \beta_\mathbf{c}\mathbf{c}(\mathbf{t}))\ \ \mathbf{s}(t)(1 + \beta_c\mathbf{c}(t))\,\right] \end{aligned} \tag{53}$$

where $\mathbf{s}(t) = \sin(4\pi t)$, $\mathbf{c}(t) = \cos(4\pi t)$, $\mathbf{s}(2t) = \sin(8\pi t)$, $\beta_b = 0.5$, and $\beta_c = 0.3$.

We simulate the LTP system with a sinusoidal input signal as the sum of different frequency cosine inputs. In order to design our input signal, we first choose the sampling frequency as $f_s = 1$ kHz. We plan to use the summation of 400 different frequency cosine signals in the range of 0.1–250 Hz for 200 s. Instead of choosing equidistant frequency values in continuous-time, we transform our limits into discrete-time frequency equivalents using the technique presented in [25], and then,

TABLE I
NRMSE OF IDENTIFICATION DATA WITH DIFFERENT NOISE REALIZATIONS

SNR	∞	40	30	20
Our Method	10^{-8}	0.1921	0.6417	2.4961
Verhaegen [15]	10^{-13}	0.9355	1.7326	3.3741

TABLE II
NRMSE FOR TEST SIGNALS

	Sinusoid	Noise	Step	Squarewave
Our Method	0.00002	0.00001	0.00003	0.00002
Verhaegen [15]	0.02020	0.00760	0.02790	0.02010

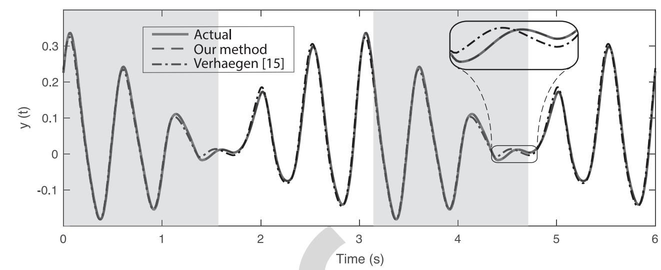

Fig. 1. Comparison of the proposed method and the method proposed in [15] for predicting the output of a square wave input signal with period π. Shaded and white regions represent the $+0.5$ and -0.5 regions of the square wave, respectively.

choose 400 equidistant frequency values in discrete-time to avoid distortion (warping) at high frequencies. Then, we transform the discrete-time frequency values back into continuous-time. This process is called *prewarping* [25].

Once we obtain the input–output data from the unknown system, we apply the proposed subspace identification method to estimate an LTP realization for the original system (see Appendix A). We estimate an equivalent representation for (52) in the form of (50) with the system matrices as follows:

$$\hat{\mathbf{A}} = \begin{bmatrix} 0 & 1 \\ -170.4848 & -2.0001 \end{bmatrix}$$
$$\hat{\mathbf{B}}(t) = \begin{bmatrix} 0 \\ 12.5671 + 6.2836\mathbf{c}(t) \end{bmatrix} \hat{\mathbf{C}}(t) \quad = \begin{bmatrix} 1 + 0.3\mathbf{c}(t) & 0 \end{bmatrix}. \tag{54}$$

Note that we neglected the sine terms with magnitude less than 10^{-8} for clarity. Since it is challenging to derive the Floquet transform for $\bar{A}(t)$ given in (53), we numerically computed a similarity transformation matrix that will give us the Floquet multipliers ($\{e^{\lambda_i T}\}|_{i=1}^{n_p}$, where $\{\lambda_i\}|_{i=1}^{n_p}$ are the eigenvalues of $\mathbf{A}$ [21] as $\mu_{1,2} = 0.5903 \pm 0.1419j$. On the other hand, Floquet multipliers of $\hat{\mathbf{A}}$, which is computed through our subspace identification method, are $\hat{\mu}_{1,2} = 0.5911 \pm 0.1360j$, which are very close to the numerical solution. In order to evaluate the prediction performance, we compute the normalized root-mean-squared error (nrmse) on identification data (see Table I). We also contaminate the output data $y(t)$ with zero mean white Gaussian noise to quantify the prediction performance with different signal-to-noise (SNR) conditions. As seen in Table I, the proposed method generates accurate output predictions for the noise-free case. For the noisy cases, we performed 100 independent noise realizations and report mean nrmse errors.

