Lines on Smooth Polarized K3-Surfaces
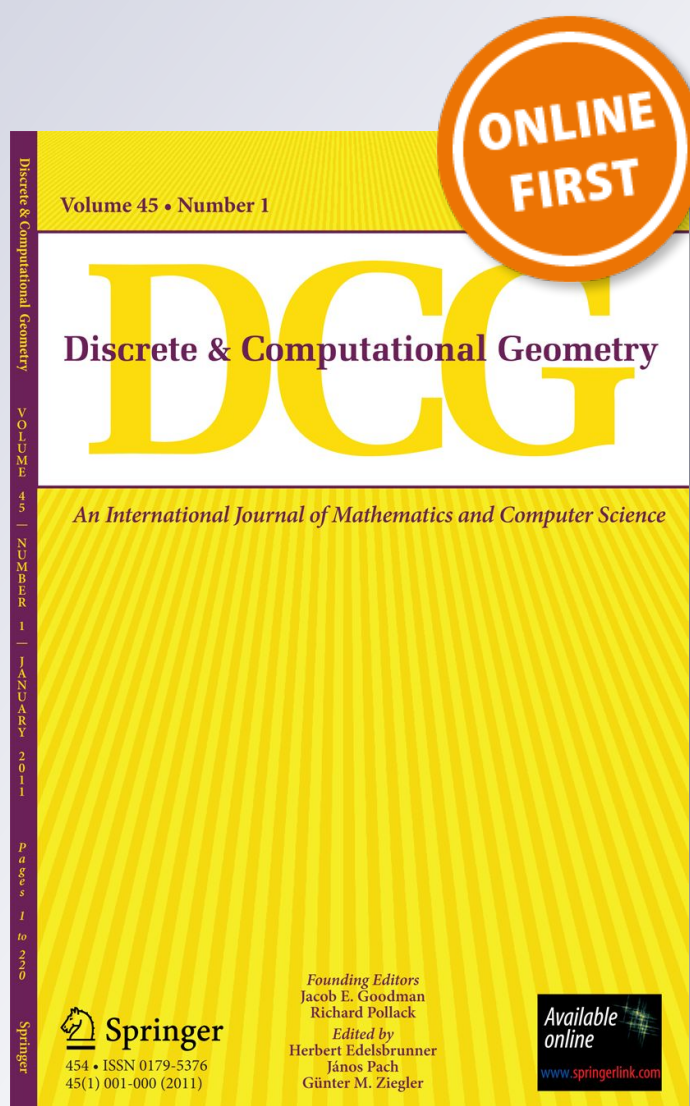
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Lines on Smooth Polarized $K3$-Surfaces

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Abstract

For each integer $D \geqslant 3$, we give a sharp bound on the number of lines contained in a smooth complex $2D$-polarized $K3$-surface in $\mathbb{P}^{D+1}$. In the two most interesting cases of sextics in $\mathbb{P}^4$ and octics in $\mathbb{P}^5$, the bounds are 42 and 36, respectively, as conjectured in an earlier paper.

1 Introduction

All algebraic varieties considered in the paper are over $\mathbb{C}$.

1.1 The Line Counting Problem

The paper deals with a very classical algebra-geometric problem, *viz.* counting straight lines in a projective surface. We confine ourselves to the smooth $2D$-polarized $K3$-surfaces $X \subset \mathbb{P}^{D+1}$, $D \geqslant 3$, and obtain sharp upper bounds on the number of lines. Our primary interest are sextics in $\mathbb{P}^4$ ($D = 3$) and octics in $\mathbb{P}^5$ ($D = 4$); however, the same approach gives us a complete answer for all higher degrees/dimensions as well.

Recall (see [14]) that a projectively normal smooth $K3$-surface $X \subset \mathbb{P}^{D+1}$ has projective degree $2D$. Given a smooth embedding $\varphi \colon X \hookrightarrow \mathbb{P}^{D+1}$, the *polarization*

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is the pull-back $\varphi^* \mathcal{O}_{\mathbb{P}^{D+1}}(1)$ regarded as a class $h \in H_2(X; \mathbb{Z})$; one has $h^2 = 2D$. Since each line $l \subset X$ is a (-2)-curve, it is uniquely determined by its homology class $[l] \in NS(X) \subset H_2(X; \mathbb{Z})$; to simplify the notation, we identify lines and their classes. In particular, the set of lines is finite; its dual incidence graph Fn X is called the *Fano graph*, and the primitive sublattice $\mathcal{F}_h(X) \subset H_2(X; \mathbb{Z})$ generated over $\mathbb{Q}$ by h and all lines $l \in$ Fn X is called the *Fano configuration* of X. This polarized lattice, subject to certain restrictions (see Theorem 2.2), defines an equilinear family of $2D$-polarized $K3$-surfaces $X \subset \mathbb{P}^{D+1}$; on the other hand, the lattice $\mathcal{F}_h(X)$ is usually recovered from the graph Fn X (see Sect. 2.4).

The case $D = 2$, i.e., that of spatial quartics, is very classical and well known. The quartic X_{64} maximizing the number of lines was constructed by Schur [15] as early as in 1882. The problem kept reappearing here and there ever since, but no significant progress had been made until 1943, when Segre [16] published a paper asserting that 64 is indeed the maximal number of lines in a smooth spatial quartic. Recently, Rams and Schütt [12] discovered a gap in Segre's argument (but not the statement); they patched the proof and extended it to algebraically closed fields of all characteristics other than 2 or 3. Finally, in [5], we gave an alternative proof and a number of refinements of the statement, including the complete classification of all quartics carrying more than 52 lines. In particular, Schur's quartic is *the only one* with the maximal number 64 of lines.

All extremal (carrying more than 52 lines) quartics found in [5] are projectively rigid, as they are the so-called *singular K3-surfaces*. Recall that a $K3$-surface X is *singular* if its Picard rank is maximal, $\rho(X) := \operatorname{rk} NS(X) = 20$. (This unfortunate term is not to be confused with singular *vs.* smooth projective models of surfaces.) Up to isomorphism, an abstract singular $K3$-surface is determined by the oriented isomorphism type of its *transcendental lattice*

$$T := NS(X)^{\perp} \subset H_2(X; \mathbb{Z}),$$

which is a positive definite even integral lattice of rank 2 (see Sect. 2.1; the orientation is given by the class $[\omega] \in T \otimes \mathbb{C}$ of a holomorphic 2-form on X); we use the notation $X(T)$, see Sect. 1.2. As a follow-up to [5] (and also motivated by [17]), in [3] I tried to study smooth projective models of singular $K3$-surfaces $X(T)$ of small discriminant $\det T$. Unexpectedly, it was discovered that Schur's quartic X_{64} can alternatively be characterized as the only smooth spatial model minimizing this discriminant: one has $X_{64} \cong X([8, 4, 8])$ (see Sect. 1.2 for the notation) and $\det T \geqslant 55$ for any other smooth quartic $X(T) \subset \mathbb{P}^3$.

After quartics, next most popular projective models of $K3$-surfaces are *sextics* $X \subset \mathbb{P}^4$ and *octics* $X \subset \mathbb{P}^5$, and the results of [3] extend to these two classes: if a singular $K3$-surface $X(T)$ admits a smooth sextic or octic model, then $\det T \geqslant 39$ or 32, respectively. In view of the alternative characterization of Schur's quartic X_{64} discovered in [3], this classification, followed by a study of the models, suggested a conjecture that a smooth sextic $X \subset \mathbb{P}^4$ (respectively, octic $X \subset \mathbb{P}^5$) may have at most 42 (respectively, 36) lines. This conjecture is proved in the present paper (the cases $D = 3, 4$ in Theorem 1.1), even though the original motivating observation that

discriminant minimizing singular $K3$-surfaces maximize the number of lines fails already for degree 10 surfaces $X \subset \mathbb{P}^6$ (see Theorem 1.7).

Each sextic in $\mathbb{P}^4$ is a regular complete intersection of a quadric and a cubic. I am not aware of any previously known interesting examples of large configurations of lines in such surfaces. The maximal number 42 of lines is attained at a 1-parameter family containing the discriminant minimizing surfaces $X([2, 1, 20])$ and $X([6, 3, 8])$.

Most octics in $\mathbb{P}^5$ are also regular complete intersections: they are the so-called *triquadrics*, i.e., intersections of three quadrics. The most well-known example is the *Kummer family*

$$\sum_{i=0}^{5} z_i = \sum_{i=0}^{5} a_i z_i = \sum_{i=0}^{5} a_i^2 z_i = 0, \quad a_i \in \mathbb{C}, \tag{1.1}$$

whose generic members contain 32 lines: famous Kummer's 16_6 configuration (see, *e.g.*, Dolgachev [6]). There are four other, less symmetric, configurations of 32 lines, either rigid or realized by 2-parameter families. However, 32 is not the maximum: there also are configurations with 33, 34, and 36 lines (see Table 1).

The space of octics contains a divisor composed by surfaces that need one cubic defining equation (see [14] and Sect. 2.2). We call these octics *special* and show that they do stand out in what concerns the line counting problem. Large Fano graphs of special octics (*vs.* triquadrics) are described by Theorem 1.3.

1.2 Common Notation

We use the following notation for particular integral lattices of small rank (see Sect. 2.1):

- $\mathbf{A}_p$, $p \geqslant 1$, $\mathbf{D}_q$, $q \geqslant 4$, $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$ are the *positive definite* root lattices generated by the indecomposable root systems of the same name (see [1]);
- $[a] := \mathbb{Z}u$, $u^2 = a$, is a lattice of rank 1;
- $[a, b, c] := \mathbb{Z}u + \mathbb{Z}v$, $u^2 = a$, $u \cdot v = b$, $v^2 = c$, is a lattice of rank 2; when it is positive definite, we assume that $0 < a \leqslant c$ and $0 \leqslant 2b \leqslant a$: then, u is a shortest vector, v is a next shortest one, and the triple (a, b, c) is unique;
- $\mathbf{U} := [0, 1, 0]$ is the *hyperbolic plane*.

Besides, $L(n)$, $n \in \mathbb{Z}$, is the lattice obtained by the scaling of a given lattice L, i.e., multiplying all values of the form by a fixed integer $n \neq 0$.

To simplify the statements, we let the girth of a forest equal to infinity, so that the inequality $\text{girth}(\Gamma) \geqslant m$ means that Γ has no cycles of length less than m.

When describing lists of integers, $a \mathbin{..} b$ means the full range $\mathbb{Z} \cap [a, b]$.

We denote by $\{a \bmod d\}$ the arithmetic progression $a + nd, \ldots, n \geqslant 0$, and use the shortcut $\{a_1, \ldots, a_r \bmod d\} := \bigcup_i \{a_i \bmod d\}$ for finite unions.

1.3 Principal Results

Given a field $K \subset \mathbb{C}$ and an integer $D \geqslant 2$, let $M_K(D)$ be the maximal number of lines defined over K that a smooth $2D$-polarized $K3$-surface $X \subset \mathbb{P}^{D+1}$ defined over K may have.

Most principal results of the paper are collected in Table 1; precise statements are found in the several theorems below. (For the sake of completeness, we also cite some results of [5] concerning quartics, i.e., $D = 2$.) In the table, we list:

- the degrees $h^2 = 2D$ for which extra information is available (marking with a [†] the two values that are special in the sense of Theorem 1.7),
- the bounds $M := M(D)$ and $\overline{M} := \overline{M}(D)$ used in Theorem 1.1 (according to which, $M(D) = M_{\mathbb{C}}(D)$ for all values of D in the table), and
- the maximal number $M_{\mathbb{R}} := M_{\mathbb{R}}(D)$ of real lines (see Theorem 1.4).

Then, for each value of D, we list, line-by-line, all Fano graphs $\Gamma := \mathrm{Fn}\, X$ containing more than $\overline{M}(D)$ lines (the notation is explained below), marking with a [‡] those realized by real lines in real surfaces (see Theorem 1.4) and indicating

- the order $|\mathrm{Aut}\, \Gamma|$ of the full automorphism group of Γ and
- the transcendental lattices $T := \mathcal{F}_h(\Gamma)^{\perp} = NS(X)^{\perp} \in H_2(X; \mathbb{Z})$ of generic smooth $2D$-polarized $K3$-surfaces X with $\mathrm{Fn}\, X \cong \Gamma$ (marking with a [*] the lattices resulting in pairs of complex conjugate equilinear families).

For the rigid configurations (rk $T = 2$), we list, in addition,

- the determinant det T (underlining the minimal ones, see Theorem 1.7),
- the numbers (r, c) of, respectively, real and pairs of complex conjugate projective isomorphism classes of surfaces X with $\mathrm{Fn}\, X \cong \Gamma$, and
- the order $|\mathrm{Aut}\, X|$ of the group of projective automorphisms of X.

Each of the few non-rigid configurations Γ appearing in the table is realized by a single connected equilinear deformation family $\mathcal{M}_D(\Gamma)$; we indicate the dimension $\dim\big(\mathcal{M}_D(\Gamma)/PGL(D + 2, \mathbb{C})\big) = \mathrm{rk}\, T - 2$ and, when known, the minimum of the discriminants det T of the singular $K3$-surfaces $X(T) \in \mathcal{M}_D(\Gamma)$.

If $D = 2$, we use the notation for the extremal configurations introduced in [5]; otherwise, we refer to the isomorphism classes of the Fano graphs introduced and discussed in more details elsewhere in the paper. In both cases, the subscript is the number of lines in the configuration. For technical reasons, we subdivide Fano graphs into several classes (see Sect. 4.1) and study them separately, obtaining more refined bounds for each class. Thus, a Fano graph $\Gamma := \mathrm{Fn}\, X$ and the configuration $\mathcal{F}_h(\Gamma) := \mathcal{F}_h(X)$ are called

- *triangular* (the Ψ_*-series, see Theorem 8.1), if $\mathrm{girth}(\Gamma) = 3$; all extremal quartics also fall into this class,
- *quadrangular* (the Θ_*-series, see Theorem 7.1), if $\mathrm{girth}(\Gamma) = 4$,
- *pentagonal* (the Φ_*-series, see Theorem 6.1), if $\mathrm{girth}(\Gamma) = 5$,
- *astral* (the Δ_*-series, see Theorem 5.1), if $\mathrm{girth}(\Gamma) \geqslant 6$ and Γ has a vertex v of valency val $v \geqslant 4$.

Table 1 Exceptional configurations (see Theorem 1.1)

D	$M, \overline{M}$	$M_{\mathbb{R}}$	Γ	$|\mathrm{Aut}\,\Gamma|$	det	(r, c)	$|\mathrm{Aut}\,X|$	$T := \mathcal{F}_h(\Gamma)^{\perp}$
2	64, 52	56	$\mathbf{X}_{64}$	4608	$\underline{48}$	(1,0)	1152	[8, 4, 8]
			$\mathbf{X}'_{60}$	480	60	(1,0)	120	[4, 2, 16]
			$\mathbf{X}''_{60}$	240	55	(0,1)	120	[4, 1, 14]*
			$\mathbf{X}_{56}$	128	64	(0,1)	32	[8, 0, 8]*
			$\mathbf{Y}_{56}{}^{\ddagger}$	64	64	(1,0)	32	[2, 0, 32]
			$\mathbf{Q}_{56}$	384	60	(1,0)	96	[4, 2, 16]
			$\mathbf{X}_{54}$	384	96	(1,0)	48	[4, 0, 24]
			$\mathbf{Q}_{54}$	48	76	(1,0)	8	[4, 2, 20]
3	42, 36	42	$\Psi_{42}{}^{\ddagger}$	432	$\underline{39}$	dim = 1		$\mathbf{A}_2 \oplus [-18]$
			$\Psi_{38}{}^{\ddagger}$	32	48	dim = 1		$\mathbf{A}_2 \oplus [-24]$
4	36, 30	34	Θ'_{36}	64	$\underline{32}$	(1,0)	16	[4, 0, 8]
			Θ''_{36}	576	36	(1,0)	144	[6, 0, 6]
			$\Theta'_{34}{}^{\ddagger}$	96		dim = 1		$\mathbf{U}(2) \oplus [12]$
			Θ_{33}	192	80	(1,0)	24	[8, 4, 12]
			Ψ_{33}	6912	36	dim = 2		$\mathbf{U}^2(3)$
			Θ_{32}	96	60	(1,0)	24	[4, 2, 16]
			$\Theta'_{32}{}^{\ddagger}$	384		dim = 2		$\mathbf{U}(2) \oplus [-4] \oplus [4]$
			$\Theta''_{32}{}^{\ddagger}$	512		dim = 2		$\mathbf{U}(2) \oplus \mathbf{U}(4)$
			Θ'''_{32}	768	36	dim = 2		$\mathbf{U}^2(3)$
			$\Theta^{\mathbf{K}}_{32}{}^{\ddagger}$	23040	$\underline{32}$	dim = 3		$\mathbf{U}^2(2) \oplus [-4]$
$5^{\dagger}$	30, 28	28	Φ'_{30}	240	100	(1,0)	60	[10, 0, 10]
			Φ''_{30}	40	75	(1,0)	10	[10, 5, 10]
			Φ'''_{30}	24	36	(1,0)	12	[4, 2, 10]
6	36, 28	36	Φ'_{30}	240	60	(1,0)	60	[8, 2, 8]
			$\Phi''_{36}{}^{\ddagger}$	1440	$\underline{15}$	(1,0)	720	[2, 1, 8]
7	30, 26	26	Φ'_{30}	240	$\underline{20}$	(1,0)	120	[4, 2, 6]
			Δ''_{27}	72	99	(1,0)	18	[6, 3, 18]
8	32, 24	24	Δ'_{32}	2304	$\underline{12}$	(1,0)	1152	[4, 2, 4]
			Δ_{26}	24	60	(1,0)	6	[4, 2, 16]
			Δ''_{25}	16	60	(1,0)	4	[4, 2, 16]
			Φ_{25}	80		dim = 1		$\mathbf{U}(5) \oplus [4]$
9	25, 24	25	$\Phi_{25}{}^{\ddagger}$	80	$\underline{15}$	dim = 1		$\mathbf{U}(5) \oplus [2]$
$10^{\dagger}$	25, 24	25	Δ'_{25}	24	96	(0,1)	6	[4, 0, 24]
			$\Phi_{25}{}^{\ddagger}$	80		dim = 2		$\mathbf{U}(5) \oplus \mathbf{U}$
			Λ_{25}	144	140	(1,0)	12	[12, 2, 12]
14	28, 24	28	$\Lambda_{28}{}^{\ddagger}$	336	$\underline{7}$	(1,0)	336	[2, 1, 4]

All other graphs are *locally elliptic* (the Λ_*-series, see Sect. 4.2), i.e., one has val $v \leqslant 3$ for each vertex $v \in \Gamma$.

In our notation for particular graphs/configurations, the subscript always stands for the number of vertices/lines. The precise description of all graphs "named" in the paper is available electronically (in the form of GRAPE records) in [4]; in most cases, the implicit reference to [4] is, in fact, *the definition* of the graph.

1.4 The Bounds

Geometrically, apart from the spatial quartics, the two most interesting projective models of $K3$-surfaces are sextics in $\mathbb{P}^4$ and octics (especially triquadrics) in $\mathbb{P}^5$. However, it turns out that the structure of the Fano graphs simplifies dramatically when the degree $h^2 = 2D$ grows, and one can easily obtain the sharp bounds $M_{\mathbb{C}}(D)$ and $M_{\mathbb{R}}(D)$ for all values of D. Below, these bounds are stated for D small and $D \to \infty$; the rather erratic precise values of

$$21 \leqslant M_{\mathbb{C}}(D) \leqslant 24, \ D > 20 \quad \text{and} \quad 19 \leqslant M_{\mathbb{R}}(D) \leqslant 24, \ D > 16$$

are postponed till Sect. 4.2 (see Corollaries 4.8 and 4.9, respectively).

For the next theorem, given a degree $h^2 = 2D$, let $M := M(D)$ and $\overline{M} := \overline{M}(D)$ be as in Table 1 or $M(D) = \overline{M}(D) = 24$ if D is not found in the table. As in [5], in addition to the upper bound $|\text{Fn } X| \leqslant M(D)$, we give a complete classification of all large (close to maximal) configurations of lines.

Theorem 1.1 (see Sect. 8.4) *Let $X \subset \mathbb{P}^{D+1}$ be a smooth $2D$-polarized $K3$-surface. Then one has $|\text{Fn } X| \leqslant M(D)$. More precisely, $|\text{Fn } X| \leqslant \overline{M}(D)$ unless $\Gamma := \text{Fn } X$ is one of the exceptional graphs (configurations) listed in Table 1.*

Both bounds are sharp whenever $2 \leqslant D \leqslant 20$ or $D \in \{1, 2, 4, 6, 7, 10 \bmod 12\}$; in particular, one has $M_{\mathbb{C}}(D) = M(D)$ for these values.

Addendum 1.2 (see Sect. 8.4) *The complete set $\{|\text{Fn } X|\}$ of values taken by the line count of a smooth $2D$-polarized $K3$-surface $X \subset \mathbb{P}^{D+1}$ is as follows*:

- *if $D = 2$, then $\{|\text{Fn } X|\} = \{0 .. 52, 54, 56, 60, 64\}$ (see [5])*;
- *if $D = 3$, then $\{|\text{Fn } X|\} = \{0 .. 36, 38, 42\}$*;
- *if $D = 4$, then $\{|\text{Fn } X|\} = \{0 .. 30, 32, 33, 34, 36\}$*.

