Ann. Henri Poincaré 19 (2018), 1787–1816

1424-0637/18/061787-30
published online March 17, 2018
https://doi.org/10.1007/s00023-018-0666-6

Annales Henri Poincaré

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Symmetries Versus Conservation Laws in Dynamical Quantum Systems: A Unifying Approach Through Propagation of Fixed Points

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Abstract. We unify recent Noether-type theorems on the equivalence of symmetries with conservation laws for dynamical systems of Markov processes, of quantum operations, and of quantum stochastic maps, by means of some abstract results on propagation of fixed points for completely positive maps on C^*-algebras. We extend most of the existing results with characterisations in terms of dual infinitesimal generators of the corresponding strongly continuous one-parameter semigroups. By means of an ergodic theorem for dynamical systems of completely positive maps on von Neumann algebras, we show the consistency of the condition on the standard deviation for dynamical systems of quantum operations, and hence of quantum stochastic maps as well, in case the underlying Hilbert space is infinite dimensional.

1. Introduction

In view of the celebrated theorem of Noether [30] on the equivalence of symmetries and conservation laws for physical systems, Baez and Fong [7] considered similar questions within the framework of "stochastic mechanics", in the sense of [6], for the dynamics of Markov processes. Letting $\{U(t)\}_{t\geq 0}$ be a (classical) dynamical stochastic system (this is called a Markov semigroup in [7]), they show that the operator of multiplication with an observable O commutes with U_t for all $t \geq 0$, an analogue for a symmetry, if and only if both its expected value $\langle O, U_t f\rangle$ and the expected value of its square $\langle O^2, U_t f\rangle$ are constant in time for every state f (probability distribution), an analogue for a conservation law. Considering the variance $\langle O^2, f\rangle - \langle O, f\rangle^2$, for f an arbitrary state, the latter condition is equivalent with both its expected value and its variance

Birkhäuser

(or standard deviation) are constant in time for every state. The approach uses an older idea of realising Markov processes in terms of closed (Hamiltonian) semigroup and is classical probability theory by its nature. The appearance of the variance makes a difference when compared to the classical Noether's theorem. Some important questions are left unanswered, among which, how is this reflected in terms of the infinitesimal generator of the semigroup.

On the other hand, questions related to Noether-type theorems have been recently considered in the context of open quantum systems in connection to adiabatic response of quantum systems undergoing unitary evolution to open quantum systems governed by Lindblad evolutions, see (1.1) from below, as seen at Avron et al. [5]. However, we are particularly interested by the setting of irreversible open quantum dynamical systems as considered by Gough et al. in [21] which explicitly refers to a point of view analogue to that considered in [7]. More precisely, let $\mathcal{T} = \{\mathcal{T}_t\}_{t\geq 0}$ denote a dynamical system in the Schrödinger picture, that is, a norm continuous semigroup of completely positive (see the definition in Sect. 2.1) trace-preserving linear maps on the trace-class $\mathcal{B}_1(\mathcal{H})$ for some fixed Hilbert space $\mathcal{H}$, for which the infinitesimal generator M takes the form, cf. [20,29],

$$M(S) = \sum_k \left(L_k S L_k^* - \frac{1}{2} S L_k^* L_k - \frac{1}{2} L_k^* L_k S \right) + \mathrm{i}[S, H], \quad S \in \mathcal{B}_1(\mathcal{H}), \quad (1.1)$$

for a collection of operators $L_k \in \mathcal{B}(\mathcal{H})$, $k = 1, 2, \ldots$, and a selfadjoint operator $H \in \mathcal{B}(\mathcal{H})$. The *constants* of $\mathcal{T}$ are the operators $A \in \mathcal{B}(\mathcal{H})$ such that $\operatorname{tr}((\mathcal{T}_t\rho)A) = \operatorname{tr}(\rho A)$ for all density operators $\rho \in \mathcal{D}(\mathcal{H})$ and all $t \geq 0$. Transferring to the Heisenberg picture, one considers the dual semigroup $\{\mathcal{J}_t\}_{t\geq 0}$ acting in $\mathcal{B}(\mathcal{H})$ whose set of fixed points, that is, all $A \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{J}_t(A) = A$ for all $t \geq 0$, coincides with the set of constants of $\mathcal{T}$. The main result in [21] says that, under the technical assumption of existence of a stationary strictly positive density operator, the set of constants of the quantum dynamical system $\{\mathcal{T}_t\}_{t\geq 0}$, which coincides with the set of fixed points of $\{\mathcal{J}_t\}_{t\geq 0}$, is a von Neumann algebra and it coincides with the commutant $\{H, L_k, L_k^* \mid k = 1, 2, \ldots\}'$. In their formulation, an analogue of the second condition on the square of the observable as in [7] does not show up and one aim of our article is to show that this happens because it is obscured by the technical assumption of existence of a stationary strictly positive density operator. In addition, the question on how are these results related to the results in [7] on dynamical stochastic systems is left unanswered and it is another aim of our article to clarify this question.

Within the same circle of ideas as in [7] and [21], Bartoszek and Bartoszek [8] recently considered a noncommutative version of dynamical stochastic system, more precisely, a strongly continuous semigroup $\{S_t\}_{t\geq 0}$ of stochastic maps with respect to some Hilbert space $\mathcal{H}$, that is, trace-preserving positive linear maps on the trace-class $\mathcal{B}_1(\mathcal{H})$, and a one-element measurement operator $M_{A^{1/2}}$, for some positive operator $A \in \mathcal{B}(\mathcal{H})$, where $M_{A^{1/2}}(T) = A^{1/2} T A^{1/2}$. In this setting, they obtain several equivalent characterisations to the compatibility (commutation) of the dynamical stochastic system $\{S_t\}_{t\geq 0}$ with the quantum measurement $M_{A^{1/2}}$: for example, one of these equivalent characterisations refers to A and A^2 being fixed by the dual semigroup $\{S_t^\sharp\}_{t\geq 0}$ and a second one refers to the commutation of the infinitesimal generator $\mathfrak{s}$ of $\{S_t\}_{t\geq 0}$ with $M_{A^{1/2}}$. The approach used in [8] combines the probability theory methods as in [7] with operator theoretical methods. There are some important questions left unanswered in [8]: for example, how are these related to the results in [7] and [21] and to what extent is the additional condition that A^2 be fixed by the dual semigroup $\{S_t^\sharp\}_{t\geq 0}$ really necessary? It is another aim of our article to provide an answer to these questions.

In this article, we show that all the results in [7,21], and [8] can be unified by means of an abstract approach within dilation theory in C^*-algebras for completely positive maps in the sense of Stinespring [34], more precisely, through the concepts of bimodule domains and multiplication domains of Choi [11]. For example, we show that the abstract results on propagation of fixed points for completely positive maps on C^*-algebras that we get in Theorem 2.2 and Corollary 2.3 short cut completely the probabilistic tools in the proofs of the main results in [7] and [8]. Also, although the results in [8] apparently refer to a more general case of positive maps that may not be completely positive, our Corollary 2.3 shows that it is exactly the complete positivity that lies behind them. In addition, in the case studied in [21], we reveal what happens if the technical assumption of existence of a stationary strictly positive density operator is removed. More precisely, we first obtain an ergodic theorem for dynamical systems of completely positive maps on von Neumann algebras, see Theorem 2.5. Then, using this theorem in combination with some techniques of injectivity of operator systems and the von Neumann algebra generated by the free group on two generators, we show the consistency of the condition on the standard deviation for dynamical systems of quantum operations, and hence for dynamical systems of quantum stochastic maps as well, in case the underlying Hilbert space is infinite dimensional. From a broader perspective, we put all these problems in the framework of analysis of quantum operations as in [2] and in closely related mathematical problems on irreversible dynamical quantum systems, e.g. as in Albeverio and Høegh-Krohn [1], Davies [14], Evans [16], Frigerio and Verri [19], Fagnola and Rebolledo [17], and Størmer [35], to quote a few. Finally, we extend most of the existing results with characterisations in terms of duals of strongly continuous one-parameter semigroups and their w^*-infinitesimal generators by a general result as in Theorem 2.4.

A few words about terminology. We have used the same names "stochastic" and, respectively, "Markov" for both the commutative (classical) case as in Sect. 3 and the noncommutative (quantum) case as in Sect. 6, hoping that there will be no danger of confusion. This way, we left the notions of quantum stochastic and, respectively, quantum Markov referring to the case of quantum operations in the Schrödinger picture and, respectively, in the Heisenberg picture, following the terminology already established in quantum physics, see [18] and [21].

We thank Marius Dădârlat for drawing our attention to the proof of Choi's Theorem in [10] obtainable solely from the Stinespring's Dilation Theorem and for many other useful discussions on these topics, to Radu Purice for clarifying some aspects from [21], and to Carlo Beenakker for indicating [13] and [25] as sources on the significance of the transpose map in quantum information theory. Last but not least, we thank the referees for a careful and critical reading of the manuscript and for providing corrections and recommendations that improved the presentation of this article.

2. Preliminary Results

2.1. Propagation of Fixed Points in C^*-Algebras

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras with unit. A linear map $\Phi\colon \mathcal{A} \to \mathcal{B}$ is *positive* if $\Phi(a) \geq 0$ for all $a \in \mathcal{A}^+$, where $\mathcal{A}^+ = \{x^*x \mid x \in \mathcal{A}\}$ denotes the cone of positive elements in $\mathcal{A}$. Any positive map is selfadjoint, in the sense that $\Phi(a^*) = \Phi(a)^*$ for all $a \in \mathcal{A}$, and bounded, more precisely, according to the Russo–Dye Theorem, $\|\Phi\| = \|\Phi(e)\|$, where by e we denote the unit of $\mathcal{A}$.

Given an arbitrary natural number n, we consider the C^*-algebra $M_n(\mathcal{A})$ of all $n \times n$ matrices with entries in $\mathcal{A}$, organised as a C^*-algebra in a canonical way, e.g. by identifying it with the C^*-algebra $\mathcal{A} \otimes M_n$. This gives rise to the nth-order *amplification map* $\Phi_{(n)}\colon M_n(\mathcal{A}) \to M_n(\mathcal{B})$ defined by

$$\Phi_{(n)}(A) = [\Phi(a_{i,j})]_{i,j=1}^n, \quad A = [a_{i,j}]_{i,j=1}^n \in M_n(\mathcal{A}). \tag{2.1}$$

Φ is called *n-positive* if $\Phi_{(n)}$ is positive. Φ is called *completely positive* if it is n-positive for all $n \in \mathbb{N}$.

Given $\mathcal{A}$ a C^*-algebra with unit, a closed linear subspace $\mathcal{S}$ of $\mathcal{A}$ is called an *operator system* if it is stable under the adjoint operation $a \mapsto a^*$ and contains the unit of $\mathcal{A}$. Note that any operator system is linearly generated by the cone of all its positive elements. Also, for any linear map $\Psi\colon \mathcal{S} \to \mathcal{B}$, for $\mathcal{B}$ an arbitrary C^*-algebra, the definitions of positive map, n-positive map, and completely positive map, as defined before, make perfectly sense. More generally, these definitions make sense if $\mathcal{S}$ is assumed to be stable under the adjoint operation only.

For an arbitrary linear map $\Phi\colon \mathcal{A} \to \mathcal{B}$, the set

$$\mathcal{M}_\Phi = \{a \in \mathcal{A} \mid \Phi(a^*a) = \Phi(a)^*\Phi(a) \text{ and } \Phi(aa^*) = \Phi(a)\Phi(a^*)\} \tag{2.2}$$

is called the *multiplicative domain* of Φ. If Φ is unital, then $\mathcal{M}_\Phi$ contains the unit of $\mathcal{A}$.

We start with the following theorem, due to Choi [11]; it is worth observing that assertion (2) is actually a property of propagation of multiplicativity which motivates the name of $\mathcal{M}_\Phi$. The Schwarz Inequality was first obtained in a special case by Kadison in [26], that's why sometimes it is called the Kadison–Schwarz Inequality. A modern and short proof is available in [10], which also points out its dilation theory substance, as a consequence of the Stinespring's Dilation Theorem [34].

Theorem 2.1. *Let $\Phi\colon \mathcal{A} \to \mathcal{B}$ be a contractive completely positive map. Then:*

(1) (The Schwarz Inequality) $\Phi(a)^*\Phi(a) \le \Phi(a^*a)$ *for all* $a \in \mathcal{A}$.

(2) (The Multiplicativity Property) *Let* $a \in \mathcal{A}$. *Then:*
   (i) $\Phi(a^*a) = \Phi(a)^*\Phi(a)$ *if and only if* $\Phi(ba) = \Phi(b)\Phi(a)$ *for all* $b \in \mathcal{A}$.
   (ii) $\Phi(aa^*) = \Phi(a)\Phi(a)^*$ *if and only if* $\Phi(ab) = \Phi(a)\Phi(b)$ *for all* $b \in \mathcal{A}$. *Consequently,*

$$\mathcal{M}_\Phi = \{a \in \mathcal{A} \mid \Phi(ab) = \Phi(a)\Phi(b),\ \Phi(ba) = \Phi(b)\Phi(a),\ \textit{for all } b \in \mathcal{A}\}. \quad (2.3)$$

(3) *The multiplicative domain* $\mathcal{M}_\Phi$ *defined at* (2.2) *is a* C^**-subalgebra of* $\mathcal{A}$ *and it coincides with the largest* C^**-subalgebra* $\mathcal{C}$ *of* $\mathcal{A}$ *such that* $\Phi|_\mathcal{C}\colon \mathcal{C} \to \mathcal{B}$ *is a* $*$*-homomorphism.*

Actually, the Schwarz Inequality is true under the more general condition that Φ is 2-positive, while the Multiplicativity Property holds for 4-positive maps: see also [31].

