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ORIGINAL PAPER

MATHEMATISCHE NACHRICHTEN

Precise inclusion relations among Bergman–Besov and Bloch–Lipschitz spaces and H^∞ on the unit ball of $\mathbb{C}^N$

H. Turgay Kaptanoğlu[1] | A. Ersin Üreyen[2]

[1]Bilkent Üniversitesi, Matematik Bölümü, 06800 Ankara, Turkey
[2]Anadolu Üniversitesi, Matematik Bölümü, 26470 Eskişehir, Turkey

Correspondence
H. Turgay Kaptanoğlu, Bilkent Üniversitesi, Matematik Bölümü, 06800 Ankara, Turkey.
Email: kaptan@fen.bilkent.edu.tr

Abstract
We describe exactly and fully which of the spaces of holomorphic functions in the title are included in which others. We provide either new results or new proofs. More importantly, we construct explicit functions in each space that show our relations are strict and the best possible. Many of our inclusions turn out to be sharper than the Sobolev imbeddings.

KEYWORDS
atomic decomposition, Bergman, Besov, Bloch, Lipschitz space, bounded holomorphic function, Hadamard gap series, inclusion, Littlewood–Paley inequality, Ryll–Wojtaszczyk polynomial, Sobolev imbedding

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1 | INTRODUCTION

The purpose of this paper is to provide a complete description of the inclusion relations among the spaces mentioned in the title by providing proofs of the missing cases and of simpler proofs of the known cases as well as exhibiting explicit examples in all cases that show that the inclusions are strict and the best possible.

Let $\mathbb{B}$ be the unit ball in $\mathbb{C}^N$ with respect to the usual hermitian inner product $\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_N\overline{w}_N$ and the associated norm $|z| = \sqrt{\langle z, z\rangle}$. Let $H(\mathbb{B})$ and H^∞ denote the spaces of all and bounded holomorphic functions on $\mathbb{B}$, respectively.

We let ν be the Lebesgue measure on $\mathbb{B}$ normalized so that $\nu(\mathbb{B}) = 1$. For $q \in \mathbb{R}$, we also define on $\mathbb{B}$ the measures

$$d\nu_q(z) := \left(1 - |z|^2\right)^q \, d\nu(z).$$

For $0 < p < \infty$, we denote the Lebesgue classes with respect to ν_q by L^p_q, using also the notation $L^p_0 = L^p$. The Lebesgue class of essentially bounded functions on $\mathbb{B}$ with respect to any ν_q is the same (see [14, Proposition 2.3]); we denote it by $\mathcal{L}^\infty$. For $\alpha \in \mathbb{R}$, we also define the weighted classes

$$\mathcal{L}^\infty_\alpha := \left\{ \varphi \text{ measurable on } \mathbb{B} \,:\, \left(1 - |z|^2\right)^\alpha \varphi(z) \in \mathcal{L}^\infty \right\}$$

so that $\mathcal{L}^\infty_0 = \mathcal{L}^\infty$, which are normed by

$$\|\varphi\|_{\mathcal{L}^\infty_\alpha} := \operatorname*{ess\,sup}_{z\in\mathbb{B}} \left(1 - |z|^2\right)^\alpha |\varphi(z)|.$$

For $q > -1$ and $0 < p < \infty$, the weighted Bergman spaces are $A_q^p = L_q^p \cap H(\mathbb{B})$. To extend this family to all real q, we resort to derivatives. Given $q \in \mathbb{R}$ and $0 < p < \infty$, let m be a nonnegative integer such that $q + pm > -1$. Then the Bergman–Besov space B_q^p consists of all $f \in H(\mathbb{B})$ for which

$$\left(1 - |z|^2\right)^m \frac{\partial^m f}{\partial z_1^{\gamma_1} \cdots \partial z_N^{\gamma_N}} \in L_q^p$$

for every multi-index $\gamma = (\gamma_1, \dots, \gamma_N)$ with $\gamma_1 + \cdots + \gamma_N = m$.

