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A Modified Gravity Theory: Null Aether

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Abstract

General quantum gravity arguments predict that Lorentz symmetry might not hold exactly in nature. This has motivated much interest in Lorentz breaking gravity theories recently. Among such models are vector-tensor theories with preferred direction established at every point of spacetime by a fixed-norm vector field. The dynamical vector field defined in this way is referred to as the "aether." In this paper, we put forward the idea of a *null* aether field and introduce, for the first time, the Null Aether Theory (NAT)–a vector-tensor theory *with a preferred null direction* at each spacetime point. This construction allows us to take the null aether field in the direction of a principal null direction of the Weyl tensor of spacetime and introduce a scalar function which we call "spin-0 aether field." We study the Newtonian limit of this theory and show that it is equivalent to that of general relativity with the meaning that the solar system phenomenology remains unaffected in this approximation. We then construct exact spherically symmetric black hole solutions in the theory in four dimensions, which contain Vaidya-type nonstationary solutions and stationary Schwarzschild-(A)dS type solutions, Reissner-Nordström-(A)dS type solutions and solutions of conformal gravity as special cases. We also study exact gravitational wave solutions–AdS-plane waves and *pp*-waves–in this theory in any dimension $D \geq 3$. Assuming the Kerr-Schild-Kundt class of metrics for such solutions, we show that the full field equations of the theory are reduced to two, in general coupled, differential equations when the background metric assumes the maximally symmetric form. In the case of AdS background, we explicitly solve these coupled differential equations and thereby construct exact AdS-plane wave solutions in the theory. The main conclusion of these computations is that the spin-0 aether field acquires a "mass" determined by the cosmological constant of the background spacetime and the Lagrange multiplier given in the theory.

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1 Introduction

Lorentz violating theories of gravity have attracted much attention recently. This is mainly due to the fact that some quantum gravity theories, such as string theory and loop quantum gravity, predict that the spacetime structure at very high energies–typically at the Planck scale–may not be smooth and continuous, as assumed by relativity. This means that the rules of relativity do not apply and Lorentz symmetry must break down at or below the Planck distance. Indeed, in recent years, astrophysical observations on high-energy cosmic rays seem to have provided evidence supporting this idea (see, e.g., [1]).

The simplest way to study Lorentz violation in the context of gravity is to assume that there is a vector field with fixed norm coupling to gravity at each point of spacetime. In other words, the spacetime is locally endowed with a metric tensor and a dynamical vector field with constant norm. The vector field defined in this way is referred to as the "aether" because it establishes a preferred direction at each point in spacetime and thereby explicitly breaks local Lorentz symmetry. The existence of such a vector field would affect the propagation of particles–such as electrons and photons–through spacetime, which manifests itself at very high energies and can be observed by studying the spectrum of high energy cosmic rays. For example, the interactions of these particles with the field would restrict the electron's maximum speed or cause polarized photons to rotate as they travel through space over long distances. Any observational evidence in these directions would be a direct indication of Lorentz violation, and therefore new physics, at or beyond the Planck scale.

So vector-tensor theories of gravity are of physical importance today because they may shed some light on the internal structure of quantum gravity theories. One such theory is Einstein-Aether theory [2, 3] in which the aether field is assumed to be timelike and therefore breaks the boost sector of the Lorentz symmetry. This theory has been investigated over the years from various respects [4–22]. There also appeared some related works [23–26] which discuss the possibility of a spacelike aether field breaking the rotational invariance of space. The internal structure and dynamics of such theories are still under examination; for example, the stability problem of the aether field has been considered in [27, 28].[1] Of course, to gain more understanding in these respects, one also needs explicit analytic solutions to the fairly complicated equations of motion that these theories possess.

In this paper, we propose yet another possibility, namely, the possibility of a *null* aether field which dynamically couples to the metric tensor of spacetime. Such a vector field picks up a preferred null direction in spacetime and violates the local Lorentz invariance explicitly. From now on, we shall refer to the theory constructed in this way as Null Aether Theory (NAT). This construction enables us to naturally introduce an extra degree of freedom, i.e. the spin-0 part of the aether field, which is a scalar field that has a mass in general. By using this freedom, we show that it is possible to construct exact black hole solutions and nonlinear wave solutions in the theory.[2] Indeed, assuming the null aether vector field (v_μ) is proportional to the one of the principal null direction(s) of the Weyl

[1]Breaking of Lorentz symmetry is discussed also in [29].

[2]In the context of Einstein-Aether theory, black hole solutions were considered in [4–13] and linearized plane wave solutions were studied in [30].

tensor (l_μ), i.e. $v_\mu = \phi(x) l_\mu$, where $\phi(x)$ is the spin-0 aether field, we first discuss the Newtonian limit of NAT and show that the theory exactly recovers general relativity with the meaning that the solar system observations put no constraints on the parameters of the theory in this approximation. We then proceed to construct exact spherically symmetric black hole solutions to the full nonlinear theory in four dimensions. Among these, there are Vaidya-type nonstationary solutions which do not need the existence of any extra matter field: the null aether field present in the very foundation of the theory behaves, in a sense, as a null matter to produce such solutions. For special values of the parameters of the theory, there are also stationary Schwarzschild-(A)dS type solutions that exist even when there is no explicit cosmological constant in the theory, Reissner-Nordström-(A)dS type solutions with some "charge" sourced by the aether, and solutions of conformal gravity that contain a term growing linearly with radial distance and so associated with the flatness of the galaxy rotation curves. Our exact solutions perfectly match the solutions in the Newtonian limit when the aether field is on the order of the Newtonian potential.

On the other hand, the same construction, $v_\mu = \phi(x) l_\mu$, also permits us to obtain exact solutions describing gravitational waves in NAT. In searching for such solutions, the Kerr-Schild-Kundt class of metrics [31–36] was shown to be a valuable tool to start with: Indeed, recently, it has been proved that these metrics are universal in the sense that they constitute solutions to the field equations of *any theory* constructed by the contractions of the curvature tensor and its covariant derivatives at any order [36]. In starting this work, one of our motivations was to examine whether such universal metrics are solutions to vector-tensor theories of gravity as well. Later on, we perceived that this is only possible when the vector field in the theory is null and aligned with the propagation direction of the waves. Taking the metric in the Kerr-Schild-Kundt class with maximally symmetric backgrounds and assuming further $l^\mu \partial_\mu \phi = 0$, we show that the AdS-plane waves and *pp*-waves form a special class of exact solutions to NAT. The whole set of field equations of the theory are reduced to two coupled differential equations, in general, one for a scalar function related to the profile function of the wave and one for the "massive" spin-0 aether field $\phi(x)$. When the background spacetime is AdS, it is possible to solve these coupled differential equations exactly in three dimensions and explicitly construct plane waves propagating in the AdS spacetime. Such constructions are possible also in dimensions higher than three but with the simplifying assumption that the profile function describing the AdS-plane wave does not depend on the transverse $D-3$ coordinates. The main conclusion of these computations is that the mass corresponding to the spin-0 aether field acquires an upper bound (the Breitenlohner-Freedman bound [37]) determined by the value of the cosmological constant of the background spacetime. In the case of *pp*-waves, where the background is flat, the scalar field equations decouple and form one Laplace equation for a scalar function related to the profile function of the wave and one massive Klein-Gordon equation for the spin-0 aether field in $(D-2)$-dimensional Euclidean flat space. Because of this decoupling, plane wave solutions, which are the subset of *pp*-waves, can always be constructed in NAT. Being exact, all these solutions might provide important insights into the internal dynamics of Lorentz violating vector-tensor theories of gravity.

The paper is structured as follows. In Sec. 2, we introduce NAT and present the field equations. In Sec. 3, we study the Newtonian limit of the theory to see the effect of the null vector field on the solar system observations. In Sec. 4, we construct exact spherically symmetric black hole solutions in their full generality in four dimensions. In Sec. 5, we study the nonlinear wave solutions of NAT propagating in nonflat backgrounds, which are assumed to be maximally symmetric, by taking the metric in the Kerr-Schild-Kundt class. In Sec. 6, we specifically consider AdS-plane waves describing plane waves moving in the AdS spacetime in $D \geq 3$ dimensions. In Sec. 7, we briefly describe the *pp*-wave spacetimes and show that they provide exact solutions to NAT. We also discuss the availability of the subclass plane waves under certain conditions. Finally, in Sec. 8, we summarize our results.