We are interested in fixed points of positive maps between C^*-algebras. Given a C^*-algebra $\mathcal{A}$ with unit e, let $\Phi\colon \mathcal{A} \to \mathcal{A}$ be a linear map that is unital and positive. We consider the set of the *fixed points* of Φ

$$\mathcal{A}^\Phi = \{a \in \mathcal{A} \mid \Phi(a) = a\}, \quad (2.4)$$

of all fixed points of Φ and it is easy to see that $\mathcal{A}^\Phi$ is an operator system. Another set of interest is the *bimodule domain*

$$\mathcal{I}(\Phi) = \{a \in \mathcal{A} \mid \Phi(ab) = a\Phi(b),\ \Phi(ba) = \Phi(b)a,\ \text{for all } b \in \mathcal{A}\}, \quad (2.5)$$

which is a C^*-subalgebra of $\mathcal{A}$ containing the unit e. Clearly,

$$\mathcal{I}(\Phi) \subseteq \mathcal{A}^\Phi \cap \mathcal{M}_\Phi. \quad (2.6)$$

On the other hand, if Φ is completely positive and contractive, by Theorem 2.1.(2) we have

$$\mathcal{A}^\Phi \cap \mathcal{M}_\Phi = \mathcal{I}_\Phi. \quad (2.7)$$

As shown in [2], even for the very particular case of a Lüders operation Φ on $\mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ denotes the von Neumann algebra of all bounded operators on a Hilbert space $\mathcal{H}$, in general we cannot expect that the set of fixed points of Φ coincides with its bimodule domain. In the following, we consider a related question: given a unital positive map $\Phi\colon \mathcal{A} \to \mathcal{A}$, we want to see whether the quality of an element $a \in \mathcal{A}$ of being fixed by Φ propagates to the whole C^*-algebra generated by e and a, denoted by $C^*(e, a)$. This question is related to the concept of multiplicative domain, that is, imposing $a^*a, aa^* \in \mathcal{A}^\Phi$ and a certain "locally complete positivity" condition on Φ as well.

Theorem 2.2. *Let* $\mathcal{A}$ *be a* C^**-algebra with unit* e, *let* $\Phi\colon \mathcal{A} \to \mathcal{A}$ *be a unital linear map, and let* $a \in \mathcal{A}$ *and a* C^**-subalgebra* $\mathcal{C}$ *of* $\mathcal{A}$ *be such that* $a, e \in \mathcal{C}$ *and* $\Phi|_\mathcal{C}\colon \mathcal{C} \to \mathcal{A}$ *is completely positive. The following assertions are equivalent:*

(i) $a, a^*a, aa^* \in \mathcal{A}^\Phi$, *that is,* $\Phi(a) = a$, $\Phi(a^*a) = a^*a$, *and* $\Phi(aa^*) = aa^*$.

(ii) $a \in \mathcal{A}^\Phi \cap \mathcal{M}_\Phi$, *that is,* $\Phi(a) = a$, $\Phi(a^*a) = \Phi(a)^*\Phi(a)$, *and* $\Phi(aa^*) = \Phi(a)\Phi(a)^*$.

(iii) $\Phi|_{\mathcal{C}}$ *has the Bimodule Property, that is,* $\Phi(ba) = \Phi(b)a$ *and* $\Phi(ab) = a\Phi(b)$ *for all* $b \in \mathcal{C}$.
(iv) $C^*(e,a) \subseteq \mathcal{A}^\Phi$*, that is,* $\Phi(b) = b$ *for all* $b \in C^*(e,a)$.

Proof. Let us first note that, since $a, e \in \mathcal{C}$ it follows that $C^*(a,e) \subseteq \mathcal{C}$.

(i)⇒(ii). By assumptions, it follows

$$\Phi(a^*a) = a^*a = \Phi(a)^*\Phi(a), \quad \Phi(aa^*) = aa^* = \Phi(a)\Phi(a)^*,$$

hence, $a \in \mathcal{A}^\Phi \cap \mathcal{M}_\Phi$.

(ii)⇒(iii). Since $\Phi|\mathcal{C}$ is unital and completely positive, by Russo-Dye Theorem it is (completely) contractive hence, by Theorem 2.1.(2), $\Phi|_{\mathcal{C}}$ has the Bimodule Property and consequently $\Phi(ba) = \Phi(b)\Phi(a) = \Phi(b)a$ and $\Phi(ab) = \Phi(a)\Phi(b) = a\Phi(b)$ for all $b \in \mathcal{C}$.

(iii)⇒(iv). By assumption and using a straightforward induction argument, it follows that, for any $n \in \mathbb{N}_0$, we have

$$\Phi(xa^n) = \Phi(x)a^n, \quad \Phi(a^nx) = a^n\Phi(x), \quad x \in C^*(e,a), \tag{2.8}$$

and, since Φ is selfadjoint, we have $\Phi(a^*) = \Phi(a)^* = a^*$, hence

$$\Phi(xa^{*n}) = \Phi(x)a^{*n}, \quad \Phi(a^{*n}x) = a^{*n}\Phi(x), \quad x \in C^*(e,a). \tag{2.9}$$

From (2.8) and (2.9), by a straightforward induction argument, it follows that for any monomial p in two noncommutive variables X and Y

$$p(X,Y) = X^{i_1}Y^{j_1}\cdots X^{i_m}Y^{j_m}, \quad i_1,\ldots,j_m \in \mathbb{N}_0,\ j_1,\ldots,j_m \in \mathbb{N}_0,\ m \in \mathbb{N},$$

it follows that

$$\Phi(p(a,a^*)) = p(a,a^*), \tag{2.10}$$

where $p(a,a^*) \in \mathcal{A}$ is the element obtained by formally replacing X with a and Y with a^*. Then, by linearity, it follows that (2.10) is true for any complex polynomials p in two noncommutative variables X and Y hence, since the collection of all elements of form $p(a,a^*)$ is dense in $C^*(e,a)$ and $\Phi|_{C^*(e,a)}$ is continuous, assertion (iv) follows.

(iv)⇒(i). This implication is clear. □

As an application of Theorem 2.2, we record the special case of a normal element a, that is, $a^*a = aa^*$, when the condition of "locally complete positivity" follows from the condition of positivity.

Corollary 2.3. *Let* $\Phi\colon \mathcal{A} \to \mathcal{A}$ *be a linear map which is positive and unital, and let* $a \in \mathcal{A}$ *be a normal element. The following assertions are equivalent:*

(i) $\Phi(a) = a$ *and* $\Phi(a^*a) = a^*a$.
(ii) $\Phi(b) = b$ *for all* $b \in C^*(e,a)$.

Proof. Only the implication (i)⇒(ii) requires a proof. Since a is normal it follows that $C^*(e,a)$ is a commutative C^*-algebra hence $\Phi|_{C^*(e,a)}\colon C^*(e,a) \to \mathcal{A}$ is completely positive, see [34], and we can apply Theorem 2.2. □

2.2. Fixed Points of w^*-Continuous One-Parameter Semigroups

Let X be a Banach space. We consider a strongly continuous one-parameter semigroup $\{\Psi_t\}_{t\geq 0}$ of linear bounded operators on X, that is,

(i) $\Psi_t\colon X \to X$ is a bounded linear operator for all $t \geq 0$.
(ii) $\Psi_s \Psi_t = \Psi_{s+t}$, for all $s, t \geq 0$.
(iii) $\Psi_0 = I$.
(iv) $\mathbb{R}_+ \ni t \mapsto \Psi_t(x) \in X$ is continuous for each $x \in X$.

Under these assumptions, from the general theory of one-parameter semigroups, e.g. see Hille and Phillips [24], Dunford and Schwartz [15], the *infinitesimal generator* ψ exists as a densely defined closed operator on X, with

$$\psi(x) = \lim_{t\to 0+} \frac{\Psi_t(x) - x}{t} = \frac{\mathrm{d}}{\mathrm{d}t}\Psi_t(x)|_{t=0}, \quad x \in \mathrm{Dom}(\psi), \tag{2.11}$$

and

$$\mathrm{Dom}(\psi) = \left\{ x \in X \mid \lim_{t\to 0+} \frac{\Psi_t(x) - x}{t} \text{ exists in } X \right\}. \tag{2.12}$$

In addition, e.g. see Corollary VIII.1.5 in [15], the limit

$$\omega = \lim_{t\to\infty} \log \|\Psi_t\|/t = \inf_{t>0} \log \|\Psi_t\|/t \tag{2.13}$$

exists with the *growth bound* $\omega < \infty$ and, e.g. see Theorem VIII.1.11 in [15], for any complex number λ with $\mathrm{Re}\,\lambda > \omega$, the operator $\lambda I - \psi$ has a bounded inverse. Also, by the proof of the Hille–Yosida–Phillips Theorem, e.g. see Theorem VIII.1.13 in [15], we have

$$\Psi_t(x) = \lim_{\lambda\to\infty} \mathrm{e}^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda^2 t)^n (\lambda I - \psi)^{-n}(x)}{n!}, \quad x \in \mathrm{Dom}(\psi),\ t \geq 0. \tag{2.14}$$

Throughout this article, $X^\sharp$ denotes the topological dual space of X. For every strongly continuous one-parameter semigroup $\{\Psi_t\}_{t\geq 0}$ of bounded linear operators on X, the dual one-parameter semigroup $\{\Psi_t^\sharp\}_{t\geq 0}$ of bounded linear operators on $X^\sharp$ exists, that is,

$$\langle \Psi_t(x), f\rangle = \langle x, \Psi_t^\sharp(f)\rangle, \quad x \in X,\ f \in X^\sharp,\ t \geq 0, \tag{2.15}$$

with the following properties

(i) $\Psi_t^\sharp\colon X^\sharp \to X^\sharp$ is a linear bounded and w^*-continuous operator for all $t \geq 0$.
(ii) $\Psi_t^\sharp \Psi_s^\sharp = \Psi_{s+t}^\sharp$, for all $s, t \geq 0$.
(iii) $\Psi_0^\sharp = I$.
(iv) $\mathbb{R}_+ \ni t \mapsto \Psi_t^\sharp(f) \in X^\sharp$ is w^*-continuous for each $f \in X^\sharp$.

Then, e.g. see [32], $\{\Psi_t^\sharp\}_{t\geq 0}$ is a w^*-continuous semigroup of operators on $X^\sharp$ and hence, the *w^*-infinitesimal generator* $\psi^\sharp$ exists as a w^*-closed operator on $X^\sharp$, hence a closed operator on $X^\sharp$, with

$$\psi^\sharp(f) = w^* - \lim_{t\to 0+} \frac{\Psi_t^\sharp(f) - f}{t} = w^* - \frac{\mathrm{d}}{\mathrm{d}t}\Psi_t^\sharp(f)|_{t=0}, \tag{2.16}$$

and

$$\operatorname{Dom}(\psi^\sharp) = \left\{ f \in X^\sharp \mid w^* - \lim_{t \to 0+} \frac{\Psi_t^\sharp(f) - f}{t} \text{ exists in } X^\sharp \right\}. \tag{2.17}$$

The notation we use for $\psi^\sharp$ looks like an abuse but actually it is not: by the Phillips's Theorem in [32],

$$\operatorname{Dom}(\psi^\sharp) = \{ f \in X^\sharp \mid X \ni f \mapsto \langle x, \psi(f) \rangle \text{ is continuous } \}, \tag{2.18}$$

and

$$\langle \psi(x), f \rangle = \langle x, \psi^\sharp(f) \rangle, \quad x \in \operatorname{Dom}(\psi),\ f \in \operatorname{Dom}(\psi^\sharp), \tag{2.19}$$

hence, the w^*-infinitesimal generator $\psi^\sharp$ of the dual w^*-continuous semigroup $\{\Psi_t^\sharp\}_{t \geq 0}$ on $X^\sharp$ is indeed the dual operator of the infinitesimal generator ψ of the strongly continuous semigroup $\{\Psi_t\}_{t \geq 0}$ on X and, consequently, the notation for $\psi^\sharp$ is fully justified.

In addition, one of the major differences between the two infinitesimal generators ψ and $\psi^\sharp$ is that $\operatorname{Dom}(\psi^\sharp)$ may not be dense in $X^\sharp$, although it is always w^*-dense, while $\operatorname{Dom}(\psi)$ is always dense in X.

The following theorem shows that joint fixed points of the dual one-parameter semigroup are exactly the elements of the null space of the dual infinitesimal generator. We think that this result might be known but we could not find any reference for it.

Theorem 2.4. *Let $\{\Psi_t\}_{t \geq 0}$ be a strongly continuous semigroup of operators on a Banach space X, let $\{\Psi_t^\sharp\}_{t \geq 0}$ be the associated dual w^*-continuous semigroup of operators on $X^\sharp$, and ψ and, respectively, $\psi^\sharp$, their infinitesimal generators. Considering $f \in X^\sharp$, the following assertions are equivalent:*

(i) $\Psi_t^\sharp(f) = f$ *for all real* $t \geq 0$.
(ii) $f \in \operatorname{Ker}(\psi^\sharp)$, *that is,* $f \in \operatorname{Dom}(\psi^\sharp)$ *and* $\psi^\sharp(f) = 0$.

Proof. (i)⇒(ii). This is a clear consequence of (2.16) and (2.17).