As mentioned at the end of Sect. 1.1, the Fano graphs of special octics differ from those of triquadrics. They are described by the following theorem, which implies, in particular, that the graphs Θ_*^* in Table 1 are realized *only* by triquadrics, whereas Ψ_{33} is realized only by special octics.

Theorem 1.3 (see Sect. 5.4) *Let $X \subset \mathbb{P}^5$ be a smooth special octic, and assume that $|\text{Fn } X| \geqslant 20$. Then one of the following statements holds*:

(1) girth($\text{Fn } X$) = 3 *and* $\text{Fn } X \cong \Psi_{33}, \Psi_{29}, \Psi'_{27}, \Psi''_{27}$ *or* $|\text{Fn } X| \leqslant 25$, *or*
(2) girth($\text{Fn } X$) = 4 *and* $\text{Fn } X \cong \Theta_{21}$ *has a biquadrangle (see Definition 5.3)*.

Conversely, if $X \subset \mathbb{P}^5$ is a smooth octic such that either girth(Fn X) = 3 *or* Fn X *has a biquadrangle, then X is special.*

The threshold $|\text{Fn } X| \geqslant 20$ in Theorem 1.3 is optimal: there are graphs with 19 lines realized by both triquadrics and special octics (see Proposition 5.9).

For completeness, in Sect. 5.5 we discuss also "lines" in the hyperelliptic models

$$X \to \Sigma_{D-2} \hookrightarrow \mathbb{P}^{D+1}, \quad D \geqslant 3,$$

which turn out very simple and quite similar to special octics (see Theorem 5.11; the case $D = 2$ is considered in [3]). The conjectural bound of 144 lines in double planes $X \to \mathbb{P}^2$ (see [3]) is left to a subsequent paper.

As a by-product of the classification of large Fano graphs, we obtain bounds on the maximal number $M_\mathbb{R}(D)$ of real lines in a real $K3$-surface.

Theorem 1.4 (see Sect. 8.5) *The number $M_\mathbb{R}(D)$ is as given by Table 1. If D is not found in the table, then $M_\mathbb{R}(D) \leqslant 24$; this bound is sharp for $D \leqslant 16$.*

1.5 Asymptotic Bounds

If D is large, all Fano graphs become very simple, *viz.* disjoint unions of (affine) Dynkin diagrams, and we have

$$\limsup_{D \to \infty} M_\mathbb{C}(D) = 24, \qquad \limsup_{D \to \infty} M_\mathbb{R}(D) = 21.$$

More precisely, we have the following two asymptotic statements.

Theorem 1.5 (see Sect. 3.4) *Let $X \subset \mathbb{P}^{D+1}$ be a smooth $2D$-polarized $K3$-surface, and assume that $D \gg 0$. Then, either*

- *Fn X is a disjoint union of Dynkin diagrams, and $|\text{Fn } X| \leqslant 19$, or*
- *all lines in X are fiber components of an elliptic pencil, and $|\text{Fn } X| \leqslant 24$.*

The precise sharp bound $M_\mathbb{C}(D)$, $D \gg 0$, is as follows:

- *if $D = 1$ mod 3 or $D = 2$ mod 4, then $M_\mathbb{C}(D) = 24$;*
- *otherwise, if $D = 0$ mod 4 or $D = 2$ mod 5, then $M_\mathbb{C}(D) = 22$;*
- *in all other cases, $M_\mathbb{C}(D) = 21$.*

Theorem 1.6 (see Sect. 3.5) *If $D \gg 0$, the sharp bound $M_\mathbb{R}(D)$ is as follows:*

- *if $D = 0$ mod 3, $D = 1$ mod 5, or $D = 0$ mod 7, then $M_\mathbb{R}(D) = 21$;*
- *otherwise, if $D \notin \{17, 113 \text{ mod } 120\}$, then $M_\mathbb{R}(D) = 20$;*
- *in all other cases, $M_\mathbb{R}(D) = 19$.*

The conclusion of Theorem 1.5 holds to full extent (all Fano graphs are elliptic or parabolic) for $D > 620$ (see Remark 3.10). The expression for $M_\mathbb{C}(D)$ is valid for $D > 45$ (see Corollary 4.8), and the expression for $M_\mathbb{R}(D)$ in Theorem 1.6 is valid for $D > 257$ (cf. Corollary 4.9).

The structure of elliptic pencils carrying many lines is given by Corollary 3.8 and Table 2.

1.6 Discriminant Minimizing Surfaces

Given a degree $h^2 = 2D$, one can pose a question on the minimal discriminant

$$\mathrm{md}(D) := \min \left\{ \det T \mid \mathrm{rk}\, T = 2 \text{ and there is a smooth model } X(T) \subset \mathbb{P}^{D+1} \right\}$$

of a singular $K3$-surface $X(T)$ admitting a smooth model of degree $2D$. In [3], this question was answered for $D = 2, 3, 4$, and the conjecture on the maximal number of lines in sextic and octic models was based on the assumption that the surfaces minimizing the discriminant should also maximize the number of lines. Ironically, although the statement of the conjecture does hold, its motivating assumption fails for the very next value $D = 5$.

The proof of the next theorem is omitted as it repeats, almost literally, the same computation as in [3].

Theorem 1.7 *For each degree $h^2 = 2D$ as in Table 1, there is a unique genus of discriminant minimizing transcendental lattices T_0. If $D \neq 5, 10$, the discriminant minimizing surfaces have smooth models $X(T_0) \subset \mathbb{P}^{D+1}$ maximizing the number of lines (underlined in the table). For the two exceptional values, one has:*

- $\mathrm{md}(5) = 32$ *and* $X([2, 0, 16]) \subset \mathbb{P}^6$ *has 28 lines (see Φ_{28} in [4]);*
- $\mathrm{md}(10) = 15$ *and* $X([4, 1, 4]) \subset \mathbb{P}^{11}$ *has 24 (see Δ_{24} in [4]) or 20 lines.* ◁

Unless $D = 3$ or 9, the discriminant minimizing transcendental lattices T_0 are shown in Table 1. If $D = 3$, one has $T_0 = [2, 1, 20]$ or $[6, 1, 8]$ and $X(T_0) \in \mathcal{M}_3(\Psi_{42})$ (see [3]); if $D = 9$, then $T_0 = [2, 1, 8]$ and $X(T_0) \in \mathcal{M}_9(\Phi_{25})$.

1.7 Contents of the Paper

In Sect. 2, after introducing the necessary definitions and results concerning lattices, polarized lattices and their relation to $K3$-surfaces, and graphs, we state the arithmetical reduction of the line counting problem that is used throughout the paper. An important statement is Proposition 2.8: it rules out most hyperbolic graph for most degrees. As an application, in Sect. 2.5 we prove several sharp bounds on the valency of a line in a smooth polarized $K3$-surface.

In Sect. 3, we discuss parabolic graphs (essentially, affine Dynkin diagrams) and their relation to elliptic pencils on a $K3$-surface. As a counterpart of Proposition 2.8, we prove Proposition 3.7 stating the periodicity of the set of geometric degrees for each parabolic graph. In Sect. 3.4, we prove Theorems 1.5 and 1.6.

Having the conceptual part settled, we start the classification of hyperbolic Fano graphs. We divide them into groups, according to the minimal fiber, and study case-by-case: locally elliptic (see Sect. 4), astral (see Sect. 5), pentagonal (see Sect. 6), quadrangular (see Sect. 7), and triangular (see Sect. 8). For each group, we obtain a finer classification of large graphs in small degrees, which is stated as a separate theorem. Combining these partial statements, we prove Theorems 1.1 and 1.4 in Sect. 8.4.

In Sect. 5, we discuss also special octics and hyperelliptic models of $K3$-surfaces.

In Appendix A, we outline a few technical details concerning the algorithms used in the proofs when enumerating large Fano graphs.

2 Preliminaries

We start with an arithmetical reduction of the line counting problem (which, in fact, is well known, see Theorems 2.2 and 2.7) and describe the technical tools used to detect geometric configurations of lines. Then, Fano graphs are subdivided into *elliptic*, *parabolic*, and *hyperbolic*. The first two classes are not very interesting, as they may contain at most 19 or 24 lines, respectively. (Parabolic graphs are treated in more detail in Sect. 3.) Thus, the rest of the paper deals mainly with hyperbolic Fano graphs. The very first results, obtained in this section, are a degree bound (see Proposition 2.8) and valency bounds (see Propositions 2.10 and 2.11).

2.1 Lattices

A *lattice* is a finitely generated free abelian group L equipped with a symmetric bilinear form $b \colon L \otimes L \to \mathbb{Z}$ (which is usually understood and omitted from the notation). We abbreviate $x \cdot y := b(x, y)$ and $x^2 := b(x, x)$. In this paper, all lattices are *even*, i.e., $x^2 = 0 \bmod 2$ for all $x \in L$.

The *determinant* $\det L$ is the determinant of the Gram matrix of b in any integral basis, and the *kernel* $\ker L$ (as opposed to the kernel $\operatorname{Ker} \varphi$ of a homomorphism φ) is the subgroup

$$\ker L := L^{\perp} = \{x \in L \mid x \cdot y = 0 \text{ for all } y \in L\}.$$

An integral lattice L is *unimodular* if $\det L = \pm 1$; it is *nondegenerate* if $\det L \neq 0$ or, equivalently, $\ker L = 0$.

The *inertia indices* $\sigma_{\pm,0}(L)$ are the classical inertia indices of the quadratic form $L \otimes \mathbb{R}$. Clearly, $\sigma_0(L) = \operatorname{rk} \ker L$. The inertia index $\sigma_+(L)$ (the sum $\sigma_+(L) + \sigma_0(L)$) is the dimension of maximal positive definite (respectively, semidefinite) subspaces in $L \otimes \mathbb{R}$; a similar statement holds for $\sigma_-(L)$ and negative subspaces. Hence, these quantities are monotonous:

$$\text{if } S \subset L, \text{ then } \sigma_{\pm}(S) \leqslant \sigma_{\pm}(L) \text{ and } \sigma_{\pm}(S) + \sigma_0(S) \leqslant \sigma_{\pm}(L) + \sigma_0(L). \quad (2.1)$$

A lattice L is called *hyperbolic* if $\sigma_+(L) = 1$.

An *extension* of a lattice L is any overlattice $S \supset L$. If L is nondegenerate, there is a canonical inclusion of L to the *dual group*

$$L^{\vee} = \{x \in L \otimes \mathbb{Q} \mid x \cdot y \in \mathbb{Z} \text{ for all } y \in L\},$$

which inherits from L a $\mathbb{Q}$-valued quadratic form. Any finite index extension of L is a subgroup of $L^{\vee}$. In general, extensions of L can be described in terms of the *discriminant group* $\operatorname{discr} L := L^{\vee}/L$: this is a finite abelian group equipped with the

nondegenerate $(\mathbb{Q}/2\mathbb{Z})$-valued quadratic form $(x \bmod L) \mapsto x^2 \bmod 2\mathbb{Z}$. Since we omit the details of the computation, we merely refer to the original paper [9].

2.2 Polarized Lattices

Given an even integer $D \geqslant 2$, a $2D$-*polarized lattice* is a nondegenerate hyperbolic lattice S equipped with a distinguished vector $h \in S$ such that $h^2 = 2D$. For such a lattice (S, h) we can define its set of *lines*

$$\mathrm{Fn}(S, h) := \big\{ v \in S \mid v^2 = -2, \ v \cdot h = 1 \big\}.$$

This set is finite. Both D and h are often omitted from the notation. A polarized lattice (S, h) is called a *configuration* if it is generated over $\mathbb{Q}$ by h and $\mathrm{Fn}(S, h)$.

Denote by $\mathbf{L}$ a fixed representative of the isomorphism class of the intersection lattice $H_2(X; \mathbb{Z})$ of a $K3$-surface X: this is the only (up to isomorphism) unimodular even lattice of rank 22 and signature -16; in other words, $\mathbf{L} \cong \mathbf{U}^3 \oplus 2\mathbf{E}_8^2(-1)$.

Definition 2.1 Depending on the geometric problem, we define "bad" vectors in a polarized lattice (S, h) as vectors $e \in S$ with one of the following properties:

(1) $e^2 = -2$ and $e \cdot h = 0$ (*exceptional divisor*),
(2) $e^2 = 0$ and $e \cdot h = 2$ (*quadric pencil*), or
(3) $e^2 = 0$ and $e \cdot h = 3$ (*cubic pencil*).

Usually, only vectors as in (1) or (2) are excluded, whereas vectors as in (3) are to be excluded if $D = 4$ and we consider triquadrics rather than all octic surfaces.

Respectively, a polarized lattice (S, h) is called

- *admissible*, if it contains no "bad" vectors,
- *geometric*, if it is admissible and has a primitive embedding to $\mathbf{L}$, and
- *subgeometric*, if it admits a geometric finite index extension.

Given a smooth $2D$-polarized $K3$-surface $X \subset \mathbb{P}^{D+1}$, we denote by $\mathrm{Fn}\, X$ the set of lines contained in X (the *Fano graph*) and define the *Fano configuration* $\mathcal{F}_h(X)$ as the primitive sublattice $S \subset H_2(X; \mathbb{Z})$ generated over $\mathbb{Q}$ by the polarization h and the classes $[l]$ of all lines $l \in \mathrm{Fn}\, X$. The next statement is standard and well known; it follows from the global Torelli theorem for $K3$-surfaces [11], surjectivity of the period map [8], and Saint-Donat's results on projective models of $K3$-surfaces [14] (cf. also [5, Thm. 3.11] or [3, Thm. 7.3]).

Theorem 2.2 *A $2D$-configuration (S, h) is isomorphic to the Fano configuration $\mathcal{F}_h(X)$ of a smooth $2D$-polarized $K3$-surface X if and only if (S, h) is geometric; in this case, an isomorphism $(S, h) \to \mathcal{F}_h(X)$ induces a bijection $\mathrm{Fn}(S, h) \to \mathrm{Fn}\, X$.*

Given a primitive embedding $S \hookrightarrow \mathbf{L}$, all $2D$-polarized surfaces X such that

$$\big(H_2(X; \mathbb{Z}), \mathcal{F}_h(X), h\big) \cong (\mathbf{L}, S, h) \tag{2.2}$$

constitute one or two (complex conjugate) equilinear deformation families; modulo the projective group $PGL(D + 2, \mathbb{C})$, their dimension equals $20 - \mathrm{rk}\, S$. ◁

Remark 2.3 The number of families in Theorem 2.2 equals one or two depending on whether **L** does or, respectively, does not admit an automorphism preserving h and S (as a set) and reversing the orientation of maximal positive definite subspaces in $\mathbf{L} \otimes \mathbb{R}$. (Such an automorphism would be induced by the complex conjugation.) A configuration S is called *rigid* if rk $S = 20$: such configurations are realized by finitely many projective equivalence classes of surfaces.

Remark 2.4 According to [14], if $D = 4$, one should also require that $h \in S$ is a primitive vector. However, this condition holds automatically if Fn $S \neq \varnothing$. Besides, in this case one can also distinguish between all octic surfaces and triquadrics; as explained above, the difference is in the definition of the admissibility: cubic pencils as in Definition 2.1(3) are either allowed (special octics) or excluded (triquadrics).

We use Theorem 2.2 in conjunction with the following two algorithms.

Algorithm 2.5 There is an effective algorithm (using the enumeration of vectors of given length in a definite lattice, implemented as `ShortestVectors` in `GAP` [7]) detecting whether a lattice (S, h) is admissible and computing the set Fn(S, h).

Algorithm 2.6 A nondegenerate lattice admits but finitely many finite index extensions, which can be effectively enumerated (see [9, Prop. 1.4.1]). Then, one can use Algorithm 2.5 to select the admissible extensions (note that this is not automatic) and, for each such extension, use [9, Corr. 1.12.3] to detect whether it admits a primitive embedding to **L**. (A necessary, but not sufficient condition is that rk $S \leqslant 20$.) This computation, using Nikulin's techniques of discriminant forms, can also be implemented in `GAP` [7].

2.3 Real Configurations

For a real (i.e., invariant under the standard complex conjugation involution) smooth $K3$-surface $X \subset \mathbb{P}^{D+1}$, denote by Fn$_\mathbb{R} X \subset$ Fn X the set of real lines contained in X. As explained in [5], X can be deformed to a real surface X' such that Fn $X' =$ Fn$_\mathbb{R} X' \cong$ Fn$_\mathbb{R} X$, i.e., all lines in X' are real. Hence, when computing the bound $M_\mathbb{R}(D)$, we can assume *all* lines real.

Theorem 2.7 (cf. [5, Lem. 3.10]) *Consider a geometric configuration (S, h) and a primitive embedding $S \hookrightarrow \mathbf{L}$. Then, family (2.2) contains a real surface with all lines real if and only if the generic transcendental lattice $T := S^\perp$ has a sublattice isomorphic to* [2] *or* $\mathbf{U}(2)$. $\triangleleft$

2.4 Configurations as Graphs

One can easily show that, if a polarized lattice S is admissible, for any two lines $u, v \in$ Fn S one has $u \cdot v = 0$ or 1; respectively, we say that u and v are *disjoint* or *intersect*. Therefore, it is convenient to regard Fn S as a graph: the vertices are the lines $v \in$ Fn S, and two vertices are connected by an edge if and only if the lines intersect.

We adopt the graph theoretic terminology; for example, we define the *valency* val v of a line $v \in$ Fn S as the number of lines $u \in$ Fn S such that $u \cdot v = 1$.

The term *subgraph* always means an induced subgraph. To simplify statements, introduce the *relative valency*

$$\mathrm{val}_\Sigma\, l := \#\left\{a \in \Sigma \mid a \cdot l = 1\right\}$$

with respect to a subgraph $\Sigma \subset \Gamma$ and, given two subgraphs $\Sigma_1, \Sigma_2 \subset \Gamma$, define

$$\Sigma_1 * \Sigma_2 := \max\left\{\mathrm{val}_{\Sigma_i}\, l_j \mid l_j \in \Sigma_j, (i,j) = (1,2) \text{ or } (2,1)\right\}.$$

For example, $\Sigma * \Sigma = 0$ if and only if the subgraph $\Sigma \subset \Gamma$ is *discrete*.

Conversely, to an abstract graph Γ (loop free and without multiple edges) we can associate three lattices

$$\mathbb{Z}\Gamma, \qquad \mathcal{F}(\Gamma) := \mathbb{Z}\Gamma/\ker, \qquad \mathcal{F}_h(\Gamma) := (\mathbb{Z}\Gamma + \mathbb{Z}h)/\ker.$$

Here, $\mathbb{Z}\Gamma$ is freely generated by the vertices $v \in \Gamma$, so that $v^2 = -2$ and $u \cdot v = 1$ or 0 if u and v are, respectively, adjacent or not. For the last lattice $\mathcal{F}_h(\Gamma)$, the sum is direct, but not orthogonal; the even integer $2D := h^2 \geqslant 4$ is assumed fixed in advance, and we have $v \cdot h = 1$ for each $v \in \Gamma$.

A graph Γ is called

- *elliptic*, if $\sigma_+(\mathbb{Z}\Gamma) = \sigma_0(\mathbb{Z}\Gamma) = 0$,
- *parabolic*, if $\sigma_+(\mathbb{Z}\Gamma) = 0$ and $\sigma_0(\mathbb{Z}\Gamma) > 0$, and
- *hyperbolic*, if $\sigma_+(\mathbb{Z}\Gamma) = 1$.

(Since we are interested in hyperbolic lattices only, we do not consider graphs with $\sigma_+(\mathbb{Z}\Gamma) > 1$.) When the degree $h^2 = 2D$ is fixed, we extend to graphs the terms *admissible*, *geometric*, and *subgeometric* introduced in Definition 2.1, referring to the corresponding properties of the lattice $\mathcal{F}_h(\Gamma)$. By Theorem 2.2, subgeometric graphs are those that can appear as induced subgraphs in the Fano graph Fn X of a smooth $2D$-polarized $K3$-surface X. Conversely, a degree D is *geometric* for Γ if $\Gamma \cong$ Fn X for some smooth $2D$-polarized $K3$-surface X; the set of geometric degrees is denoted by gd Γ. Similarly, for a subfield $K \subset \mathbb{C}$, we introduce the set $\mathrm{gd}_K \Gamma$ of the values of D for which $\Gamma \cong$ Fn$_K X$ for some smooth $2D$-polarized $K3$-surface $X \hookrightarrow \mathbb{P}^{D+1}$ defined over K.

The correspondence between graphs and lattices is not exactly one-to-one: in general, $S \supset \mathcal{F}_h(\mathrm{Fn}\, S)$ is a finite index extension and $\Gamma \subset \mathrm{Fn}\, \mathcal{F}_h(\Gamma)$ is an induced subgraph (assuming that $\mathcal{F}_h(\Gamma)$ is hyperbolic).

If Γ is elliptic, the lattice $\mathcal{F}_h(\Gamma)$ is obviously hyperbolic. Parabolic graphs are treated separately in Sect. 3.1. If Γ is hyperbolic, we can define the *intrinsic polarization* as a vector $h_\Gamma \in \mathbb{Z}\Gamma \otimes \mathbb{Q}$ with the property that $v \cdot h_\Gamma = 1$ for each vertex $v \in \Gamma$. If such a vector exists, it is unique modulo ker $\mathbb{Z}\Gamma$; in particular, $h_\Gamma^2 \in \mathbb{Q}$ is well defined. (Note that h_Γ^2 may be negative.)