We shall use the metric signature $(-,+,+,+,\ldots)$ throughout the paper.

2 Null Aether Theory

The theory we shall consider is defined in D dimensions and described by, in the absence of matter fields, the action

$$I = \frac{1}{16\pi G}\int d^D x\sqrt{-g}\,[R - 2\Lambda - K^{\mu\nu}{}_{\alpha\beta}\nabla_\mu v^\alpha \nabla_\nu v^\beta + \lambda(v_\mu v^\mu + \varepsilon)], \tag{1}$$

where

$$K^{\mu\nu}{}_{\alpha\beta} = c_1 g^{\mu\nu} g_{\alpha\beta} + c_2 \delta^\mu_\alpha \delta^\nu_\beta + c_3 \delta^\mu_\beta \delta^\nu_\alpha - c_4 v^\mu v^\nu g_{\alpha\beta}. \tag{2}$$

Here Λ is the cosmological constant and v^μ is the so-called aether field which dynamically couples to the metric tensor $g_{\mu\nu}$ and has the fixed-norm constraint

$$v_\mu v^\mu = -\varepsilon, \quad (\varepsilon = 0, \pm 1) \tag{3}$$

which is introduced into the theory by the Lagrange multiplier λ in (1). Accordingly, the aether field is a timelike (spacelike) vector field when $\varepsilon = +1$ ($\varepsilon = -1$), and it is a null vector field when $\varepsilon = 0$.[3] The constant coefficients c_1, c_2, c_3 and c_4 appearing in (2) are the dimensionless parameters of the theory.[4]

The equations of motion can be obtained by varying the action (1) with respect to the independent variables: Variation with respect to λ produces the constraint equation (3) and variation with respect to $g^{\mu\nu}$ and v^μ produces the respective, dynamical field equations

$$\begin{aligned} G_{\mu\nu} + \Lambda g_{\mu\nu} = {} & \nabla_\alpha \left[J^\alpha{}_{(\mu} v_{\nu)} - J_{(\mu}{}^\alpha v_{\nu)} + J_{(\mu\nu)} v^\alpha \right] \\ & + c_1 \left(\nabla_\mu v_\alpha \nabla_\nu v^\alpha - \nabla_\alpha v_\mu \nabla^\alpha v_\nu \right) \\ & + c_4 \dot{v}_\mu \dot{v}_\nu + \lambda v_\mu v_\nu - \frac{1}{2} L g_{\mu\nu}, \end{aligned} \tag{4}$$

$$c_4 \dot{v}^\alpha \nabla_\mu v_\alpha + \nabla_\alpha J^\alpha{}_\mu + \lambda v_\mu = 0, \tag{5}$$

[3] The case with $\varepsilon = +1$ is associated with Einstein-Aether theory [2,3].

[4] In Einstein-Aether theory, these parameters are constrained by some theoretical and observational arguments [2,3,16,38–43].

where $\dot{v}^\mu \equiv v^\alpha \nabla_\alpha v^\mu$ and

$$J^\mu{}_\alpha \equiv K^{\mu\nu}{}_{\alpha\beta} \nabla_\nu v^\beta, \tag{6}$$
$$L \equiv J^\mu{}_\alpha \nabla_\mu v^\alpha. \tag{7}$$

In writing (4), we made use of the constraint (3). From now on, we will assume that the aether field v^μ is null (i.e., $\varepsilon = 0$) and refer to the above theory as Null Aether Theory, which we have dubbed NAT. This fact enables us to obtain λ from the aether equation (5) by contracting it by the vector $u^\mu = \delta^\mu_0$; that is,

$$\lambda = -\frac{1}{u^\nu v_\nu} \left[c_4 u^\mu \dot{v}^\alpha \nabla_\mu v_\alpha + u^\mu \nabla_\alpha J^\alpha{}_\mu \right], \tag{8}$$

since it is always the case that $u^\nu v_\nu \neq 0$ for a null vector. It is obvious that flat Minkowski metric ($\eta_{\mu\nu}$) and a constant null vector ($v_\mu = const.$), together with $\lambda = 0$, constitute a solution to NAT.

Null Aether Theory, to our knowledge, is introduced for the first time in this paper. There are some number of open problems to be attacked such as Newtonian limit, black holes, exact solutions, stability, etc. In this work, we investigate the Newtonian limit, the spherically symmetric black hole solutions (in $D = 4$) and the AdS wave and *pp*-wave solutions of NAT. In all these cases, we assume that $v_\mu = \phi(x) l_\mu$, where l_μ is a null vector defining the principal null direction(s) of the Weyl tensor and $\phi(x)$ is a scalar field defined as the spin-0 aether field that has a mass in general. The covariant derivative of the null vector l_μ can always be decomposed in terms of the optical scalars: expansion, twist, and shear [44].

3 Newtonian Limit of Null Aether Theory

Now we shall examine the Newtonian limit of NAT to see whether there are any contributions to the Poisson equation coming from the null aether field. For this purpose, as usual, we shall assume that the gravitational field is weak and static and produced by a nonrelativistic matter field. Also, we know that the cosmological constant–playing a significant role in cosmology–is totally negligible in this context.

Let us take the metric in the Newtonian limit as

$$ds^2 = -[1 + 2\Phi(\vec{x})]dt^2 + [1 - 2\Psi(\vec{x})][(dx^1)^2 + (dx^2)^2 + \ldots + (dx^{D-1})^2], \tag{9}$$

where $x^\mu = (t, \vec{x})$ with $\vec{x} = (x^1, x^2, \ldots, x^{D-1})$. In this spacetime, a null vector at linear order can be defined, up to a multiplicative function of $\vec{x}$, as

$$l_\mu = \delta^0_\mu + (1 - \Phi - \Psi)\frac{x^i}{r}\delta^i_\mu, \tag{10}$$

where $r^2 = \vec{x} \cdot \vec{x} = \delta_{ij} x^i x^j = x^i x^i$ with $i = 1, \ldots, D-1$. Now we write the null aether field as $v_\mu = \phi(\vec{x}) l_\mu$ (since we are studying with a null vector, we always have this freedom) and assume that $\phi(\vec{x})$ is some arbitrary function at the same order as Φ and/or Ψ. Then the zeroth component of the aether equation (5) gives, at the linear order,

$$c_1 \nabla^2 \phi + \lambda \phi = 0, \tag{11}$$

where $\nabla^2 \equiv \partial_i\partial_i$, and the ith component gives, at the linear order,

$$\begin{aligned}(c_2+c_3)r^2x^j\partial_j\partial_i\phi - (2c_1+c_2+c_3)x^ix^j\partial_j\phi \\ + [2c_1+(D-1)(c_2+c_3)]r^2\partial_i\phi - (D-2)(c_1+c_2+c_3)x^i\phi = 0,\end{aligned} \quad (12)$$

after eliminating λ using (11). It can also be shown that at the linear order the aether contribution to the equation (4) is zero and so the only contribution comes from the nonrelativistic matter for which

$$T^{matter}_{\mu\nu} = \rho_m(\vec{x})\delta^0_\mu\delta^0_\nu. \quad (13)$$

Here we are assuming that the matter fields do not couple to the aether field at the linear order. Therefore, the only nonzero components of (4) are the 00 and the ij component (the $0i$ component is satisfied identically). Taking the trace of the ij component produces

$$\nabla^2(\Phi - \Psi) = 0, \quad (14)$$

which enforces

$$\Phi = \Psi, \quad (15)$$

for the spacetime to be asymptotically flat. Using this fact, we can write, from the 00 component of (4),

$$\nabla^2\Phi = 4\pi G\rho_m. \quad (16)$$

Thus we see that the Poisson equation is unaffected by the null aether field at the linear order in G.