(ii)⇒(i). Let $\lambda > \max\{\omega, 0\}$, where ω is defined as in (2.13). Since $\psi^\sharp$ is the dual operator of ψ, as in (2.19) and (2.18), and $\lambda I - \psi$ is boundedly invertible, it follows that $\lambda I - \psi^\sharp$ is boundedly invertible, e.g. see Theorem 1.5 in [32]. Consequently, for any $x \in \operatorname{Dom}(\psi)$ and any $g \in X^\sharp$ we have

$$\begin{aligned}
\left\langle x, \mathrm{e}^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda^2 t)(\lambda I - \psi^\sharp)^{-n}(g)}{n!} \right\rangle &= \left\langle x, \mathrm{e}^{-\lambda t} \Big(\sum_{n=0}^{\infty} \frac{(\lambda^2 t)(\lambda I - \psi)^{-n}}{n!}\Big)^\sharp (g) \right\rangle \\
&= \left\langle \mathrm{e}^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda^2 t)(\lambda I - \psi)^{-n}(x)}{n!}, g \right\rangle
\end{aligned}$$

hence, by (2.14) it follows that

$$\lim_{\lambda \to \infty} \left\langle x, \mathrm{e}^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda^2 t)(\lambda I - \psi^\sharp)^{-n}(g)}{n!} \right\rangle = \langle \Psi_t(x), g \rangle. \tag{2.20}$$

On the other hand, from $\psi^\sharp(f) = 0$ it follows that $(\lambda I - \psi^\sharp)(f) = \lambda f$ hence $(\lambda I - \psi^\sharp)^{-1}(f) = \frac{1}{\lambda} f$. By induction we obtain

$$(\lambda I - \psi^\sharp)^{-n}(f) = \frac{1}{\lambda^n} f, \quad n \geq 0. \tag{2.21}$$

Consequently, it follows that

$$\sum_{n=0}^{\infty} \frac{(\lambda^2 t)^n (\lambda I - \psi^\sharp)^{-n}(f)}{n!} = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} f = \mathrm{e}^{\lambda t} f,$$

hence, letting $g = f$ in (2.20), it follows that

$$\langle x, \Psi_t^\sharp(f)\rangle = \langle \Psi_t(x), f\rangle = \lim_{\lambda\to\infty} \langle x, \mathrm{e}^{-\lambda t}\mathrm{e}^{\lambda t} f\rangle = \langle x, f\rangle,$$

and then, since $\mathrm{Dom}(\psi)$ is dense in X, it follows that $\Psi_t^\sharp(f) = f$ for all $t \geq 0$. ☐

2.3. An Ergodic Theorem in von Neumann Algebras

We first recall some definitions, in addition to those in Sect. 2.1. For details, see, e.g. [31]. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and let $\mathcal{V} \subseteq \mathcal{A}$ and $\mathcal{W} \subseteq \mathcal{B}$ be subspaces. For any linear map $\Phi\colon \mathcal{V} \to \mathcal{W}$ and any natural number n, the *nth-order amplification* $\Phi_{(n)}\colon \mathcal{V}\otimes M_n \to \mathcal{W}\otimes M_n$ can be defined as $\Phi_{(n)} = \Phi\otimes I_n$, where I_n denotes the identity operator on M_n. Explicitly, by means of the canonical identifications $M_n(\mathcal{V}) = \mathcal{V}\otimes M_n$ and $M_n(\mathcal{W}) = \mathcal{W}\otimes M_n$, this means

$$\Phi_{(n)}([v_{i,j}]_{i,j=1}^n) = [\Phi(v_{i,j})]_{i,j=1}^n, \quad [v_{i,j}]_{i,j=1}^n \in M_n(\mathcal{V}). \tag{2.22}$$

Note that, by the embeddings $M_n(\mathcal{V}) \subseteq M_n(\mathcal{A})$ and $M_n(\mathcal{W}) \subseteq M_n(\mathcal{B})$, it follows that $M_n(\mathcal{V})$ and, respectively, $M_n(\mathcal{W})$ have canonical norms induced by the C^*-norms on $M_n(\mathcal{A})$ and $M_n(\mathcal{B})$. Consequently, we can let $\|\Phi_{(n)}\|$ denote the corresponding operator norm. Clearly,

$$\|\Phi\| = \|\Phi_{(1)}\| \leq \|\Phi_{(2)}\| \leq \cdots \leq \|\Phi_{(n)}\| \leq \|\Phi_{(n+1)}\| \leq \cdots . \tag{2.23}$$

The map Φ is called *completely bounded* if

$$\|\Phi\|_{\mathrm{cb}} = \sup_{n\geq 1} \|\Phi_{(n)}\| < \infty. \tag{2.24}$$

Let $\mathcal{CB}(\mathcal{V},\mathcal{W})$ denote the vector space of all completely bounded maps $\Phi\colon \mathcal{V} \to \mathcal{W}$. Also, such a map Φ is called *completely contractive* if $\|\Phi\|_{\mathrm{cb}} \leq 1$. A linear map $\Phi\colon \mathcal{V} \to \mathcal{V}$ is called an *idempotent* if $\Phi^2 = \Phi\Phi = \Phi$ and, it is called a *projection* if it is completely contractive and idempotent. A subspace $\mathcal{V} \subseteq \mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, is called *injective* if there exists a projection $\Phi\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ with range equal to $\mathcal{V}$.

A linear map $\Phi\colon \mathcal{A} \to \mathcal{A}$ is called a *conditional expectation* if it is positive, idempotent, and it has the following bimodule property: $\Phi(ar) = \Phi(a)r$ and $\Phi(ra) = r\Phi(a)$, for all $a \in \mathcal{A}$ and all $r \in \mathrm{Ran}(\Phi)$. By a classical result of Tomyama [36], a C^*-algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is injective if and only if there is a conditional expectation in $\mathcal{B}(\mathcal{H})$ with range equal to $\mathcal{A}$.

For a semigroup $\mathbf{\Phi} = \{\Phi_t\}_{t\geq 0}$ of unital, completely positive maps on a C^*-algebra $\mathcal{M}$, we consider $\mathcal{M}^{\mathbf{\Phi}}$ the set of joint fixed points of $\mathbf{\Phi}$, that is,

$$\mathcal{M}^{\mathbf{\Phi}} = \bigcap_{t\geq 0} \mathcal{M}^{\Phi_t} = \{a \in \mathcal{M} \mid \Phi_t(a) = a, \text{ for all } t \geq 0\}, \tag{2.25}$$

see Sect. 2.1, which is an operator system, and the joint bimodule domain

$$\begin{aligned}\mathcal{I}(\mathbf{\Phi}) &= \bigcap_{t\geq 0} \mathcal{I}(\Phi_t) \\ &= \{a \in \mathcal{M} \mid \Phi_t(ab) = a\Phi_t(b),\ \Phi_t(ba) = \Phi_t(b)a, \text{ for all } b \in \mathcal{A},\ t \geq 0\},\end{aligned} \tag{2.26}$$

which is clearly a C^*-subalgebra of $\mathcal{M}$ and included in $\mathcal{M}^{\mathbf{\Phi}}$. In case $\mathcal{M}$ is a von Neumann algebra and each Φ_t is w^*-continuous, $\mathcal{M}^{\mathbf{\Phi}}$ is w^*-closed and $\mathcal{I}(\mathbf{\Phi})$ is a von Neumann subalgebra of $\mathcal{M}$.

Theorem 2.5. *Let $\mathcal{M}$ be a von Neumann algebra and $\mathbf{\Phi} = \{\Phi_t\}_{t\geq 0}$ be a w^*-continuous semigroup of w^*-continuous, unital, completely positive maps on $\mathcal{M}$. Then:*

(a) *There exists a completely positive, unital, idempotent map $\Psi\colon \mathcal{M} \to \mathcal{M}$ such that the set of joint fixed points $\mathcal{M}^{\mathbf{\Phi}}$ is the range of Ψ.*
(b) *The following assertions are equivalent:*
  (i) *$\mathcal{M}^{\mathbf{\Phi}}$ is stable under multiplication.*
  (ii) *$\mathcal{M}^{\mathbf{\Phi}}$ is a von Neumann algebra.*
  (iii) *$\mathcal{M}^{\mathbf{\Phi}} = \mathcal{I}(\mathbf{\Phi})$.*
  (iv) *Ψ is a conditional expectation.*
(c) *If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, and $\mathcal{B}(\mathcal{H})^{\mathbf{\Psi}}$ is stable under multiplication, then $\mathcal{B}(\mathcal{H})^{\mathbf{\Psi}}$ is an injective von Neumann algebra.*

Proof. (a) For each real number $t > 0$, let $\Psi_t\colon \mathcal{M} \to \mathcal{M}$ be defined by

$$\Psi_t = \frac{1}{t}\int_0^t \Phi_s \mathrm{d}s. \tag{2.27}$$

The integral converges with respect to the point-w^*-topology, that is, for all $a \in \mathcal{M}$ and all $f \in \mathcal{M}_*$, we have

$$\langle \Psi_t(a), f\rangle = \frac{1}{t}\int_0^t \langle \Phi_s(a), f\rangle \mathrm{d}s.$$

It is easy to see that Ψ_t is w^*-continuous, unital, and completely positive and hence, by Russo–Dye's Theorem, a completely contractive map for each $t > 0$. By Alaoglu's Theorem, the closed unit ball of $\mathcal{M}$ is w^*-compact, hence by Tychonov's Theorem the closed unit ball of $\mathcal{CB}(\mathcal{M})$ is compact with respect to the point-w^*-topology. Consequently, considering the sequence $\{\Psi_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{\Psi_{k_n}\}_{n\in\mathbb{N}}$ such that

$$w^*\text{-}\lim_{n\to\infty} \Psi_{k_n}(a) = \Psi(a), \quad a \in \mathcal{M},$$

for some linear map $\Psi\colon \mathcal{M} \to \mathcal{M}$. Clearly, Ψ is unital and completely positive. Let $t \geq 0$ be an arbitrary real number and $n \in \mathbb{N}$ be large enough such that $t \leq n$. Then

$$\begin{aligned}\Psi_n - \Phi_t\Psi_n &= \frac{1}{n}\left(\int_0^n \Phi_s \mathrm{d}s - \int_0^n \Phi_{t+s}\mathrm{d}s\right)\\ &= \frac{1}{n}\left(\int_0^n \Phi_s \mathrm{d}s - \int_t^{t+n} \Phi_s\mathrm{d}s\right)\\ &= \frac{1}{n}\left(\int_0^t \Phi_s \mathrm{d}s - \int_n^{t+n} \Phi_s\mathrm{d}s\right)\end{aligned}$$

hence

$$\|\Psi_n - \Phi_t\Psi_n\| \leq \frac{1}{n}\Big(\int_0^t \|\Phi_s\|\mathrm{d}s + \int_n^{t+n}\|\Phi_s\|\mathrm{d}s\Big) = \frac{2t}{n} \xrightarrow[n\to\infty]{} 0. \tag{2.28}$$

On the other hand, using the representation

$$\Phi_t\Psi - \Psi = (\Phi_t\Psi - \Phi_t\Psi_{k_n}) + (\Phi_t\Psi_{k_n} - \Psi_{k_n}) + (\Psi_{k_n} - \Psi), \quad n \in \mathbb{N}, \tag{2.29}$$

and taking into account that, for all $a \in \mathcal{M}$, by the defining property of the subsequence $(\Psi_{k_n})_{n\in\mathbb{N}}$, we have

$$(\Phi_t\Psi - \Phi_t\Psi_{k_n})(a) = \Phi_t(\Psi(a) - \Psi_{k_n}(a)) \xrightarrow[n\to\infty]{w^*} 0,$$

and then of (2.28), it follows that $\Phi_t\Psi = \Psi$, for all $t \geq 0$. Similarly we obtain $\Psi\Phi_t = \Psi$ for all $t \geq 0$, hence

$$\Phi_t\Psi = \Psi\Phi_t = \Psi, \text{ for all } t \geq 0. \tag{2.30}$$

From (2.30) we get

$$\Psi_{k_n}(\Psi(a)) = \frac{1}{k_n}\int_0^{k_n} \Phi_s(\Psi(a))\mathrm{d}s = \Psi(a), \quad a \in \mathcal{M},\ n \in \mathbb{N},$$

and then letting $n \to \infty$ it follows that $\Psi\Psi = \Psi$, hence Ψ is an idempotent. If $a \in \mathcal{M}^{\boldsymbol{\Phi}}$ is arbitrary, then $\Psi_{k_n}(a) = a$ for all $n \in \mathbb{N}$ whence, letting $n \to \infty$ it follows $\Psi(a) = a$. We have proven that $\mathcal{M}^{\boldsymbol{\Phi}} \subseteq \operatorname{Ran}(\Psi)$. Since, by (2.30), $\operatorname{Ran}(\Psi) \subseteq \mathcal{M}^{\boldsymbol{\Phi}}$, we have $\mathcal{M}^{\boldsymbol{\Phi}} = \operatorname{Ran}(\Psi)$.

(b) Only the equivalence of (i) and (iv) requires a proof.

Assume firstly that $\mathcal{M}^{\boldsymbol{\Phi}}$ is stable under multiplication. By the result at item (a), it follows that $\operatorname{Ran}(\Psi) = \mathcal{M}^{\boldsymbol{\Phi}}$ is a von Neumann algebra. Then, for arbitrary $a \in \operatorname{Ran}(\Psi)$,

$$\Psi(a)^*\Psi(a) = a^*a = \Psi(a^*a), \quad \Psi(a)\Psi(a)^* = aa^* = \Psi(aa^*),$$

hence, by Theorem 2.1, for any $b \in \mathcal{M}$ we have

$$\Psi(ab) = \Psi(a)\Psi(b) = a\Psi(b), \quad \Psi(ba) = \Psi(b)\Psi(a) = \Psi(b)a,$$

consequently Ψ is a conditional expectation.

Conversely, if Ψ is a conditional expectation, then $\mathcal{M}^{\boldsymbol{\Phi}} = \operatorname{Ran}(\Psi)$ is a C^*-algebra, hence stable under multiplication.

(c) This is a consequence of the results proven at items (a) and (b). $\square$

3. Dynamics for Markov Processes: The Real Commutative Case

In this section, we consider the setting of dynamics of Markov processes in the framework of "stochastic mechanics" in the sense of [7] and [6]. As explained there, many concepts are obtained by analogy with quantum systems and here we show that the same mathematical tools we use for the analysis of quantum systems, as in Sects. 2.1 and 2.2, can be used as well for "stochastic mechanics".

Let $(X;\mu)$ be a σ-finite measure space. A *probability distribution* p is an element in $L^1_{\mathbb{R}}(X;\mu)$ which is positive and $\|p\|_1 = 1$. An *observable* O is an element in $L^\infty_{\mathbb{R}}(X;\mu)$, identified with the operator of multiplication $O\colon L^1_{\mathbb{R}}(X;\mu) \to L^1_{\mathbb{R}}(X;\mu)$

$$(Og)(x) = O(x)g(x), \quad g \in L^1_{\mathbb{R}}(X;\mu),\ x \in X.$$

The *expected value* of the observable O with respect to a probability distribution g is

$$E(O;g) = \langle O, g\rangle = \int_X O(x)g(x)\mathrm{d}\mu(x), \tag{3.1}$$

the *variance* of O with respect to g is

$$V(O;g) = \langle O^2, g\rangle - \langle O, g\rangle^2, \tag{3.2}$$

while the *standard deviation* of O with respect to g is

$$\sigma(O;g) = \sqrt{\langle O^2, g\rangle - \langle O, g\rangle^2}. \tag{3.3}$$

A *stochastic operator* is a bounded linear operator $U\colon L^1_{\mathbb{R}}(X;\mu) \to L^1_{\mathbb{R}}(X;\mu)$ that maps probability distributions to probability distributions, equivalently, U is *positive*, that is,

$$\text{if } g \in L^1_{\mathbb{R}}(X;\mu) \text{ and } g \geq 0 \text{ then } Ug \geq 0,$$

and

$$\int_X (Ug)(x)\mathrm{d}\mu(x) = \int_X g(x)\mathrm{d}\mu(x), \qquad \text{for all } g \in L^1_{\mathbb{R}}(X;\mu).$$

The latter condition can also be written as

$$\langle 1, Ug\rangle = \langle 1, g\rangle, \quad g \in L^1_{\mathbb{R}}(X;\mu).$$

A bounded linear operator $T\colon L^\infty_{\mathbb{R}}(X;\mu) \to L^\infty_{\mathbb{R}}(X;\mu)$ is called a *Markov map* if it is w^*-continuous, *positive*, in the sense that for any $f \in L^\infty_{\mathbb{R}}(X;\mu)$ with $f \geq 0$ it follows $Tf \geq 0$, and *unital*, that is, $T1 = 1$.