The spaces B_q^2 are reproducing kernel Hilbert spaces whose kernels occupy a large part in our study of all B_q^p spaces. Consequently, even to define the spaces of interest in this work, it is more advantageous to use certain radial differential operators that are compatible with the kernels. So we follow [14,16] and resort to invertible radial differential operators D_s^t of order $t \in \mathbb{R}$ for any $s \in \mathbb{R}$ that map $H(\mathbb{B})$ to itself. These are described in detail in Section 2. Consider the linear transformation I_s^t defined for $f \in H(\mathbb{B})$ by

$$I_s^t f(z) := \left(1 - |z|^2\right)^t D_s^t f(z).$$

Definition 1.1. For $q \in \mathbb{R}$ and $0 < p < \infty$, we define the *Bergman–Besov space* B_q^p to consist of all $f \in H(\mathbb{B})$ for which $I_s^t f$ belongs to L_q^p for some s, t satisfying

$$q + pt > -1. \tag{1.1}$$

The quantity $\|f\|_{B_q^p} := \|I_s^t f\|_{L_q^p}$ for any such s, t defines a norm on B_q^p for $p \geq 1$ and a quasinorm for $0 < p < 1$.

Definition 1.2. For $\alpha \in \mathbb{R}$, we define the *Bloch–Lipschitz space*$\mathcal{B}_\alpha^\infty$ to consist of all $f \in H(\mathbb{B})$ for which $I_s^t f$ belongs to $\mathcal{L}_\alpha^\infty$ for some s, t satisfying

$$\alpha + t > 0. \tag{1.2}$$

The quantity $\|f\|_{\mathcal{B}_\alpha^\infty} := \|I_s^t f\|_{\mathcal{L}_\alpha^\infty}$ for any such s, t defines a norm on $\mathcal{B}_\alpha^\infty$.

Remark 1.3. By now, it is well-known that Definitions 1.1 and 1.2 are independent of s, t under (1.1) and (1.2), respectively, and also of the particular type of the derivative. Further, the norms on a given space depending on s, t are equivalent to each other under (1.1) or (1.2). For these, see, for example, [3, Theorem 5.12 (i)], [14,16,29]. So given a pair s, t, I_s^t imbeds B_q^p isometrically into L_q^p if and only if (1.1) holds, and I_s^t imbeds $\mathcal{B}_\alpha^\infty$ isometrically into $\mathcal{L}_\alpha^\infty$ if and only if (1.2) holds.

If $q > -1$, we can take $t = 0$ in (1.1) and obtain the weighted Bergman spaces $B_q^p = A_q^p$. Further, B_{-1}^2 is the Hardy space H^2, $B_{-(1+N)}^2$ is the Dirichlet space, and B_{-N}^2 is the Drury–Arveson space. If $\alpha > 0$, we can take $t = 0$ in (1.2) and obtain the weighted Bloch spaces. If $\alpha < 0$, then the corresponding spaces are the holomorphic Lipschitz spaces $\Lambda_{-\alpha} = \mathcal{B}_\alpha^\infty$; see, for example, [21, Section 6.4].

Our use of α follows [16] and [7], which is more logical in view of the operators I_s^t and conforms well with the notation of B_q^p, so the *usual Bloch space* $\mathcal{B}_0^\infty = \mathcal{B}^\infty$ corresponds to $\alpha = 0$. Most other authors use $\alpha + 1$ while [29] uses $-\alpha$ where we use α. There is no discussion of little Bloch spaces in this paper.

The following three theorems in increasing intricacy are our main results. Unless otherwise specified, we use the full ranges of the parameters, $0 < p < \infty$ and $q, \alpha, s, t \in \mathbb{R}$, and all our results cover the standard weighted Bergman spaces as special cases.

Notation 1.4. If X_a is a family of spaces indexed by $a \in \mathbb{R}$, the symbol $X_{<a}$ denotes any one of the spaces X_b with $b < a$. For functions, $h_{<a}$ has a similar meaning.