Outside of a spherically symmetric mass distribution, the Poisson equation (16) reduces to the Laplace equation which gives

$$\Phi(r) = -\frac{GM}{r^{D-3}}, \quad (17)$$

for dimensions $D > 3$. Here we have absorbed all the constants into G. In $D = 3$, since the force goes with inverse r, the Newtonian potential should go with the logarithm of r. On the other hand, for spherical symmetry, the condition (12) can be solved and yields

$$\phi(r) = a_1r^{\alpha_1} + a_2r^{\alpha_2}, \quad (18)$$

where a_1 and a_2 are arbitrary constants and

$$\alpha_{1,2} = -\frac{1}{2}\left[(D-3) \pm \sqrt{(D-1)^2 + 4(D-2)\frac{c_1}{c_2+c_3}}\right]. \quad (19)$$

This solutions immediately puts the following condition on the parameters of the theory

$$\frac{c_1}{c_2+c_3} \geq -\frac{(D-1)^2}{4(D-2)}, \quad (20)$$

since we always have $D > 2$. Specifically, when $c_1 = -(D-1)^2(c_2+c_3)/4(D-2)$, we have

$$\phi(r) = \frac{a_1+a_2}{r^{(D-3)/2}}; \quad (21)$$

when $c_1 = 0$, we have

$$\phi(r) = \frac{a_1}{r^{D-2}} + a_2 r; \tag{22}$$

or when $c_1 = -(c_2 + c_3)$, we have

$$\phi(r) = \frac{a_1}{r^{D-3}} + a_2. \tag{23}$$

To sum up, in $D = 4$, we have exactly the same Newtonian approximation as that of general relativity. Hence our NAT is in agreement with the solar system observations and this fact puts no constraints on the parameters $\{c_1, c_2, c_3, c_4\}$ of the theory.

4 Black Hole Solutions in Null Aether Theory

In this section, we shall construct spherically symmetric black hole solutions to NAT in $D = 4$. Let us start with the generic spherically symmetric metric in the following form with $x^\mu = (u, r, \theta, \vartheta)$:

$$ds^2 = -\left(1 - \frac{\Lambda}{3} r^2\right) du^2 + 2dudr + r^2 d\theta^2 + r^2 \sin^2\theta d\vartheta^2 + 2f(u,r)du^2, \tag{24}$$

where Λ is the cosmological constant. For $f(u,r) = 0$, this becomes the metric of the usual (A)dS spacetime. Since the aether field is null, we take it to be $v_\mu = \phi(u,r) l_\mu$ with $l_\mu = \delta^u_\mu$ being the null vector of the geometry.

With the metric ansatz (24), from the u component of the aether equation (5), we obtain

$$\begin{aligned}\lambda = -\frac{1}{3r^2\phi}\Bigg\{ & 3(c_1+c_3)\left[\Lambda r^2 + (r^2 f')'\right]\phi + c_1\left[(3 - \Lambda r^2 - 6f)(r^2\phi')' + 6r(r\dot{\phi})'\right] \\ & + 3(c_2+c_3)(r^2\dot{\phi})' - 3c_4\left[2r^2\phi'^2 + \phi(r^2\phi')'\right]\phi \Bigg\},\end{aligned} \tag{25}$$

and from the r component, we have

$$(c_2 + c_3)(r^2\phi'' + 2r\phi') - 2(c_1 + c_2 + c_3)\phi = 0, \tag{26}$$

where the prime denotes differentiation with respect to r and the dot denotes differentiation with respect to u. The equation (26) can easily be solved and the generic solution is

$$\phi(u,r) = a_1(u) r^{\alpha_1} + a_2(u) r^{\alpha_2}, \tag{27}$$

for some arbitrary functions $a_1(u)$ and $a_2(u)$, where

$$\alpha_{1,2} = -\frac{1}{2}\left[1 \pm \sqrt{9 + 8\frac{c_1}{c_2 + c_3}}\right]. \tag{28}$$

Note that when $c_1 = -9(c_2 + c_3)/8$, the square root vanishes and the roots coincide to give $\alpha_1 = \alpha_2 = -1/2$. Inserting this solution into the Einstein equations (4) yields, for the ur component,

$$(1 + 2\alpha_1)a_1(u)^2 b_1 r^{2\alpha_1} + (1 + 2\alpha_2)a_2(u)^2 b_2 r^{2\alpha_2} - (rf)' = 0, \tag{29}$$

with the identifications

$$b_1 \equiv -\frac{1}{4}[2c_2 + (c_2 + c_3)\alpha_1], \quad b_2 \equiv -\frac{1}{4}[2c_2 + (c_2 + c_3)\alpha_2]. \tag{30}$$

Thus we obtain

$$f(u, r) = \begin{cases} a_1(u)^2 b_1 r^{2\alpha_1} + a_2(u)^2 b_2 r^{2\alpha_2} + \dfrac{\tilde{\mu}(u)}{r}, & \text{for } \alpha_1 \neq -\dfrac{1}{2} \;\&\; \alpha_2 \neq -\dfrac{1}{2}, \\ \dfrac{\mu(u)}{r}, & \text{for } \alpha_1 = \alpha_2 = -\dfrac{1}{2}, \end{cases} \tag{31}$$

where $\tilde{\mu}(u)$ and $\mu(u)$ are arbitrary functions. Notice that the last case occurs only when $c_1 = -9(c_2 + c_3)/8$. If we plug (31) into the other components, we identically satisfy all the equations except for the uu component which, together with λ from (25), produces

$$[2c_2 + (c_2 + c_3)\alpha_1]\dot{a}_1 a_2 + [2c_2 + (c_2 + c_3)\alpha_2]a_1\dot{a}_2 + 2\dot{\tilde{\mu}} = 0, \tag{32}$$

for $\alpha_1 \neq -\frac{1}{2}$ and $\alpha_2 \neq -\frac{1}{2}$, and

$$(3c_2 - c_3)\dot{\overline{(a_1 + a_2)^2}} + 8\dot{\mu} = 0, \tag{33}$$

for $\alpha_1 = \alpha_2 = -\frac{1}{2}$. The last case immediately leads to

$$\mu(u) = \frac{1}{8}(c_3 - 3c_2)(a_1 + a_2)^2 + m, \tag{34}$$

where m is the integration constant. Thus we see that Vaidya-type solutions can be obtained in NAT without introducing any extra matter fields, which is unlike the case in general relativity. Observe also that when $f(u, r) = 0$, we should obtain the (A)dS metric as a solution to NAT [see (24)]. Then it is obvious from (29) that this is the case only if $\alpha_1 = \alpha_2 = -\frac{1}{2}$ corresponding to

$$\phi(u, r) = \begin{cases} \dfrac{d}{\sqrt{r}}, & \text{for } c_1 = -\dfrac{9}{8}(c_2 + c_3), \\ \dfrac{a(u)}{\sqrt{r}}, & \text{for } c_1 = -\dfrac{9}{8}(c_2 + c_3) \;\&\; c_3 = 3c_2, \end{cases} \tag{35}$$

where d is an arbitrary constant and $a(u)$ is an arbitrary function.

Defining a new time coordinate t by the transformation

$$du = g(t, r)dt + \frac{dr}{1 - \frac{\Lambda}{3}r^2 - 2f(t, r)}, \tag{36}$$

one can bring the metric (24) into the Schwarzschild coordinates

$$ds^2 = -\left(1 - \frac{\Lambda}{3}r^2 - 2f\right) g^2 dt^2 + \frac{dr}{\left(1 - \frac{\Lambda}{3}r^2 - 2f\right)} + r^2 d\theta^2 + r^2 \sin^2\theta d\vartheta^2, \tag{37}$$

where the function $g(t, r)$ should satisfy

$$\frac{\partial g}{\partial r} = 2\left(1 - \frac{\Lambda}{3}r^2 - 2f\right)^{-2} \frac{\partial f}{\partial t}. \tag{38}$$

When $a_1(u)$ and $a_2(u)$ are constants, since $f = f(r)$ then, the condition (38) says that $g = g(t)$ and so it can be absorbed into the time coordinate t, meaning that $g(t, r)$ can be set equal to unity in (36) and (37). In this case, the solution (37) will describe a spherically symmetric stationary black hole spacetime. The horizons of this solution should then be determined by solving the equation

$$0 = h(r) \equiv 1 - \frac{\Lambda}{3}r^2 - 2f = \begin{cases} 1 - \frac{\Lambda}{3}r^2 - \frac{2}{r}\left(a_1^2 b_1 r^{-q} + a_2^2 b_2 r^q\right) - \frac{2\tilde{m}}{r} & (\text{for } q \neq 0), \\ 1 - \frac{\Lambda}{3}r^2 - \frac{2m}{r} & (\text{for } q = 0), \end{cases} \tag{39}$$

where $\tilde{m} = const.$, $m = const.$, and

$$q \equiv \sqrt{9 + 8\frac{c_1}{c_2 + c_3}}, \quad b_1 = \frac{1}{8}[c_3 - 3c_2 + (c_2 + c_3)q], \quad b_2 = \frac{1}{8}[c_3 - 3c_2 - (c_2 + c_3)q]. \tag{40}$$

The last case ($q = 0$) is the usual Schwarzschild-(A)dS spacetime. At this point, it is important to note that when a_1 and a_2 are in the order of the Newton's constant G, i.e. $a_1 \sim G$ and $a_2 \sim G$, since $h(r)$ depends on the squares of a_1 and a_2, we recover the Newtonian limit discussed in Sec. 3 for $\Lambda = 0$, $\tilde{m} = GM$ and $D = 4$. For special values of the parameters of the theory, the first case ($q \neq 0$) of (39) becomes a polynomial of r; for example,

- When $c_1 = 0$ ($q = 3$), $h(r) \equiv 1 - A/r^4 - Br^2 - 2\tilde{m}/r$: This is a Schwarzschild-(A)dS type solution if $A = 0$. Solutions involving terms like A/r^4 can be found in, e.g., [9,45].