Given any bounded linear operator $U\colon L^1_{\mathbb{R}}(X;\mu) \to L^1_{\mathbb{R}}(X;\mu)$, there exists its *dual operator* $U^\sharp\colon L^\infty_{\mathbb{R}}(X;\mu) \to L^\infty_{\mathbb{R}}(X;\mu)$, which is linear and bounded, defined by

$$\begin{aligned}\langle Ug, f\rangle &= \int_X (Ug)(x)f(x)\mathrm{d}\mu(x) = \int_X g(x)(U^\sharp f)(x)\mathrm{d}\mu(x)\\ &= \langle g, U^\sharp f\rangle,\ f \in L^1_{\mathbb{R}}(X;\mu),\ g \in L^\infty_{\mathbb{R}}(X;\mu).\end{aligned}$$

In addition, $U^\sharp$ is w^*-continuous. If $U\colon L^1_{\mathbb{R}}(X;\mu) \to L^1_{\mathbb{R}}(X;\mu)$ is a stochastic operator, then its dual $U^\sharp\colon L^\infty_{\mathbb{R}}(X;\mu) \to L^\infty_{\mathbb{R}}(X;\mu)$ is a Markov operator.

A *discrete stochastic semigroup* with respect to the measure space $(X;\mu)$ is a sequence $\{U_n\}_{n\geq 0}$ subject to the following conditions:

(ms1) $U_n\colon L^1_{\mathbb{R}}(X;\mu)\to L^1_{\mathbb{R}}(X;\mu)$ is stochastic for all $n\geq 0$.
(ms2) $U_{n+m}=U_nU_m$ for all $n,m\geq 0$.
(ms3) $U_0=I$.

Clearly, any discrete stochastic semigroup is of the form $U_n = U^n$, $n\geq 0$, where $U=U_1$ is a stochastic operator. Considering the dual operator $U^\sharp\colon L^\infty_{\mathbb{R}}(X;\mu)\to L^\infty_R R(X;\mu)$, which is actually a Markov operator, we can equivalently discuss *discrete Markov semigroups.*

The equivalence of assertions (i), (ii), (i)$'$, and (ii)$'$ in the following theorem has been obtained in [7], for which we provide a proof based on Theorem 2.2 of propagation of fixed points for completely positive maps, as well as complete their theorem with two more equivalent assertions in terms of duals of stochastic operators.

Theorem 3.1. *Let $(X;\mu)$ be a σ-finite measure space, $U\colon L^1_{\mathbb{R}}(X;\mu)\to L^1_{\mathbb{R}}(X;\mu)$ a stochastic operator and $O\in L^\infty_{\mathbb{R}}(X;\mu)$ an observable. The following assertions are equivalent:*

- (i) $[O,U]=0$.
- (ii) *For any probability distribution g on X, we have $\langle O,Ug\rangle=\langle O,g\rangle$ and $\langle O^2,Ug\rangle=\langle O^2,g\rangle$.*
- (i)$'$ $[O,U^n]=0$ *for all $n\geq 0$.*
- (ii)$'$ *For any probability distribution g on X, the expected values of O and O^2 with respect to U^ng do not depend on $n\geq 0$.*
- (i)$''$ $[O,U^\sharp]=0$.
- (ii)$''$ $U^\sharp(O)=O$ *and* $U^\sharp(O^2)=O^2$.

Proof. The equivalences (i)$\Leftrightarrow$(i)$'$, (ii)$\Leftrightarrow$(ii)$'$, (i)$\Leftrightarrow$(i)$''$, and (ii)$\Leftrightarrow$(ii)$''$ are clear.

(i)$''$ $\Rightarrow$(ii)$''$. Assume that $[O,U^\sharp]=0$ hence, for any $f\in L^\infty_{\mathbb{R}}(X;\mu)$ we have $OU^\sharp(f)=U^\sharp(Of)$. Letting $f=1$ and taking into account that $U^\sharp(1)=1$, it follows $U^\sharp(O)=O$, and then letting $f=O$, we have $U^\sharp(O^2)=OU^\sharp(O)=O^2$.

(ii)$''$ $\Rightarrow$(i)$''$. The spaces $L^1_{\mathbb{R}}(X;\mu)$ and $L^\infty_{\mathbb{R}}(X;\mu)$ are naturally embedded in $L^1_{\mathbb{C}}(X;\mu)$ and, respectively, in $L^\infty_{\mathbb{C}}(X;\mu)$. The real stochastic operator U can be naturally lifted to a complex stochastic operator $U\colon L^1_{\mathbb{C}}(X;\mu)\to L^1_{\mathbb{C}}(X;\mu)$. More precisely, since

$$L^1_{\mathbb{C}}(X;\mu)=L^1_{\mathbb{R}}(X;\mu)\oplus \mathrm{i}L^1_{\mathbb{R}}(X;\mu),$$

we can define $\widetilde{U}\colon L^1_{\mathbb{C}}(X;\mu)\to L^1_{\mathbb{C}}(X;\mu)$ by

$$\widetilde{U}(g+\mathrm{i}f)=Ug+\mathrm{i}Uf,\quad f,g\in L^1_{\mathbb{R}}(X;\mu),$$

and observe that $\widetilde{U}$ has the following two properties:

$$\text{if } g\in L^1_{\mathbb{C}}(X;\mu) \text{ and } g\geq 0 \text{ then } \widetilde{U}g\geq 0,$$

and

$$\int_X(\widetilde{U}g)(x)\mathrm{d}\mu(x)=\int_X g(x)\mathrm{d}\mu(x),\qquad \text{for all } g\in L^1_{\mathbb{C}}(X;\mu).$$

Then, $\widetilde{U}^\sharp\colon L^\infty_{\mathbb{C}}(X;\mu) \to L^\infty_{\mathbb{C}}(X;\mu)$ is unital and positive. Since $L^\infty_{\mathbb{C}}(X;\mu)$ is a commutative C^*-algebra, $\widetilde{U}^\sharp$ is completely positive, cf. [34].

On the other hand, the observable O can be naturally viewed as a real valued function in $L^\infty_{\mathbb{C}}(X;\mu)$ and, if $U^\sharp(O) = O$ and $U^\sharp(O^2) = O^2$, it follows that $\widetilde{U}^\sharp(O) = O$ and $\widetilde{U}^\sharp(O^2) = O^2$. Now, we can use Theorem 2.2 and conclude that $\widetilde{U}^\sharp(Of) = O\widetilde{U}^\sharp(f)$ for all $f \in L^\infty_C C(X;\mu)$, hence $[O,\widetilde{U}] = 0$ and then $[O,U] = 0$. □

A *continuous stochastic semigroup* on $(X;\mu)$ is a strongly continuous semigroup of stochastic operators on $L^1_{\mathbb{R}}(X;\mu)$. The *infinitesimal generator* of $\{U_t\}_{t\geq 0}$ is the closed and densely defined operator H in $L^1_{\mathbb{R}}(X;\mu)$, see Sect. 2.2. Let $\{U_t\}_{t\geq 0}$ be a continuous stochastic semigroup with respect to $(X;\mu)$ and H its infinitesimal generator. Then, $\{U_t^\sharp\}_{t\geq 0}$ is a w^*-continuous semigroup of Markov maps. The *w^*-infinitesimal generator* of $\{U_t^\sharp\}_{t\geq 0}$ is the w^*-closed, hence closed, and w^*-densely defined (but, in general, not densely defined) operator $H^\sharp$ in $L^\infty_{\mathbb{R}}(X;\mu)$ which, by Phillips Theorem [32], can be described by

$$H^\sharp f = w^*\text{-}\lim_{t\to 0+} \frac{U_t f - f}{t}, \quad f \in \operatorname{Dom}(H^\sharp),$$

where

$$\operatorname{Dom}(H^\sharp) = \left\{ f \in L^\infty_{\mathbb{R}}(X;\mu) \mid w^*\text{-}\lim_{t\to 0+} \frac{U_t^\sharp f - f}{t} \text{ exists in } L^\infty_{\mathbb{R}}(X;\mu) \right\}.$$

The equivalence of assertions (i) and (ii) in the next theorem has been obtained in [7], which we now obtain as a consequence of Theorem 2.2, via Theorem 3.1. We complete their theorem with four more equivalent assertions in terms of infinitesimal generators and their duals. The proofs are very similar with those in Theorem 6.4, and we prefer to provide the details for the more general theorem, in particular, the equivalence of assertions (ii)′ and (iii)′ follows from Theorem 2.4.

Theorem 3.2. *Let $(X;\mu)$ be a σ-finite measure space, $\{U_t\}_{t\geq 0}$ a continuous stochastic semigroup with respect to $(X;\mu)$, H its infinitesimal generator, and $O \in L^\infty_{\mathbb{R}}(X;\mu)$ an observable. The following assertions are equivalent:*

- (i) $[O, U_t] = 0$ *for all real* $t \geq 0$.
- (ii) *For every probability distribution g on $(X;\mu)$, both the expected value and the standard deviation of O with respect to $U_t g$ are constant with respect to $t \geq 0$.*
- (iii) $[O,H] = 0$, *in the sense that the operator of multiplication with O leaves* $\operatorname{Dom}(H)$ *invariant and $OHg = HOg$ for all $g \in \operatorname{Dom}(H)$.*
- (i)′ $[O, U_t^\sharp] = 0$ *for all real* $t \geq 0$.
- (ii)′ $U_t^\sharp(O) = O$ *and* $U_t^\sharp(O^2) = O^2$ *for all real* $t \geq 0$
- (iii)′ *Both O and O^2 are in the kernel of $H^\sharp$, that is, $O, O^2 \in \operatorname{Dom}(H^\sharp)$ and* $H^\sharp(O) = H^\sharp(O^2) = 0$.

4. Constants of Dynamical Quantum Systems

We now consider the setting of dynamical quantum systems closer to the setting in [21]. Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{B}(\mathcal{H})$ be the von Neumann algebra of all bounded linear operators $T\colon \mathcal{H} \to \mathcal{H}$ and let $\mathcal{B}_1(\mathcal{H})$ be the *trace-class*, that is, the collection of all operators $T \in \mathcal{B}(\mathcal{H})$ subject to the condition $\|T\|_1 = \operatorname{tr}(|T|) < +\infty$, where $|T| = (T^*T)^{1/2}$ denotes the module of T and tr denotes the usual normal faithful semifinite *trace* on $\mathcal{B}(\mathcal{H})$. Let $\mathcal{D}(\mathcal{H})$ denote the set of *states*, or *density operators*, with respect to $\mathcal{H}$, that is, the set of all positive elements $\rho \in \mathcal{B}_1(\mathcal{H})$ with $\operatorname{tr}(\rho) = \|\rho\|_1 = 1$.

The map $\Psi\colon \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ is called a *quantum operation*, if it is completely positive, see Sect. 2.1 for definition, and trace-preserving. Note that the trace-class $\mathcal{B}_1(\mathcal{H})$ is considered here as a $*$-subspace of the C^*-algebra $\mathcal{B}(\mathcal{H})$ and, consequently, the concept of completely positive map on $\mathcal{B}_1(\mathcal{H})$ makes perfectly sense.

We note that the definition of a quantum operation we adopt here is a bit more restrictive than usual. In quantum information theory, they use the term of a *quantum communication channel*, or briefly a *quantum channel*, for what we call here a quantum operation.

For a fixed Banach space X, recall that we denote its topological dual space by $X^\sharp$ and the duality map by $X \times X^\sharp \ni (x, f) \mapsto \langle x, f\rangle$, see Sect. 2.2. The topics of this article refer to the Banach space $(\mathcal{B}_1(\mathcal{H}), \|\cdot\|_1)$ and its topological dual Banach space $(\mathcal{B}(\mathcal{H}), \|\cdot\|)$ with the duality map $\mathcal{B}_1(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \ni (T, S) \mapsto \langle T, S\rangle = \operatorname{tr}(TS)$, e.g. see Theorem 19.2 in [12]. In particular, for a quantum operation Ψ when viewed as a trace-preserving completely positive map $\Psi\colon \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$, one usually refers to the Schrödinger picture, to which the Heisenberg picture is corresponding by duality: the dual map $\Psi^\sharp\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is defined by

$$\langle \Psi(T), S\rangle = \operatorname{tr}(\Psi(T)S) = \operatorname{tr}(T\Psi^\sharp(S)) = \langle T, \Psi^\sharp(S)\rangle, \quad T \in \mathcal{B}_1(\mathcal{H}),\ S \in \mathcal{B}(\mathcal{H}),$$

and it is a ultraweakly continuous (w^*-continuous) completely positive and unital linear map.

There are many quantum operations. For example, if $\{A_k \mid k \in \mathbb{N}\}$ is a collection of operators in $\mathcal{B}(\mathcal{H})$ such that $\sum_{k=1}^\infty A_k A_k^* = I$, then the linear map $\mathcal{B}_1(\mathcal{H}) \ni T \mapsto \sum_{k=1}^\infty A_k^* T A_k \in \mathcal{B}_1(\mathcal{H})$ is a quantum operation. According to Kraus [27,28], if $\mathcal{H}$ is separable, then any quantum operation with respect to $\mathcal{H}$ has this form.