Proposition 2.8 *Given a hyperbolic graph Γ, the $2D$-polarized lattice $\mathcal{F}_h(\Gamma)$ is hyperbolic if and only if h_Γ exists and $2D \leqslant h_\Gamma^2$.*

Proof Assume that $\mathcal{F}_h(\Gamma)$ is hyperbolic. Then, by (2.1), the image of the projection $\varphi \colon \mathbb{Z}\Gamma \to \mathcal{F}_h(\Gamma)$ is nondegenerate and we have the orthogonal projection

$$\mathrm{pr} \colon \mathcal{F}_h(\Gamma) \to \varphi(\mathbb{Z}\Gamma) \otimes \mathbb{Q}.$$

Thus, h_Γ exists (as any pull-back of $\mathrm{pr}\, h$) and, since $\varphi(\mathbb{Z}\Gamma)$ itself is hyperbolic, its orthogonal complement must be negative definite, i.e., $h^2 \leqslant h_\Gamma^2$ (and, if the equality holds, we must have $h \in \varphi(\mathbb{Z}\Gamma) \otimes \mathbb{Q}$).

The converse statement is straightforward: if h_Γ exists, we have an orthogonal direct sum decomposition $\mathcal{F}_h(\Gamma) \otimes \mathbb{Q} = (\mathcal{F}(\Gamma) \otimes \mathbb{Q}) \oplus \mathbb{Q}v$, where $v := h - \varphi(h_\Gamma)$, in which either $v = 0$ (if $2D = h_\Gamma^2$) or $v^2 < 0$. □

Corollary 2.9 (of the proof) *Let Γ be a hyperbolic graph and* $\mathrm{rk}\,\mathcal{F}(\Gamma) = 20$. *Then, the $2D$-polarized lattice $\mathcal{F}_D(\Gamma)$ can be geometric only if $2D = h_\Gamma^2$.* ◁

Thus, each hyperbolic graph Γ can be contained in the Fano graph of a smooth $2D$-polarized $K3$-surface X only for finitely many values of D. Some of the values allowed by Proposition 2.8 can further be eliminated by requiring that Γ should be subgeometric and using Algorithms 2.5 and 2.6. Below, for each graph Γ used, we merely describe the set $\mathrm{gd}\,\Gamma$, referring to Proposition 2.8 and omitting further details of the computation.

2.5 Valency Bounds

Applying Proposition 2.8 to a star-shaped graph with a "central" vertex v of valency $\mathrm{val}\,v \geqslant 5$, we obtain the following statement.

Proposition 2.10 *Let $X \subset \mathbb{P}^{D+1}$ be a smooth $2D$-polarized $K3$-surface, fix a line $v \in \mathrm{Fn}\,X$, and assume that all lines intersecting v are pairwise disjoint. Then, we have the following sharp bounds on* $\mathrm{val}\,v$:

$D =$	2	3	4	5	6	7 .. 11	else
$\mathrm{val}\,v \leqslant$	12	9	8	7	6	5	4

If $D = 4$ and X is a triquadric, then $\mathrm{val}\,v \leqslant 7$ (sharp bound). ◁

In Proposition 2.10, if $D = 3$ and v intersects ten disjoint lines $a_1, \ldots a_{10}$, then there is an eleventh line b intersecting v and one of a_i, cf. Proposition 2.11.

Without the assumption that the star $\mathrm{star}\,v \subset \mathrm{Fn}\,X$ is discrete, the bound for quartics ($D = 2$) is $\mathrm{val}\,v \leqslant 20$, and $\mathrm{star}\,v$ may be rather complicated (see [5,12]). The star simplifies dramatically if $D \geqslant 3$: the following statement is also proved by applying Proposition 2.8 to a few simple test configurations.

Proposition 2.11 *Let $X \subset \mathbb{P}^{D+1}$ be a smooth $2D$-polarized $K3$-surface, $D \geqslant 3$, and let $v, a_1, a_2 \in \mathrm{Fn}\,X$ be three lines such that $v \cdot a_1 = v \cdot a_2 = a_1 \cdot a_2 = 1$. Then, all other lines intersecting v are disjoint from a_1, a_2, and each other and we have the following sharp bounds on* $\mathrm{val}\,v$:

$$D = \quad 3 \quad 4, 5 \quad 6 \,.. \, 11 \quad \text{else}$$
$$\text{val } v \leqslant \quad 11 \quad 5 \quad \quad 3 \quad \quad 2$$

Furthermore, if $D = 3$, then val $v \neq 10$. ◁

3 Elliptic Pencils

In this section, we establish a relation between parabolic Fano graphs Fn X and elliptic pencils on the $K3$-surface X. In particular, we establish a certain periodicity for such graphs (see Proposition 3.7) and prove Theorems 1.5 and 1.6.

3.1 Parabolic Graphs

It is well known that any parabolic or elliptic graph Γ is a disjoint union of simply laced Dynkin diagrams and affine Dynkin diagrams, with at least one component affine if Γ is parabolic. We describe the combinatorial type of Γ as a formal sum of the corresponding **A–D–E** types of its components.

If Σ is a connected parabolic graph (affine Dynkin diagram), the kernel ker $\mathbb{Z}\Sigma$ is generated by a single vector $k_\Sigma = \sum n_v v$, $v \in \Sigma$; this generator can be chosen so that all $n_v > 0$ and, under this assumption, it is unique. We call the coefficient sum $\deg \Sigma := \sum_v n_v$ the *degree* of Σ. We have

$$\deg \widetilde{\mathbf{A}}_p = p + 1, \ p \geqslant 1, \quad \deg \widetilde{\mathbf{D}}_q = 2q - 2, \ q \geqslant 4,$$
$$\deg \widetilde{\mathbf{E}}_6 = 12, \quad \deg \widetilde{\mathbf{E}}_7 = 18, \quad \deg \widetilde{\mathbf{E}}_8 = 30. \tag{3.1}$$

For a connected elliptic graph (Dynkin diagram) Σ, we can define the *degree set* ds Σ as the set $\{\sum_v m_v\}$ of the coefficient sums of all positive roots $\sum_v m_v v \in \mathbb{Z}\Sigma$. A simple computation shows that

$$\text{ds } \mathbf{A}_p = \{1 \,.. \, p\}, \ p \geqslant 1, \quad \text{ds } \mathbf{D}_q = \{1 \,.. \, 2q - 3\}, \ q \geqslant 4,$$
$$\text{ds } \mathbf{E}_6 = \{1 \,.. \, 11\}, \quad \text{ds } \mathbf{E}_7 = \{1 \,.. \, 13, 17\}, \quad \text{ds } \mathbf{E}_8 = \{1 \,.. \, 16, 23\}. \tag{3.2}$$

The *Milnor number* of a parabolic or elliptic graph Γ is $\mu(\Gamma) := \text{rk } \mathcal{F}(\Gamma)$. We have

$$|\Gamma| = \mu(\Gamma) + \#\,(\text{parabolic components of } \Gamma).$$

Lemma 3.1 *Let Σ be a parabolic component of an admissible parabolic graph Γ. Then, the image of ker $\mathbb{Z}\Gamma$ under the projection $\varphi \colon \mathbb{Z}\Gamma \to \mathcal{F}_h(\Gamma)$ is generated by the image $\varphi(k_\Sigma) \neq 0$, which is primitive in any admissible extension $S \supset \mathcal{F}_h(\Gamma)$.*

Proof Since $\mathcal{F}_h(\Gamma)$ is assumed hyperbolic, by (2.1) one has rk $\varphi(\text{ker } \mathbb{Z}\Gamma) \leqslant 1$. On the other hand, $\varphi(k_\Sigma) \cdot h = \deg \Sigma \neq 0$; hence, $\varphi(k_\Sigma) \neq 0$ and this vector generates $\varphi(\text{ker } \mathbb{Z}\Gamma)$ over $\mathbb{Q}$. To show that $\varphi(k_\Sigma) \neq 0$ is primitive, consider a maximal elliptic subgraph $\Sigma_0 \subset \Sigma$: it is a Dynkin diagram of the same name. If $a := \frac{1}{m} k_\Sigma \in S$ for some $m \geqslant 2$, then, by (3.1) and (3.2), $a \cdot h \in \text{ds } \Sigma_0$; hence, there is a root $e \in \mathbb{Z}\Sigma_0$ with $e \cdot h = a \cdot h$, and $e - a$ is an exceptional divisor as in Definition 2.1(1). □

Lemma 3.1 implies that φ maps elements $k_\Sigma \in \mathbb{Z}\Sigma$ corresponding to all parabolic components Σ of Γ to the same element of $\mathcal{F}_h(\Gamma)$, *viz.* positive primitive generator of the kernel $\ker \varphi(\mathbb{Z}\Gamma)$.

Corollary 3.2 *In any admissible graph Γ, one has $v \cdot k_{\Sigma'} = v \cdot k_{\Sigma''}$ for any two disjoint connected parabolic subgraphs Σ', $\Sigma'' \subset \Gamma$ and any vector $v \in \mathcal{F}_h(\Gamma)$.* ◁

Corollary 3.3 *All parabolic components of an admissible parabolic graph Γ have the same degree; this common degree* deg Γ *is called the* degree *of* Γ. ◁

An admissible graph Γ is called *saturated* if $\Gamma = \mathrm{Fn}\, \mathcal{F}_h(\Gamma)$. (*A priori*, this notion depends on the choice of a degree $h^2 = 2D$.) The following statement is proved similar to the primitivity of $\varphi(k_\Sigma)$ in Lemma 3.1.

Corollary 3.4 *If Γ is an admissible parabolic graph and Σ is an elliptic component of Γ, then* deg $\Gamma \notin$ ds Σ. *If Γ is saturated, then also* $(\deg \Gamma - 1) \notin$ ds Σ. ◁

3.2 Parabolic Graphs as Elliptic Pencils

Let X be a smooth $2D$-polarized $K3$-surface, and consider a connected parabolic subgraph $\Sigma \subset \mathrm{Fn}\, X$. (Recall that Σ is an affine Dynkin diagram.) The class k_Σ, regarded as a divisor, is nef and, since also $k_\Sigma^2 = 0$ and k_Σ is primitive in $H_2(X; \mathbb{Z})$ (see Lemma 3.1), this class is a fiber of a certain elliptic pencil $\pi : X \to \mathbb{P}^1$. We denote by $\mathcal{P}(\Sigma) \subset \mathrm{Fn}\, X$ the subgraph spanned by all linear components of the reducible fibers of π. Clearly,

$$\mathcal{P}(\Sigma) = \left\{ v \in \mathrm{Fn}\, X \mid v \cdot k_\Sigma = 0 \right\};$$

alternatively, $\mathcal{P}(\Sigma)$ consists of Σ and all vertices $v \in \mathrm{Fn}\, X$ that are not adjacent to any of the vertices of Σ. In this form, the notion of pencil can be extended to any geometric configuration S and affine Dynkin diagram $\Sigma \subset \mathrm{Fn}\, S$; we use the same notation $\mathcal{P}(\Sigma)$, or $\mathcal{P}_S(\Sigma)$ if S is to be specified.

Note that $\mathcal{P}(\Sigma)$ is a parabolic graph and $\mathcal{P}(\Sigma) = \mathcal{P}(\Sigma')$ for any other parabolic component $\Sigma' \subset \mathcal{P}(\Sigma)$. In this language, deg $\mathcal{P}(\Sigma)$ is the common projective degree of the fibers of π.

Thus, any maximal geometric parabolic graph Γ is the set of linear components of the fibers of an elliptic pencil. By Lemma 3.1 and (2.1), we have

$$\mu(\Gamma) = |\Gamma| - \#\,(\text{parabolic components of } \Gamma) \leqslant 18. \tag{3.3}$$

Besides, the usual bound on the topological Euler characteristic yields

$$|\Gamma| \leqslant \chi(X) = 24, \tag{3.4}$$

with the inequality strict whenever Γ has at least one elliptic component. Under the additional assumption that deg $\Gamma \leqslant 7$, this can be refined to

$$\mu(\Gamma) + b_0(\Gamma) = |\Gamma| + \#\,(\text{elliptic components of } \Gamma) \leqslant 24 \tag{3.5}$$

by taking into account the fiber components of higher projective degree. (Indeed, considering singular elliptic fibers one by one, one can easily see that, under the assumption $\deg \Gamma \leqslant 7$, the number of elliptic components of Γ does not exceed the number of fiber components of π that are not lines and, thus, are not present in Γ. On the other hand, the total number of fiber components is at most 24.) The two latter bounds are less obvious arithmetically; their proof would require considering graphs satisfying (3.3) one by one and using Algorithm 2.6 (cf. [5]).

Remark 3.5 Without the assumption $\deg \Gamma \leqslant 7$, the *a priori* possible pencils Γ of a given degree d can be enumerated as follows. Consider a collection $\Sigma_1, \ldots, \Sigma_n$ of affine Dynkin diagrams such that $\sum_i |\Sigma_i| \leqslant 24$ and assign an integral weight (projective degree) $w_v \geqslant 1$ to each vertex v of the union so that $\sum_{v \in \Sigma} n_v w_v = d$ for each component $\Sigma = \Sigma_i$, where $\sum_{v \in \Sigma} n_v v = k_\Sigma$. Then, take for Γ the induced subgraph of the union $\bigcup_i \Sigma_i$ spanned by all vertices of weight 1. (Note that we have $\deg \Sigma_i \leqslant d$ for each i and, for Γ to be parabolic, the equality must hold for at least one value of i.) This description does *not* guarantee that Γ is geometric, but it rules out many graphs that are not.

To emphasize the relation between parabolic graphs and elliptic pencils, we adopt the following terminology. A maximal parabolic subgraph $\Gamma \subset \mathrm{Fn}\,X$ is called a *pencil*, and its components are called *fibers*; a pencil is uniquely determined by any of its *parabolic* fibers. (Strictly speaking, only parabolic components of Γ are whole fibers of π, whereas elliptic ones are parts of fibers containing irreducible curves of higher projective degree; it may even happen that several elliptic components of Γ are contained in the same fiber of π.) All other lines in X are called *sections* of Γ: they are indeed (multi-)sections of π, and we disregard all multisections of higher projective degree. We denote by $\sec \Gamma := \mathrm{Fn}\,X \smallsetminus \Gamma$ the set of all sections, and

$$\sec_n \Gamma := \left\{ v \in \mathrm{Fn}\,X \mid v \cdot k_\Sigma = n \text{ for some/any parabolic fiber } \Sigma \text{ of } \Gamma \right\}$$

stands for the set of *n-sections*, $n \geqslant 1$ (see Corollary 3.2). For $l \in \Gamma$, we also use

$$\sec l := \left\{ v \in \mathrm{Fn}\,X \smallsetminus \Gamma \mid v \cdot l = 1 \right\} \quad \text{and} \quad \sec_n l := \sec l \cap \sec_n \Gamma.$$

Sometimes, 1- and 2-sections are called *simple sections* and *bisections*, respectively. Any section of Γ intersects each parabolic fiber, but it may miss elliptic ones.

As an immediate consequence of these definitions, we have

$$|\mathrm{Fn}\,X| = |\Gamma| + |\sec \Gamma| \tag{3.6}$$

for each pencil $\Gamma \subset \mathrm{Fn}\,X$.

Convention 3.6 A section $s \in \sec \Gamma$ can be described by its *coordinates*

$$\bar{s} := \left\{ l \in \Gamma \mid s \cdot l = 1 \right\} \subset \Gamma.$$

If s is an *n*-section, then $|\bar{s} \cap \Sigma'| = n$ and $|\bar{s} \cap \Sigma''| \leqslant n$ for each parabolic fiber $\Sigma' \subset \Gamma$ and elliptic fiber $\Sigma'' \subset \Gamma$, respectively. Often, the map $s \mapsto \bar{s}$ is injective, but

we do not assert this in general. If a configuration S is spanned by h, Γ, and a number of *pairwise disjoint* sections $s_1, \ldots, s_n \in \sec \Gamma$, the Gram matrix of S is determined by the coordinates $\bar{s}_1, \ldots, \bar{s}_n$. That is why, in the rest of the paper, we pay special attention to finding as many *a priori* disjoint sections as possible.

3.3 The Periodicity

Denote by $\pi_D(\Gamma)$ the set of connected components of the equilinear stratum Fn $X \cong \Gamma$ of smooth $2D$-polarized $K3$-surfaces. The following statement is in sharp contrast with Proposition 2.8.

Proposition 3.7 *Let Γ be a parabolic graph, $d := \deg \Gamma$, and let $D \geqslant d^2 + d$. Then, for any $D' = D \bmod d$, there is a canonical inclusion $\pi_{D'}(\Gamma) \hookrightarrow \pi_D(\Gamma)$. This inclusion is a bijection if also $D' \geqslant d^2 + d$.*

Proof Consider the sublattice $H := \mathbb{Z}h + \mathbb{Z}k \subset \mathcal{F}_h(\Gamma)$, where k is the common image of the vectors $k_\Sigma \in \mathbb{Z}\Sigma$ corresponding to all parabolic components Σ of Γ. The change of variables $h \mapsto h \pm k$ shows that the abstract isomorphism type of the pair of lattices $H \subset \mathcal{F}_h(\Gamma)$ depends on $D \bmod d$ only. Since any polarized automorphism of $\mathcal{F}_h(\Gamma)$ preserves H pointwise, the set of isomorphism classes of embeddings $\mathcal{F}_h(\Gamma) \hookrightarrow \mathbf{L}$ also depends only on $D \bmod d$; hence, by Theorem 2.2, we only need to investigate the dependence on D of the admissibility of a finite index extension $S \supset \mathcal{F}_h(\Gamma)$ and the existence of extra lines.

Let $E := H^\perp$; this is a negative definite lattice. We have $S \subset \mathcal{F}_h(\Gamma)^\vee \subset H^\vee \oplus E^\vee$ (cf. [9]), and any vector $e \in S$ decomposes as $e = e_H + e_E$, $e_H \in H^\vee$, $e_E \in E^\vee$. The sublattice H is primitive: if $a := \alpha h + \beta k \in \mathbf{L}$, we have $\alpha = a \cdot l \in \mathbb{Z}$ for any $l \in \Gamma$, and then $\beta \in \mathbb{Z}$ by Lemma 3.1. Hence, $e_E^2 < 0$ unless $e \in H$. Note also that $H^\vee$ is generated by $h^* = k/d$ and $k^* = (dh - 2Dk)/d^2$.

Let $D \geqslant d^2 + d$. If e is an exceptional divisor as in Definition 2.1(1), then

$$e_H = mk^*, \ m \in \mathbb{Z}, \qquad e_H^2 = -\frac{2m^2 D}{d^2} < -2$$

unless $m = 0$. It follows that $e \in E^\vee$ is independent of D. If e is a quadric pencil as in Definition 2.1(2), then

$$e_H = 2h^* + mk^*, \ m \in \mathbb{Z}, \qquad e_H^2 = -\frac{2m(mD - 2d)}{d^2} \leqslant 0.$$

Hence, $m = 0$ and $e = e_H = 2h^*$, but this vector cannot be in S by the primitivity of H. Finally, if e is a line, then

$$e_H = h^* + mk^*, \ m \in \mathbb{Z}, \qquad e_H^2 = -\frac{2m(mD - d)}{d^2} < -2$$

unless $m = 0$ or $m = 1$ and $D = d^2 + d$. In the latter case, $e_H^2 = -2$ and, hence, $e = e_H = h^* + k^* \notin S$. In the former case, $e = h^* + e_E$, where $e_E \in E^\vee$, $e_E^2 = -2$

Table 2 Large subgeometric parabolic graphs

$|\Gamma|$	Γ	Polarizations D	$T := \mathcal{F}_h(\Gamma)^\perp$
24	$8\widetilde{\mathbf{A}}_2$	4^+ or $\{7 \bmod 3\}$	$\mathbf{U}^2(3)$
	$6\widetilde{\mathbf{A}}_3$	2^+ or $\{6 \bmod 4\}$	$[8, 0, 8]$
22	$7\widetilde{\mathbf{A}}_2 + \mathbf{A}_1$	4^+ or $\{7 \bmod 3\}$	$\mathbf{U}^2(3) \oplus [-6]$
	$5\widetilde{\mathbf{A}}_3 + \mathbf{A}_2$	2^+ or $\{6 \bmod 4\}$	$\mathbf{A}_2(-4) \oplus [-16]$
	$5\widetilde{\mathbf{A}}_3 + 2\mathbf{A}_1$	2^+ or $\{4 \bmod 2\}$	$[8, 0, 8] \oplus [-4]$
	$4\widetilde{\mathbf{A}}_4 + 2\mathbf{A}_1$	2^+ or $\{7 \bmod 5\}$	$[10, 0, 10]$
21	$7\widetilde{\mathbf{A}}_2$	$\{4 \bmod 3\}$	$\mathbf{U}^2(3) \oplus \mathbf{A}_2(-3)$
		5^+ or $\{8 \bmod 3\}$	$\mathbf{U}^2(3) \oplus \mathbf{A}_2(-1)$
	$6\widetilde{\mathbf{A}}_2 + 3\mathbf{A}_1$	3^+ or $\{6 \bmod 3\}^\ddagger$	$\mathbf{A}_2 \oplus [-6]^3$
		$\{4 \bmod 3\}$	$\mathbf{A}_2(3) \oplus [-6]^3$
	$5\widetilde{\mathbf{A}}_3 + \mathbf{A}_1$	$\{2 \bmod 2\}$	$[12, 4, 12] \oplus [-4]^2$
	$4\widetilde{\mathbf{A}}_4 + \mathbf{A}_1$	$\{2, 5 \bmod 5\}$	$[50] \oplus \mathbf{U}(5)$
		6^+ or $\{11 \bmod 5\}^\ddagger$	$[2] \oplus \mathbf{U}(5)$
	$3\widetilde{\mathbf{A}}_5 + \mathbf{A}_3$	$\{4 \bmod 6\}$	$[24, 0, 36]$
		2^+ or $\{8 \bmod 6\}$	$[4, 0, 24]$
	$3\widetilde{\mathbf{A}}_5 + 3\mathbf{A}_1$	$\{4 \bmod 6\}$	$[24, 12, 24]$
		$\{6 \bmod 6\}$	$[8, 4, 8]$
	$3\widetilde{\mathbf{A}}_6$	14^+ or $\{21 \bmod 7\}^\ddagger$	$[2, 1, 4]$
		$\{2, 4, 8 \bmod 7\}$	$[14, 7, 28]$
	$2\widetilde{\mathbf{D}}_5 + \widetilde{\mathbf{A}}_7 + \mathbf{A}_1$	$\{2 \bmod 4\}$	$[32, 16, 40]$
	$3\widetilde{\mathbf{D}}_6$	$\{10 \bmod 10\}$	$[4, 0, 4], [20, 0, 20]$
		$\{2, 8 \bmod 10\}$	$[4, 0, 100]$
		$\{4, 6 \bmod 10\}$	$[8, 4, 52]$
	$3\widetilde{\mathbf{E}}_6$	$\{4 \bmod 12\}$	$[6, 0, 72]$
		$\{6 \bmod 12\}^\ddagger$	$[2, 0, 24]$
		$\{7 \bmod 12\}$	$[6, 0, 18], [6, 3, 6]$
		$\{10 \bmod 12\}$	$[18, 0, 24]$
		$\{12 \bmod 12\}$	$[6, 0, 8]$
		$\{15 \bmod 12\}^\ddagger$	$[2, 0, 6], [2, 1, 2]$

does not depend on D. Thus, oll "undesirable" vectors are independent of the choice of $D \geqslant d^2 + d$ within the same class $D \bmod d$. If $D < d^2 + d$, these vectors are still present, but new ones can appear, ruling out some of the configurations. □

Corollary 3.8 *Table 2 lists all subgeometric pencils Γ such that $|\Gamma| \geqslant 21$.*

In the table, for each pencil Γ, we list the values of D (marking with a $^+$ those for which any geometric finite index extension of $\mathcal{F}_h(\Gamma)$ contains sections of Γ) and generic transcendental lattices $T := \mathcal{F}_h(\Gamma)^\perp$, which depend on $D \bmod \deg \Gamma$ only.