- When $c_1 = -(c_2 + c_3)$ ($q = 1$), $h(r) \equiv 1 - A - \Lambda r^2/3 - B/r^2 - 2\tilde{m}/r$: This is a Reissner-Nordström-(A)dS type solution if $A = 0$.

- When $c_1 = -5(c_2 + c_3)/8$ ($q = 2$), $h(r) \equiv 1 - \Lambda r^2/3 - A/r^3 - Br - 2\tilde{m}/r$: This solution with $A = 0$ has been obtained by Mannheim and Kazanas [46] in conformal gravity who also argue that the linear term Br can explain the flatness of the galaxy rotation curves.

Here A and B are the appropriate combinations of the constants appearing in (39). For such cases, the equation $h(r) = 0$ may have at least one real root corresponding to the event horizon of the black hole. For generic values of the parameters, however, the existence of the real roots of $h(r) = 0$ depends on the signs and values of the constants Λ, b_1, b_2, and $\tilde{m}$ in (39). When q is an integer, the roots can be found by solving the polynomial equation $h(r) = 0$, as in the examples given above. When q is not an integer, finding the roots of $h(r)$ is not so easy, but when the signs of $\lim_{r\to 0^+} h(r)$ and $\lim_{r\to\infty} h(r)$ are opposite, we can say that there must be at least one real root of this function. Since the signs of these limits depends on the signs of the constants Λ, b_1, b_2, and $\tilde{m}$, we have the following cases in which $h(r)$ has at least one real root:

- If $0 < q < 3$, $\Lambda < 0$, $b_1 > 0 \quad \Rightarrow \quad \lim\limits_{r\to 0^+} h(r) < 0 \quad \& \quad \lim\limits_{r\to\infty} h(r) > 0$;
- If $0 < q < 3$, $\Lambda > 0$, $b_1 < 0 \quad \Rightarrow \quad \lim\limits_{r\to 0^+} h(r) > 0 \quad \& \quad \lim\limits_{r\to\infty} h(r) < 0$;
- If $q > 3$, $b_1 > 0$, $b_2 < 0 \quad \Rightarrow \quad \lim\limits_{r\to 0^+} h(r) < 0 \quad \& \quad \lim\limits_{r\to\infty} h(r) > 0$;
- If $q > 3$, $b_1 < 0$, $b_2 > 0 \quad \Rightarrow \quad \lim\limits_{r\to 0^+} h(r) > 0 \quad \& \quad \lim\limits_{r\to\infty} h(r) < 0$.

Of course, these are not the only possibilities, but we give these examples to show the existence of black hole solutions of NAT in the general case.

5 Plane Wave Solutions in Null Aether Theory: Kerr-Schild-Kundt Class of Metrics

Now we shall construct exact plane wave solutions to NAT by studying in generic $D \geq 3$ dimensions. For this purpose, we first write the aether field in the form $v^\mu = \phi(x) l^\mu$, where $\phi(x)$ is a scalar function representing the spin-0 aether field and l^μ is a null vector, and assume the relations

$$l_\mu l^\mu = 0, \quad \nabla_\mu l_\nu = \frac{1}{2}(l_\mu \xi_\nu + l_\nu \xi_\mu), \quad l_\mu \xi^\mu = 0, \tag{41}$$

$$l^\mu \partial_\mu \phi = 0, \tag{42}$$

where ξ^μ is an arbitrary vector field for the time being. It should be noted that l^μ is not a Killing vector. From these relations it follows that

$$l^\mu \nabla_\mu l_\nu = 0, \quad l^\mu \nabla_\nu l_\mu = 0, \quad \nabla_\mu l^\mu = 0, \quad \dot{v}_\mu = 0, \tag{43}$$

and (6) and (7) are worked out to be

$$J^\mu{}_\nu = c_1 l_\nu \nabla^\mu \phi + c_3 l^\mu \nabla_\nu \phi + (c_1 + c_3) \phi \nabla^\mu l_\nu, \quad L = 0. \tag{44}$$

Then one can compute the field equations (4) and (5) as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \Bigg[-c_3 \nabla_\alpha \phi \nabla^\alpha \phi + (c_1 - c_3) \phi \Box \phi - 2 c_3 \phi \xi^\alpha \partial_\alpha \phi + \left(\lambda - \frac{c_1 + c_3}{4} \xi_\alpha \xi^\alpha \right) \phi^2 \Bigg] l_\mu l_\nu - (c_1 + c_3) \phi^2 R_{\mu\alpha\nu\beta} l^\alpha l^\beta, \tag{45}$$

$$[c_1(\Box \phi + \xi^\alpha \partial_\alpha \phi) + \lambda \phi] l_\mu + (c_1 + c_3) \phi R_{\mu\nu} l^\nu = 0, \tag{46}$$

where $\Box \equiv \nabla_\mu \nabla^\mu$ and use has been made of the identity $[\nabla_\mu, \nabla_\nu] l_\alpha = R_{\mu\nu\alpha\beta} l^\beta$.

Now suppose that the spacetime metric has the generalized Kerr-Schild form [47]

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + 2V l_\mu l_\nu, \tag{47}$$

where $\bar{g}_{\mu\nu}$ is the background metric assumed to be maximally symmetric; i.e. its curvature tensor has the form

$$\bar{R}_{\mu\alpha\nu\beta} = K(\bar{g}_{\mu\nu}\bar{g}_{\alpha\beta} - \bar{g}_{\mu\beta}\bar{g}_{\nu\alpha}) \tag{48}$$

with

$$K = \frac{\bar{R}}{D(D-1)} = const. \tag{49}$$

It is therefore either Minkowski, de Sitter (dS), or anti-de Sitter (AdS) spacetime, depending on whether $K = 0$, $K > 0$, or $K < 0$. The vector l^μ in (47) satisfies (41) and (42) and additionally

$$l^\mu \partial_\mu V = 0. \tag{50}$$

Kerr-Schild metrics with l_μ satisfying the relations (41) and (50) are called Kerr-Schild-Kundt metrics [31–36]. All the properties (41), (42), and (50), together with the inverse metric

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - 2V l^\mu l^\nu, \tag{51}$$

imply that (see, e.g., [32])

$$\Gamma^\mu_{\mu\nu} = \bar{\Gamma}^\mu_{\mu\nu}, \quad l_\mu \Gamma^\mu_{\alpha\beta} = l_\mu \bar{\Gamma}^\mu_{\alpha\beta}, \quad l^\alpha \Gamma^\mu_{\alpha\beta} = l^\alpha \bar{\Gamma}^\mu_{\alpha\beta}, \tag{52}$$

$$\bar{g}^{\alpha\beta}\Gamma^\mu_{\alpha\beta} = \bar{g}^{\alpha\beta}\bar{\Gamma}^\mu_{\alpha\beta}, \tag{53}$$

$$R_{\mu\alpha\nu\beta} l^\alpha l^\beta = \bar{R}_{\mu\alpha\nu\beta} l^\alpha l^\beta = -K l_\mu l_\nu, \tag{54}$$

$$R_{\mu\nu} l^\nu = \bar{R}_{\mu\nu} l^\nu = (D-1) K l_\mu, \tag{55}$$

$$R = \bar{R} = D(D-1)K, \tag{56}$$

and the Einstein tensor is calculated as

$$G_{\mu\nu} = -\frac{(D-1)(D-2)}{2} K \bar{g}_{\mu\nu} - \rho l_\mu l_\nu, \tag{57}$$

with

$$\rho \equiv \bar{\Box} V + 2\xi^\alpha \partial_\alpha V + \left[\frac{1}{2}\xi_\alpha \xi^\alpha + (D+1)(D-2)K\right] V, \tag{58}$$

where $\bar{\Box} \equiv \bar{\nabla}_\mu \bar{\nabla}^\mu$ and $\bar{\nabla}_\mu$ is the covariant derivative with respect to the background metric $\bar{g}_{\mu\nu}$.