To provide a comparative analysis, we implemented the time-domain subspace identification method proposed in [15] for the same example defined in (52). We simulated (52) with a white noise sequence and collected sampled input–output data. Note that the method proposed in [15] works with discrete-time LTP systems. However, since we are working with sampled data, it is fair to compare the input–output data of the two methods. Note that the nrmse results presented in Table I for [15] are based on the prediction performance of its own identification signal (noise sequence). Our method works slightly better than that presented in [15] for predicting the identification signals under different noise realizations. In addition, we tested both methods with different test signals, such as a sinusoidal noise sequence and step and square wave input signals (see Table II). Again, the proposed method works slightly better for the prediction of different test signals as compared with the method proposed in [15]. To illustrate, we show a comparison plot for the square wave test signal prediction performance of the two methods in Fig. 1. The minor difference in prediction performance can be spotted in this comparison plot.

The comparison of our method with that proposed in [15] reveals that both methods are accurate in predicting identification and test signals. However, we emphasize certain points for a complete discussion. First, the LTP state-space model generated by our method is more intuitive than the model obtained using [15], which seeks to find a time-invariant state-space quadruple for each discrete-time step. Therefore, for an N-periodic discrete-time LTP system, the model obtained using [15] generates N different state-space quadruples, which are much more difficult to interpret than the form in (54) generated by our method. Moreover, the Floquet form in (54) is more preferable due to the time-invariant state matrix. Nevertheless, even though both methods work with a single input–output data pair, the model obtained using [15] finds and works with the smallest data length. Therefore, the method proposed in [15] is more advantageous in terms of using less data.

VI. CONCLUSION

In this note, we propose a new method for subspace-based state-space identification of LTP systems using frequency response data. Our solution is based on the fact that LTP systems can be transformed into equivalent discrete-time LTI systems. To accomplish this, we utilize a bilinear (Tustin) transformation and a frequency-domain lifting method available in the literature. Then, we estimate an LTI system representation that can predict the input–output data of the original system.

We, then, introduce a novel method to obtain a time-periodic realization for the estimated equivalent lifted LTI system. Note that the proposed LTP realization method works with the complexity of a standard subspace identification procedure. Finally, the estimated LTP system has a time-invariant state matrix. Therefore, our method allows finding Floquet transforms for known LTP systems via system identification.

APPENDIX A

Following is a summary of implementation details.

1) Simulate (1) with a sum-of-cosines input, selecting the frequencies as defined in [25].
2) Obtain the sampled data $\mathbf{u}_d[k]$ and $\mathbf{y}_d[k]$ for (3).
3) Use (13) and (7) to obtain $\mathcal{U}_d$ and $\mathcal{Y}_d$.
4) Process each frequency separately; choose $\bar{\mathcal{U}}_d = K$ and use (25) to obtain $\mathcal{Y}$.
5) Combine $\bar{\mathcal{U}}_d$ and $\mathcal{Y}$ for each frequency in vectors, and use the CVA [18] to obtain (30) (backsubstitute $\mathcal{T}_x$ and $\mathcal{T}_y$).
6) Perform eigenvalue decomposition on $\hat{A}$, and perform the similarity transformation in (32).

7) Construct the constraint equation in (42), and use the SVD to find the nullspace vectors.
8) Choose a solution from the nullspace, and do the similarity transformation in (48) to obtain (49).
9) Use (50) as the inverse bilinear (Tustin) transform.

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