Proof of Corollary 3.8 If $|\Gamma| = 24$, then, by (3.3), Γ has at least six parabolic fibers and, hence, the minimal Milnor number μ_0 of such a fiber is subject to the inequality $6\mu_0 \leqslant 18$. Taking (3.5) into account, we arrive at the two combinatorial types listed in the table. Similarly, one can show that $|\Gamma| \neq 23$ and classify all pencils with $|\Gamma| = 22$ or 21. (The combinatorial types that are never geometric are not shown in the table.) Finally, the set gd Γ for each combinatorial type Γ is determined using Proposition 3.7 and Algorithm 2.6. $\square$

Remark 3.9 The bound $d^2 + d$ in Proposition 3.7 is closely related to the intrinsic polarization in Proposition 2.8. If Γ consists of an affine Dynkin diagram of degree d and a single n-section, one has $h_\Gamma^2 < 2(d/n)^2 + 2(d/n)$. By the obvious monotonicity of h_Γ^2, if $D \geqslant d^2 + d$, any (sub-)geometric graph containing an affine Dynkin diagram of degree d is parabolic. (This conclusion can as well be derived from the proof of Proposition 3.7.)

3.4 Proof of Theorem 1.5

If the Fano graph Fn X is hyperbolic, it contains a pencil Γ. There are but finitely many parabolic graphs satisfying (3.3) and (3.4), each of them has finitely many hyperbolic 1-vertex extensions, and each extension is geometric for finitely many values of D by Proposition 2.8. Thus, for $D \gg 0$, the graph Fn X is either elliptic, and then $|\text{Fn } X| \leqslant 19$ by (2.1), or parabolic. In the latter case, the bound $M_{\mathbb{C}}(D)$ on $|\text{Fn } X|$ is given by Corollary 3.8. $\square$

Remark 3.10 The precise lower bound on D in Theorem 1.5 is rather high: the conclusion holds to full extent for $D \geqslant 621$. There exists a one-vertex extension of the affine Dynkin diagram $\widetilde{\mathbf{E}}_8$ (*viz.* $\widetilde{\mathbf{E}}_8$ itself and one simple section, cf. Fig. 1 in Sect. 4.1) which is geometric for all $D \leqslant 620$.

Remark 3.11 Corollary 3.8 shows also that, for each $D \geqslant 6$, there exists a smooth $2D$-polarized $K3$-surface X with Fn X parabolic and $|\text{Fn } X| = 21$: namely, one can have Fn $X \cong 7\widetilde{\mathbf{A}}_2$ or $6\widetilde{\mathbf{A}}_2 + 3\mathbf{A}_1$.

3.5 Proof of Theorem 1.6

In view of Theorem 1.5, the bound $|\text{Fn}_{\mathbb{R}} X| \leqslant 21$ and the values of D for which it is sharp follow immediately from Theorem 2.7 and Corollary 3.4 (see Table 2, where real configurations are marked). For the lower bound $M_{\mathbb{R}}(D) \geqslant 19$ one can use a pencil $\Gamma \cong 6\widetilde{\mathbf{A}}_2 + \mathbf{A}_1$ or $2\widetilde{\mathbf{E}}_8 + \mathbf{A}_1$. The values of $D \gg 0$ for which $M_{\mathbb{R}}(D) = 20$ are given by the classification of pencils of size 20 (using Remark 3.5): the complete list, which is too long, is found in [4]. $\square$

4 The Taxonomy of Hyperbolic Graphs

In [5] and [2], the Fano graphs of spatial quartics are subdivided into triangular (all large graphs), quadrangular, and quadrangle free. In this paper, we adopt a more consistent

taxonomy based on the type of a minimal fiber (see Definition 4.1): this approach allows a more refined classification and stronger bounds.

Till the rest of the paper, *we consider hyperbolic graphs only*. (Some of the statements do not make use of this assumption, merely asserting that a certain linear combination of h and lines would have to be a bad vector in the sense of Definition 2.1. However, we do not single out these statements, as from our point of view graphs with $\sigma_+ > 1$ have no geometric meaning.) In the proofs, we implicitly refer to Corollary 3.8 or, if necessary, more refined classification of the parabolic graphs of a given type based directly on Proposition 3.7 (see also [4]).

4.1 The Taxonomy

Recall that any hyperbolic graph Γ contains a connected parabolic subgraph: this obvious statement is part of the classification of elliptic and parabolic graphs.

Definition 4.1 A *minimal fiber* in a hyperbolic or parabolic graph Γ is a connected parabolic subgraph (affine Dynkin diagram) $\Sigma \subset \Gamma$ minimal with respect to the following lexicographic order: we compare the Milnor numbers and, in the case of equal Milnor number μ, let $\widetilde{\mathbf{A}}_\mu < \widetilde{\mathbf{D}}_\mu < \widetilde{\mathbf{E}}_\mu$. The graph Γ itself is referred to as a Σ-*graph*, where Σ is the (obviously well-defined) common isomorphism type of the minimal fibers in Γ. A *minimal pencil* in Γ is any pencil of the form $\mathcal{P}(\Sigma) \subset \Gamma$, where $\Sigma \subset \Gamma$ is a minimal fiber. This terminology also applies to configurations (S, h), according to the Fano graph $\mathrm{Fn}(S, h)$.

The $\widetilde{\mathbf{A}}_2$-, $\widetilde{\mathbf{A}}_3$-, and $\widetilde{\mathbf{A}}_4$-graphs/configurations are alternatively called *triangular*, *quadrangular*, and *pentagonal*, respectively; these graphs Γ are characterized by the property that $\mathrm{girth}(\Gamma) = 3$, 4, and 5, respectively.

The $\widetilde{\mathbf{D}}_4$-graphs/configurations are called *astral*; such graphs Γ are characterized by the property that $\mathrm{girth}(\Gamma) \geqslant 6$ and Γ has a vertex of valency at least four. Since $\widetilde{\mathbf{D}}_4$ is the only affine Dynkin diagram with a vertex of valency 4, any Σ-graph Γ with $\mu(\Sigma) \geqslant 5$ is *locally elliptic*, i.e., one has $\mathrm{val}\, v \leqslant 3$ for any vertex $v \in \Gamma$. Analyzing such affine Dynkin diagrams Σ one by one, we easily arrive at the following bound on the number and positions of the sections of a minimal pencil.

Proposition 4.2 *Let $\Sigma \subset \Gamma$ be a minimal fiber, and assume that $\mu(\Sigma) \geqslant 5$. Then, the pencil $\mathcal{P}(\Sigma)$ has at most three parabolic fibers and its sections are as shown in Fig. 1. Furthermore, all sections are simple and, unless $\Sigma \cong \widetilde{\mathbf{A}}_5$, pairwise disjoint. If $\Sigma \cong \widetilde{\mathbf{A}}_5$, two sections may intersect only if they are adjacent to opposite corners of the hexagon Σ.* ◁

Remark 4.3 Certainly, Fig. 1 shows maximal sets of sections. If $\Sigma \cong \widetilde{\mathbf{A}}_7$, the condition is that the "distance" between the two sections in the octagon Σ must be at least 3 (as otherwise the graph would contain $\widetilde{\mathbf{D}}_5$ or $\widetilde{\mathbf{D}}_6$); hence, there are two distinct maximal sets.
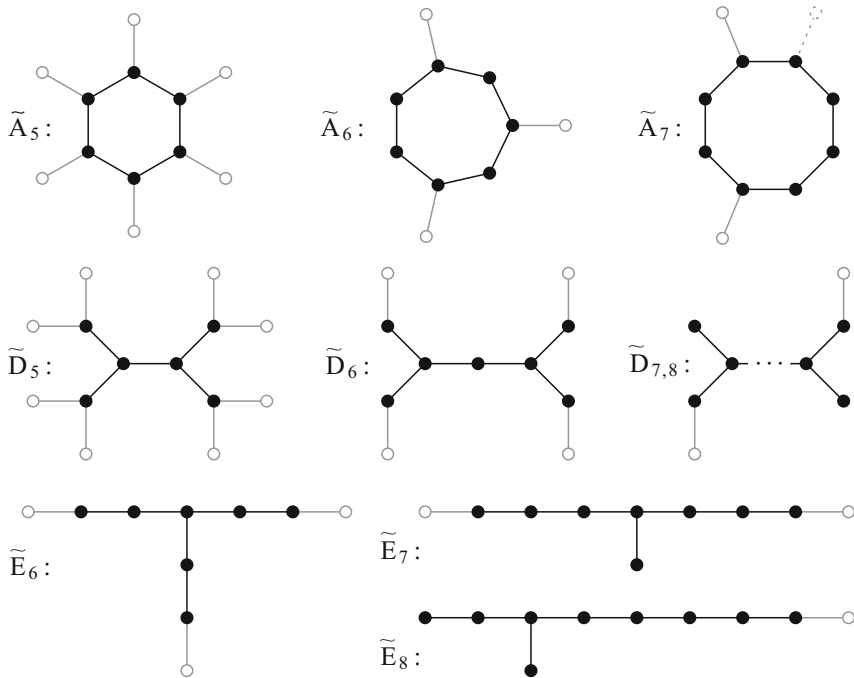
Fig. 1 Sections (white) at a minimal fiber (black) (see Proposition 4.2)

4.2 Locally Elliptic Graphs

In view of Theorems 1.5 and 1.6, most interesting are the configurations with twenty or more lines. In the locally elliptic case, one can easily list such configurations. (If D is small, the number of configurations may be huge and we confine ourselves to those close to maximal.) The results are collected in the several statements below. In Theorem 4.6, we skip the values of D for which $\overline{M}(D) = 20$, as they are too erratic; these values, as well as a complete description of all graphs, are found in [4].

Theorem 4.4 (see Sect. 4.4) *Let X be a smooth $2D$-polarized $K3$-surface, and assume that* Fn X *is a hyperbolic* $\widetilde{\mathbf{A}}_5$-*graph. Then, with one exception* Fn $X \cong \Lambda_{25}$ *shown in Table 3, one has a sharp bound* $|\mathrm{Fn}\,X| \leqslant \overline{M}$, *where* $\overline{M} := \overline{M}(D)$ *is as follows*:

$$
\begin{array}{ccccccc}
D: & 2\mathinner{.\,.}12 & 13 & 14 & 15 & 16\mathinner{.\,.}20,\,22 & \text{else} \\
\overline{M}: & 24 & 22 & 23 & 21 & 20 & \leqslant 19
\end{array}
$$

The notation in Table 3 is similar to Table 1 (see Sect. 1.3), with most fields skipped.

Theorem 4.5 (see Sect. 4.4) *Let X be a smooth $2D$-polarized $K3$-surface, and assume that* Fn X *is a hyperbolic* $\widetilde{\mathbf{D}}_5$-*graph. Then, with one exception* Fn $X \cong \Lambda_{28}$ *shown in Table 3, one has a sharp bound* $|\mathrm{Fn}\,X| \leqslant \overline{M}$, *where* $\overline{M} := \overline{M}(D)$ *is as follows*:

Table 3 Large locally elliptic graphs (see Theorems 4.4 and 4.5)

D	Γ	$|\text{Aut }\Gamma|$	det	(r, c)	$|\text{Aut }X|$	$T := \mathcal{F}_h(\Gamma)^{\perp}$
2	Λ_{25}	144	780	(2,0)	12	[4, 2, 196]
4	Λ_{25}	144	620	(2,0)	12	[4, 2, 156], [20, 10, 36]
6	Λ_{25}	144	460	(1,0)	12	[20, 10, 28]
$10^{\dagger}$	Λ_{25}	144	140	(1,0)	12	[12, 2, 12]
14	$\Lambda_{28}{}^{\ddagger}$	336	$\underline{7}$	(1,0)	336	[2, 1, 4]

$$
\begin{array}{c|ccccc}
D: & 2\,..\,20 & 21, 26, 32 & 22, 24 & 23, 25, 27, 28, 30, 34, 36 & \text{else} \\
\overline{M}: & 24 & 21 & 22 & 20 & \leqslant 19
\end{array}
$$

The extremal graph Λ_{28} is 3-regular, 3-arc transitive, and distance regular; it is isomorphic to the so-called *Coxeter graph* (the only symmetric 3-regular graph on 28 vertices). The configuration has 21 minimal pencils, all of type $2\widetilde{\mathbf{D}}_5 + \widetilde{\mathbf{A}}_7$.

Theorem 4.6 (see Sect. 4.4) *Let X be a smooth $2D$-polarized $K3$-surface, and assume that* Fn X *is a hyperbolic Σ-graph, $\mu(\Sigma) \geqslant 6$. Then $|\text{Fn } X| \leqslant \overline{M} := \overline{M}(D)$, where*

- *if $\Sigma \cong \widetilde{\mathbf{E}}_8$ and $D = 309$, then $\overline{M} = 21$;*
- *if $\Sigma \cong \widetilde{\mathbf{E}}_7$ and $D \in \{2\,..\,99\}$, then $\overline{M} = 21$;*
- *if $\Sigma \cong \widetilde{\mathbf{A}}_6$ and $D \in \{18, 21, 22\}$, then $\overline{M} = 21$;*
- *if $\Sigma \cong \widetilde{\mathbf{D}}_6$, then $\overline{M}$ is as follows:*

$$
\begin{array}{c|cc}
D: & 2\,..\,10, 12\,..\,16, 18\,..\,28 & 11, 17, 29\,..\,34, 36\,..\,38, 40, 50 \\
\overline{M}: & 22 & 21
\end{array}
$$

- *if $\Sigma \cong \widetilde{\mathbf{E}}_6$, then $\overline{M}$ is as follows:*

$$
\begin{array}{c|cccc}
D: & 2\,..\,38, 40\,..\,45 & 39 & 48\,..\,50, 64 \\
\overline{M}: & 22 & 24 & 21
\end{array}
$$

- *in all other cases, $\overline{M} \leqslant 20$.*

These bounds are sharp. (*See* [4] *for the graphs and the values* $\overline{M}(D) = 20$.)

Addendum 4.7 (see Sect. 4.4) *There are but six hyperbolic locally elliptic Fano graphs with* 24 *vertices:*

- Λ_{24}^{A}: $|\text{Aut }\Lambda_{24}^{\text{A}}| = 144$, 3-*regular*, gd $= \{2\,..\,12\}$, $\text{gd}_{\mathbb{R}} = \{3\,..\,6, 8\,..\,12\}$;
- Λ_{24}^{E}: $|\text{Aut }\Lambda_{24}^{\text{E}}| = 72$, gd $= \text{gd}_{\mathbb{R}} = \{39\}$;
- Λ_{24}: $|\text{Aut }\Lambda_{24}| = 48$, gd $= \{2\,..\,20\}$, $\text{gd}_{\mathbb{R}} = \{5, 10, 13, 18\}$;
- Λ'_{24}: $|\text{Aut }\Lambda'_{24}| = 32$, gd $= \{2\,..\,16\}$, $\text{gd}_{\mathbb{R}} = \{7, 12, 15, 16\}$;
- Λ''_{24}: $|\text{Aut }\Lambda''_{24}| = 12$, gd $= \{2\,..\,13\}$, $\text{gd}_{\mathbb{R}} = \{5, 7, 12\}$;
- Λ'''_{24}: $|\text{Aut }\Lambda'''_{24}| = 8$, gd $= \{14\}$, $\text{gd}_{\mathbb{R}} = \varnothing$.

As a consequence, for each $D \in \{2 .. 20\}$, there exists a geometric hyperbolic locally elliptic configuration with 24 *lines, and for each $D \in \{3 .. 13, 15, 16\}$, there exists a real configuration with these properties (cf. the sharpness in Theorems* 1.1 *and* 1.4*).*

In the subsequent sections we show that, if $D > 16$ and $\mu(\Sigma) \leqslant 4$, the number of lines in geometric Σ-configurations (S, h), $h^2 = 2D$, is maximized by parabolic graphs. Combining this observation with Theorems 4.4–4.6, we obtain precise sharp bounds on the number of lines.

Corollary 4.8 *If $D > 20$ (cf. Theorem* 1.1*), the maximum $M_{\mathbb{C}}(D)$ is as given by Theorem* 1.5*, with the exception of the following eight values: $M_{\mathbb{C}}(39) = 24$ and $M_{\mathbb{C}}(D) = 22$ for $D \in \{21, 23, 29, 33, 35, 41, 45\}$.* □

Corollary 4.9 *If $D > 16$ (cf. Theorem* 1.4*), one has*

$$M_{\mathbb{R}}(D) = \max\{M_p(D), M_{\mathrm{h}}(D)\},$$

where $M_p(D)$ is the maximum over the parabolic graphs given by Theorem 1.6*, and $M_{\mathrm{h}}(D)$ is the maximum over the hyperbolic graphs, which is as follows:*

- *$M_{\mathrm{h}}(D) = 24$ if $D \in \{18, 39\}$,*
- *$M_{\mathrm{h}}(D) = 22$ if $D \in \{17, 19, 21..25, 27, 30, 33, 34, 37, 41, 42, 45\}$,*
- *$M_{\mathrm{h}}(D) = 21$ if $D \in \{20, 26, 28, 29, 31, 32, 35, 36, 38, 40, 43, 44, 46 .. 99, 309\}$,*
- *$M_{\mathrm{h}}(D) = 20$ if $D \in \{100 .. 308, 342, 396, 565, 576, 601\}$,*

and $M_{\mathrm{h}}(D) \leqslant 19$ for all other values $D > 16$. ◁

4.3 Idea of the Proof

The approach outlined in this section will be used in most proofs till the end of the paper.

We attempt to prove a bound $|\mathrm{Fn}\, X| \leqslant \overline{M}$ for graphs $\mathrm{Fn}\, X$ satisfying certain conditions, subject to certain exceptions that are to be determined. To this end, we choose an appropriate pencil $\Gamma := \mathcal{P}(\Sigma)$ with a distinguished fiber Σ and try to estimate separately the two terms in (3.6). Typically, the bound given by the sum of the two estimates is too rough. Therefore, we choose appropriate thresholds $\overline{M}_\Gamma$ and $\overline{M}_{\mathrm{sec}}$, so that $\overline{M}_\Gamma + \overline{M}_{\mathrm{sec}} = \overline{M}$, and try to *classify*, separately, the following two types of geometric graphs:

- pencils $\Gamma \supset \Sigma$ of size $|\Gamma| > \overline{M}_\Gamma$ with at least a few sections without which the count $|\mathrm{Fn}\, X| > \overline{M}$ is not reachable (see Sect. A.2), and
- section sets $\mathrm{sec}\,\Gamma$ (in the presence of just the single distinguished fiber Σ) of size $|\mathrm{sec}\,\Gamma| > \overline{M}_{\mathrm{sec}}$ (see Sect. A.5).