Theorem 1.5. *Given B_q^p, we have the inclusions*

$$\mathcal{B}_{<\frac{1+q}{p}}^\infty \subset B_q^p \subset \mathcal{B}_{\frac{1+N+q}{p}}^\infty.$$

Theorem 1.6. *Let B_q^p be given.*

(i) *If $p \leq P$, then $B_q^p \subset B_Q^P$ if and only if*

$$\frac{1 + N + q}{p} \leq \frac{1 + N + Q}{P}. \tag{1.3}$$

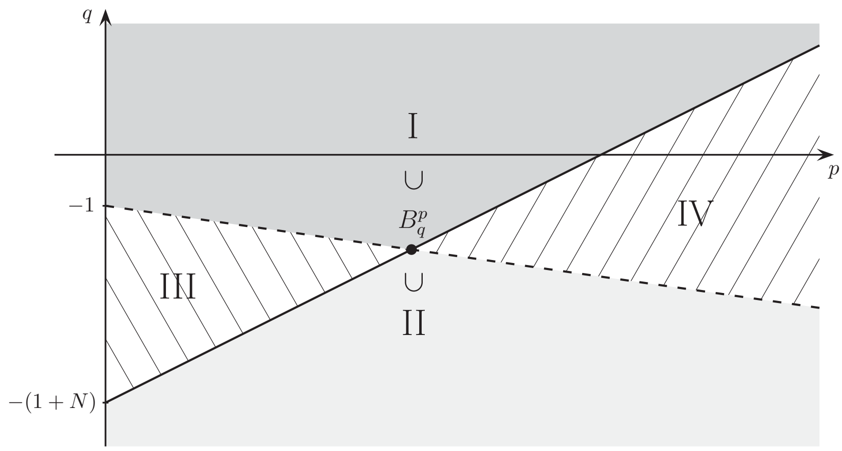

FIGURE 1 If $(P, Q) \in \mathrm{I}$, then $B^p_q \subset B^P_Q$; if $(P, Q) \in \mathrm{II}$, then $B^P_Q \subset B^p_q$; if $(P, Q) \in \mathrm{III} \cup \mathrm{IV}$, then neither B^p_q nor B^P_Q contains the other, but $B^p_q \cap B^P_Q \neq \emptyset$

(ii) *If $P < p$, then $B^p_q \subset B^P_Q$ if and only if*

$$\frac{1+q}{p} < \frac{1+Q}{P}. \tag{1.4}$$

Theorem 1.7.

(i) $B^p_q \subset H^\infty$ *if and only if* $q < -(1+N)$, *or* $q = -(1+N)$ *and* $0 < p \le 1$.

(ii) $H^\infty \subset B^p_q$ *if and only if* $q > -1$, *or* $q = -1$ *and* $p \ge 2$.

Theorem 1.5 can be equivalently stated from the point of view of the Bloch–Lipschitz spaces: Given $\mathcal{B}^\infty_\alpha$, the inclusions $B^p_{\alpha p-(1+N)} \subset \mathcal{B}^\infty_\alpha \subset B^p_{>\alpha p-1}$ hold.

Note that both parts of Theorem 1.6 state if-and-only-if conditions, and there is no third alternative. Thus Theorem 1.6 covers all possible inclusion relations between two members of the B^p_q family of spaces.

There is a one-to-one correspondence between the points (p, q) in the right half plane of the pq-plane and the Bergman–Besov family of spaces B^p_q. The inclusions of Theorem 1.6 are shown graphically in Figure 1. There, the space B^p_q is included in all the spaces in region I and includes all the spaces in region II. The space B^p_q does not contain nor is contained in the spaces in regions III and IV, but has nonempty intersection with them since all B^p_q spaces contain all holomorphic polynomials. A very rudimentary version of this figure is in [13, p. 731]. In Figure 1, we call the quadrant $\{q > -1\}$ the Bergman zone and its complementary quadrant $\{q \le -1\}$ the proper Besov zone. We show in Corollary 7.2 that the spaces in the proper Besov zone require some kind of a derivative in their integral norms.

The proofs of the inclusions are often known, but we simplify them, give new ones, and complete the missing cases. The real contribution and the strength of this paper is in finding categorical examples and counterexamples of functions that lie in some spaces but not in some others, whose proofs turn out to be considerably more difficult than those of inclusions.