For a fixed Hilbert space $\mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H})$, we have the *left multiplication operator* $L_A\colon \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ defined by $L_A(T) = AT$, for all $T \in \mathcal{B}_1(\mathcal{H})$, and the *right multiplication operator* $R_A\colon \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ defined by $R_A(T) = TA$, for all $T \in \mathcal{B}_1(\mathcal{H})$. Observe that, exactly with the same formal definition, we may have the left multiplication operator $L_A\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ and, respectively, $R_A\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$. We will not use different notations for these operators, hoping that which is which will be clear from the context. For example, when viewing $L_A\colon \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$, its dual $L_A^\sharp\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ coincides with the operator $R_A\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$. Also, considering $M_A(T) = A^*TA$, the one-element quantum measurement operator, then $M_A = L_{A^*}R_A$.

A family indexed on the set of nonnegative real numbers $\mathbf{\Psi} = \{\Psi_t\}_{t\geq 0}$ is called a *dynamical quantum system*, sometimes called a *dynamical quantum stochastic system*, with respect to a Hilbert space $\mathcal{H}$, if it is a strongly continuous semigroup of quantum operations $\Psi_t\colon \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$, $t \geq 0$. For a dynamical quantum system $\mathbf{\Psi}$, we consider its infinitesimal generator ψ, see Sect. 2.2 for the general setting, which is a densely defined closed operator on the Banach space $\mathcal{B}_1(\mathcal{H})$. This definition makes a representation of the dynamical quantum system $\mathbf{\Psi}$ into the Schrödinger picture. Transferring a dynamical quantum system $\mathbf{\Psi}$ into the Heisenberg picture, we get its dual, usually called *dynamical quantum Markov system*, $\mathbf{\Psi}^\sharp = \{\Psi_t^\sharp\}_{t\geq 0}$ which is a w^*-continuous one-parameter semigroup of w^*-continuous, unital, completely positive linear maps $\Psi_t^\sharp\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ to which one associates its w^*-infinitesimal generator $\psi^\sharp$, as in (2.16) and (2.17). Here, an important issue is that by Phillips's Theorem [32], $\psi^\sharp$ is indeed the dual of ψ.

Note that our definitions are more general than those usually considered in most mathematical models of quantum open systems, e.g. see [18,21] and the rich bibliography cited there, which instead of strong continuity requires the (operator) norm continuity, that is, the mapping $\mathbb{R}_+ \ni t \mapsto \Psi_t \in \mathcal{L}(\mathcal{B}_1(\mathcal{H}))$ should be continuous with respect to the operator norm of $\mathcal{L}(\mathcal{B}_1(\mathcal{H}))$.

An operator $A \in \mathcal{B}(\mathcal{H})$ is called a *constant* of the dynamical quantum system $\mathbf{\Psi} = \{\Psi_t\}_{t\geq 0}$, if, for any density operator $\rho \in \mathcal{D}(\mathcal{H})$, $\operatorname{tr}(\Psi_t(\rho)A)$ does not depend on $t \geq 0$, equivalently, $\operatorname{tr}(\Psi_t(\rho)A) = \operatorname{tr}(\rho A)$ for all $t \geq 0$. Clearly, A is a constant of $\mathbf{\Psi}$ if and only if for any $T \in \mathcal{B}_1(\mathcal{H})$ we have $\operatorname{tr}(\Psi_t(T)A) = \operatorname{tr}(TA)$ for all $t \geq 0$, equivalently, $\operatorname{tr}(T\Psi_t^\sharp(A)) = \operatorname{tr}(TA)$ for all $T \in \mathcal{B}_1(\mathcal{H})$ and all $t \geq 0$. Consequently, $A \in \mathcal{B}(\mathcal{H})$ is a constant of Ψ if and only if $\Psi_t^\sharp(A) = A$ for all $t \geq 0$, that is, A is a fixed point of $\Psi_t^\sharp$ for all $t \geq 0$. Formally, letting $\mathcal{C}^{\mathbf{\Psi}}$ denote the *set of constants* of $\mathbf{\Psi}$

$$\begin{aligned} \mathcal{C}^{\mathbf{\Psi}} &= \{A \in \mathcal{B}(\mathcal{H}) \mid \text{for all } \rho \in \mathcal{D}(\mathcal{H}),\ \operatorname{tr}(\Psi_t(\rho)A) \text{ is independent of } t\} \\ &= \{A \in \mathcal{B}(\mathcal{H}) \mid \operatorname{tr}(\Psi_t(\rho)A) = \operatorname{tr}(\rho A) \text{ for all } \rho \in \mathcal{D}(\mathcal{H}) \text{ and all } t \geq 0\} \\ &= \{A \in \mathcal{B}(\mathcal{H}) \mid \Psi_t^\sharp(A) = A \text{ for all } t \geq 0\} = \mathcal{B}(\mathcal{H})^{\mathbf{\Psi}^\sharp}, \end{aligned} \tag{4.1}$$

where the last equality is actually the definition of $\mathcal{B}(\mathcal{H})^{\mathbf{\Psi}^\sharp}$ as the set of all joint fixed points of $\Psi_t^\sharp$, $t \geq 0$. In addition, as a consequence of Theorem 2.4, we have

$$\mathcal{C}^{\mathbf{\Psi}} = \mathcal{B}(\mathcal{H})^{\mathbf{\Psi}^\sharp} = \operatorname{Ker}(\psi^\sharp) = \{T \in \mathcal{B}(\mathcal{H}) \mid T \in \operatorname{Dom}(\psi^\sharp),\ \psi^\sharp(T) = 0\}. \tag{4.2}$$

Theorem 4.1. *Let $\mathbf{\Psi} = \{\Psi_t\}_{t\geq 0}$ be a dynamical quantum stochastic system with respect to the Hilbert space $\mathcal{H}$, let ψ denote its infinitesimal generator, and let $A \in \mathcal{B}(\mathcal{H})$. The following assertions are equivalent:*

(i) $[L_A, \Psi_t] = 0$ *for all* $t \geq 0$.

(ii) *A and A^*A are constants of $\mathbf{\Psi}$.*

(iii) $[R_A, \Psi_t^\sharp] = 0$ *for all* $t \geq 0$.
(iv) A *and* A^*A *are joint fixed points of* $\Psi_t^\sharp$ *for all* $t \geq 0$.
(v) $[L_A, \psi] = 0$*, that is,* L_A *leaves* $\mathrm{Dom}(\psi)$ *invariant and* $A\psi(T) = \psi(AT)$ *for all* $T \in \mathrm{Dom}(\psi)$.
(vi) $A, A^*A \in \mathrm{Ker}(\psi^\sharp)$*, i.e.,* $A, A^*A \in \mathrm{Dom}(\psi^\sharp)$ *and* $\psi^\sharp(A) = \psi^\sharp(A^*A) = 0$.

Proof. In order to prove the equivalence of (i) through (iv), we show that, by fixing $\Psi = \Psi_t$ for some $t \in [0, +\infty)$, the following assertions are equivalent:
(i) $[L_A, \Psi] = 0$.
(ii) A and A^*A are constants of Ψ.
(iii) $[R_A, \Psi^\sharp] = 0$.
(iv) A and A^*A are fixed points of $\Psi^\sharp$.

(i)$\Leftrightarrow$(iii) and (ii)$\Leftrightarrow$(iv) are clear.

(iii)$\Rightarrow$(iv). If $[R_A, \Psi^\sharp] = 0$, then $\Psi^\sharp(SA) = \Psi^\sharp(S)A$ for all $S \in \mathcal{B}(\mathcal{H})$. Letting $S = I$, we get $\Psi^\sharp(A) = A$ and, since $\Psi^\sharp$ is positive, hence selfadjoint, it follows that $\Psi^\sharp(A^*) = A^*$. Then, letting $S = A^*$, we get $\Psi^\sharp(A^*A) = \Psi^\sharp(A^*)A = A^*A$.

(iv)$\Rightarrow$(iii). Assume that $\Psi^\sharp(A) = A$ and $\Psi^\sharp(A^*A) = A^*A$. Then, $\Psi^\sharp(A^*) = A^*$ and $\Psi^\sharp(A^*A) = A^*A = \Psi^\sharp(A^*)\Psi(A)$. By Theorem 2.1.(2).(i), we have $\Psi^\sharp(TA) = \Psi^\sharp(T)\Psi^\sharp(A) = \Psi^\sharp(T)A$ for all $T \in \mathcal{B}(\mathcal{H})$, hence $[R_A, \Psi^\sharp] = 0$.

The equivalence of assertions (iv) and (vi) follows from (4.2). Finally, the equivalence of assertions (i) and (v) is a straightforward consequence of the definition of the infinitesimal generator ψ. □

There is a symmetric variant to Theorem 4.1, in which L_A and R_A are interchanged and, correspondingly, A^*A and AA^* are interchanged. We leave the reader to formulate it.

In order to substantiate further definitions and questions, we record some natural definitions from quantum probability in analogy with those from classical probability, compared with (3.1)–(3.3). Let A be a bounded observable with respect to the Hilbert space $\mathcal{H}$, that is, $A \in \mathcal{B}(\mathcal{H})$ and $A = A^*$. For any state $\rho \in \mathcal{D}(\mathcal{H})$, one considers the *expected value* of A in the state ρ,

$$E(A; \rho) = \langle \rho, A \rangle = \mathrm{tr}(\rho A), \tag{4.3}$$

the *variation* of A in the state ρ,

$$V(A; \rho) = \langle \rho, A^2 \rangle - \langle \rho, A \rangle^2 = \mathrm{tr}(\rho A^2) - \mathrm{tr}(\rho A)^2, \tag{4.4}$$

and its *standard deviation*,

$$\sigma(A; \rho) = \sqrt{\langle \rho, A^2 \rangle - \langle \rho, A \rangle^2} = \sqrt{\mathrm{tr}(\rho A^2) - \mathrm{tr}(\rho A)^2}. \tag{4.5}$$

In case of a bounded observable $A \in \mathcal{B}(\mathcal{H})^+$, with expected value, variation, and standard deviation to an arbitrary state $\rho \in \mathcal{D}(\mathcal{H})$ as in (4.3) through (4.5), Theorem 4.1 can be reformulated to a noncommutative analogue of the Noether-type theorem as in [7], see Theorem 3.1.

Corollary 4.2. *Let* $\mathbf{\Psi} = \{\Psi_t\}_{t \geq 0}$ *be a dynamical quantum stochastic system with respect to the Hilbert space* $\mathcal{H}$*, let* ψ *denote its infinitesimal generator,* *and let $A \in \mathcal{B}(\mathcal{H})$, $A = A^*$, be a bounded observable. The following assertions are equivalent:*

(i) $[L_A, \Psi_t] = 0$ *for all* $t \geq 0$.
(i)′ $[R_A, \Psi_t] = 0$ *for all* $t \geq 0$.
(ii) *In any state* $\rho \in \mathcal{D}(\mathcal{H})$, A *and* A^2 *have expected values with respect to* Ψ_t *independent of* $t \geq 0$.
(ii)′ *In any state* $\rho \in \mathcal{D}(\mathcal{H})$, A *has expected value and standard deviation with respect to* Ψ_t *independent of* $t \geq 0$.
(iii) $[R_A, \Psi_t^\sharp] = 0$ *for all* $t \geq 0$.
(iii)′ $[L_A, \Psi_t^\sharp] = 0$ *for all* $t \geq 0$.
(iv) A *and* A^2 *are joint fixed points of* $\Psi_t^\sharp$ *for all* $t \geq 0$.
(v) $[L_A, \psi] = 0$, *that is,* L_A *leaves* $\mathrm{Dom}(\psi)$ *invariant and* $A\psi(T) = \psi(AT)$ *for all* $T \in \mathrm{Dom}(\psi)$.
(v′) $[R_A, \psi] = 0$, *that is,* R_A *leaves* $\mathrm{Dom}(\psi)$ *invariant and* $\psi(T)A = \psi(TA)$ *for all* $T \in \mathrm{Dom}(\psi)$.
(vi) $A, A^2 \in \mathrm{Ker}(\psi^\sharp)$, *that is,* $A, A^2 \in \mathrm{Dom}(\psi^\sharp)$ *and* $\psi^\sharp(A) = \psi^\sharp(A^2) = 0$.

In order to put the investigations from [21] in a perspective closer to our approach, we now consider a scale of sets of constants of Ψ, more precisely, let

$$\mathcal{C}_2^\Psi = \{A \in \mathcal{B}(\mathcal{H}) \mid A, A^*A, AA^* \in \mathcal{C}^\Psi\}, \tag{4.6}$$

$$\mathcal{C}_{\mathrm{p}}^\Psi = \{A \in \mathcal{B}(\mathcal{H}) \mid p(A, A^*) \in \mathcal{C}^\Psi \text{ for all complex polynomials } p \text{ in two noncommutative variables}\}, \tag{4.7}$$

$$\mathcal{C}_{\mathrm{c}}^\Psi = \{A \in \mathcal{B}(\mathcal{H}) \mid C^*(I, A) \subseteq \mathcal{C}^\Psi\}, \tag{4.8}$$

$$\mathcal{C}_{\mathrm{w}}^\Psi = \{A \in \mathcal{B}(\mathcal{H}) \mid W^*(A) \subseteq \mathcal{C}^\Psi\}, \tag{4.9}$$

where $C^*(I, A)$ denotes the C^*-algebra generated by I and A, while $W^*(A)$ denotes the von Neumann algebra generated by A. Transferring these classes in the Heisenberg picture, we have

$$\mathcal{C}_2^\Psi = \{A \in \mathcal{B}(\mathcal{H}) \mid A, A^*A, AA^* \in \mathcal{B}(\mathcal{H})^{\Psi^\sharp}\} = \mathcal{B}(\mathcal{H})_2^{\Psi^\sharp}, \tag{4.10}$$

$$\mathcal{C}_{\mathrm{p}}^\Psi = \{A \in \mathcal{B}(\mathcal{H}) \mid p(A, A^*) \in \mathcal{B}(\mathcal{H})^{\Psi^\sharp} \text{ for all complex polynomials } p \text{ in two noncommutative variables}\} = \mathcal{B}(\mathcal{H})_{\mathrm{p}}^{\Psi^\sharp}, \tag{4.11}$$

$$\mathcal{C}_{\mathrm{c}}^\Psi = \{A \in \mathcal{B}(\mathcal{H}) \mid C^*(I, A) \subseteq \mathcal{B}(\mathcal{H})^{\Psi^\sharp}\} = \mathcal{B}(\mathcal{H})_{\mathrm{c}}^{\Psi^\sharp}, \tag{4.12}$$

$$\mathcal{C}_{\mathrm{w}}^\Psi = \{A \in \mathcal{B}(\mathcal{H}) \mid W^*(A) \subseteq \mathcal{B}(\mathcal{H})^{\Psi^\sharp}\} = \mathcal{B}(\mathcal{H})_{\mathrm{w}}^{\Psi^\sharp}. \tag{4.13}$$

It is easy to see that $\mathcal{C}^\Psi$ is an operator system, that is, a vector space stable under taking adjoints and containing the identity I, and w^*-closed, hence closed with respect to the operator norm as well. As any other operator system, $\mathcal{C}^\Psi$ is linearly generated by the set of its positive elements but, in general, not stable under multiplication, cf. [3,4,9].