The thresholds are chosen so that, on the one hand, the computation is feasible and, on the other hand, the partial configurations S obtained have maximal or close to maximal rank. In the former case, $\mathrm{rk}\, S = 20$, all geometric extensions of S are found by Algorithm 2.6; in the latter case, we either add a few (usually, up to two) more lines to increase the rank to 20 or prove that S has no further extensions.

The graphs $\mathrm{Fn}\, X$ with $|\mathrm{Fn}\, X| > \overline{M}$ are found in the process of the classification, and all others satisfy the inequality $|\mathrm{Fn}\, X| \leqslant \overline{M}_\Gamma + \overline{M}_{\mathrm{sec}} = \overline{M}$ given by (3.6).

4.4 Proof of Theorems 4.4–4.6 and Addendum 4.7

Given a configuration S as in the statements, fix a minimal pencil $\Gamma := \mathcal{P}(\Sigma) \subset \text{Fn } S$. By Proposition 4.2, this pencil has at most three parabolic fibers, hence $|\Gamma| \leqslant 21$ by (3.3), and at most eight sections; thus, by (3.6), we have a bound $|\text{Fn } S| \leqslant 29$, which holds in any degree and over any field (including positive characteristic).

For the more precise bounds as in the statements, we classify the configurations, running the modified version (see Sect. A.3) of the algorithm of Sect. A.2. For Theorem 4.6, we take $\overline{M} = 19$, obtaining all graphs with 20 or more vertices. For Theorems 4.4, 4.5 and Addendum 4.7, in order to avoid too large output, we choose $19 \leqslant \overline{M} \leqslant 23$ close to the expected maximum, depending on D. The other thresholds used are $\overline{M}_\Gamma = \overline{M} - \overline{M}_{\text{sec}}$ and $\overline{M}_{\text{sec}} = \max |\text{sec } \Gamma|$, where the latter maximum, depending on Σ and D, is computed using Proposition 2.8. (If $\Sigma \cong \mathbf{A}_5$, we need to take into account the possible intersections of pairs of sections adjacent to opposite corner; up to automorphism, this results in 23 sets of sections.) A few technical observations (in terms of the coordinates, see Convention 3.6) reduce the number of candidates:

- $|\bar{s}| \leqslant 3$ for each $s \in \text{sec } \Gamma$ (as the graph is locally elliptic);
- $|\bar{s}' \cap \bar{s}''| \leqslant 1$ for $s' \neq s'' \in \text{sec } \Gamma$ (as the graph is quadrangle free);
- $|\bar{s}| \leqslant 2$ for each $s \in \text{sec } \Gamma$, if $\Sigma \cong \widetilde{\mathbf{D}}_7$ (no subgraphs isomorphic to $\widetilde{\mathbf{D}}_6$);
- $|\bar{s}| \leqslant 1$ for each $s \in \text{sec } \Gamma$, if $\Sigma \cong \widetilde{\mathbf{D}}_8$ (no subgraphs isomorphic to $\widetilde{\mathbf{E}}_7$).

Finally, the defining property $\mathfrak{P}$ is that S should be a Σ-configuration: since we always have $\Sigma \subset \text{Fn } S$, this property is obviously hereditary. □

5 Other Pencils with Disjoint Sections

There are two other (unrelated) classes of pencils in Fano graphs whose sections are *a priori* known to be disjoint: astral configurations and Fano graphs of special octics. At the end of this section, in Sect. 5.5, we consider also the toy problem of counting lines in hyperelliptic models of degree $2D \geqslant 6$.

5.1 Astral Configurations

Recall that *astral* are the $\widetilde{\mathbf{D}}_4$-graphs. These graphs Γ are characterized by the property that $\text{girth}(\Gamma) \geqslant 6$ and $\max \text{val } v \geqslant 4$, $v \in \Gamma$.

Theorem 5.1 (see Sect. 5.2) *Let X be a smooth $2D$-polarized $K3$-surface, and assume that the Fano graph* $\text{Fn } X$ *is astral. Let $\overline{M} := \overline{M}(D)$ be as in Table 4 or as follows*:

D :	3, 5, 9	11, 13	15	16	{17 mod 6}	{18 mod 2}	{19, 21 mod 6}
$\overline{M}$:	24	22	20	21	18	20	19

Then, with the exceptions listed in Table 4, one has a sharp bound $|\text{Fn } X| \leqslant \overline{M}$.

The graph Δ'_{32} in Table 4 is 4-regular, 3-arc transitive, distance regular, and bipartite. According to D. Pasechnik (private communication), this is the graph of points and

Table 4 Extremal astral configurations (see Theorem 5.1)

D	$\overline{M}$	Γ	$|\text{Aut}\,\Gamma|$	det	(r, c)	$|\text{Aut}\,X|$	$T := \mathcal{F}_h(\Gamma)^\perp$
2	26	Δ'_{27}	72	300	(2,0)	12	[4, 2, 76]
4	26	Δ'_{28}	576	44	(1,0)	96	[4, 2, 12]
		Δ'_{27}	72	204	(0,1)	12	[12, 6, 20]
		Δ''_{27}	72	207	(1,1)	18	[6, 3, 36], [12, 3, 18]*
6	26	Δ'_{28}	576	28	(1,0)	96	[4, 2, 8]
7	24	Δ''_{27}	72	99	(1,0)	18	[6, 3, 18]
8	24	Δ'_{32}	2304	$\underline{12}$	(1,0)	1152	[4, 2, 4]
		Δ_{26}	24	60	(1,0)	6	[4, 2, 16]
		Δ''_{25}	16	60	(1,0)	4	[4, 2, 16]
$10^\dagger$	24	Δ'_{25}	24	96	(0,1)	6	[4, 0, 24]
12	23	$\Delta'_{24}{}^\ddagger$	1152		dim = 1		$\mathbf{U}(2) \oplus [4]$
14	21	$\Delta'_{24}{}^\ddagger$	1152		dim = 2		$\mathbf{U}^2(2)$

lines in the affine plane $\mathbb{F}_4^2$, with one pencil of parallel lines removed. The graph contains eight minimal pencils, all of type $4\widetilde{\mathbf{D}}_4$.

Two other sufficiently symmetric graphs are Δ'_{28} and Δ'_{24}, containing three and two minimal pencils, all of type $4\widetilde{\mathbf{D}}_4$, respectively. Each graph Γ is bipartite, the bicomponents C_1, C_2 are orbits of $\text{Aut}\,\Gamma$, and the action of $\text{Aut}\,\Gamma$ on $C_1 \times C_2$ has two orbits, distinguished by whether vertices are adjacent or not.

The bipartite graphs Δ'_{28}, Δ'_{27}, and Δ'_{24} are subgraphs of Δ'_{32}.

5.2 Proof of Theorem 5.1

Consider an astral configuration S, fix a minimal fiber $\Sigma \subset \text{Fn}\,X$ of type $\widetilde{\mathbf{D}}_4$, and let $\Gamma := \mathcal{P}(\Sigma)$: this pencil may have parabolic fibers of type $\widetilde{\mathbf{D}}_4$ or $\widetilde{\mathbf{A}}_5$ and elliptic fibers of type $\mathbf{A}_p$, $p \leqslant 4$. Since we assume that $\text{girth}(\text{Fn}\,S) \geqslant 6$, all sections of Γ are pairwise disjoint, and their number is bounded by Proposition 2.8 as follows:

$$\begin{array}{c|ccccccccc} D = & 2, 4, 6, 8 & 3, 5, 7 & 9 & 10 & 11 & 12 .. 14 & 15, 16 & 17 .. 20 & 21 .. 34 & \text{else} \\ |\sec \Gamma| \leqslant & 12 & 11 & 9 & 8 & 6 & 4 & 3 & 2 & 1 & 0 \end{array}$$

(Strictly speaking, for large values of D these bounds use an additional assumption that the pencil admits at least one extra line $l \in \Gamma \setminus \Sigma$.)

To simplify the further computation, we eliminate several large sets of sections. The following lemma is proved in Sect. A.8, after the necessary terminology is developed.

Lemma 5.2 (see Sect. A.8) *In the settings of the theorem, one has*:

(1) *if* $|\sec \Gamma| = 12$, *then* $|\text{Fn}\,S| \leqslant 24$ *or* $\text{Fn}\,S \cong \Delta'_{27}$ ($D = 2, 4$) *or* Δ'_{32} ($D = 8$);
(2) *if* $D = 5$ *and* $|\sec \Gamma| \geqslant 11$, *then* $|\text{Fn}\,S| \leqslant 22$.

There remains to run the algorithm of Sect. A.3 (using a few further restrictions on the fibers of Γ, which are also stated in Sect. A.8) and list geometric configurations S satisfying the inequality $|\text{Fn } S| > \overline{M}$, where the threshold $\overline{M}$ is as in the statement of the theorem and $\overline{M}_{\text{sec}}$ is given by the table above, with 11 reduced down to 10 for $D = 5$ and 12 reduced down to 11, both due to Lemma 5.2. $\qquad \square$

5.3 Geometry of Special Octics

In this section, we assume $D = 4$. Recall that an octic $X \subset \mathbb{P}^5$ is special if and only if its Néron–Severi lattice has a cubic pencil as in Definition 2.1(3), i.e., a class $e \in NS(X)$ such that $e^2 = 0$ and $e \cdot h = 3$.

Let $l \in \text{Fn } X$ be a line. The determinant of the lattice spanned by h, l, and e is $-8t^2 + 6t + 18$, where $t := l \cdot e$; hence, this lattice is hyperbolic if and only if $l \cdot e = 0, \pm 1$. If $l \cdot e = -1$, then the difference $e - l$ is a quadric pencil as in Definition 2.1(2). Thus, we conclude that

$$l \cdot e = 0 \text{ or } 1 \quad \text{for any line } l \in \text{Fn } X. \qquad (5.1)$$

Similarly, if a cubic pencil $e \in NS(X)$ exists, it is unique. Indeed, if there were two such classes $e_1 \neq e_2$, the requirement that $NS(X)$ should be hyperbolic would imply that $e_1 \cdot e_2 = 1$ or 2. In the former case, $e_1 - e_2$ would be an exceptional divisor as in Definition 2.1(1); in the latter, $h - e_1 - e_2$ would be a quadric pencil as in Definition 2.1(2). Thus, below we denote this unique class by $e_X \in NS(X)$ and refer to it as *the* cubic pencil in X. As an immediate consequence, the graph Fn X splits canonically into two sets

$$C_n = C_n(X) := \{l \in \text{Fn } X \mid l \cdot e_X = n\}, \quad n = 0, 1. \qquad (5.2)$$

Definition 5.3 A *biquadrangle* in a graph Γ is a complete bipartite subgraph $K_{2,3}$. In other words, a biquadrangle consists of a quadrangle (type $\widetilde{\mathbf{A}}_3$ fiber) $\Sigma = \{l_1, l_2, l_3, l_4\}$ and a bisection s_{13} adjacent to two opposite corners $l_1, l_3 \in \Sigma$.

Definition 5.4 A bipartite graph Γ is called *principal* if it has a bicomponent, also called *principal*, containing all vertices of valency greater than two.

Proposition 5.5 *Let $X \subset \mathbb{P}^5$ be a smooth octic. If X is special, then either*

(1) $\text{girth}(\text{Fn } X) = 3$, *or*
(2) $\text{girth}(\text{Fn } X) = 4$ *and* Fn X *has a biquadrangle* (*then,* Fn X *is bipartite*), *or*
(3) $\text{girth}(\text{Fn } X) \geqslant 6$ *and* $C_1(X)$ *is a principal bicomponent of* Fn X.

Conversely, if Fn X *is as in items* (1) *or* (2) *above, then X is special.*

Proof Assume that X is special and let $e := e_X$.

Consider two lines $l_1, l_2 \in \text{Fn } X$, $l_1 \cdot l_2 = 1$. If $l_1 \cdot e = l_2 \cdot e = 0$, then the class $l_3 := e - l_1 - l_2$ is also a line, and $\{l_1, l_2, l_3\}$ is a triangle. Conversely, if $\{l_1, l_2, l_3\}$ is a triangle, then the sum $e := l_1 + l_2 + l_3$ is a cubic pencil and X is special.

In the rest of the proof, we assume that girth(Fn X) $\geqslant 4$.

If $l_1 \cdot e = l_2 \cdot e = 1$, then $l_1 + l_2 + 2e - h$ is an exceptional divisor, contradicting to the assumption that X is smooth. Thus, C_0 and C_1 are a pair of distinguished complimentary bicomponents and Fn X is bipartite. Since h is fixed, in the relation

$$h = e + l_1 + l_2 + l_3 + l_4 + s_{13},$$

the presence of any five classes implies the presence of the sixth. (This relation holds whenever $l_1, l_2, l_3, l_4, s_{13}$ are as in Definition 5.3 and $l_1, l_3 \in C_0$.) Hence, the following statements are equivalent:

- the graph Fn X has a biquadrangle, consisting of a quadrangle $\{l_1, l_2, l_3, l_4\}$ and a bisection s_{13} intersecting l_1 and l_3;
- X is special and the graph Fn X has a quadrangle $\{l_1, l_2, l_3, l_4\}$; here, we can assume that $l_1, l_3 \in C_0$ and $l_2, l_4 \in C_1$;
- X is special and there is a vertex $l_1 \in C_0$ such that val $l_1 > 2$, i.e., incident to at least three other vertices l_2, l_4, s_{13} (and thus, C_1 is not principal).

These observations complete the proof. $\qquad\square$

Corollary 5.6 (of the proof and the uniqueness of e_X) *In Proposition 5.5(2), one has* val $v \leqslant 3$ *for each vertex* $v \in C_0$. *In particular, each minimal pencil has exactly one bisection* (*cf. Definition 5.3*). $\qquad\triangleleft$

Proposition 5.7 *Assume that X is special and* girth(Fn X) $\geqslant 4$, *so that the graph* Fn $X = C_0 \cup C_1$ *is bipartite. Then the following statements hold*:

(1) $|C_1| \leqslant 7$ *or* $|C_1| = 9$; *in the latter case,* val $v = 3$ *for each vertex* $v \in C_0$;
(2) $|C_0| \leqslant 12$ *and, if* $|C_0| \geqslant 9$, *then* val $v \leqslant 6$ *for each vertex* $v \in C_1$.

Proof For Statement (1), it suffices to observe that any nine lines $l_1, \ldots, l_9 \in C_1$ are subject to the relation $3h = l_1 + \ldots + l_9 + 5e_X$.

The bound $|C_0| \leqslant 12$ follows from the fact that each line $l \in C_0$ is a component of a type $\widetilde{\mathbf{A}}_1$ (or $\widetilde{\mathbf{A}}_1^*$) fiber of the elliptic pencil $\pi : X \to \mathbb{P}^1$ defined by e_X (the other component of this fiber being the conic $e_X - l$). The valency bound is mainly given by the inertia index of the lattice; in the border case $C_0 = \{l_1, \ldots, l_9\}$ and a line $l \in C_1$ intersecting $l_1, \ldots, l_7$, the class
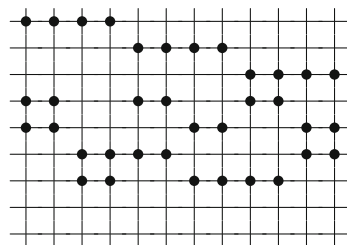
$$l_1 + \cdots + l_7 - l_8 - l_9 + 3l - h$$

is an exceptional divisor as in Definition 2.1(1). (In fact, the valency bound does not use the assumption that X should be special.) $\qquad\square$

5.4 Proof of Theorem 1.3

In view of Proposition 5.5, there only remains to prove the bounds on the number of lines in a special octic X.

Fig. 2 The configuration Θ_{21} (see Theorem 1.3)

If girth(Fn X) $= 3$, we pick a triangle $\Sigma \subset$ Fn X and consider $\Gamma := \mathcal{P}(\Sigma)$. The sections of Γ are pairwise disjoint by Proposition 2.11, and Proposition 2.8 yields $|\sec \Gamma| \leqslant 9$; hence, $|$Fn $X| \leqslant 33$, as stated. For the more refined statement, we run the algorithm of Sect. A.3 and list all configurations with more than $\overline{M} := 26$ lines.

Assume that girth(Fn X) $\geqslant 4$ and, hence, Fn $X = C_0 \cup C_1$ is bipartite. By Proposition 5.7, we have either $|$Fn $X| \leqslant 19$ or $|C_1| = 9$ and $|C_0| \geqslant 11$; in the latter case, $|$Fn $X| = 20$ or 21 and, since C_1 is not a principal bicomponent, Fn X has a biquadrangle by Proposition 5.5. For the refined classification, we run an algorithm similar to Sect. A.3, using a "pencil" $C_0 = \{l_1, \ldots, l_n\}$, $n = 11$, 12, and adding "sections" $s \in C_1$ (regarded as subsets of C_0, cf. Convention 3.6). To form a biquadrangle, we start with a triple of sections s_1, s_2, s_3 such that

$$\bar{s}_1 \cap \bar{s}_2 = \bar{s}_2 \cap \bar{s}_3 = \bar{s}_3 \cap \bar{s}_1 = \{l_1, l_2\};$$

there are but three such triples. Then, at most three more sections can be added increasing the rank of the lattice, and we arrive at the dichotomy $|$Fn $X| \leqslant 19$ or Fn $X \cong \Theta_{21}$ (see Fig. 2). $\qquad\square$

Remark 5.8 The computation at the end of the proof can be taken to a more combinatorial level if one takes into account the intrinsic structure on the set C_0 of size 12 or 11 arising from the imprimitive embedding $\mathbb{Z}C_0 + \mathbb{Z}h + \mathbb{Z}c_X \hookrightarrow \mathbf{L}$ (cf. pencils of type $(0, 12)$ and $(0, 11)$ in [5]). We leave this exercise to the reader.

One can use Propositions 5.5 and 5.7 to take the classification one step further and find the extremal configurations that are neither triangular nor quadrangular. For such a graph $\Gamma = C_0 \cup C_1$, we have $|\Gamma| < 19$ unless $|C_0| = 12$ and $|C_1| = 7$. This time, the class c_X does not need to belong to $\mathcal{F}_h(X)$; hence, we replace this lattice with the primitive hull of $\mathcal{F}_h(X) + \mathbb{Z}c_X$. The result of the straightforward computation using Sect. A.3 is as follows.

Proposition 5.9 *Let* $X \subset \mathbb{P}^5$ *be a smooth special octic such that* girth(Fn X) $\geqslant 5$. *Then* Fn $X \cong \Delta'_{19}$, Δ''_{19} *(see Fig. 3) or* $|$Fn $X| \leqslant 18$. *The extremal Fano graphs* Δ'_{19}, Δ''_{19} *can also be realized by smooth triquadrics.* $\qquad\triangleleft$

5.5 Hyperelliptic Models

A *hyperelliptic model* of degree $2D$ of a $K3$-surface X is a two-to-one map $X \to Y \hookrightarrow \mathbb{P}^{D+1}$ splitting through a rational surface Y. Due to [14], a degree $2D$ model
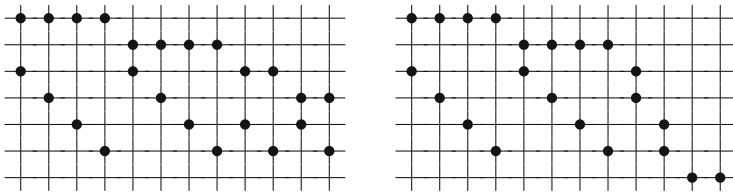
Fig. 3 The configurations Δ'_{19} and Δ''_{19} (see Proposition 5.9)

$X \to \mathbb{P}^{D+1}$ defined by a class $h \in NS(X)$ is hyperelliptic if and only if one of the following conditions holds:

(1) $D = 1$: any model $X \to \mathbb{P}^2$ is hyperelliptic;
(2) $D = 4$ and h is divisible by 2 in $NS(X)$: the model is the composition of a double covering $X \to \mathbb{P}^2$ and the Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$;
(3) $D \geqslant 2$ and there is a quadric pencil $e \in NS(X)$ as in Definition 2.1(2): the rational surface $Y \subset \mathbb{P}^{D+1}$ is a scroll in this case (see also [13]).

A hyperelliptic model $X \to Y \hookrightarrow \mathbb{P}^{D+1}$ is considered *smooth* if so is its ramification locus in Y. A *line* in a hyperelliptic model is a rational curve $L \subset X$ that is mapped one-to-one onto a line in $\mathbb{P}^{D+1}$. Thus, arithmetically, a line can still be described as a class $l \in NS(X)$ such that $l^2 = -2$ and $l \cdot h = 1$.

We will treat the case $D = 1$ in a subsequent paper: the conjectural bound for such models is 144 lines (see [3]).

In Case (2), where the model splits through the Veronese embedding, it has no lines, as $l \cdot h = 0 \mod 2$ for any $l \in NS(X)$.

Thus, we are left with Case (3), where the image Y is a scroll. To study such models, we need to change the definition of the admissibility of a configuration S (see Definition 2.1): the exceptional divisors as in (1) are still excluded, but this time we *assume* the presence of a quadric pencil $e \in S$ as in (2), requiring instead that there should be no class $c \in S$ such that

(2′) $c^2 = 0$ and $c \cdot h = 1$ (*fixed component*, see [10]).