It turns out that whenever a space is included in H^∞ in this paper, then it is also included in the *ball algebra* $A(\mathbb{B})$ of holomorphic functions on $\mathbb{B}$ that extend continuously to $\overline{\mathbb{B}}$. This fact is inherent in our proofs, but we make a note of it each time. Both these spaces are normed with $\|f\|_\infty = \sup_{z\in\mathbb{B}} |f(z)|$.

Each inclusion in these results is strict and the best possible. *Strict* means that the two spaces in an inclusion are not equal. *Best possible* means either a space that contains a given one is the smallest possible in the family, or the inclusion result is an if-and-only-if condition. We make sure of these by exhibiting explicit functions that lie in one space but not in the other.

Moreover, each inclusion of ours is *continuous*, that is, if $X \subset Y$, then the *inclusion map* $i : X \to Y$ is continuous. This can be checked by $\|\cdot\|_Y \lesssim \|\cdot\|_X$. Such an inequality is inherent in the proof of every inclusion we claim.

We prove Theorem 1.5 in Section 5, Theorem 1.6 in Section 6, and Theorem 1.7 in Section 7. We prove two other elementary inclusions in Section 4. Our approach to Theorems 1.5 and 1.6 is to prove each inclusion first for one value of q covering all values of p and then apply differentiation to pass to other spaces. Our proof of Theorem 1.7 is highly nontrivial and here we supply the missing cases. It uses techniques varying from atomic decomposition to Littlewood–Paley inequalities and to

Ryll–Wojtaszczyk polynomials. We also show that in essence the norm of B^p_q requires a derivative of order specified by (1.1) in Section 7. In Section 3, we construct the example functions we use repeatedly; they have the general property that each lies in one space but not in a "nearby" space. In the last Section 8, we make a comparison of our inclusions with the holomorphic counterparts of Sobolev imbeddings. It so happens that in most cases our imbeddings are sharper than those dictated by the Sobolev imbedding theorem.

We do not make any comparisons with the Hardy spaces, because along with the Hardy Sobolev spaces and BMOA, those should be the topic of a different work. In this respect, H^∞ is not a Hardy space, because the correct $p = \infty$ version of the Hardy spaces is BMOA.

2 | PRELIMINARIES

In multi-index notation, $\gamma = (\gamma_1, \ldots, \gamma_N) \in \mathbb{N}^N$ is an N-tuple of nonnegative integers with $|\gamma| = \gamma_1 + \cdots + \gamma_N$, $\gamma! = \gamma_1! \cdots \gamma_N!$, $0^0 = 1$, and $z^\gamma = z_1^{\gamma_1} \cdots z_N^{\gamma_N}$. The number of distinct multi-indices γ with $|\gamma| = m$, that is, the dimension of the space of holomorphic homogeneous polynomials of degree m in N variables is $\delta_m = \binom{N-1+m}{N-1}$.

The standard basis vectors of $\mathbb{C}^N$ are $e_j = (0, \ldots, 0, 1, 0 \ldots, 0)$ with 1 in the jth position, $j = 1, \ldots, N$. An overbar $\overline{(\ )}$ indicates complex conjugate for functions and closure for sets. A *quasinorm* is given by the inequality $\|f + g\| \leq C(\|f\| + \|g\|)$ for some constant $C > 1$ in place of the triangle inequality. We use the term norm even when we mean quasinorm. The inner product of a space of functions X is denoted $[\,\cdot\,, \cdot\,]_X$. The pth power summable sequence spaces are denoted ℓ^p.

Let $\mathbb{S}$ be the unit sphere in $\mathbb{C}^N$. When $N = 1$, $\mathbb{B}$ is the unit disc $\mathbb{D}$ and $\mathbb{S}$ is the unit circle $\mathbb{T}$. We let σ be the Lebesgue measure on $\mathbb{S}$ normalized so that $\sigma(\mathbb{S}) = 1$. For $0 < p \leq \infty$, we denote the Lebesgue classes with respect to σ by $L^p(\sigma)$. The polar coordinates formula that relates σ and ν is the one in [21, § 1.4.3].