On the other hand, as in (4.6)–(4.9), we have the joint versions of the scale of sets of constants

$$\mathcal{C}_{\mathrm{w}}^{\boldsymbol{\Psi}} \subseteq \mathcal{C}_{\mathrm{c}}^{\boldsymbol{\Psi}} \subseteq \mathcal{C}_{\mathrm{p}}^{\boldsymbol{\Psi}} \subseteq \mathcal{C}_2^{\boldsymbol{\Psi}} \subseteq \mathcal{C}^{\boldsymbol{\Psi}},$$

in the Schrödinger picture, more precisely,

$$\mathcal{C}_{\bullet}^{\boldsymbol{\Psi}} = \bigcap_{t\geq 0} \mathcal{C}_{\bullet}^{\Psi_t}, \text{ where } \bullet = 2, \mathrm{p}, \mathrm{c}, \mathrm{w}, \tag{4.14}$$

and, as in (4.10)–(4.13), the sets of joint fixed points

$$\mathcal{B}(\mathcal{H})_{\mathrm{w}}^{\boldsymbol{\Psi}^\sharp} \subseteq \mathcal{B}(\mathcal{H})_{\mathrm{c}}^{\boldsymbol{\Psi}^\sharp} \subseteq \mathcal{B}(\mathcal{H})_{\mathrm{p}}^{\boldsymbol{\Psi}^\sharp} \subseteq \mathcal{B}(\mathcal{H})_{2}^{\boldsymbol{\Psi}^\sharp} \subseteq \mathcal{B}(\mathcal{H})^{\boldsymbol{\Psi}^\sharp},$$

in the Heisenberg picture,

$$\mathcal{B}(\mathcal{H})_{\bullet}^{\boldsymbol{\Psi}^\sharp} = \bigcap_{t\geq 0} \mathcal{B}(\mathcal{H})_{\bullet}^{\Psi_t^\sharp}, \text{ where } \bullet = 2, \mathrm{p}, \mathrm{c}, \mathrm{w}. \tag{4.15}$$

$\mathcal{C}^{\boldsymbol{\Psi}} = \mathcal{B}(\mathcal{H})^{\boldsymbol{\Psi}}$ is a w^*-closed operator system and w^*-closed, hence closed with respect to the operator norm on $\mathcal{B}(\mathcal{H})$ as well, linearly generated by the set of its positive elements but, in general, not stable under multiplication.

Theorem 4.3. *Let $\boldsymbol{\Psi}$ be a dynamical quantum system with respect to the Hilbert space $\mathcal{H}$.*

(a) *For any dynamical quantum system $\boldsymbol{\Psi}$, we have $\mathcal{C}_2^{\boldsymbol{\Psi}} = \mathcal{C}_{\mathrm{p}}^{\boldsymbol{\Psi}} = \mathcal{C}_{\mathrm{c}}^{\boldsymbol{\Psi}} = \mathcal{C}_{\mathrm{w}}^{\boldsymbol{\Psi}}$ and this set is a von Neumann algebra.*

(b) *The following assertions are equivalent:*
   - (i) *$\mathcal{C}^{\boldsymbol{\Psi}}$ is stable under multiplication.*
   - (ii) *$\mathcal{C}^{\boldsymbol{\Psi}} = \mathcal{C}_2^{\boldsymbol{\Psi}}$.*
   - (iii) *$\mathcal{C}^{\boldsymbol{\Psi}}$ is a C^*-algebra.*
   - (iv) *$\mathcal{C}^{\boldsymbol{\Psi}}$ is a von Neumann algebra.*

Proof. Clearly, without loss of generality, it is sufficient to prove these equivalences for the case of a single quantum operation Ψ.

(a) Clearly, $\mathcal{C}_2^{\Psi} \supseteq \mathcal{C}_{\mathrm{p}}^{\Psi} \supseteq \mathcal{C}_{\mathrm{c}}^{\Psi} \supseteq \mathcal{C}_{\mathrm{w}}^{\Psi}$. Due to the density of the set of all operators $p(A, A^*)$, where p is an arbitrary complex polynomial in two noncommutative variables, in $C^*(I, A)$, the w^*-density of $C^*(I, A)$ in $W^*(A)$, as well as the continuity and w^*-continuity of the map $A \mapsto \operatorname{tr}(\Psi(\rho)A)$, we have the equality $\mathcal{C}_{\mathrm{p}}^{\Psi} = \mathcal{C}_{\mathrm{c}}^{\Psi} = \mathcal{C}_{\mathrm{w}}^{\Psi}$. On the other hand, using the dual representations as in (4.10) and (4.12), from Theorem 2.2 we obtain $\mathcal{C}_2^{\Psi} = \mathcal{C}_{\mathrm{c}}^{\Psi}$.

In order to prove that this set is a von Neumann algebra, it is preferable to use its representation in the Heisenberg picture as $\mathcal{B}(\mathcal{H})_2^{\Psi^\sharp}$, see (4.10). Since $\Psi^\sharp$ is positive it is selfadjoint, hence $\mathcal{B}(\mathcal{H})_2^{\Psi^\sharp}$ is stable under taking the involution $A \mapsto A^*$. If $A, B \in \mathcal{B}(\mathcal{H})_2^{\Psi^\sharp}$, by Theorem 2.2 we have

$$\Psi^\sharp(AB) = A\Psi^\sharp(B) = AB, \tag{4.16}$$

hence $\mathcal{B}(\mathcal{H})_2^{\Psi^\sharp}$ is stable under multiplication. On the other hand,

$$\begin{aligned}\Psi^\sharp((A+B)^*(A+B)) &= \Psi^\sharp(A^*A + A^*B + B^*A + B^*A)\\ &= \Psi^\sharp(A^*A) + \Psi^\sharp(A^*B) + \Psi^\sharp(B^*A) + \Psi^\sharp(B^*A)\end{aligned}$$

hence, taking into account of (4.16),

$$= A^*A + A^*B + B^*A + B^*A = (A+B)^*(A+B).$$

Similarly, we prove that $(A+B)(A+B)^*$ is a fixed point of $\Psi^\sharp$. Since clearly $A+B$ is a fixed point of $\Psi^\sharp$, it follows that $\mathcal{B}(\mathcal{H})_2^{\Psi^\sharp}$ is stable under addition as well. On the other hand, since $\Psi^\sharp$ is w^*-continuous, it follows that $\mathcal{B}(\mathcal{H})_2^{\Psi^\sharp}$ is a von Neumann algebra.

(b) This is actually a reformulation of Lemma 2.2 in [2]. □

Remarks 4.4. (a) The main theorem in [21] states that, for a dynamical quantum (stochastic) system $\mathbf{\Psi}$ under two additional constraints, namely, that the semigroup is (operator) norm continuous and that there exists a stationary strictly positive density operator, that is, there exists $\rho \in \mathcal{B}_1(\mathcal{H})^+$ that is strictly positive and such that $\Psi_t(\rho) = \rho$ for all real $t \geq 0$, then $\mathcal{C}^{\mathbf{\Psi}} = \mathcal{B}(\mathcal{H})^{\mathbf{\Psi}^\sharp}$ is a von Neumann algebra. This theorem remains true under the general assumption that the semigroup $\mathbf{\Psi}$ is strongly continuous: we use Theorem 4.3 while the existence of a stationary strictly positive density operator ρ implies the existence of a normal faithful stationary state $\omega(T) = \operatorname{tr}(\rho T)$, $T \in \mathcal{B}(\mathcal{H})$, and then Theorem 2.3 in [2].

(b) In case the dynamical quantum system $\mathbf{\Psi}$ is (operator) norm continuous, the infinitesimal generator ψ is bounded and, by a result of Lindblad [29] (and, in the finite-dimensional case, of Gorini et al. [20]), it takes the form

$$\psi(S) = \sum_{k=1}^{\infty}\left(L_k S L_k^* - \frac{1}{2} S L_k^* L_k - \frac{1}{2} L_k^* L_k S\right) + \mathrm{i}[S,H], \quad S \in \mathcal{B}_1(\mathcal{H}), \tag{4.17}$$

for a collection of operators $L_k \in \mathcal{B}(\mathcal{H})$, $k = 1, 2, \ldots$, and a selfadjoint operator $H \in \mathcal{B}(\mathcal{H})$. It is easy to see that its adjoint, which is the infinitesimal generator of the dual quantum Markov semigroup $\{\Psi_t^\sharp\}_{t \geq 0}$, is

$$\psi^\sharp(T) = \sum_{k=1}^{\infty}\left(L_k^* T L_k - \frac{1}{2} L_k^* L_k T - \frac{1}{2} T L_k^* L_k\right) - \mathrm{i}[T,H], \quad T \in \mathcal{B}(\mathcal{H}). \tag{4.18}$$

Consequently, using (4.2), it follows that the constants of $\mathbf{\Psi}$ are exactly the solutions $T \in \mathcal{B}(\mathcal{H})$ of the equation

$$\sum_{k=1}^{\infty}\left(L_k^* T L_k - \frac{1}{2} L_k^* L_k T - \frac{1}{2} T L_k^* L_k\right) - \mathrm{i}[T,H] = 0, \tag{4.19}$$

which is an operator Riccati equation.

(c) In case the dynamical quantum system $\mathbf{\Psi}$ is (operator) norm continuous, hence (4.17) and (4.18) hold, and $\mathbf{\Psi}$ has a stationary strictly positive density operator, it is proven in [21] that the set $\mathcal{C}_{\mathrm{w}}^{\mathbf{\Psi}}$ coincides with the commutant $\{H, L_k, L_k^* \mid k = 1, 2, \ldots\}'$, in particular, it is a von Neumann algebra.

5. Are the Conditions on A and A^2 (AA^* and A^*A) Independent?

We are in a position to approach the following question: to what extent are the latter conditions on A^*A or AA^* as in Theorem 4.1.(ii), and the latter condition on A^2 as in Corollary 4.2, really necessary? Note that a positive answer to this question will answer the similar question asked for the more general case of dynamical stochastic systems as in Sect. 6, see Remark 6.7.(b).

Example 5.1. As in [2], let $\mathbb{F}_2$ denote the free group on two generators g_1 and g_2, and let $\ell^2(\mathbb{F}_2)$ denote the Hilbert space of all square summable functions $f\colon \mathbb{F}_2 \to \mathbb{C}$. In $\ell^2(\mathbb{F}_2)$, a canonical orthonormal basis is made up by $\{\delta_x\}_{x\in\mathbb{F}_2}$, where $\delta_x(y) = 0$ for all $y \in \mathbb{F}_2$, $y \neq x$, and $\delta_x(x) = 1$. Since $\mathbb{F}_2$ is infinitely countable, it follows that $\ell^2(\mathbb{F}_2)$ is infinite dimensional and separable. Let $U_j \in \mathcal{B}(\ell_2(\mathbb{F}_2))$ denote the unitary operators $U_j\delta_x = \delta_{g_j x}$, $x \in \mathbb{F}_2$ and $j = 1, 2$.

We consider the linear bounded operator $\psi\colon \mathcal{B}_1(\ell^2(\mathbb{F}_2)) \to \mathcal{B}_1(\ell^2(\mathbb{F}_2))$ defined by

$$\psi(S) = U_1 S U_1^* + U_2 S U_2^* - 2S, \quad S \in \mathcal{B}_1(\ell^2(\mathbb{F}_2)), \tag{5.1}$$

and then let

$$\Psi_t(S) = \exp(t\psi(S)), \quad S \in \mathcal{B}_1(\ell^2(\mathbb{F}_2)),\ t \geq 0. \tag{5.2}$$

From [29], see Remark 4.4.(b), it follows that $\mathbf{\Psi} = \{\Psi_t\}_{t\geq 0}$ is a (operator) norm continuous semigroup of quantum operations with respect to $\ell^2(\mathbb{F}_2)$.

Also, let $L(\mathbb{F}_2) = W^*(U_1, U_2)$ denote the group von Neumann algebra of $\mathbb{F}_2$. We observe, e.g. by means of (4.19), that the commutant von Neumann algebra $L(\mathbb{F}_2)'$ is included in the set of constants $\mathcal{C}^{\mathbf{\Psi}}$.

Lemma 5.2. *Let $\mathbf{\Psi}$ be the dynamical quantum system as in Example 5.1. Then, $\mathcal{C}^{\mathbf{\Psi}}$ is stable under multiplication if and only if it coincides with $L(\mathbb{F}_2)'$.*

Proof. It is sufficient to prove that, if $\mathcal{C}^{\mathbf{\Psi}}$ is stable under multiplication then it coincides with $L(\mathbb{F}_2)'$. To see this, assume that $\mathcal{C}^{\mathbf{\Psi}}$ is stable under multiplication hence, by Theorem 4.3.(b), it is a von Neumann algebra. By Remark 4.4.(b), it follows that for any orthogonal projection $E \in \mathcal{C}^{\mathbf{\Psi}}$ equation (4.19) holds which, in our special case, is

$$U_1^* E U_1 + U_2^* E U_2 = 2E. \tag{5.3}$$

Consequently, for each vector $h \in \ell^2(\mathbb{F}_2)$ that lies in the range of E, we have

$$\|EU_1 h\|^2 + \|EU_2 h\|^2 = \langle U_1^* E U_1 h, h\rangle + \langle U_2^* E U_2 h, h\rangle = 2\langle Eh, h\rangle = 2\|h\|^2,$$

from which, after a moment of thought, we see that $U_j h$ should lie in the range of E for $j = 1, 2$. We have shown that U_j leaves the range of E invariant, $j = 1, 2$. Since the same is true for the range of $I - E$, it follows that U_j commutes with all orthogonal projections in the von Neumann algebra $\mathcal{C}^{\mathbf{\Psi}}$, hence $\mathcal{C}^{\mathbf{\Psi}} \subseteq \{U_1, U_1^*, U_2, U_2^*\}' = L(\mathbb{F}_2)'$. The converse inclusion was observed at the end of Example 5.1. $\square$

During the proof of the next theorem, we use terminology as in Sect. 2.3.