Now, arguing as in Sect. 5.3 and using an obvious analogue of Proposition 2.8, we can easily prove the following statement.

Lemma 5.10 *Consider a smooth hyperelliptic model $X \to \mathbb{P}^{D+1}$, $D \geqslant 2$, and let $e \in NS(X)$ be a quadric pencil. Then, for any line $l \in \mathrm{Fn}\, X$, we have $l \cdot e = 0$ or 1. Furthermore, if $D \geqslant 3$, the quadric pencil $e := e_X \in NS(X)$ is unique.* □

If $D = 2$, the obvious maximum of 48 lines (24 over each of the two rulings) is realized by a connected equilinear family containing, in particular, the discriminant minimizing singular surfaces (see [3]).

Thus, from now on, we assume that $D \geqslant 3$, and Lemma 5.10 gives us a natural splitting $\mathrm{Fn}\, X = C_0 \cup C_1$ as in (5.2). The set C_0 consists of the components of the reducible fibers of an elliptic pencil—the pullback of the ruling of Y; these fibers are of type $\widetilde{\mathbf{A}}_1$ (or $\widetilde{\mathbf{A}}_1^*$), we have $l' \cdot l'' = 2$ for two lines l', l'' in the same fiber, and each line $l \in C_1$ intersects exactly one of l', l''. Further computation shows that the set C_1 is empty with the following few exceptions:

- $D = 3$ and $C_1 = \{l', l''\}$, $l' \cdot l'' = 1$ and $h = l' + l'' + 2e$;
- $D = 4$ and $C_1 = \{l', l''\}$, $l' \cdot l'' = 0$ and $h = l' + l'' + 3e$;
- $D = 6$ and $C_1 = \{l\}$ so that $h = 2l + 5e$; hence, $C_0 = \varnothing$.

Geometrically, C_1 projects to the exceptional section $E \subset Y$; if $D = 6$ ($E^2 = -4$), this section is a component of the ramification locus.

Theorem 5.11 *Let $X \to \mathbb{P}^{D+1}$ be a hyperelliptic model, $D \geqslant 3$. Then, unless $D = 6$ and $|\mathrm{Fn}\, X| = 1$, we have $|\mathrm{Fn}\, X| = 0 \bmod 2$ and $|\mathrm{Fn}\, X| \leqslant 24 - 4\varepsilon$, where we let $\varepsilon := D \bmod 2 \in \{0, 1\}$. This bound is sharp.*

Proof If $C_1 = \varnothing$, the configuration consists of a single pencil C_0 and, by an obvious analogue of Proposition 3.7, the maximal number of lines depends on $D \bmod 2$ only. Using Algorithm 2.6, we arrive at $|\mathrm{Fn}\, X| \leqslant 24 - 4\varepsilon$. The other case $|C_1| = 2$ is treated similarly: for each value $D = 3, 4$ and each cardinality $|C_0|$, there is a single candidate, which is analyzed using Algorithm 2.6. $\square$

6 Pentagonal Configurations

In the remaining three sections, we consider configurations with minimal pencils whose sections may intersect each other.

6.1 Statements

Recall that a graph Γ is *pentagonal* if $\mathrm{girth}(\Gamma) = 5$. All large geometric pentagonal graphs are described by the following theorem.

Theorem 6.1 (see Sect. 6.3) *Let X be a smooth $2D$-polarized $K3$-surface, and assume that the Fano graph $\mathrm{Fn}\, X$ is pentagonal, i.e., $\mathrm{girth}(\mathrm{Fn}\, X) = 5$. Let $\overline{M} := \overline{M}(D)$ be as in Table 5 or as follows*:

$$
\begin{array}{llll}
D: & \{10, 11 \bmod 5\} & \{12 \bmod 5\} & \text{else} \\
\overline{M}: & 21 & 22 & 20
\end{array}
$$

Then, with the exceptions listed in Table 5, one has a sharp bound $|\mathrm{Fn}\, X| \leqslant \overline{M}$.

The notation used in Table 4 is similar to Table 1 (with some columns skipped); it is explained in Sect. 1.3. It is remarkable that the same graph Φ'_{30} is a Fano graph of $2D$-polarized $K3$-surfaces for all $2 \leqslant D \leqslant 7$. The graph Φ''_{30} works for $2 \leqslant D \leqslant 5$; if $D = 6$, this graph is still geometric, but not saturated (pencils acquire extra fibers and/or sections), and $\Phi''_{36} \supset \Phi''_{30}$ is its saturation.

The graph Φ''_{36} is 5-regular, 2-arc transitive, and distance regular; according to D. Pasechnik (private communication), it is the *Sylvester graph*. The graph contains 36 minimal pencils, all of type $4\widetilde{\mathbf{A}}_4 + \mathbf{A}_1$.

The graph Φ'_{30} is 4-regular and 1-arc transitive; it contains six copies of $4\widetilde{\mathbf{A}}_4$.

Table 5 Extremal pentagonal configurations (see Theorem 6.1)

D	$\overline{M}$	Γ	$|\text{Aut }\Gamma|$	det	(r, c)	$|\text{Aut }X|$	$T := \mathcal{F}_h(\Gamma)^\perp$
2	29	Φ'_{30}	240	220	(2,0)	60	[4, 2, 56], [16, 6, 16]
		Φ''_{30}	40	255	(1,1)	10	[4, 1, 64]*, [16, 1, 16]
3	29	Φ'_{30}	240	180	(1,0)	60	[14, 4, 14]
		Φ''_{30}	40	195	(1,0)	10	[14, 1, 14]
4	29	Φ'_{30}	240	140	(1,1)	60	[12, 2, 12]
		Φ''_{30}	40	135	(1,1)	10	[12, 3, 12]
$5^\dagger$	28	Φ'_{30}	240	100	(1,0)	60	[10, 0, 10]
		Φ''_{30}	40	75	(1,0)	10	[10, 5, 10]
		Φ'''_{30}	24	36	(1,0)	12	[4, 2, 10]
6	28	Φ'_{30}	240	60	(1,0)	60	[8, 2, 8]
		$\Phi''_{36}{}^\ddagger$	1440	$\underline{15}$	(1,0)	720	[2, 1, 8]
7	26	Φ'_{30}	240	$\underline{20}$	(1,0)	120	[4, 2, 6]
8	24	Φ_{25}	80		dim = 1		$\mathbf{U}(5) \oplus [4]$
9	23	$\Phi_{25}{}^\ddagger$	80	$\underline{15}$	dim = 1		$\mathbf{U}(5) \oplus [2]$
$10^\dagger$	21	$\Phi_{25}{}^\ddagger$	80		dim = 2		$\mathbf{U}(5) \oplus \mathbf{U}$

6.2 Sections at a Fiber Σ of Type $\widetilde{\mathbf{A}}_4$

Consider a geometric pentagonal configuration S and pick a distinguished pentagon $\Sigma := \{l_1, \ldots, l_5\} \subset \text{Fn } S$.

Convention 6.2 Here and below, we always assume that the n lines $l_1, \ldots, l_n$ of a type $\widetilde{\mathbf{A}}_{n-1}$ fiber in a configuration are numbered cyclically, so that

$$l_{i \pm n} = l_i, \qquad l_i \cdot l_j = 1 \quad \text{if and only if } i - j = \pm 1 \bmod n.$$

Thus, we regard the indices as elements of the cyclic group $\mathbb{Z}/n$.

Consider the pencil $\Gamma := \mathcal{P}(\Sigma)$. Since Fn S has no triangles or quadrangles, we immediately obtain the following lemma.

Lemma 6.3 *For $\Sigma \subset \Gamma \subset \text{Fn } S$ as above, the following statements hold:*

- *all sections of Γ are simple;*
- $(\text{sec } l_i) * (\text{sec } l_j) = 0$ *unless $i - j = \pm 2 \bmod 5$;*
- $(\text{sec } l_i) * (\text{sec } l_j) \leqslant 1$ *if $i - j = \pm 2 \bmod 5$.*

In particular, Σ is suitable for enumeration in the sense of Sect. A.5.

Lemma 6.4 *For $\Sigma \subset \Gamma \subset \text{Fn } S$ as above, one has the following bounds on $|\text{sec } \Gamma|$:*

$D =$	2, 3	4	5	6	7	8	9, 10	11, 12	13 .. 15	16 .. 25	else
$|\text{sec } \Gamma| \leqslant$	16	17	18	15	10	7	5	3	2	1	0

Furthermore, the following statements hold:

(1) *if* $2 \leqslant D \leqslant 4$ *and* $|\sec \Gamma| \geqslant 14$*, then* $|\mathrm{Fn}\, S| \leqslant 29$;
(2) *if* $D = 5$ *and* $|\sec \Gamma| \geqslant 14$*, then* $\mathrm{Fn}\, S \cong \Phi'''_{30}$ *or* $|\mathrm{Fn}\, S| \leqslant 28$;
(3) *if* $D = 6$ *and* $|\sec \Gamma| \geqslant 12$*, then* $\mathrm{Fn}\, S \cong \Phi''_{36}$ *or* $|\mathrm{Fn}\, S| \leqslant 28$;
(4) *if* $7 \leqslant D \leqslant 9$*, then there is at most one pair of intersecting sections*;
(5) *if* $D \geqslant 10$*, then all sections of* Γ *are pairwise disjoint*.

Besides, if $D = 5$*, then* $\mathrm{val}\, l \in \{0 .. 3, 5\}$ *for each line* $l \in \Sigma$.

Proof Since Σ is suitable for enumeration, we can run the algorithm of Sect. A.5.

If $D \geqslant 7$, we can easily enumerate *all* sets of sections, obtaining, in particular, Statements (4) and (5). (These lists are also used in Sect. 6.3.)

If $D \leqslant 6$, we only list the sets $\sec \Gamma$ with $|\sec \Gamma| \geqslant 14$ ($|\sec \Gamma| \geqslant 12$ if $D = 6$). Then, Statements (1)–(3) are proved by applying Step 3 (see Sect. A.5) in order to classify large configurations. The technical details are explained in Sect. A.7.

The last statement is straightforward. □

6.3 Proof of Theorem 6.1

Following the idea explained in Sect. 4.3, we can still use the algorithm of Sect. A.3 based on discrete sets of sections.

Let $\overline{M}$ be as in the statement of the theorem. If $D \geqslant 14$, there are relatively few sets of sections $\sec \Gamma$ (see Lemma 6.4 and its proof), and we can use the algorithm directly, taking into account (if present) the one pair of sections that intersect.

If $D \leqslant 12$, we use the following trick. Let $\overline{M}_{\mathrm{sec}} = 13$ ($\overline{M}_{\mathrm{sec}} = 11$ if $D = 12$). Then, by Lemma 6.4, it suffices to consider pencils Γ with $|\Gamma| > \overline{M}_\Gamma := \overline{M} - \overline{M}_{\mathrm{sec}}$. Pick such a pencil Γ and let $n_i := \mathrm{val}\, l_i$, $i \in \mathbb{Z}/5$, be a desired set of goals, such that

$$\overline{M} - |\Gamma| < \sum_{i \in \mathbb{Z}/5} n_i \leqslant \overline{M}_{\mathrm{sec}}. \tag{6.1}$$

(In addition, we can require that each pair (n_i, n_{i+1}), $i \in \mathbb{Z}/5$, is realized by the valencies of a pair of adjacent lines of Σ in a geometric configuration: this condition is easily checked as all sections involved are pairwise disjoint, see Lemma 6.3.) Let, further, $m := \max\{n_i + n_{i+1} \mid i \in \mathbb{Z}/5\}$, and consider the set

$$\mathcal{N}(n_i) := \big\{(p, q) \mid p \geqslant q,\, p + q = m,\, (p, q) = (n_i, n_{i\pm1}) \text{ for some } i \in \mathbb{Z}/5\big\}.$$

Then, denoting by $\mathcal{N}$ the union of $\mathcal{N}(n_i)$ over all sets of goals (n_i) satisfying (6.1), we can assert that, in any geometric pentagonal extension $S \supset \mathcal{F}_h(\Gamma)$ such that $|\mathrm{Fn}\, X| > \overline{M}$, there is a pair $(p, q) \in \mathcal{N}$ such that Γ has at least $(p + q)$ disjoint sections: up to automorphism of Σ, we can assume that $\mathrm{val}\, l_1 \geqslant p$ and $\mathrm{val}\, l_2 \geqslant q$.

If $D \in \{2 .. 4, 6\}$, the computation based on this observation completes the proof, as all configurations obtained by adding to the pencil $(p + q)$ disjoint sections as above have the maximal rank 20 and can be analyzed using Algorithm 2.6.

If $D = 5$, then, occasionally, we may need to add an extra section. Namely, let (n_i) be a set of goals such that $2m \leqslant \sum_i n_i$: this is the case for all configurations of low rank. Then, a simple argument shows that we can reindex the lines so that $n_1 \geqslant n_2$, $n_1 + n_2 = m$ is maximal, and $n_3 > 0$, i.e., Γ has at least one section adjacent to l_3. Taking this extra section into account and allowing it to intersect one of those adjacent to l_1, we always obtain a configuration of the maximal rank 20, and there remains to apply Algorithm 2.6. $\qquad\square$

7 Quadrangular Configurations

Recall that a graph Γ and the corresponding configuration are *quadrangular* if $\mathrm{girth}(\Gamma) = 4$, i.e., Γ has no triangles, but it does contain a quadrangle.

7.1 Statements

In this section, we mostly ignore the case $D = 2$, although the maximal cardinality of a triangle free configuration of lines in a spatial quartic remains an open problem (see the discussion in Sect. 7.5). The next statement is the principal result of the section.

Theorem 7.1 (see Sect. 7.4) *Let $X \subset \mathbb{P}^{D+1}$ be a smooth $2D$-polarized $K3$-surface, $D \geqslant 3$, and assume that the Fano graph* Fn X *is quadrangular, i.e.,* $\mathrm{girth}(\mathrm{Fn}\, X) = 4$. *Then, with few exceptions, we have the following sharp bounds*:

$D =$	3	4	5, 6	7	9	$\{8 \bmod 4\}$	$\{10 \bmod 4\}$	$\{11 \bmod 4\}$	$\{13 \bmod 4\}$
$\lvert\mathrm{Fn}\,X\rvert \leqslant$	36	30	28	22	20	22	24	20	18

The exception is the case $D = 4$, Fn $X \cong \Theta_*^*$ *(see Table 1 and comments below).*

Addendum 7.2 (see Sect. 7.4) *The complete set $\{\lvert\mathrm{Fn}\,X\rvert\}$ of values taken by the line count of a quadrangular configuration is as follows*:

- *if $D = 3$, then $\{\lvert\mathrm{Fn}\,X\rvert\} = \{4 .. 36\}$;*
- *if $D = 4$, then $\{\lvert\mathrm{Fn}\,X\rvert\} = \{4 .. 30, 32 .. 34, 36\}$.*

According to Theorem 1.3, the exceptional Fano graphs Θ_*^* in Table 1 are realized by triquadrics only. The next statement is proved by a simple computation.

Addendum 7.3 *There are inclusions $\Theta_{32}' \subset \Theta_{34}' \subset \Theta_{36}'$ and, hence, respective specializations of the corresponding families of triquadrics. All other graphs Θ_*^* in Theorem 7.1 are maximal with respect to inclusion.* $\qquad\square$

The graphs Θ_{36}'', Θ_{32}', Θ_{32}'', Θ_{32}''', and Θ_{32}^{K} are 6-regular and vertex transitive. The last one, Θ_{32}^{K}, is the famous *Kummer configuration* 16_6: it is 2-arc transitive, distance regular, and bipartite. The graphs Θ_{36}' and Θ_{32}^{K} can be represented by the discriminant minimizing singular $K3$-surface $X([4, 0, 8])$ (see [3]). More details on these graphs and corresponding triquadrics are found in Table 1 (see also [4]).

7.2 Sections at a Fiber Σ of Type $\widetilde{A}_3$

Till the end of this section, we consider a geometric quadrangular configuration S and pick a quadrangle $\Sigma := \{l_1, l_2, l_3, l_4\}$. We assume the lines ordered cyclically (see Convention 6.2) and, usually, so that

$$\text{val } l_1 \geqslant \text{val } l_3, \quad \text{val } l_1 \geqslant \text{val } l_2 \geqslant \text{val } l_4, \quad \text{val } l_3 \geqslant \text{val } l_4 \text{ if } \text{val } l_1 = \text{val } l_2. \quad (7.1)$$

Consider also the pencil $\Gamma := \mathcal{P}(\Sigma)$; since $k_\Sigma = l_1 + l_2 + l_3 + l_4$ and $\deg \Gamma = 4$, all fibers of Γ are of types $\widetilde{A}_3$, A_2, or A_1.

We subdivide quadrangular configurations according to whether they do or do not have a biquadrangle (see Definition 5.3).

Lemma 7.4 *For $\Sigma \subset \text{Fn } S$ as above, one has*:

(1) *each subgraph $\sec l_i$, $1 \leqslant i \leqslant 4$, is discrete*;
(2) $\sec l_i \cap \sec l_j = \varnothing$ *whenever $j - i = 1 \bmod 2$*;
(3) *if $D = 3$, then $(\sec l_i) * (\sec l_{i\pm 1}) \leqslant 2$ and $(\sec l_i) * (\sec l_{i+2}) \leqslant 3$*;
(4) *if $D \geqslant 4$, then $(\sec l_i) * (\sec l_j) \leqslant 1$ for any pair (i, j)*;
(5) *if $D \geqslant 8$, then the subgraph $\sec \Gamma$ is discrete*.

If, in addition, S has no biquadrangle, then also

(6) $\sec l_i \cap \sec l_j = \varnothing$ *for any pair $1 \leqslant i < j \leqslant 4$*;
(7) $(\sec l_i) * (\sec l_{i\pm 1}) \leqslant 1$ *and $(\sec l_i) * (\sec l_{i+2}) \leqslant 2$*.

(In view of Statement (4), the last restriction is meaningful only for $D = 2, 3$.)

Proof Most assertions are immediate from the assumption that S should have no triangle or, for the last two statements, biquadrangle. Statements (3), (4), and (5) are proved by applying Proposition 2.8 to simple test configurations. $\qquad\qquad$ $\square$

Lemma 7.5 *If S has no biquadrangle, one has the following bounds on $|\sec \Gamma|$*:

$D =$	2, 3	4	5	6	7, 8	9, 10	11 .. 16	else
$|\sec \Gamma| \leqslant$	21	20	12	8	4	2	1	0

Furthermore, the following statements hold:

- *if $D = 2, 3$ and $|\sec \Gamma| = 21$, then $|\text{Fn } S| = 34$ (a unique configuration)*;
- *if $D = 4$ and $|\sec \Gamma| \geqslant 15$, then either $|\text{Fn } S| \leqslant 30$ or $\text{Fn } S$ is isomorphic to one of the exceptional graphs $\Theta'_{36}, \Theta''_{36}, \Theta'_{34}, \Theta''_{32}, \Theta''_{32}, \Theta'''_{32}, \Theta^K_{32}$.*

Proof If $D \geqslant 4$, then, due to Lemma 7.4, the fiber Σ is suitable for enumeration in the sense of Sect. A.5. Applying the algorithm of Sect. A.5 (in the order l_1, l_3, l_2, l_4), assuming (7.1) and taking for the defining property

$$\mathfrak{P}\colon S \longmapsto S \text{ has neither a triangle nor a biquadrangle,} \qquad (7.2)$$

we arrive at the statement of the lemma.

Fig. 4 A biquadrangle for $D = 3$
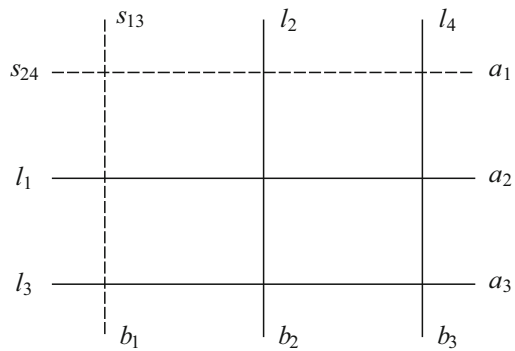
If $D = 4$ and $|\sec \Gamma| \geqslant 15$, the configurations are classified using Step 3 of the algorithm (see Sect. A.5.4); we postpone the proof until Sect. A.6, after the algorithm is explained in more details.

If $D = 2, 3$, the situation is more complicated, see Lemma 7.4(7). However, large configurations can still be enumerated similar to Sect. A.5. Assuming that $\operatorname{val} l_1 \geqslant 6$ and $\operatorname{val} l_1 + \operatorname{val} l_3 \geqslant 11$, see (7.1), we arrive at a unique configuration S satisfying the inequality $|\sec \Gamma| > 20$. This configuration has rank 20, contains 34 lines, and has no nontrivial $\mathfrak{P}$-geometric finite index extensions. $\qquad\square$

7.3 Configurations with a Biquadrangle

Assume that the pencil $\Gamma = \mathcal{P}(\Sigma)$ has a bisection s_{13} intersecting l_1 and l_3. Using Proposition 2.8, one can show that $D \leqslant 4$ and, unless $D = 2$, such a bisection is unique. We ignore the case $D = 2$ (cf. Example 7.9), whereas the case $D = 4$ was considered in Sect. 5.3, resulting in the bound $|\operatorname{Fn} S| \leqslant 21$ (see Theorem 1.3).