Let's also recall the definition of the *Hardy spaces* on $\mathbb{B}$. For $0 < p < \infty$, we say an $f \in H(\mathbb{B})$ belongs to H^p whenever

$$\|f\|^p_{H^p} = \sup_{0<r<1} \int_{\mathbb{S}} |f(r\zeta)|^p \, d\sigma(\zeta) < \infty.$$

Since σ is finite, clearly $H^\infty \subset H^p$.

The *Pochhammer symbol* $(a)_b$ is given by

$$(a)_b := \frac{\Gamma(a+b)}{\Gamma(a)}$$

when a and $a + b$ are off the pole set $-\mathbb{N}$ of the gamma function Γ. In particular, $(a)_0 = 1$ and for k a positive integer, we have $(a)_k = a(a+1)\cdots(a+k-1)$. The Stirling formula yields

$$\frac{\Gamma(c+a)}{\Gamma(c+b)} \sim c^{a-b}, \qquad \frac{(a)_c}{(b)_c} \sim c^{a-b}, \qquad \text{and} \qquad \frac{(c)_a}{(c)_b} \sim c^{a-b} \qquad (\operatorname{Re} c \to \infty), \tag{2.1}$$

where $x \sim y$ means both $x = \mathcal{O}(y)$ and $y = \mathcal{O}(x)$ for all x, y in question. If only $x = \mathcal{O}(y)$, we write $x \lesssim y$.

An $f \in H(\mathbb{B})$ can be written in terms of its *homogeneous expansion* and its Taylor series as

$$f(z) = \sum_{k=0}^\infty f_k(z) = \sum_{|\gamma|=0}^\infty f_\gamma z^\gamma$$

in which f_k is a holomorphic homogeneous polynomial in $z_1, \ldots, z_N$ of degree k.

Definition 2.1. For $q \in \mathbb{R}$ and $z, w \in \mathbb{B}$, the *Bergman–Besov kernels* are

$$K_q(z,w) := \begin{cases} \dfrac{1}{(1 - \langle z,w\rangle)^{1+N+q}} = \displaystyle\sum_{k=0}^\infty \frac{(1+N+q)_k}{k!} \langle z,w\rangle^k, & q > -(1+N), \\ {}_2F_1(1,1;1-(N+q);\langle z,w\rangle) = \displaystyle\sum_{k=0}^\infty \frac{k! \ \langle z,w\rangle^k}{(1-(N+q))_k}, & q \leq -(1+N), \end{cases}$$

where ${}_2F_1 \in H(\mathbb{D})$ is the usual hypergeometric function.

These kernels for $q < -(1+N)$ appear in the literature first in [3, p. 13]. Let the coefficient of $\langle z, w\rangle^k$ in the series expansion of $K_q(z,w)$ be $c_k(q)$. Note that $c_0(q) = 1$, $c_k(q) > 0$ for any k, and by (2.1),

$$c_k(q) \sim k^{N+q} \qquad (k \to \infty),$$

for all q. The kernel K_q is the reproducing kernel of the Hilbert space B^2_q. These facts, coupled with the binomial expansion of $\langle z, w\rangle^k$ give us another norm

$$\|z^\gamma\|^2_{B^2_q} = \frac{1}{c_{|\gamma|}(q)} \frac{\gamma!}{|\gamma|!} \sim \frac{1}{|\gamma|^{N+q}} \frac{\gamma!}{|\gamma|!}$$

for B^2_q that is equivalent to the ones given in Definition 1.1. It also follows either from here by polarization or by [21, Proposition 1.4.8] that $[z^{\gamma_1}, z^{\gamma_2}]_{B^2_q} = 0$ if $\gamma_1 \neq \gamma_2$. Thus $f \in H(\mathbb{B})$ belongs to B^2_q if and only if

$$\sum_\gamma \frac{|f_\gamma|^2}{|\gamma|^{N+q}} \frac{\gamma!}{|\gamma|!} < \infty. \tag{2.2}$$

Definition 2.2. For any $s, t \in \mathbb{R}$, we define the *radial differential operator* D^t_s on $H(\mathbb{B})$ by

$$D^t_s f := \sum_{k=0}^\infty d_k(s,t) f_k := \sum_{k=0}^\infty \frac{c_k(s+t)}{c_k(s)} f_k.$$