Theorem 5.3. *On any infinite dimensional separable Hilbert space $\mathcal{H}$, there exists a (operator) norm continuous semigroup of quantum operations $\mathbf{\Phi} = \{\Phi_t\}_{t\geq 0}$ with respect to $\mathcal{H}$, for which:*

(a) *The set of constants* $\mathcal{C}^{\mathbf{\Phi}}$ *is not a von Neumann algebra, equivalently, it is not stable under multiplication.*
(b) *There exists* $A \in \mathcal{B}(\mathcal{H})^+$ *which is a constant of* $\mathbf{\Phi}$*, but* A^2 *is not.*

Proof. We first show that the (operator) norm continuous dynamical quantum system $\mathbf{\Psi}$ as in Example 5.1 has all the required properties. To this end, it is sufficient to prove assertion (b), then assertion (a) will follow from Theorem 4.3. By a classical result of Hakeda and Tomiyama [23], a von Neumann algebra $\mathcal{M}$ is injective if and only if its commutant $\mathcal{M}'$ is injective. By another classical result of Schwartz [33], see also Tomiyama [37], the von Neumann algebra $L(\mathbb{F}_2)$ is not injective hence, its commutant $L(\mathbb{F}_2)'$ is not injective either. Consequently, by Lemma 5.2 and Theorem 2.5, the set of joint fixed points $\mathcal{B}(\ell^2(\mathbb{F}_2))^{\mathbf{\Psi}^\sharp} = \mathcal{C}^{\mathbf{\Psi}}$ is strictly larger than the joint bimodule set $\mathcal{I}(\mathbf{\Psi}^\sharp)$. Since $\mathcal{B}(\ell^2(\mathbb{F}_2))^{\mathbf{\Psi}^\sharp}$ is an operator system, hence linearly generated by its positive cone, there exits $A \in \mathcal{B}(\ell^2(\mathbb{F}_2))^{\mathbf{\Psi}^\sharp} \setminus \mathcal{I}(\mathbf{\Phi})$ with $A \geq 0$. In view of Theorem 2.1, this implies $A^2 \notin \mathcal{B}(\ell^2(\mathbb{F}_2))^{\mathbf{\Psi}^\sharp}$.

In general, if $\mathcal{H}$ is an infinite dimensional separable Hilbert space, then there exists a unitary operator $U\colon \ell^2(\mathbb{F}_2) \to \mathcal{H}$ and let $\Phi_t = U^*\Psi_t U$, for all real $t \geq 0$. Then, $\mathbf{\Phi} = \{\Phi_t\}_{t\geq 0}$ has all the required properties. □

Theorem 5.3 answers, in the negative, also the question on whether the condition that A^2 is a joint fixed point, as in Theorem 6.4.(i), is a consequence of the condition that A is a joint fixed point.

6. Dynamical Systems of Stochastic/Markov Maps: The Noncommutative Case

Notation is as in Sect. 4. A linear map $\Psi\colon \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ is called *stochastic* if it maps states into states, equivalently, if it is positive, that is, $\Psi(A) \geq 0$ for all $A \in \mathcal{B}_1(\mathcal{H})^+$, and *trace-preserving*, that is, $\operatorname{tr}(\Psi(T)) = \operatorname{tr}(T)$ for all $T \in \mathcal{B}_1(\mathcal{H})$. Clearly, any quantum operation is a stochastic map.

Similarly as in Sect. 4, if Ψ is a stochastic linear map, then its dual $\Psi^\sharp$ is a ultraweakly continuous positive and unital linear map on $\mathcal{B}(\mathcal{H})$, called a *Markov map*.

The following example shows that there exist stochastic maps that are not quantum operations. The idea of using the transpose map for this kind of examples can be tracked back to Arveson [3,4]. Stochastic maps that are not quantum operations, in particular, the transpose map, play an important role in entanglement detectors in quantum information theory, e.g. see Chruscinski and Kossakowski [13], Horodecki et al. [25] and the rich bibliography cited there.

Example 6.1. Let $\mathcal{H}$ be an arbitrary Hilbert space with dimension at least 2, for which we fix an orthonormal basis $\{e_j\}_{j\in\mathcal{J}}$. We consider the *conjugation* operator $J\colon \mathcal{H} \to \mathcal{H}$ defined by $Jh = \overline{h}$ where, for arbitrary $h = \sum_{j\in\mathcal{J}} h_j e_j$, we let $\overline{h} = \sum_{j\in\mathcal{J}} \overline{h}_j e_j$. Then, J is conjugate linear, conjugate selfadjoint, that is, it has the following property

$$\langle Jh, k\rangle = \langle Jk, h\rangle, \quad h, k \in \mathcal{H}, \tag{6.1}$$

isometric, and $J^2 = I$.

Further on, let $\tau\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be defined by $\tau(S) = JS^*J$, for all $T \in \mathcal{B}(\mathcal{H})$. It is easy to see that τ is isometric, that is, $\|\tau(S)\| = \|S\|$ for all $S \in \mathcal{B}(\mathcal{H})$, and that $\tau(I) = I$. On the other hand, if $S \in \mathcal{B}(\mathcal{H})^+$, then

$$\langle \tau(S)h, h\rangle = \langle JSJh, h\rangle = \langle Jh, SJh\rangle = \langle SJh, Jh\rangle \geq 0, \quad h \in \mathcal{H},$$

hence τ is positive. Let us also observe that, with respect to the matrix representation of operators in $\mathcal{B}(\mathcal{H})$ associated with the orthonormal basis $\{e_j\}_{j\in\mathcal{J}}$, τ is the *transpose map*: if T has the matrix representation $[t_{i,j}]_{i,j\in\mathcal{J}}$, then $\tau(T)$ has the matrix representation $[t_{j,i}]_{j,i\in\mathcal{J}}$.

We claim now that τ leaves $\mathcal{B}_1(\mathcal{H})$ invariant and the corresponding restriction map $\mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ is stochastic. To see this, we first observe that if $T \in \mathcal{B}_1(\mathcal{H})^+$, we have $\tau(T) \in \mathcal{B}_1(\mathcal{H})^+$, e.g. using that τ is the transpose map with respect to the matrix representations of operators in $\mathcal{B}_1(\mathcal{H})$ associated with the orthonormal basis $\{e_j\}_{j\in\mathcal{J}}$, and the definition of the trace in terms of any orthonormal basis of $\mathcal{H}$. Also, $\|\tau(T)\|_1 = \operatorname{tr}(\tau(T)) = \operatorname{tr}(T) = \|T\|_1$. Since any operator $T \in \mathcal{B}_1(\mathcal{H})$ is a linear combination of four positive trace-class operators, the claim follows.

Finally, we show that τ is not completely positive, more precisely, it is not 2-positive. To see this, we consider the *matrix units* $\{E_{i,j}\}_{i,j\in\mathcal{J}}$, that is, for any $i, j \in \mathcal{J}$, $E_{i,j}$ denote the rank 1 operator on $\mathcal{H}$ with $E_{i,j}e_j = e_i$ and $E_{i,j}e_k = 0$ for all $k \neq j$ and observe that $\tau(E_{i,j}) = E_{j,i}$. Since $\dim \mathcal{H} \geq 2$, there exist $i, j \in \mathcal{J}$ with $i \neq j$. Then, consider the positive finite rank operator in $M_2(\mathcal{B}_1(\mathcal{H}))$ defined by

$$E = \begin{bmatrix} E_{i,i} & E_{i,j} \\ E_{j,i} & E_{j,j} \end{bmatrix}$$

and observe that

$$\tau_2(E) = \begin{bmatrix} \tau(E_{i,i}) & \tau(E_{i,j}) \\ \tau(E_{j,i}) & \tau(E_{j,j}) \end{bmatrix} = \begin{bmatrix} E_{i,i} & E_{j,i} \\ E_{i,j} & E_{j,j} \end{bmatrix}$$

which is not positive, e.g. see [31], p. 5. Therefore, τ is a stochastic map but not a quantum operation.

Remarks 6.2. (1) By means of the matrix transpose interpretation of τ as in Example 6.1, it follows easily that its dual $\tau^\sharp\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ has the same formal definition: $\tau(S) = JS^*J$, for all $S \in \mathcal{B}(\mathcal{H})$, and the same matrix transpose interpretation with respect to a fixed orthonormal basis of $\mathcal{H}$.

(2) The stochastic map τ described in Example 6.1 is invertible, $\tau^{-1} = \tau$, and antimultiplicative, that is, $\tau(ST) = \tau(T)\tau(S)$ for all $S, T \in \mathcal{B}_1(\mathcal{H})$. The same properties are shared by its dual $\tau^\sharp$. In particular, both τ and $\tau^\sharp$ are $*$-antihomomorphisms.

(3) In addition to the map τ described in Example 6.1, many other stochastic maps that are not quantum operations can be obtained by considering

convex combinations of linear maps of type $\tau\circ\Psi$ or $\Psi\circ\tau$, where Ψ are quantum operations.

With notation as in the previous section, we consider a strongly continuous one-parameter semigroup $\mathbf{\Psi}=\{\Psi_t\}_{t\geq 0}$ of stochastic maps with respect to some Hilbert space $\mathcal{H}$. Under these assumptions, we observe that $\{\Psi_t\}_{t\geq 0}$ is uniformly bounded on $\mathcal{B}_1(\mathcal{H})$. Most of the following facts that we briefly recall refer to a particular situation of the general theory of one-parameter semigroup theory on Banach spaces, e.g. see [24] and [15], see Sect. 2.2. Given a strongly continuous semigroup $\mathbf{\Psi}=\{\Psi_t\}_{t\geq 0}$ of stochastic maps with respect to some Hilbert space $\mathcal{H}$, the *infinitesimal generator* ψ exists as a densely defined closed operator on $\mathcal{B}_1(\mathcal{H})$. For every strongly continuous one-parameter semigroup $\mathbf{\Psi}=\{\Psi_t\}_{t\geq 0}$ of stochastic maps, the dual one-parameter semigroup $\mathbf{\Psi}^\sharp=\{\Psi_t^\sharp\}_{t\geq 0}$ of Markov maps exists, that is,

$$\begin{aligned}\langle\Psi_t(T),S\rangle&=\operatorname{tr}(\Psi_t(T)S)=\operatorname{tr}(T\Psi_t^\sharp(S))\\&=\langle T,\Psi_t^\sharp(S)\rangle,\ T\in\mathcal{B}_1(\mathcal{H}),\ S\in\mathcal{B}(\mathcal{H}),\ t\geq 0.\end{aligned}\tag{6.2}$$

Then $\{\Psi_t^\sharp\}_{t\geq 0}$ is a w^*-continuous semigroup of contractions on $\mathcal{B}(\mathcal{H})$ and hence, the *w^*-infinitesimal generator* $\psi^\sharp$ exists as a w^*-closed operator on $\mathcal{B}(\mathcal{H})$, hence a closed operator on $\mathcal{B}(\mathcal{H})$. By Phillips' Theorem [32], the w^*-infinitesimal generator $\psi^\sharp$ of the dual w^*-continuous semigroup $\{\Psi_t^\sharp\}_{t\geq 0}$ of Markov maps is indeed the dual operator of the infinitesimal generator ψ of the strongly continuous semigroup $\{\Psi_t\}_{t\geq 0}$ of stochastic maps and, consequently, the notation for $\psi^\sharp$ is fully justified.

Also, let us observe that, since $\Psi_t^\sharp(I)=I$, it follows that

$$I\in\operatorname{Dom}(\psi^\sharp)\text{ and }\psi^\sharp(I)=0.\tag{6.3}$$

In addition, one of the major differences between the two infinitesimal generators ψ and $\psi^\sharp$ is that $\operatorname{Dom}(\psi^\sharp)$ may not be dense in $\mathcal{B}(\mathcal{H})$, although it is always w^*-dense, while $\operatorname{Dom}(\psi)$ is always dense in $\mathcal{B}_1(\mathcal{H})$.

From the quantum measurements point of view, given a quantum operation Ψ, it is of interest to characterise those elements $A\in\mathcal{B}(\mathcal{H})$ with the property that $[\Psi,M_A]=0$, that is, $\Psi(A^*XA)=A^*\Psi(X)A$ for all $X\in\mathcal{B}_1(\mathcal{H})$, where $M_A\colon\mathcal{B}_1(\mathcal{H})\to\mathcal{B}_1(\mathcal{H})$ denotes the one-element measurement, that is, the linear map $M_A(X)=A^*XA$ for all $X\in\mathcal{B}_1(\mathcal{H})$ and the commutator is defined as usually $[\Phi,\Psi]=\Phi\Psi-\Psi\Phi$. Note that, since $\mathcal{B}_1(\mathcal{H})$ is a two-sided ideal of $\mathcal{B}(\mathcal{H})$, M_A can be defined either as a linear map $\mathcal{B}_1(\mathcal{H})\to\mathcal{B}_1(\mathcal{H})$ or as a linear map $\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{H})$. Actually, if we consider $M_A\colon\mathcal{B}_1(\mathcal{H})\to\mathcal{B}_1(\mathcal{H})$, then its dual map $M_A^\sharp\colon\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{H})$ is the one-element measurement map M_{A^*}.

Remark 6.3. Let $\Psi\colon\mathcal{B}_1(\mathcal{H})\to\mathcal{B}_1(\mathcal{H})$ be a bounded linear map and $A\in\mathcal{B}(\mathcal{H})$. Then, $[\Psi,M_A]=0$ if and only if $[\Psi^\sharp,M_{A^*}]=0$.