Thus, for the metric ansatz (47), the field equations (45) and (46) become

$$\begin{aligned} &\left[-\frac{(D-1)(D-2)}{2} K + \Lambda\right] \bar{g}_{\mu\nu} - (\rho - 2\Lambda V) l_\mu l_\nu \\ &\quad = \Bigg\{ -c_3 \bar{\nabla}_\alpha \phi \bar{\nabla}^\alpha \phi + (c_1 - c_3)\phi \bar{\Box}\phi - 2c_3 \phi \xi^\alpha \partial_\alpha \phi \\ &\qquad + \left[\lambda + (c_1 + c_3)\left(K - \frac{1}{4}\,\xi_\alpha \xi^\alpha\right)\right] \phi^2 \Bigg\} l_\mu l_\nu, \end{aligned} \tag{59}$$

$$\{c_1(\bar{\Box}\phi + \xi^\alpha \partial_\alpha \phi) + [\lambda + (c_1 + c_3)(D-1)K]\,\phi\} l_\mu = 0. \tag{60}$$

From these, we deduce that

$$\Lambda = \frac{(D-1)(D-2)}{2}K, \tag{61}$$

$$\begin{aligned}
&\bar{\Box}V + 2\xi^\alpha\partial_\alpha V + \left[\frac{1}{2}\xi_\alpha\xi^\alpha + 2(D-2)K\right]V \\
&\quad = c_3\left[\bar{\nabla}_\alpha\phi\bar{\nabla}^\alpha\phi - \frac{\lambda}{c_1}\phi^2\right] + (c_1+c_3)\phi\xi^\alpha\partial_\alpha\phi \\
&\qquad + \frac{c_1+c_3}{c_1}\left\{[c_1(D-2) - c_3(D-1)]K + \frac{c_1}{4}\xi_\alpha\xi^\alpha\right\}\phi^2,
\end{aligned} \tag{62}$$

$$c_1(\bar{\Box}\phi + \xi^\alpha\partial_\alpha\phi) + [\lambda + (c_1+c_3)(D-1)K]\phi = 0, \tag{63}$$

where we eliminated the $\phi\bar{\Box}\phi$ term that appears in (59) by using the aether equation (63) and assuming $c_1 \neq 0$.

Now let us make the ansatz

$$V(x) = V_0(x) + \alpha\phi(x)^2, \tag{64}$$

for some arbitrary constant α. With this, we can write (62) as

$$\begin{aligned}
&\bar{\Box}V_0 + 2\xi^\alpha\partial_\alpha V_0 + \left[\frac{1}{2}\xi_\alpha\xi^\alpha + 2(D-2)K\right]V_0 \\
&\quad = (c_3 - 2\alpha)\left\{\bar{\nabla}_\alpha\phi\bar{\nabla}^\alpha\phi - \frac{1}{c_1}[\lambda + (c_1+c_3)(D-1)K]\phi^2\right\} \\
&\quad + (c_1 + c_3 - 2\alpha)\left\{\phi\xi^\alpha\partial_\alpha\phi + \left[(D-2)K + \frac{1}{4}\xi_\alpha\xi^\alpha\right]\phi^2\right\}.
\end{aligned} \tag{65}$$

Here there are two possible choices for α. The first one is $\alpha = c_3/2$, as in the previous section, for which (65) becomes

$$\begin{aligned}
&\bar{\Box}V_0 + 2\xi^\alpha\partial_\alpha V_0 + \left[\frac{1}{2}\xi_\alpha\xi^\alpha + 2(D-2)K\right]V_0 \\
&\qquad = c_1\left\{\phi\xi^\alpha\partial_\alpha\phi + \left[(D-2)K + \frac{1}{4}\xi_\alpha\xi^\alpha\right]\phi^2\right\},
\end{aligned} \tag{66}$$

and reduces to

$$\bar{\Box}V_0 = 0 \tag{67}$$

when $K = 0$ and $\xi^\mu = 0$, which is the *pp*-wave case to be discussed in Sec. 7. The other choice, $\alpha = (c_1 + c_3)/2$, drops the second term in (65) and produces

$$\begin{aligned}
&\bar{\Box}V_0 + 2\xi^\alpha\partial_\alpha V_0 + \left[\frac{1}{2}\xi_\alpha\xi^\alpha + 2(D-2)K\right]V_0 \\
&\qquad = -c_1\bar{\nabla}_\alpha\phi\bar{\nabla}^\alpha\phi + [\lambda + (c_1+c_3)(D-1)K]\phi^2.
\end{aligned} \tag{68}$$

Here it should be stressed that this last case is present only when the background metric is nonflat (i.e. $K \neq 0$) and/or $\xi^\mu \neq 0$.

On the other hand, the aether equation (63) can be written as

$$(\bar{\Box} + \xi^\alpha \partial_\alpha)\phi - m^2\phi = 0, \tag{69}$$

where, assuming λ is constant, we defined

$$m^2 \equiv -\frac{1}{c_1}\left[\lambda + (c_1 + c_3)(D-1)K\right] \tag{70}$$

since $c_1 \neq 0$. The equation (69) can be considered as the equation of the spin-0 aether field ϕ with m being the "mass" of the field. The definition (70) requires that

$$\frac{1}{c_1}\left[\lambda + (c_1 + c_3)(D-1)K\right] \leq 0, \tag{71}$$

the same constraint as in (130) when $K = 0$. Obviously, the field ϕ becomes "massless" if

$$\lambda = -(c_1 + c_3)(D-1)K. \tag{72}$$

Thus we have shown that, for any solution ϕ of the equation (69), there corresponds a solution V_0 of the equation (66) for $\alpha = c_3/2$ or of the equation (68) for $\alpha = (c_1 + c_3)/2$, and we can construct an exact wave solution with nonflat background given by (47) with the profile function (64) in NAT.

6 AdS-Plane Waves in Null Aether Theory

In this section, we shall specifically consider AdS-plane waves for which the background metric $\bar{g}_{\mu\nu}$ is the usual D-dimensional AdS spacetime with the curvature constant

$$K \equiv -\frac{1}{\ell^2} = -\frac{2|\Lambda|}{(D-1)(D-2)}, \tag{73}$$

where ℓ is the radius of curvature of the spacetime. We shall represent the spacetime by the conformally flat coordinates for simplicity; i.e. $x^\mu = (u, v, x^i, z)$ with $i = 1, \ldots, D-3$ and

$$d\bar{s}^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu = \frac{\ell^2}{z^2}(2dudv + dx_i dx^i + dz^2), \tag{74}$$

where u and v are the double null coordinates. In these coordinates, the boundary of the AdS spacetime lies at $z = 0$.