Lemma 7.6 *Assume that $D = 3$ and Γ has a bisection s_{13} intersecting l_1 and l_3. Then $|\operatorname{Fn} S| \leqslant 36$.*

Proof It is immediate that Γ has another bisection, *viz.*

$$s_{24} := h - l_1 - l_2 - l_3 - l_4 - s_{13},$$

intersecting l_2 and l_4, so that $s_{13} \cdot s_{24} = 1$, and any other line $l \in \operatorname{Fn} S$ intersects exactly one of $l_1, l_2, l_3, l_4, s_{13}, s_{24}$. Thus, renaming the lines to $a_1, a_2, a_3, b_1, b_2, b_3$ as in Fig. 4, we have

$$|\operatorname{Fn} S| = \sum_{i=1}^{3} \operatorname{val} a_i + \sum_{j=1}^{3} \operatorname{val} b_j - 12 = \frac{1}{3} \sum_{i,j=1}^{3} (\operatorname{val} a_i + \operatorname{val} b_j) - 12.$$

Since $\operatorname{val} l \leqslant 9$ for each line $l \in \operatorname{Fn} S$ (see Proposition 2.10), we have $|\operatorname{Fn} S| \leqslant 42$.

From now on, assume that $|\mathrm{Fn}\, S| > 36$. Reindexing the lines (and choosing, if necessary, another quadrangle Σ), we can also assume that $\mathrm{val}\, a_1 \geqslant \mathrm{val}\, b_1$ and the pair $(i, j) = (1, 1)$ maximizes the sum $\mathrm{val}\, a_i + \mathrm{val}\, b_j$, $i, j = 1, 2, 3$. By the count above, $\mathrm{val}\, a_1 = 9$ and $\mathrm{val}\, b_1 = 9$ or 8; hence, $|\Gamma| = \mathrm{val}\, a_1 + \mathrm{val}\, b_1 - 2 = 16$ or 15, respectively. In the former case, (7.1) implies $\mathrm{val}\, l_1 \geqslant 8$. In the latter case, we can order the lines so that $\mathrm{val}\, l_4 \leqslant \mathrm{val}\, l_2 \leqslant 8$ and $\mathrm{val}\, l_3 \leqslant \mathrm{val}\, l_1$; then again $\mathrm{val}\, l_1 \geqslant 8$.

Now, we can run an algorithm similar to that of Sect. A.2, taking "triangle free" for the defining property $\mathfrak{P}$ and starting with a pencil Γ of size $|\Gamma| = 16$ or 15 and a pair of bisections s_{13}, s_{24}. Recall that each line in $\mathrm{Fn}\, S$ intersects exactly one of the bisections s_{13}, s_{24}; together with the assumption that S should be triangle free, this observation determines the pair of bisections uniquely up to $\mathrm{Aut}\, \Gamma$. Furthermore, all other sections of Γ are simple and disjoint from s_{13}, s_{24}. Thus, taking l_1 for l, at Steps 1 and 2 we add at least five disjoint sections $s_i \in \sec l$ (recall that $\mathrm{val}\, l_1 \geqslant 8$). Configurations of low rank (Step 3) are analyzed as described in Sect. A.4.2. □

7.4 Proof of Theorem 7.1 and Addendum 7.2

Let $S := \mathcal{F}_h(X)$ be as in the theorem. If S has a biquadrangle, the statement is given by Lemma 7.6 and Theorem 1.3. Thus, assume that S is biquadrangle free.

We can use (3.3) and (3.5) to list all combinatorial types of pencils Γ of degree 4; then, Propositions 3.7 and 2.8 imply the following lemma.

Lemma 7.7 *The cardinality* $|\Gamma|$ *is bounded as follows*:

$D \in$	{6 mod 4}	{4 mod 4}	{7 mod 4}	{5 mod 4}
max $|\Gamma| =$	24	22	20	18

If Γ *has at least one simple section, the values of* D *are bounded as follows*:

- *if* $|\Gamma| \geqslant 22$, *then* $D = 4$;
- *if* $|\Gamma| = 21$, *then* $D \in \{2, 4, 6\}$;
- *if* $|\Gamma| = 19, 20$, *then* $D \in \{2, 4, 6, 8\}$;
- *if* $|\Gamma| = 17, 18$, *then* $D \in \{2 .. 10\}$. ◁

Lemmas 7.5 and 7.7 imply the statement of Theorem 7.1 for $D \geqslant 9$.

For the remaining few values of D, including $D = 2$, we run the algorithm of Sect. A.2, taking (7.2) for the defining property $\mathfrak{P}$. The threshold is $|\mathrm{Fn}\, S| > \overline{M}$, where $\overline{M} := \overline{M}(D)$ is the bound stated in the theorem; then, in view of Lemma 7.5, we can assume that $|\Gamma| > \overline{M}_\Gamma := 16$. (For $D = 6$ or 8 we can increase $\overline{M}_\Gamma$ to 20 or 18, respectively.) Furthermore, assuming (7.1) and taking l_1 for l, we have

$$|\sec l| = |\sec_1 l| \geqslant v_{\min} := \left\lceil \tfrac{1}{4}(\overline{M} + 1 - |\Gamma|) \right\rceil.$$

Configurations of low rank (Step 3) are analyzed as described in Sect. A.4.2. The result is the conclusion that $|\mathrm{Fn}\, S| \leqslant \overline{M}(D)$ unless $D = 4$ and $\mathrm{Fn}\, S \cong \Theta'_{36}, \Theta''_{36}, \Theta'_{34}, \Theta_{33}, \Theta_{32}$, as stated. As a by-product (cf. Sect. A.1), we obtain Addendum 7.2. □

7.5 Spatial Quartis

If $X \subset \mathbb{P}^3$ is a smooth quartic and Fn X is triangle free, the best known bound is $|\text{Fn } X| \leqslant 52$, whereas the best example so far had 33 lines. (Example 7.9 has 37 lines.) The last proof applies to $D = 2$, and we state this fact separately (combining it with the results of the previous sections). This leaves configurations with a biquadrangle as the only case still open.

Proposition 7.8 *Let* $X \subset \mathbb{P}^3$ *be a smooth quartic, and assume that* Fn X *has no triangles or biquadrangles. Then* $|\text{Fn } X| \leqslant 36$, *and this bound is sharp.*   ◁

Example 7.9 There exists a smooth spatial quartic $X \subset \mathbb{P}^3$ with the graph Fn X triangle free and $|\text{Fn } X| = 37$. This configuration, found in [4], was obtained during the experiments leading to the proof of Lemma 7.6.

8 Triangular Configurations

Recall that a graph Γ and the corresponding configuration are called *triangular* if $\text{girth}(\Gamma) = 3$. Large triangular configurations of lines in smooth quartics have been studied in [5] (see $D = 2$ in Table 1). Therefore, we assume that $D \geqslant 3$.

8.1 Statements

The following theorem is the principal result of this section.

Theorem 8.1 (see Sect. 8.3) *Let* $X \subset \mathbb{P}^{D+1}$ *be a smooth* $2D$-*polarized* $K3$-*surface,* $D \geqslant 3$, *and assume that the Fano graph* Fn X *is triangular, i.e.,* $\text{girth}(\text{Fn } X) = 3$. *Then, with few exceptions, we have the following sharp bounds*:

$$
\begin{array}{ccccc}
D = & 3 & 4 & 5 & \{7 \bmod 3\} & \text{else} \\
|\text{Fn } X| \leqslant & 36 & 29 & 24 & 24 & 21
\end{array}
$$

The exceptions are $D = 3$, Fn $X \cong \Psi_{42}, \Psi_{38}$ *and* $D = 4$, Fn $X \cong \Psi_{33}$ *(see Table 1).*

Addendum 8.2 (see Sect. 8.3) *If* $D = 3$, *the number of lines in triangular configurations takes all values in* $\{3 \mathinner{\ldotp\ldotp} 36, 38, 42\}$.

The graph Ψ_{42} is 9-regular; it contains two minimal pencils, which are both of type $3\widetilde{\mathbf{A}}_2 + 3\mathbf{A}_1$. The corresponding equilinear family contains models of the two discriminant minimizing surfaces $X([2, 1, 20])$ and $X([6, 3, 8])$ (see [3]).

8.2 Lines Intersecting a Triangle

Consider a geometric configuration S and a triangle $\Sigma = \{l_1, l_2, l_3\} \subset \text{Fn } S$. Let $\Gamma := \mathcal{P}(\Sigma)$. The next lemma is an application of Proposition 2.8 to simple test configurations (cf. also Proposition 2.11).

Table 6 Pencils of degree 3 with a section

$|\Gamma|$	Γ	D, remarks
24	$8\widetilde{\mathbf{A}}_2$	4, $s \in \mathcal{F}_h(\Gamma) \otimes \mathbb{Q}$
22	$7\widetilde{\mathbf{A}}_2 + \mathbf{A}_1$	4, $s \in \mathcal{F}_h(\Gamma) \otimes \mathbb{Q}$
21	$7\widetilde{\mathbf{A}}_2$	5, $s \in \mathcal{F}_h(\Gamma) \otimes \mathbb{Q}$
	$6\widetilde{\mathbf{A}}_2 + 3\mathbf{A}_1$	3, 4
20	$6\widetilde{\mathbf{A}}_2 + 2\mathbf{A}_1$	2, 3, 4
19	$6\widetilde{\mathbf{A}}_2 + \mathbf{A}_1$	2, 3, 4, 5
	$5\widetilde{\mathbf{A}}_2 + 4\mathbf{A}_1$	3, 4

Lemma 8.3 *Assume that $D \geqslant 3$. Then any section of Γ is simple. Furthermore,*

(1) *each subgraph $\sec l_i$, $1 \leqslant i \leqslant 3$, is discrete;*
(2) $(\sec l_i) * (\sec l_j) \leqslant 1$ *for $1 \leqslant i \neq j \leqslant 3$;*
(3) *if $D \geqslant 4$, then $\sec \Gamma$ is discrete.*   ◁

Lemma 8.4 *We have the following sharp bounds on the cardinality $|\sec \Gamma|$:*

$$
\begin{array}{c|ccccc}
D = & 3 & 4 & 5, 6 & 7 .. 11 & \text{else} \\
|\sec \Gamma| \leqslant & 21 & 9 & 3 & 1 & 0
\end{array}
$$

If $D = 3$ and $|\sec \Gamma| \geqslant 20$, then either $\mathrm{Fn}\, S \cong \Psi_{42}$ or $|\mathrm{Fn}\, S| \leqslant 36$.

Proof If $D \geqslant 4$, then, in view of Lemma 8.3(3), all statements easily follow from Proposition 2.8. If $D = 3$, then Lemma 8.3(1), (2) means that Σ is suitable for enumeration (see Sect. A.5), and we use the algorithm of Sect. A.5.   □

8.3 Proof of Theorem 8.1 and Addendum 8.2

Using Proposition 2.8 and bounds (3.3), (3.5), we can compile a list of large pencils $\Gamma := \mathcal{P}(\Sigma)$ admitting a section; the result is shown in Table 6. (In the first three lines, we indicate that any section lies in $\mathcal{F}_h(\Gamma) \otimes \mathbb{Q}$ and, hence, $\mathcal{F}_h(\Gamma)$ has no geometric extensions of positive corank.) Comparing this with Corollary 3.8 and Lemma 8.4, we conclude that, for $D \geqslant 6$, the maximal cardinality $|\mathrm{Fn}\, X|$ is realized by a parabolic configuration and is as stated in the theorem. Similarly, if $D = 5$, the maximum $|\mathrm{Fn}\, X| = 24$ is given by the only geometric finite index extension of $7\widetilde{\mathbf{A}}_2$, which has three sections.

The case $D = 4$ is considered in Sect. 5.4, resulting in Theorem 1.3. Thus, we assume that $D = 3$ and $|\mathrm{Fn}\, X| > \overline{M} := 36$. In view of Lemma 8.4, we can also assume that $|\sec \Gamma| \leqslant \overline{M}_{\sec} := 19$ and, hence, $|\Gamma| > \overline{M}_\Gamma := 17$. Indexing the lines so that $|\sec l_1| \geqslant |\sec l_2| \geqslant |\sec l_3|$, we apply the algorithm of Sect. A.2, taking for l the line l_1 and estimating the number of sections *via*

$$
|\sec l| \geqslant v_{\min} := \left\lceil \tfrac{1}{3}(37 - |\Gamma|) \right\rceil.
$$

The final step of the algorithm (see Sect. A.2.3) is explained in Sect. A.4.1.   □

8.4 Proof of Theorem 1.1 and Addendum 1.2

The case of spatial quartics is treated in [5]. For the other degrees, Theorem 1.1 is proved by comparing the bounds given by Theorems 4.4–4.6 (locally elliptic graphs), 4.5 (astral graphs), and 6.1, 7.1, and 8.1 (pentagonal, quadrangular, and triangular graphs, respectively).

Addendum 1.2 is given by [5] (for $D = 2$) or Addendum 7.2 (for $D = 3, 4$). □

8.5 Proof of Theorem 1.4

We use Theorem 2.7 and detect the extremal graphs in Table 1 realized by configurations of real lines in real surfaces (marked with a ‡ in the table). If such a graph does not exist, we merely state that $M_{\mathbb{R}}(D) = \overline{M}(D)$ is the auxiliary bound in Theorem 1.1. The sharpness of this bound for $D = 8$ and $D > 10$ is given by Addendum 4.7.

If $D = 5$, the Fano graph Φ_{28} of the discriminant minimizing surface $X([2, 0, 16])$ (see Theorem 1.7 and [4]) is real. If $D = 7$, a certain real configuration Φ_{26} (see [4]) is found in the course of the proof of Theorem 6.1. □

Appendix A: Algorithms

This appendix outlines a few technical details concerning the algorithms used to enumerate large pencils or sets of sections. All computations were done in GAP [7].

A.1 Defining Properties

In the enumeration routines, we are interested in the configurations satisfying a certain hereditary defining property $\mathfrak{P}$ (such as locally elliptic, triangle free, quadrangle free, *etc.*) We incorporate this property as part of Algorithm 2.5: upon establishing the admissibility of a configuration S, we compute the graph Fn S and disregard S as invalid if $\mathfrak{P}(S)$ does not hold.

Since Algorithm 2.6 relies upon Algorithm 2.5, we will essentially speak about $\mathfrak{P}$-(sub-) *geometric* configurations, i.e., those admitting a geometric finite index extension satisfying the defining property $\mathfrak{P}$.

For statistical purposes (*e.g.*, for establishing the sharpness of the bounds), in Algorithm 2.6 we keep track of the sizes of all geometric configurations: as soon as a $\mathfrak{P}$-geometric configuration S is discovered, its cardinality |Fn S| is recorded.

A.2 Pencils

As an essential part of most proofs, we enumerate all $\mathfrak{P}$-geometric configurations S spanned by a pencil $\Gamma := \mathcal{P}(\Sigma)$ with a distinguished fiber Σ and a number of sections $s_i \in \sec_1 \Gamma$ such that

$$|\Gamma| > \overline{M}_\Gamma, \qquad |\text{Fn } S| > \overline{M}, \tag{A.1}$$

where $\overline{M}_\Gamma$ and $\overline{M}$ are certain thresholds fixed in advance. The types of the fibers of Γ are given by Corollaries 3.3 and 3.4, and the possible combinatorial types of pencils are listed using (3.3)–(3.5) (see also Remark 3.5).

Thus, we fix a combinatorial type $\Gamma \supset \Sigma$ and change the defining property $\mathfrak{P}$ to

$$\mathfrak{P}_\Gamma \colon S \longmapsto \mathfrak{P}(S) \text{ and } \Gamma \subset \mathrm{Fn}\, S \text{ is a maximal parabolic subgraph.}$$

We also fix a line $l \in \Sigma$ and, assuming that $|\sec_1 l| \geqslant |\sec_1 l'|$, $l' \in \Sigma$, in the resulting configuration S, determine a lower bound $v_{\min} \leqslant |\sec_1 l|$ on the number of simple sections through l that is necessary for the inequality $|\mathrm{Fn}\, S| > \overline{M}$ in (A.1).

A.2.1 Step 1

We start with the lattice S_0 and stabilizer G_0:

$$S_0 := \mathcal{F}_h(\Gamma), \qquad G_0 := \mathrm{stab}\{l\} \subset \mathrm{Aut}\,\Gamma.$$

Then, sections $s_1, s_2, \ldots$ intersecting l are added to Γ one by one, and we consider consecutive overlattices and subgroups

$$S_i := (S_{i-1} + \mathbb{Z}s_i)/\ker \supset S_{i-1}, \qquad G_i := \mathrm{stab}\{s_1, \ldots, s_i\} \subset G_0.$$

At each step, we require that

$$S_i \text{ should be } \mathfrak{P}_\Gamma\text{-subgeometric (cf. Sect. } A.1) \text{ and } \mathrm{rk}\, S_i > \mathrm{rk}\, S_{i-1}. \qquad \text{(A.2)}$$

This process continues until no other section satisfying (A.2) can be added.

In all cases, the set $\sec l$ is discrete (cf. Sect. A.5(1)) and, hence, a section s is described by its coordinates $\bar{s} \subset \Gamma$ (see Convention 3.6), so that $l \in \bar{s}$. At each step, we try for s_i a single representative of each G_{i-1}-orbit. We also check partially the defining property $\mathfrak{P}$, leaving the ultimate validation to Algorithm 2.6, where the group G_i is used to reduce the number of candidates to be analyzed.

Remark A.1 As a technical tool reducing the computation and overcounting, we fix an integer $2 \leqslant p \leqslant 4$ (depending on the size $|G_0|$) and store, during the i-th step, the G_0-orbits of all i-tuples $\{s_1, \ldots, s_i\}$, $i \leqslant p$, of sections satisfying (A.2). These pools P_i are used two-fold: first, during the i-th step (if $i \leqslant p$), we disregard the orbits that have already been stored; second, when collecting the candidates s_{i+1} to extend a set $\{s_1, \ldots, s_i\}$, we select only those $s_{i+1} \in P_1$ for which each subset $s \subset \{s_i, \ldots, s_{i+1}\}$ of cardinality $n \leqslant \min(i, p)$ is in the pool P_n.

A.2.2 Step 2

We use Algorithm 2.6 to compute all $\mathfrak{P}_\Gamma$-geometric finite index extensions of all configurations S obtained *at all stages* of Step 1. Most configurations violating the hypothesis $|\mathrm{Fn}\, S| \leqslant \overline{M}$ are found at this point. Then, disregarding the configurations

of rank 20 (as admitting no further extensions) and those in which $|\sec l| < v_{\min}$ (as not meeting the goal), we replace each remaining geometric extension with the respective lattice $\mathcal{F}_h(\Gamma \cup \sec l)$. The sets $\sec l$ are sorted again, retaining a single representative of each G_0-orbit.

A.2.3 Step 3

The remaining configurations have rank 19 or 18. We pick another line $l' \in \Sigma$ and try to add up to two independent extra sections $e_i \in \sec l'$. (The choice of l' is explained and justified in Sect. A.4.) Thus, we start with

$$S_0 := \mathcal{F}_h(\Gamma \cup \{s_1, \dots, s_n\}), \qquad G_0 := \mathrm{stab}\{s_1, \dots, s_n\} \subset \mathrm{stab}(l, l') \times \mathbb{S}(\sec l)$$

and add one or two extra sections as in Step 1, obtaining consecutive lattices S_i and groups G_i. In addition to (A.2), we require that the set $\sec_1 l$ should remain constant (as otherwise S_i is among the configurations considered at Steps 1 and 2). Finally, an analogue of Step 2 is applied to all new lattices obtained.

Remark A.2 The extra sections e_i may intersect the original sections s_k; hence, an extra section e is determined by its coordinates $\bar{e} \subset \Gamma$ (cf. Convention 3.6) and a subset of $\{s_1, \dots, s_n\}$. The cardinality of the latter is usually bounded by the defining property $\mathfrak{P}$ (cf. Sect. A.5(2)).

A.3 Pencils with Disjoint Sections

If *all* sections of interest are known to be pairwise disjoint (*e.g.*, in the treatment of the locally elliptic configurations, see Sect. 4.4), the algorithm of Sect. A.2 can be modified. Namely, we start with a pencil Γ and a distinguished fiber $\Sigma = \{l_1, \dots, l_n\}$ and fix the *goals* $v_i \leqslant |\sec l_i|$, $i = 1, \dots, n$. Then, disjoint sections are added one by one as in Sect. A.2.1, except that we do *not* require that the rank of the configuration should increase at each step. As soon as a configuration S of the maximal rank 20 is obtained, it is analyzed as in Sect. A.2.2. Configurations of lower rank *that do not meet the goals* are discarded as small.

A.4 Triangular and Quadrangular Configurations

A posteriori, it turns out that it is only triangular (for $D = 3$) and quadrangular configurations that need the algorithm of Sect. A.2 to full extent. Below, we explain and justify the choice of the additional line(s) $l', l'' \in \Sigma$ for Step 3 (see Sect. A.2.3).

A.4.1 Triangular Configurations (See Sect. 8.3)

Let $\Sigma := (l_1, l_2, l_3)$ be the triangle used in the proof, so that $l = l_1$. Left to Step 3 are four configurations S_0 of rank 19 and three configurations of rank 18. In the former case, a single extra section is to be added and, by the obvious symmetry, we can use $l' = l_2$.