The one-element measurement operator M_A is usually associated to a positive operator A. In this case, due to a certain mathematical model for quantum measurements, one rather considers the one-element measurement in the Lüders form $M_{A^{1/2}}$ for some positive operator A, e.g. see [2] or Gudder [22]. In order to briefly explain this, let us recall that in yes–no experiments *quantum effects* are modelled by operators $A \in \mathcal{B}(\mathcal{H})^+$ with $A \leq I$, for some Hilbert space $\mathcal{H}$. In case A is a projection, it corresponds to a *sharp* quantum measurement, while, in general, A corresponds to a *unsharp* (more realistic) quantum measurement. Given a state $\rho \in \mathcal{D}(\mathcal{H})$, that is, the probability that the quantum effect A occurs (has a yes outcome) in the state ρ is $P_\rho(A) = \operatorname{tr}(\rho A) = \operatorname{tr}(A^{1/2}\rho A^{1/2}) = \operatorname{tr}(M_{A^{1/2}}(\rho))$ and, consequently, the postmeasurement state is $M_{A^{1/2}}(\rho)/P_\rho(A)$.

The equivalence of (ii)–(v) in the following theorem has been obtained in [8], compared with Theorem 3.2. Here, we show that these equivalences can be obtained as a direct consequence of Corollary 2.3. We add two more equivalent characterisations, assertions (vi) and (vii), in terms of the dual infinitesimal generator, which actually make the proofs simpler, while assertion (i) is an equivalent formulation that points out the Noether's type theorem character. In this respect, assertions (iii) and (vi) express a symmetry property, while assertions (i), (ii), and (vii) express a conservation law of the system.

Theorem 6.4. *Let $\mathbf{\Psi} = \{\Psi_t\}_{t\geq 0}$ be a strongly continuous one-parameter semigroup of stochastic maps on $\mathcal{B}_1(\mathcal{H})$, ψ its infinitesimal generator, and let $A \in \mathcal{B}(\mathcal{H})^+$. With notation as before, the following assertions are equivalent:*

- (i) *For any density operator ρ, both the expected value and the standard deviation of A with respect to $\Psi_t(\rho)$ are constant with respect to $t \geq 0$.*
- (ii) $\Psi_t^\sharp(A) = A$ *and* $\Psi_t^\sharp(A^2) = A^2$ *for all* $t \geq 0$.
- (iii) $[M_{A^{1/2}}, \Psi_t] = 0$ *for all* $t \geq 0$.
- (iv) $[M_{A^{1/2}}, \Psi_t^\sharp] = 0$ *for all* $t \geq 0$.
- (v) $[M_{A^{1/2}}, \psi] = 0$ *that is, for all* $T \in \operatorname{Dom}(\psi)$ *we have* $A^{1/2}TA^{1/2} \in \operatorname{Dom}(\psi)$ *and* $\psi(A^{1/2}TA^{1/2}) = A^{1/2}\psi(T)A^{1/2}$.
- (vi) $[M_{A^{1/2}}, \psi^\sharp] = 0$ *that is, for all* $S \in \operatorname{Dom}(\psi^\sharp)$ *we have* $A^{1/2}SA^{1/2} \in \operatorname{Dom}(\psi^\sharp)$ *and* $\psi^\sharp(A^{1/2}TA^{1/2}) = A^{1/2}\psi^\sharp(T)A^{1/2}$.
- (vii) $A, A^2 \in \operatorname{Ker}(\psi^\sharp)$, *that is,* $A, A^2 \in \operatorname{Dom}(\psi^\sharp)$ *and* $\psi^\sharp(A) = \psi^\sharp(A^2) = 0$.

Before proceeding to the proof of this theorem, we prove two preliminary results. The first one is essentially Remark 5.4 in [8] for which we provide a coordinate free proof.

Lemma 6.5. *If E is a projection and $C \in \mathcal{B}_1(\mathcal{H})^+$ such that* $\operatorname{tr}(C) = \operatorname{tr}(ECE)$, *then $C = CE = EC$.*

Proof. Taking into account that $C^{1/2}EC^{1/2} \leq C$ and that

$$0 \leq \operatorname{tr}(C - C^{1/2}EC^{1/2}) = \operatorname{tr}(C) - \operatorname{tr}(C^{1/2}EC^{1/2}) = \operatorname{tr}(C) - \operatorname{tr}(ECE) = 0,$$

it follows that $C = C^{1/2}EC^{1/2}$ hence,

$$0 = C^{1/2}(I-E)C^{1/2} = C^{1/2}(I-E)(I-E)C^{1/2} = ((I-E)C^{1/2})^*((I-E)C^{1/2}),$$

which implies $(I - E)C^{1/2} = 0$ hence $(I - E)C = 0$. From here, it follows $EC = C$ and then taking adjoints we have $CE = C$ as well. □

The second preliminary result is a short cut of Corollary 5.5, Corollary 5.6, and Lemma 5.7 in [8].

Lemma 6.6. *Let Ψ be a stochastic map with respect to a Hilbert space $\mathcal{H}$ and let E be a projection such that $\Psi^\sharp(E) = E$. Then,*

(i) $\Psi(ETE) = E\Psi(ETE) = \Psi(ETE)E$ *for all* $T \in \mathcal{B}_1(\mathcal{H})$.
(ii) $E\Psi^\sharp(ESE) = \Psi^\sharp(ESE)E = \Psi^\sharp(ESE)$ *for all* $S \in \mathcal{B}(\mathcal{H})$.
(iii) $\Psi^\sharp(ESE) = E\Psi^\sharp(S)E$ *for all* $S \in \mathcal{B}(\mathcal{H})$.

Proof. (i) It is sufficient to prove this for all $T \in \mathcal{B}_1(\mathcal{H})^+$. With this assumption, we have

$$\begin{aligned}\operatorname{tr}(E\Psi(ETE)E) &= \operatorname{tr}(E\Psi(ETE)) = \langle E, \Psi(ETE)\rangle \\ &= \langle \Psi^\sharp(E), ETE\rangle = \langle E, ETE\rangle = \operatorname{tr}(ETE) = \operatorname{tr}(\Psi(ETE)),\end{aligned}$$

and, consequently, applying Lemma 6.5 for $C = \Psi(ETE)$, the conclusion follows.

(ii) To see this, without loss of generality it is sufficient to assume that $S \in \mathcal{B}(\mathcal{H})^+$ is a contraction, that is, $0 \leq S \leq I$. Then, $0 \leq ESE \leq E$ and hence $0 \leq \Psi^\sharp(ESE) \leq \Psi^\sharp(E) = E$, which implies that the range of $\Psi^\sharp(ESE)$ is contained in the range of E. This implies $E\Psi^\sharp(ESE) = \Psi^\sharp(ESE)$, and then by taking adjoints, we have $\Psi^\sharp(ESE)E = \Psi^\sharp(ESE)$ as well.

(iii) Let $T \in \mathcal{B}_1(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$ be arbitrary. Using assertion (ii), we have

$$\begin{aligned}\langle \Psi^\sharp(ESE), T\rangle &= \langle E\Psi^\sharp(ESE)E, T\rangle = \langle \Psi^\sharp(ESE), ETE\rangle \\ &= \langle ESE, \Psi(ETE)\rangle = \langle S, E\Psi(ETE)E\rangle\end{aligned}$$

and then, using assertion (i), we have

$$= \langle S, \Psi(ETE)\rangle = \langle \Psi^\sharp(S), ETE\rangle = \langle E\Psi^\sharp(S)E, T\rangle,$$

hence assertion (iii) follows. □

Proof of Theorem 6.4. In order to prove the equivalence of assertions (i), (ii), and (iii), we actually prove an equivalent reformulation, taking into account (4.3)–(4.5), namely that the following assertions are mutually equivalent:

(i) $[\Psi, M_{A^{1/2}}] = 0$, that is, $\Psi(A^{1/2}TA^{1/2}) = A^{1/2}\Psi(T)A^{1/2}$ for all $T \in \mathcal{B}_1(\mathcal{H})$.
(ii) $[\Psi^\sharp, M_{A^{1/2}}] = 0$, that is, $\Psi^\sharp(A^{1/2}SA^{1/2}) = A^{1/2}\Psi^\sharp(S)A^{1/2}$ for all $S \in \mathcal{B}(\mathcal{H})$.
(iii) $\Psi^\sharp(A) = A$ and $\Psi^\sharp(A^2) = A^2$.

(i)$\Leftrightarrow$(ii). This is a consequence of Remark 6.3.

(ii)$\Rightarrow$(iii). Since $\Psi^\sharp$ is unital it follows that $\Psi^\sharp(A) = \Psi^\sharp(A^{1/2}IA^{1/2}) = A^{1/2}\Psi^\sharp(I)A^{1/2} = A^{1/2}A^{1/2} = A$ and then $\Psi^\sharp(A^2) = \Psi^\sharp(A^{1/2}AA^{1/2}) = A^{1/2}\Psi^\sharp(A)A^{1/2} = A^{1/2}AA^{1/2} = A^2$.

(iii)$\Rightarrow$(ii). Letting $\Psi^\sharp = \Phi$ in Corollary 2.3, it follows that $\Psi^\sharp(S) = S$ for all $S \in C^*(I, A)$. Since $\Psi^\sharp$ is w^*-continuous, by functional calculus with bounded Borel functions on $\sigma(A)$, it follows that $\Psi^\sharp(S) = S$ for all $S \in W^*(A)$, the von Neumann algebra generated by A in $\mathcal{B}(\mathcal{H})$. In particular, for any spectral projection E of A we have $\Psi^\sharp(E) = E$. From Lemma 6.6, it follows

$$\Psi^\sharp(ETE) = E\Psi^\sharp(T)E, \quad T \in \mathcal{B}(\mathcal{H}). \tag{6.4}$$

From here, by the Spectral Theorem for A, it follows that for any function f that is continuous on $\sigma(A)$, we have

$$\Psi^\sharp(f(A)Tf(A)) = f(A)\Psi^\sharp(T)f(A), \quad T \in \mathcal{B}(\mathcal{H}). \tag{6.5}$$

Letting $f(t) = \sqrt{t}$, $t \in \sigma(A)$, the assertion follows.

(iii)$\Leftrightarrow$(iv) is a direct consequence of the duality.

(iii)$\Rightarrow$(v). For arbitrary $T \in \mathrm{Dom}(\psi)$ and $t \geq 0$, we have

$$\begin{aligned}\frac{\Psi_t(A^{1/2}TA^{1/2}) - A^{1/2}TA^{1/2}}{t} &= \frac{A^{1/2}\Psi_t(T)A^{1/2} - A^{1/2}TA^{1/2}}{t} \\ &= A^{1/2}\frac{\Psi_t(T) - T}{t}A^{1/2} \xrightarrow[t\to 0+]{} A^{1/2}\psi(T)A^{1/2},\end{aligned}$$

hence $A^{1/2}TA^{1/2} \in \mathrm{Dom}(\psi)$ and $\psi(A^{1/2}TA^{1/2}) = A^{1/2}\psi(T)A^{1/2}$.

(v)$\Rightarrow$(vi). Let $S \in \mathrm{Dom}(\psi^\sharp)$. Then, for any $T \in \mathrm{Dom}(\psi)$ we have $A^{1/2}TA^{1/2} \in \mathrm{Dom}(\psi)$ and $\psi(A^{1/2}TA^{1/2}) = A^{1/2}\psi(T)A^{1/2}$, hence

$$\langle \psi(T), A^{1/2}SA^{1/2}\rangle = \langle A^{1/2}\psi(T)A^{1/2}, S\rangle = \langle \psi(A^{1/2}TA^{1/2}), S\rangle$$

whence, taking into account of the continuity of the map $\mathcal{B}_1(\mathcal{H}) \ni T \mapsto A^{1/2}TA^{1/2} \in \mathcal{B}_1(\mathcal{H})$, it follows that $A^{1/2}SA^{1/2} \in \mathrm{Dom}(\psi^\sharp)$. Consequently,

$$\begin{aligned}\langle T, \psi^\sharp(A^{1/2}SA^{1/2})\rangle &= \langle \psi(T), A^{1/2}SA^{1/2}\rangle = \langle \psi(A^{1/2}TA^{1/2}), S\rangle \\ &= \langle A^{1/2}TA^{1/2}, \psi^\sharp(S)\rangle = \langle T, A^{1/2}\psi^\sharp(S)A^{1/2}\rangle,\end{aligned}$$

hence, $\psi^\sharp(A^{1/2}SA^{1/2}) = A^{1/2}\psi^\sharp(S)A^{1/2}$.

(vi)$\Rightarrow$(vii). By (6.3), we have $A = A^{1/2}IA^{1/2} \in \mathrm{Dom}(\psi^\sharp)$ and $\psi^\sharp(A) = A^{1/2}\psi^\sharp(I)A^{1/2} = 0$. Then, $A^2 = A^{1/2}AA^{1/2} \in \mathrm{Dom}(\psi^\sharp)$ and $\psi^\sharp(A^2) = A^{1/2}$ $\psi^\sharp(A)A^{1/2} = 0$.

(vii)$\Rightarrow$(ii). This is a consequence of Theorem 2.4. □

Remarks 6.7. (a) Under the assumptions of Theorem 6.4, each of the assertions (i)–(vii) is equivalent with each of the following assertions, cf. [8]:

(viii) $\frac{\mathrm{d}}{\mathrm{d}t}\langle \Psi_t(T), A\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle \Psi_t(T), A^2\rangle = 0$ for all $T \in \mathcal{B}_1(\mathcal{H})$.

(ix) $\frac{\mathrm{d}}{\mathrm{d}t}\langle \Psi_t(T), A^n\rangle = 0$ for all $T \in \mathcal{B}_1(\mathcal{H})$ and all $n \geq 0$.

(x) For every spectral projection E of A we have $[M_E, \psi] = 0$, that is, for any $T \in \mathrm{Dom}(\psi)$ we have $ETE \in \mathrm{Dom}(\psi)$ and $\psi(ETE) = E\psi(T)E$.

The equivalence of assertion (x) is short cut in our proof, but it is an important step during the proof provided in [8]. Assertion (viii) is clearly equivalent with assertion (ii), while assertion (ix) is equivalent with assertion (viii) in view of Corollary 2.3.

(b) A natural question is whether the condition that A^2 is a joint fixed point of $\mathbf{\Psi}$, as in Theorem 6.4.(i), is a consequence of the condition that A is a joint fixed point of $\mathbf{\Psi}$. The answer is negative, in general, and it is obtained as a consequence of Theorem 5.3, since any quantum operation is a stochastic map as well.

Acknowledgements

Work supported by the grant PN-III-P4-ID-PCE-2016-0823 Dynamics and Differentiable Ergodic Theory from UEFISCDI, Romania.

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Communicated by Claude Alain Pillet.
Received: November 12, 2017.
Accepted: February 26, 2018.