Now if we take the null vector in the full spacetime of the Kerr-Schild form (47) as $l_\mu = \delta_\mu^u$, then using (51) along with $l_\mu l^\mu = 0$,

$$l^\mu = g^{\mu\nu}l_\nu = \bar{g}^{\mu\nu}l_\nu = \frac{z^2}{\ell^2}\delta_v^\mu \quad \Rightarrow \quad l^\alpha\partial_\alpha V = \frac{z^2}{\ell^2}\frac{\partial V}{\partial v} = 0 \;\; \& \;\; l^\alpha\partial_\alpha\phi = \frac{z^2}{\ell^2}\frac{\partial\phi}{\partial v} = 0, \tag{75}$$

so the functions V and ϕ are independent of the coordinate v; that is, $V = V(u, x^i, z)$ and $\phi = \phi(u, x^i, z)$. Therefore the full spacetime metric defined by (47) will be

$$ds^2 = [\bar{g}_{\mu\nu} + 2V(u, x^i, z) l_\mu l_\nu] dx^\mu dx^\nu = d\bar{s}^2 + 2V(u, x^i, z) du^2, \tag{76}$$

with the background metric (74). It is now straightforward to show that (see also [32])

$$\nabla_\mu l_\nu = \bar{\nabla}_\mu l_\nu = \frac{1}{z}(l_\mu \delta^z_\nu + l_\nu \delta^z_\mu), \tag{77}$$

where we used the second property in (52) to convert the full covariant derivative ∇_μ to the background one $\bar{\nabla}_\mu$, and $l_\mu = \delta^u_\mu$ with $\partial_\mu l_\nu = 0$. Comparing (77) with the defining relation in (41), we see that

$$\left.\begin{aligned} \xi_\mu &= \frac{2}{z}\delta^z_\mu, \\ \xi^\mu &= g^{\mu\nu}\xi_\nu = \bar{g}^{\mu\nu}\xi_\nu = \frac{2z}{\ell^2}\delta^\mu_z, \end{aligned}\right\} \quad \Rightarrow \quad \xi_\mu \xi^\mu = \frac{4}{\ell^2}, \tag{78}$$

where we again used (51) together with $l_\mu \xi^\mu = 0$.

Thus, for the AdS-plane wave ansatz (76) with the profile function

$$V(u, x^i, z) = V_0(u, x^i, z) + \alpha\, \phi(u, x^i, z)^2 \tag{79}$$

to be an exact solution of NAT, the equations that must be solved are the aether equation (69), which takes the form

$$z^2 \hat{\partial}^2 \phi + (4 - D) z\, \partial_z \phi - m^2 \ell^2 \phi = 0, \tag{80}$$

where $\hat{\partial}^2 \equiv \partial_i \partial^i + \partial_z^2$ and

$$m^2 \equiv -\frac{1}{c_1}\left[\lambda - (c_1 + c_3)\frac{D-1}{\ell^2}\right], \tag{81}$$

and the equation (66) for $\alpha = c_3/2$, which becomes

$$z^2 \hat{\partial}^2 V_0 + (6 - D) z\, \partial_z V_0 + 2(3 - D) V_0 = c_1[2z\phi\partial_z\phi + (3 - D)\phi^2], \tag{82}$$

or the equation (68) for $\alpha = (c_1 + c_3)/2$, which becomes

$$z^2 \hat{\partial}^2 V_0 + (6 - D) z\, \partial_z V_0 + 2(3 - D) V_0 = -c_1[z^2(\hat{\partial}\phi)^2 + m^2 \ell^2 \phi^2], \tag{83}$$

where $(\hat{\partial}\phi)^2 \equiv \partial_i \phi \partial^i \phi + (\partial_z \phi)^2$.

6.1 AdS-Plane Waves in Three Dimensions

It is remarkable that the equations (80), (82), and (83) can be solved exactly in $D = 3$. In that case $x^\mu = (u, v, z)$, and so, $V_0 = V_0(u, z)$ and $\phi = \phi(u, z)$. Then (80) becomes

$$z^2 \partial_z^2 \phi + z \partial_z \phi - m^2 \ell^2 \phi = 0, \tag{84}$$

with

$$m^2 \equiv -\frac{1}{c_1}\left[\lambda - \frac{2(c_1+c_3)}{\ell^2}\right], \tag{85}$$

and has the general solution, when $m \neq 0$,

$$\phi(u,z) = a_1(u) z^{m\ell} + a_2(u) z^{-m\ell}, \tag{86}$$

where $a_1(u)$ and $a_2(u)$ are arbitrary functions. With this solution, (82) and (83) can be written compactly as

$$z^2 \partial_z^2 V_0 + 3z\partial_z V_0 = E_1(u) z^{2m\ell} + E_2(u) z^{-2m\ell}, \tag{87}$$

where

$$\left.\begin{array}{l} E_1(u) \equiv 2c_1 m\ell\, a_1(u)^2, \\ E_2(u) \equiv -2c_1 m\ell\, a_2(u)^2, \end{array}\right\} \text{ for } \alpha = \frac{c_3}{2}, \tag{88}$$

$$\left.\begin{array}{l} E_1(u) \equiv -2c_1 m^2\ell^2\, a_1(u)^2, \\ E_2(u) \equiv -2c_1 m^2\ell^2\, a_2(u)^2, \end{array}\right\} \text{ for } \alpha = \frac{c_1+c_3}{2}. \tag{89}$$

The general solution of (87) is

$$V_0(u,z) = b_1(u) + b_2(u) z^{-2} + \frac{1}{4m\ell}\left[\frac{E_1(u)}{m\ell+1}\, z^{2m\ell} + \frac{E_2(u)}{m\ell-1}\, z^{-2m\ell}\right], \tag{90}$$

with the arbitrary functions $b_1(u)$ and $b_2(u)$. Note that the second term $b_2(u)z^{-2}$ can always be absorbed into the AdS part of the metric (76) by a redefinition of the null coordinate v, which means that one can always set $b_2(u) = 0$ here and in the following solutions without loosing any generality. In obtaining (90), we assumed that $m\ell \pm 1 \neq 0$. If, on the other hand, $m\ell + 1 = 0$, then the above solution becomes

$$V_0(u,z) = b_1(u) + b_2(u) z^{-2} - \frac{E_1(u)}{2}\, z^{-2}\ln z + \frac{E_2(u)}{8}\, z^2, \tag{91}$$

and if $m\ell - 1 = 0$, it becomes

$$V_0(u,z) = b_1(u) + b_2(u) z^{-2} + \frac{E_1(u)}{8}\, z^2 - \frac{E_2(u)}{2}\, z^{-2}\ln z. \tag{92}$$

At this point, a physical discussion must be made about the forms of the solutions (86) and (90): As we pointed out earlier, the point $z = 0$ represents the boundary of the background AdS spacetime; so, in order to have an asymptotically AdS behavior as we approach $z = 0$, we should have (the Breitenlohner-Freedman bound [37])

$$-1 < m\ell < 1. \tag{93}$$

Since $\ell^2 = 1/|\Lambda|$ in three dimensions, this restricts the mass to the range

$$0 < m < \sqrt{|\Lambda|}, \tag{94}$$

which, in terms of λ through (85), becomes

$$(c_1 + 2c_3)|\Lambda| < \lambda < 2(c_1 + c_3)|\Lambda| \quad \text{if } c_1 > 0, \tag{95}$$

$$2(c_1 + c_3)|\Lambda| < \lambda < (c_1 + 2c_3)|\Lambda| \quad \text{if } c_1 < 0. \tag{96}$$

Thus we have shown that the metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \frac{\ell^2}{z^2}(2dudv + dz^2) + 2V(u,z)du^2, \tag{97}$$

with the profile function

$$V(u,z) = V_0(u,z) + \alpha\phi(u,z)^2, \tag{98}$$

describes an exact plane wave solution, propagating in the three-dimensional AdS background, in NAT.

Up to now, we considered the case $m \neq 0$. The case $m = 0$, which corresponds to the choice $\lambda = 2(c_1 + c_3)/\ell^2$ in (85), needs special handling. The solution of (84) when $m = 0$ is

$$\phi(u,z) = a_1(u) + a_2(u)\ln z, \tag{99}$$

with the arbitrary functions $a_1(u)$ and $a_2(u)$. Inserting this into (82) and (83) for $D = 3$ produces

$$z^2\partial_z^2 V_0 + 3z\partial_z V_0 = E_1(u) + E_2(u)\ln z, \tag{100}$$

where

$$\left.\begin{array}{l} E_1(u) \equiv 2c_1 a_1(u)a_2(u), \\ E_2(u) \equiv 2c_1 a_2(u)^2, \end{array}\right\} \text{ for } \alpha = \frac{c_3}{2}, \tag{101}$$

$$\left.\begin{array}{l} E_1(u) \equiv -c_1 a_2(u)^2, \\ E_2(u) \equiv 0, \end{array}\right\} \text{ for } \alpha = \frac{c_1 + c_3}{2}. \tag{102}$$

The general solution of (100) can be obtained as

$$V_0(u,z) = b_1(u) + b_2(u)z^{-2} + \frac{E_1(u)}{2}\ln z + \frac{E_2(u)}{4}\ln z(\ln z - 1). \tag{103}$$

6.2 AdS-Plane Waves in D Dimensions: A Special Solution

Let us now study the problem in D dimensions. Of course, in this case, it is not possible to find the most general solutions of the coupled differential equations (80), (82), and (83). However, it is possible to give a special solution, which may be thought of as the higher-dimensional generalization of the previous three-dimensional solution (98).