If rk $S_0 = 18$, then, in all three cases, $\sec l_2 = \sec l_3 = \varnothing$ in each geometric finite index extension $S_0' \supset S_0$. In two cases, $\sec l_2 \neq \varnothing$ and $\sec l_3 \neq \varnothing$ in each geometric corank 1 extension $S_1' \supset S_0$; hence, any geometric corank 2 extension is generated over S_0 and $\mathbb{Q}$ by two sections adjacent to the same line, which, by symmetry, can be chosen to be l_2. In the exceptional case, we have $\Gamma \cong 5\widetilde{\mathbf{A}}_2 + 3\mathbf{A}_1$ and

$$|\sec l_1| = 7, \quad |\sec l_2| = 1, \quad |\sec l_3| = 0;$$

hence, if a geometric corank 2 extension $S_2' \supset S_0$ is not generated by two sections in the same set, we have $|\sec \Gamma| \leqslant 9$ and $|\mathrm{Fn}\, S| \leqslant 27$. $\qquad\square$

A.4.2 Quadrangular Configurations (See Sects. 7.3 and 7.4)

Let $\Sigma := (l_1, l_2, l_3, l_4)$ be the quadrangle used in the proofs, with the numbering satisfying (7.1) except that, for Lemma 7.6, we lift the restriction $\mathrm{val}\, l_1 \geqslant \mathrm{val}\, l_2$. Each configuration S_0 to be considered at Step 3 has rk $S_0 \geqslant 18$, but the number of configurations is quite large. In order to avoid tedious case-by-case analysis, we merely add up to two extra sections $e' \in \sec l'$, $e'' \in \sec l''$ trying *all* pairs (l', l''); since $\mathrm{val}\, l_2 \geqslant \mathrm{val}\, l_4$ by (7.1), it suffices to consider

$$(l', l'') = (l_2, l_2), \quad (l_2, l_3), \quad (l_2, l_4), \quad (l_3, l_3).$$

(If rk $S_0 = 19$, i.e., only one extra section is to be added, we try $l' = l_2$ or l_3.) The maximal number of original sections $s_i \in \sec l_1$ that an extra section $s^* \in \sec l^*$ may intersect (see Remark A.2) is given by Lemma 7.4(3)–(5), (7), depending on the degree $h^2 = 2D$, line $l^* = l_2$ or l_3, and whether the resulting configuration is or is not allowed to have biquadrangles. If $l \neq l''$, we also need to take into account the intersection $e' \cdot e'' = 0$ or 1. These complications slow the computation down, but it still remains reasonably feasible. $\qquad\square$

A.5 Sets of Sections

Another part of the proofs is the classification of sets of sections in the presence of a fixed parabolic subgraph (affine Dynkin diagram) Σ and a hereditary defining property $\mathfrak{P}$.

Fix an ordering $\Sigma = \{l_1, \ldots, l_n\}$ of the lines constituting Σ. We say that Σ is *suitable for enumeration* if its sections in any $\mathfrak{P}$-geometric configuration have the following properties:

(1) each subgraph $\sec l_i$, $1 \leqslant i \leqslant n$, is discrete, and
(2) $(\sec l_i) * (\sec l_j) \leqslant 1$ for $1 \leqslant i \neq j \leqslant n$.

A.5.1 Steps 1_1, 1_2

Assuming these properties, we fix the *goals* $v_i = |\sec l_i|$ and start with the lattice

$$S_2 := S_{2,0} := \mathcal{F}_h(\Sigma \cup \sec l_1 \cup \sec l_2),$$

where $\sec l_i = \{s_{i1}, \ldots, s_{iv_i}\}$ consists of v_i pairwise disjoint sections, $i = 1, 2$, and $s_{1k} \cdot s_{2k} = 1$ for $1 \leqslant k \leqslant r$ (another parameter fixed in advance) and $s_{1p} \cdot s_{2q} = 0$ for all other pairs (p, q). Let $G_2 = \mathbb{S}_r \times \mathbb{S}_{v_1 - r} \times \mathbb{S}_{v_2 - r}$ be the group of symmetries of this configuration.

A.5.2 Step 1_m, $3 \leqslant m \leqslant n$

At this step, the previously constructed configuration $S_{m-1} = S_{m-1,0}$ is considered frozen, i.e., we change the defining property $\mathfrak{P}$ to

$$\mathfrak{P}_{m-1}: S \longmapsto \mathfrak{P}(S) \text{ and } |\sec l_k| = v_k \text{ for } k < m.$$

We add to S_{m-1}, one by one, *exactly* v_m pairwise disjoint sections $s_{mi} \in \sec l_m$ and consider consecutive overlattices $S_{m-1,i} := (S_{m-1,i-1} + \mathbb{Z}s_{m,i})/\ker$. Unlike (A.2), we do *not* require that the ranks should increase. Each section s_{mi} is determined by an $(m - 1)$-tuple of at most one-element sets $[s_{mi}]_k \subset \sec l_k$, $k < m$. Since the symmetry groups involved are typically much smaller than those in Sect. A.2, we use a more aggressive algorithm: at each step, we collect the candidates s_{mi} extending all previously constructed sets $\{s_{m1}, \ldots, s_{m,i-1}\}$, compute G_{m-1}-orbits of *all* i-tuples $\{s_{m1}, \ldots, s_{mi}\}$ obtained, and choose one representative of each orbit. We also use pools P_i, $i \leqslant 4$, as in Remark A.1.

Upon completion of this step, we obtain a collection of lattices $S_m := S_{m-1,v_m}$ and stabilizers G_m,

$$S_m := (S_{m-1} + \mathbb{Z} \sec l_m)/\ker, \qquad G_m := \operatorname{stab} \sec l_m \subset G_{m-1},$$

to be used at Step 1_{m+1}. (We retain only those configurations S_m which admit a $\mathfrak{P}$-geometric extension with *exactly* v_k sections intersecting l_k, $k \leqslant m$.)

If only sets of sections are to be classified (*e.g.*, in order to determine their maximal number), we stop at this point. Otherwise, if we are interested in all $\mathfrak{P}_n$-geometric extensions S of the lattices S_n obtained satisfying an inequality

$$|\operatorname{Fn} S| > \overline{M} \tag{A.3}$$

fixed in advance, we proceed similar to Sect. A.2.

A.5.3 Step 2

For each lattice S_n obtained at the *final* step 1_n, compute all $\mathfrak{P}_n$-geometric finite index extensions (by Algorithm 2.6); then, select those satisfying (A.3). After this step, we disregard all lattices of rank 20.

A.5.4 Step 3

For the remaining configurations $S_n := S_{n,0}$, we increase the rank by adding, one by one, $(20 - \operatorname{rk} S_n)$ disjoint *extra lines* (i.e., lines e_i disjoint from all $l_1, \ldots, l_n$) and

considering the configurations

$$S_{n,i} := (S_{n,i-1} + \mathbb{Z}e_i)/\ker$$

and all their $\mathfrak{P}_n$-geometric finite index extensions computed by Algorithm 2.6. At each step, we require that rk $S_{n,i} >$ rk $S_{n,i-1}$. An extra line e can be regarded as a subset $[e] \subset \sec \Sigma$ (cf. Convention 3.6; the cardinality of this subset and/or its intersections with $\sec l_i$ is usually bounded by geometric arguments), and we use an aggressive algorithm similar to Sect. A.5.2.

In the rest of this section, we justify the fact that, if rk $S_n = 18$, it suffices to add pairs of *disjoint* extra lines e_1, e_2, keeping the computation feasible. The other cases, where rk $S_n \leqslant 17$, are considered below when they appear.

We say that a $\mathfrak{P}_n$-geometric corank r extension $S' \supset S_n$ is *well generated* if it has a *good basis*, i.e., a collection of *pairwise disjoint* lines $e_1, \ldots, e_r \in \mathcal{P}_{S'}(\Sigma) \smallsetminus \Sigma$ generating S' over S_n and $\mathbb{Q}$. Any line appearing in a good basis is called *generating*. Given a good basis $\{e_1, \ldots, e_r\}$, we denote by $E_i := E(e_i)$ the connected component of the pencil $\mathcal{P}_{S'}(\Sigma)$ containing the vertex e_i, $i = 1, \ldots, r$. If $r = 1$, we consider also the union $E_{\mathrm{gen}}(S') = \bigcup_e E(e)$ over all generating lines e.

Let $\mathcal{T}(\Sigma)$ be the set of all types of fibers that can occur in the pencil $\mathcal{P}(\Sigma)$ (in any $\mathfrak{P}_n$-geometric configuration), see Corollaries 3.3 and 3.4.

Lemma A.3 *Given S_n as above and a $\mathfrak{P}_n$-geometric corank 1 extension $S' \supset S_n$, assume that, for any inclusion $E_{\mathrm{gen}}(S') \hookrightarrow \Sigma'$ to a graph $\Sigma' \in \mathcal{T}(\Sigma)$, we have*

$$|\Sigma'| - |E_{\mathrm{gen}}(S')| \leqslant \overline{M} - |\mathrm{Fn}\, S'|.$$

Then, any $\mathfrak{P}_n$-geometric corank 2 extension $S'' \supset S' \supset S_n$ satisfying (A.3) is well generated over S_n.

Proof An extension $S'' \supset S'$ is well generated over S_n if and only if there is a line $e_2 \in \mathcal{P}_{S''}(\Sigma) \smallsetminus \mathcal{P}_{S'}(\Sigma)$ disjoint from at least one generating line e_1 of S'. If this is *not* so, all new lines e_2 and all generating lines e_1 must be in the same fiber Σ' of $\mathcal{P}_{S''}(\Sigma)$, and we have $|\mathrm{Fn}\, S''| \leqslant |\mathrm{Fn}\, S'| + |\Sigma'| - |E_{\mathrm{gen}}(S')| \leqslant \overline{M}$. □

Corollary A.4 *Under the assumptions of Lemma A.3, if $E_{\mathrm{gen}}(S')$ does not admit an embedding into $\Sigma' \in \mathcal{T}(\Sigma)$, then any $\mathfrak{P}_n$-geometric extension $S'' \supset S' \supset S_n$ of corank 2 is well generated over S_n.* □

A.6 Quadrangular Configurations

In the proof of Lemma 7.5, for $D = 4$ we need to use Step 3 of the previous algorithm to list all geometric configurations S containing a pencil $\Gamma := \mathcal{P}(\Sigma)$, $\Sigma \cong \widetilde{\mathbf{A}}_3$, and such that $|\mathrm{Fn}\, S| > 30$ and $|\sec \Gamma| \geqslant 15$. In this case, the rank rk S_4 can be as low as 15, so that we may need to add up to five extra lines, and Lemma A.3 does not always apply.

Fix a configuration S_4 obtained at Step 1_4. Recall that we consider triangle free configurations and insist that the set of sections of the pencil does not increase; in

other words, we take for the defining property (depending on S_4 or, more precisely, on the fixed goals $v_i = |\sec l_i|$ in S_4, $1 \leqslant i \leqslant 4$)

$$\mathfrak{P}_4 \colon S \longmapsto S \text{ is triangle free and } \sec l_i = v_i \text{ in } S, 1 \leqslant i \leqslant 4.$$

In the notation introduced prior to Lemma A.3, we have $E_i \cong \widetilde{\mathbf{A}}_3$, $\mathbf{A}_2$, or $\mathbf{A}_1$ for any good basis $\{e_1, \ldots . e_n\}$, and the following two statements are proved by a simple analysis of the modifications of the components under $\mathfrak{P}_4$-geometric extensions. (The hypotheses of both statements are to be checked for *geometric* extensions, i.e., those obtained after applying Algorithm 2.6.)

Lemma A.5 *Assume that a $\mathfrak{P}_4$-geometric corank r extension $S' \supset S_4$ has a good basis $e_1, \ldots, e_r$, and let $\bar{r} := \#\{i \leqslant r | E_i \cong \mathbf{A}_1\}$. Then any $\mathfrak{P}_4$-geometric corank 1 extension $S'' \supset S'$ such that $|\mathrm{Fn}\, S''| > |\mathrm{Fn}\, S'| + 2\bar{r}$ is well generated over S_4.*

Proof In the worst case scenario, a component $E_i \cong \mathbf{A}_1$ extends to $E_i'' \cong \widetilde{\mathbf{A}}_3$, so that the line $e_i'' \in E''$ opposite to e_i is already in $\mathcal{P}_{S'}(\Sigma)$, and all new components are of this form. In this case, $|\mathrm{Fn}\, S''| \leqslant |\mathrm{Fn}\, S'| + 2\bar{r}$. □

Corollary A.6 *Assume that a $\mathfrak{P}_4$-geometric corank r extension $S' \supset S_4$ has a good basis such that $|E_i| \geqslant 2$, $1 \leqslant i \leqslant r$. Then any $\mathfrak{P}_4$-geometric corank 1 extension $S'' \supset S'$ is well generated over S_4.* ◁

End of the proof of Lemma 7.5 We start from a configuration S_4, rk $S_4 \geqslant 15$, and add extra lines one by one, verifying, at each step, that all subsequent corank 1 extensions are still well generated.

If rk $S_4 = 19$, there is nothing to prove: all extensions are well generated.

If rk $S_4 = 18$, then every $\mathfrak{P}_4$-geometric corank 1 extension $S' \supset S_4$ either satisfies the hypotheses of Corollary A.4 or A.6 or has $|\mathrm{Fn}\, S'| \leqslant 28$, and then we can use Lemma A.5; thus, any $\mathfrak{P}_4$-geometric corank 2 extension $S'' \supset S_4$ that satisfies the inequality $|\mathrm{Fn}\, S''| > 30$ is well generated.

If rk $S_4 = 15$ or 16, then, step-by-step, one shows that, for *each* $\mathfrak{P}_4$-geometric extension $S'^{,'} \supset S_4$, one has $|E| \geqslant 2$ for each connected component $E \subset \mathcal{P}_{S'^{,'}}(\Sigma)$; then, by Corollary A.6, all extensions at the next step are also well generated.

Finally, let rk $S_4 = 17$ and pick a $\mathfrak{P}_4$-geometric corank 1 extension $S' \supset S_4$. The computation shows that, if S' does *not* satisfy the hypotheses of Lemma A.3 or Corollary A.6, then $E_{\mathrm{gen}}(S') \cong \mathbf{A}_1$ or $2\mathbf{A}_1$ and $|\mathrm{Fn}\, S'| \leqslant 26$. By Lemma A.5, any extension $S'^{,'} \supset S'$ such that $|\mathrm{Fn}\, S'^{,'}| > 30$ contains a well-generated $\mathfrak{P}_4$-geometric corank 2 extension $S'' \supset S' \supset S$ (i.e., there is a line $l \in \mathcal{P}_{S'^{,'}}(\Sigma) \smallsetminus \mathcal{P}_{S'}(\Sigma)$ disjoint from at least one good generator of S').

Thus, let $S'' \supset S_4$ be a well-generated $\mathfrak{P}_4$-geometric corank 2 extension. In all but a few cases, Lemma A.5 implies that any further extension $S''' \supset S''$ is well generated whenever $|\mathrm{Fn}\, S'''| > 30$. In each exceptional case, the graph $G \subset \Gamma_{S''}(\Sigma)$ of generating lines splits, $G = G_1 \cup G_2$, $G_1 \cap G_2 = \varnothing$, so that

$$G_1 = \{e_1', e_1''\} \cong 2\mathbf{A}_1, \quad G_2 \cong 2\mathbf{A}_1, \ \widetilde{\mathbf{A}}_3 + \mathbf{A}_2, \ \text{or} \ 2\widetilde{\mathbf{A}}_3$$

and good are the bases of the form $\{e_1, e_2\}$, where $e_1 \in G_1$, $e_2 \in G_2$. (If $G_2 \cong 2\mathbf{A}_1$, then we also have $|\text{Fn } S''| = 28$.) A simple argument shows that any offending extension $S''' \supset S''$ is generated by an extra line e_2 such that $e_2 \cdot e_1' = e_2 \cdot e_1'' = 1$, so that e_1', e_2, e_1'' are part of a quadrangle. This observation limits the number of choices for e_2; we do list all such extensions (with the intersections of the extra lines prescribed) and find out that they are never $\mathfrak{P}_4$-geometric. □

A.7 Pentagonal Configurations

In the proof of Lemma 6.4, where the fiber $\Sigma \cong \widetilde{\mathbf{A}}_4$ is a pentagon, we encounter configurations S_5 of rank $\text{rk } S_5 \geqslant 17$. The following statement is a refinement of Lemma A.3 using the known structure of Σ. We assume that the defining property is $\mathfrak{P}\colon S \mapsto (\text{girth}(\text{Fn } S) = 5)$ and consider a subgeometric configuration S_5 obtained after Step 1_5 of the algorithm of Sect. A.5. The other notations and terminology are introduced prior to Lemma A.3.

Lemma A.7 *Let S_5 be as above and $S' \supset S_5$ a $\mathfrak{P}_5$-geometric corank 1 extension. Define $\delta(S')$ as follows*:

- $\delta(S') = 2$ *if $E_{\text{gen}} \cong \mathbf{A}_1$*;
- $\delta(S') = 1$ *if $E_{\text{gen}} \cong 2\mathbf{A}_1$ or $E_{\text{gen}} \cong \mathbf{A}_2$ with only one line generating*;
- $\delta(S') = 0$ *in all other cases.*

Then, any $\mathfrak{P}_5$-geometric corank 2 extension $S'' \supset S' \supset S_5$ satisfying the inequality $|\text{Fn } S''| > |\text{Fn } S'| + \delta(S')$ is well generated over S_5. ◁

***Proof of Lemma 6.4**(1)–(3)* For the computation, we need to show that Step 3 of the algorithm applies, i.e., that any sufficiently large $\mathfrak{P}_5$-geometric extension of any configuration S_5 is well generated. If $\text{rk } S_5 = 18$, we apply Lemma A.7, checking the hypotheses case by case. If $D = 5$, there are four configurations S_5 of rank 17, and Lemma A.7 implies that any sufficiently large extension $S'' \supset S_5$ of rank 19 is well generated. A further extension $S''' \supset S''$ is well generated whenever

$$|\text{Fn } S'''| > |\text{Fn } S''| + 4, \tag{A.4}$$

the worst case scenario being that of S'' having but two disjoint generating lines. The computation shows that $|\text{Fn } S''| \leqslant 25$ and, if $|\text{Fn } S''| = 25$, the pencil $\mathcal{P}_{S''}(\Sigma)$ has a type $\mathbf{A}_2$ fiber with both lines generating, reducing (A.4) to $|\text{Fn } S'''| > 27$. It follows that any $\mathfrak{P}_5$-geometric extension $S \supset S_5$ with $|\text{Fn } S| > 28$ is well generated, and Step 3 of the algorithm results in a unique such extension, *viz.* Φ_{30}'''. □

A.8 Astral Configurations

Let $\Sigma \cong \widetilde{\mathbf{D}}_4$ and $\Gamma := \mathcal{P}(\Sigma)$. The *central line* of a type $\widetilde{\mathbf{D}}_4$ fiber $\Sigma' \subset \Gamma$ is the only line $l \in \Sigma'$ of relative valency 4. The following simple observations follow from the assumption that $\text{girth}(\Gamma) \geqslant 6$:

- all sections of Γ are simple or double (i.e., bisections);

- a section $s \in \sec \Gamma$ is double if and only if it intersects the central line of any (equivalently, each) type $\widetilde{\mathbf{D}}_4$ fiber;
- if Γ has a bisection, then it has no fibers of type $\widetilde{\mathbf{A}}_5$;
- if Γ has two bisections, then it has no parabolic fibers other than Σ.

Proof of Lemma 5.2 We proceed as in Sect. A.5.4, mainly using Lemma A.3. Below, we consider extensions of a configuration S_5 spanned by h and $\Sigma \cup \sec \Gamma$.

Each of the three sets $\sec \Gamma$ of cardinality 12 is covered by Lemma A.3: one has rk $S_5 = 18$ and any $\mathfrak{P}_5$-geometric extension $S'' \supset S_5$ such that $|\mathrm{Fn}\, S''| > 22$ is well generated. Using Sect. A.5.4, we immediately obtain Statement (1).

There are several sets of cardinality 11, but we only consider the three that are geometric for $D = 5$. Two of these sets have more than one bisection; hence, Γ has no parabolic fibers other than Σ. We have rk $S_5 = 16$ and the computation shows that each well-generated geometric extension $S'' \supset S' \supset S_5$ of corank $c = 2$ or 1 has exactly c extra lines. Since all fibers of Γ other than Σ are of types $\mathbf{A}_p$, $p \leqslant 4$, each generating line can intersect at most two other lines $l \in \Gamma$. Hence, any further extension $S \supset S_5$ such that $|\mathrm{Fn}\, S| > |\mathrm{Fn}\, S_5| + 2 + 4 = 22$ must be well generated; using Sect. A.5.4, we show that such extensions do not exist.

In the third case, there is a single bisection s and rk $S_5 = 17$. Well-generated geometric extensions $S'' \supset S' \supset S_5$ of corank $c = 2$ or 1 have up to $c + 1$ extra lines, which are all generating, pairwise disjoint, and *disjoint from s*, implying that an extra line e cannot be the central line of a type $\widetilde{\mathbf{D}}_4$ fiber of Γ; hence, as in the previous case, e intersects at most two other lines $l \in \Gamma$. As above, we conclude that any geometric extension $S \supset S_5$ such that $|\mathrm{Fn}\, S| > 21$ must be well generated; the classification of such extensions using Sect. A.5.4 proves that $|\mathrm{Fn}\, S| \leqslant 21$. $\square$

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