The D-dimensional spacetime has the coordinates $x^\mu = (u, v, x^i, z)$ with $i = 1, \ldots, D-3$. Now assume that the functions V_0 and ϕ are homogeneous along the transverse coordinates x^i; i.e., take

$$V_0 = V_0(u,z) \;\;\&\;\; \phi = \phi(u,z) \quad \Rightarrow \quad V(u,z) = V_0(u,z) + \alpha\phi(u,z)^2. \tag{104}$$

In that case, the differential equation (80) becomes

$$z^2\partial_z^2\phi + (4 - D)z\partial_z\phi - m^2\ell^2\phi = 0, \tag{105}$$

where m is given by (81), whose general solution is, for $D \neq 3$,

$$\phi(u,z) = a_1(u) z^{r_+} + a_2(u) z^{r_-}, \tag{106}$$

where $a_1(u)$ and $a_2(u)$ are two arbitrary functions and

$$r_\pm = \frac{1}{2}\left[D - 3 \pm \sqrt{(D-3)^2 + 4m^2\ell^2}\right]. \tag{107}$$

Inserting (106) into (82) and (83) yields

$$z^2 \partial_z^2 V_0 + (6-D) z \partial_z V_0 + 2(3-D) V_0 = E_1(u) z^{2r_+} + E_2(u) z^{2r_-}, \tag{108}$$

where

$$\left.\begin{array}{l} E_1(u) \equiv c_1(2r_+ + 3 - D)\, a_1(u)^2, \\ E_2(u) \equiv c_1(2r_- + 3 - D)\, a_2(u)^2, \end{array}\right\} \text{ for } \alpha = \frac{c_3}{2}, \tag{109}$$

$$\left.\begin{array}{l} E_1(u) \equiv -c_1(r_+^2 + m^2\ell^2)\, a_1(u)^2, \\ E_2(u) \equiv -c_1(r_-^2 + m^2\ell^2)\, a_2(u)^2, \end{array}\right\} \text{ for } \alpha = \frac{c_1 + c_3}{2}. \tag{110}$$

The general solution of (108) can be obtained as

$$V_0(u,z) = b_1(u) z^{D-3} + b_2(u) z^{-2} + \frac{E_1(u)}{d_+} z^{2r_+} + \frac{E_2(u)}{d_-} z^{2r_-}, \tag{111}$$

where $b_1(u)$ and $b_2(u)$ are arbitrary functions. This solution is valid only if

$$d_+ \equiv 4r_+^2 + 2(5-D) r_+ + 2(3-D) \neq 0, \tag{112}$$
$$d_- \equiv 4r_-^2 + 2(5-D) r_- + 2(3-D) \neq 0. \tag{113}$$

When $d_+ = 0$, we have

$$V_0(u,z) = b_1(u) z^{D-3} + b_2(u) z^{-2} + \frac{E_1(u)}{4r_+ + 5 - D} z^{2r_+} \ln z + \frac{E_2(u)}{d_-} z^{2r_-}, \tag{114}$$

and, when $d_- = 0$, we have

$$V_0(u,z) = b_1(u) z^{D-3} + b_2(u) z^{-2} + \frac{E_1(u)}{d_+} z^{2r_+} + \frac{E_2(u)}{4r_- + 5 - D} z^{2r_-} \ln z. \tag{115}$$

For $m \neq 0$, all these expressions reduce to the corresponding ones in the previous section when $D = 3$.

As we discussed in the previous subsection, these solutions should behave like asymptotically AdS as we approach $z = 0$. This means that

$$r_- > -1. \tag{116}$$

With (107) and (73), this condition gives

$$m < \sqrt{\frac{2|\Lambda|}{D-1}}, \tag{117}$$

where $D > 3$. For $D = 4$ and taking the present value of the cosmological constant, $|\Lambda| < 10^{-52}$ m$^{-2} \approx 10^{-84}$ (GeV)^2, we obtain the upper bound $m < 10^{-42}$ GeV for the mass of the spin-0 aether field ϕ.

Therefore the metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \frac{\ell^2}{z^2}(2dudv + dx_i dx^i + dz^2) + 2V(u,z)du^2, \tag{118}$$

with the profile function

$$V(u,z) = V_0(u,z) + \alpha\phi(u,z)^2, \tag{119}$$

describes an exact plane wave, propagating in the D-dimensional AdS background, in NAT.

7 *pp*-Waves in Null Aether Theory

As a last example of Kerr-Schild-Kundt class of metrics, we shall consider *pp*-waves, *plane-fronted waves with parallel rays.* These are defined to be spacetimes that admit a covariantly constant null vector field l^μ; i.e.,

$$\nabla_\mu l_\nu = 0, \quad l_\mu l^\mu = 0. \tag{120}$$

These spacetimes are of great importance in general relativity in that they constitute exact solutions to the full nonlinear field equations of the theory, which may represent gravitational, electromagnetic, or some other forms of matter waves [44].

In the coordinate system $x^\mu = (u, v, x^i)$ with $i = 1, \ldots, D-2$ adapted to the null Killing vector $l_\mu = \delta^u_\mu$, the *pp*-wave metrics take the Kerr-Schild form [47, 48]

$$ds^2 = 2dudv + 2V(u, x^i)du^2 + dx_i dx^i, \tag{121}$$

where u and v are the double null coordinates and $V(u, x^i)$ is the profile function of the wave. For such metrics, the Ricci tensor and the Ricci scalar become

$$R_{\mu\nu} = -(\nabla^2_\perp V)l_\mu l_\nu \quad \Rightarrow \quad R = 0, \tag{122}$$

where $\nabla^2_\perp \equiv \partial_i\partial^i$. A particular subclass of *pp*-waves are plane waves for which the profile function $V(u, x^i)$ is quadratic in the transverse coordinates x^i, that is,

$$V(u, x^i) = h_{ij}(u)x^i x^j, \tag{123}$$

where the symmetric tensor $h_{ij}(u)$ contains the information about the polarization and amplitude of the wave. In this case the Ricci tensor takes the form

$$R_{\mu\nu} = -2\mathrm{Tr}(h)l_\mu l_\nu, \tag{124}$$

where $\mathrm{Tr}(h)$ denotes the trace of the matrix $h_{ij}(u)$.

Now we will show that *pp*-wave spacetimes described above constitute exact solutions to NAT. As before, we define the null aether field as $v^\mu = \phi(x)l^\mu$, but this time we let the scalar function $\phi(x)$ and the vector field l^μ satisfy the following conditions

$$l_\mu l^\mu = 0, \quad \nabla_\mu l_\nu = 0, \quad l^\mu \partial_\mu V, \quad l^\mu \partial_\mu \phi = 0. \tag{125}$$

Note that this is a special case of the previous analysis achieved by taking the background is flat (i.e. $K = 0$) and $\xi^\mu = 0$ there. Then it immediately follows from (44), (45), and (46) that

$$J^\mu{}_\nu = c_1 l_\nu \nabla^\mu \phi + c_3 l^\mu \nabla_\nu \phi, \quad L = 0, \tag{126}$$

and the field equations are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = -c_3 \left[\nabla_\alpha \phi \nabla^\alpha \phi - \frac{\lambda}{c_1} \phi^2 \right] l_\mu l_\nu, \tag{127}$$

$$(c_1 \Box \phi + \lambda \phi) l_\mu = 0, \tag{128}$$

where we have eliminated the $\phi \Box \phi$ term that should appear in (127) by using the aether equation (128) assuming $c_1 \neq 0$. The right-hand side of the equation (127) is in the form of the energy-momentum tensor of a null dust, i.e. $T_{\mu\nu} = \mathcal{E} l_\mu l_\nu$ with

$$\mathcal{E} \equiv -c_3 \left[\nabla_\alpha \phi \nabla^\alpha \phi - \frac{\lambda}{c_1} \phi^2 \right]. \tag{129}$$

The condition $\mathcal{E} \geq 0$ requires that[5]

$$c_3 \leq 0, \quad \frac{\lambda}{c_1} \leq 0. \tag{130}$$

On the other hand, the equation (128) gives Klein-Gordon equation for the field $\phi(x)$:

$$\Box \phi - m^2 \phi = 0, \tag{131}$$

where we defined the "mass" by

$$m^2 \equiv -\frac{\lambda}{c_1}, \tag{132}$$

which is consistent with the constraint (130).

With the *pp*-wave ansatz (121), the field equations (127) and (128) become

$$-\left(\nabla_\perp^2 V - 2\Lambda V \right) l_\mu l_\nu + \Lambda \eta_{\mu\nu} = -c_3 \left[\partial_i \phi \partial^i \phi + m^2 \phi^2 \right] l_\mu l_\nu, \tag{133}$$

$$\nabla_\perp^2 \phi - m^2 \phi = 0. \tag{134}$$

[5]At this point, it is worth mentioning that, although the Null Aether Theory being discussed here is inherently different from the Einstein-Aether theory [2,3] with a unit timelike vector field, the constraint $c_3 \leq 0$ in (130) is not in conflict with the range given in the latter theory. Indeed, imposing that the PPN parameters of Einstein-Aether theory are identical to those of general relativity, the stability against linear perturbations in Minkowski background, vacuum-Čerenkov, and nucleosynthesis constraints require that (see, e.g., [40])

$$0 < c_+ < 1, \qquad 0 < c_- < \frac{c_+}{3(1 - c_+)},$$

where $c_+ \equiv c_1 + c_3$ and $c_- \equiv c_1 - c_3$. Thus, for any fixed value c_+ in the range $2/3 < c_+ < 1$, c_3 is restricted to the range

$$-\frac{c_+(3c_+ - 2)}{6(1 - c_+)} < c_3 < \frac{c_+}{2}.$$

So there is always a region where c_3 is negative; for example, when $c_+ = 4/5$, we have $-4/15 < c_3 < 2/5$.

Therefore, the profile function of *pp*-waves should satisfy

$$\nabla_\perp^2 V = c_3 \left[\partial_i\phi\partial^i\phi + m^2\phi^2\right], \tag{135}$$

since it must be that $\Lambda = 0$. At this point, we can make the following ansatz

$$V(u, x^i) = V_0(u, x^i) + \alpha\phi(u, x^i)^2, \tag{136}$$

where α is an arbitrary constant. Now plugging this into (135), we obtain

$$\nabla_\perp^2 V_0 = (c_3 - 2\alpha)\left[\partial_i\phi\partial^i\phi + m^2\phi^2\right], \tag{137}$$

and since we are free to choose any value for α, we get

$$\nabla_\perp^2 V_0 = 0 \;\; \text{for} \;\; \alpha = \frac{c_3}{2}. \tag{138}$$

Thus, any solution $\phi(u, x^i)$ of the equation (134) together with the solution $V_0(u, x^i)$ of the Laplace equation (138) constitutes a *pp*-wave metric (121) with the profile function $V(u, x^i)$ given by (136).

Let us now consider the plane wave solutions described by the profile function (123). In that case, we can investigate the following two special cases.

The $c_3 = 0$ case:

When $c_3 = 0$ [or, $\alpha = 0$ through (138)], it is obvious from (136) that the function ϕ, satisfying (134), detaches from the function V and we should have $V = V_0$. This means that the profile function satisfies the Laplace equation, i.e.,

$$\nabla_\perp^2 V = 0, \tag{139}$$

which is solved by $V(u, x^i) = h_{ij}(u)x^ix^j$ only if $\mathrm{Tr}(h) = 0$. Thus we have shown that plane waves are solutions in NAT provided the equation (134) is satisfied independently. For example, in four dimensions with the coordinates $x^\mu = (u, v, x, y)$, the metric

$$ds^2 = 2dudv + 2[h_{11}(u)(x^2 - y^2) + 2h_{12}(u)xy]du^2 + dx^2 + dy^2 \tag{140}$$

describes a plane wave propagating along the null coordinate v [related to the aether field through $v^\mu = \phi\delta^\mu_v$ with $\phi(u, x^i)$ satisfying (134)] in flat spacetime. Here the function $h_{12}(u)$ is related to the polarization of the wave and, for a wave with constant linear polarization, it can always be set equal to zero by performing a rotation in the transverse plane coordinates x and y.

The $c_3 \neq 0$ & $V_0(u, x^i) = t_{ij}(u)x^ix^j$ case:

In this case, the Laplace equation (138) says that $\mathrm{Tr}(t) = 0$, and from (136) we have

$$\phi = \sqrt{\frac{2}{c_3}[h_{ij}(u) - t_{ij}(u)]x^ix^j}. \tag{141}$$

Inserting this into (134), we obtain

$$\left[h^k{}_k(h_{ij}-t_{ij})-(h_{ki}-t_{ki})(h^k{}_j-t^k{}_j)\right]x^ix^j-m^2\left[(h_{ij}-t_{ij})x^ix^j\right]^2=0. \qquad (142)$$

This condition is trivially satisfied if $h_{ij}=t_{ij}$, but this is just the previous $c_3=0$ case in which $V=V_0$. Nontrivially, however, the condition (142) can be satisfied by setting the coefficient of the first term and the mass m (or, equivalently, the Lagrange multiplier λ) equal to zero. Then again plane waves occur in NAT.

8 Conclusion

In this work, we introduced the Null Aether Theory (NAT) which is a vector-tensor theory of gravity in which the vector field defining the aether is assumed to be null at each point of spacetime. This construction allows us to take the aether field (v_μ) to be proportional to a principal null direction of the Weyl tensor (l_μ), i.e. $v_\mu=\phi(x)l_\mu$ with $\phi(x)$ being the spin-0 part of the aether field. We first investigated the Newtonian limit of this theory and showed that it is exactly the same as that of general relativity; that is to say, NAT reproduces the solar system phenomenology in this approximation and this fact puts no constraints on the free parameters $\{c_1,c_2,c_3,c_4\}$ of the theory. We then constructed exact spherically symmetric black hole solutions in $D=4$ and nonlinear wave solutions in $D\geq 3$ in the theory. Among the black hole solutions, we have Vaidya-type nonstationary solutions which do not need any extra matter fields for their very existence: the aether behaves in a sense as a null matter field to produce such solutions. Besides these, there are also (i) Schwarzschild-(A)dS type solutions with $h(r)\equiv 1-Br^2-2m/r$ for $c_1=0$ that exist even when there is no explicit cosmological constant in the theory, (ii) Reissner-Nordström-(A)dS type solutions with $h(r)\equiv 1-\Lambda r^2/3-B/r^2-2m/r$ for $c_1=-(c_2+c_3)$, (iii) solutions with $h(r)\equiv 1-\Lambda r^2/3-Br-2m/r$ for $c_1=-5(c_2+c_3)/8$, which were also obtained and used to explain the flatness of the galaxy rotation curves in conformal gravity, and so on. All these solutions have at least one event horizon and describe stationary black holes in NAT. We also discussed the existence of black hole solutions for arbitrary values of the parameters $\{c_1,c_2,c_3,c_4\}$.

As for the wave solutions, we specifically studied the Kerr-Schild-Kundt class of metrics in this context and showed that the full field equations of NAT reduce to just two, in general coupled, partial differential equations when the background spacetime takes the maximally symmetric form. One of these equations describes the massive spin-0 aether field $\phi(x)$. When the background is AdS, we solved these equations explicitly and thereby constructed exact AdS-plane wave solutions of NAT in three dimensions and in higher dimensions than three if the profile function describing the wave is independent of the transverse $D-3$ coordinates. When the background is flat, on the other hand, the *pp*-wave spacetimes constitute exact solutions, for generic vaules of the coupling constants, to the theory by reducing the whole set of field equations to two decoupled differential equations: one Laplace equation for a scalar function related to the profile function of the wave and one massive Klein-Gordon equation for the spin-0 aether field in $(D-2)$-dimensional Euclidean flat space. We also showed that the plane waves, subset of *pp*-waves, are solutions to the field equations of NAT provided that the parameter c_3 vanishes. When

c_3 is nonvanishing, however, the solution of the Laplace equation should satisfy certain conditions and the spin-0 aether field must be massless, i.e., $\lambda = 0$. The main conclusion of these computations is that the spin-0 part of the aether field has a mass in general determined by the cosmological constant and the Lagrange multiplier given in the theory and in the case of AdS background this mass acquires an upper bound (the Breitenlohner-Freedman bound) determined by the value of the background cosmological constant.

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