Note that $d_0(s,t) = 1$, $d_k(s,t) > 0$ for any k, and

$$d_k(s,t) \sim k^t \qquad (k \to \infty), \tag{2.3}$$

for any s, t by (2.1). So D^t_s is a continuous operator on $H(\mathbb{B})$ and is of order t; for a proof of a similar continuity result, see [9, Theorem 3.2]. In particular, $D^t_s z^\gamma = d_{|\gamma|}(s,t) z^\gamma$ for any multi-index γ, and hence $D^t_s(1) = 1$. More importantly,

$$D^0_s = I, \qquad D^u_{s+t} D^t_s = D^{t+u}_s, \qquad \text{and} \qquad \left(D^t_s\right)^{-1} = D^{-t}_{s+t} \tag{2.4}$$

for any s, t, u. Thus any D^t_s maps $H(\mathbb{B})$ onto itself.

Explicit forms of the norms of B^p_q and $\mathcal{B}^\infty_\alpha$ given in Definitions 1.1 and 1.2 are

$$\|f\|_{B^p_q} = \int_\mathbb{B} |D^t_s f(z)|^p \left(1 - |z|^2\right)^{q+pt} d\nu(z) \qquad (q + pt > -1), \tag{2.5}$$

$$\|f\|_{\mathcal{B}^\infty_\alpha} = \sup_{z\in\mathbb{B}} |D^t_s f(z)| \left(1 - |z|^2\right)^{\alpha+t} \qquad (\alpha + t > 0). \tag{2.6}$$

One of the best things about the D^t_s is that they allow us to pass easily from one kernel to the other and from one space to the other in the same family. First, it is immediate that

$$D^t_q K_q(z,w) = K_{q+t}(z,w) \tag{2.7}$$

for any q, t, where differentiation is performed on the holomorphic variable z. But the more versatile result is the following.

Theorem 2.3. *Let q, p, α, s, t be arbitrary. Then the maps $D^t_s : B^p_q \to B^p_{q+pt}$ and $D^t_s : \mathcal{B}^\infty_\alpha \to \mathcal{B}^\infty_{\alpha+t}$ are isomorphisms, and isometries when the parameters of the norms of the spaces are chosen appropriately.*

Proof. See [17, Proposition 3.1] and [15, Corollary 8.5]. The proofs require no more than Remark 1.3 and (2.4). See also [20, Corollary 3.9] for a different proof.

Note that this theorem works both ways since $t < 0$ is a possibility. □

Remark 2.4. Invertibility of D_s^t implies that only the zero function has zero norm in B_q^p or $\mathcal{B}_\alpha^\infty$. The other types of derivatives mentioned in Remark 1.3 that can be used in place of the D_s^t are powers of the *holomorphic gradient* and the usual *radial derivative* given by

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_N}\right) \qquad \text{and} \qquad Rf(z) = \langle \nabla f(z), \overline{z}\rangle.$$

Integrals of these derivatives define seminorms for the spaces B_q^p or $\mathcal{B}_\alpha^\infty$.

The holomorphic automorphism of $\mathbb{B}$ that exchanges 0 and z is the map

$$\varphi_z(w) = \frac{z - P_z(w) - \sqrt{1-|z|^2}(I - P_z)(w)}{1 - \langle w, z\rangle} \qquad (w \in \mathbb{B}), \tag{2.8}$$

where $P_z(w) = \langle w, z\rangle z/|z|^2$ is the projection on the complex line passing through 0 and z. It reduces to the well-known function $\varphi_z(w) = (z - w)/(1 - \overline{z}w)$ for $w \in \mathbb{D}$ when $N = 1$. The *Bergman metric* on $\mathbb{B}$ is

$$d(z, w) = \frac{1}{2}\log\frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|} = \tanh^{-1}|\varphi_z(w)|.$$

This metric is invariant under compositions with the automorphisms of $\mathbb{B}$. We denote the balls centered at a with radius r in the Bergman metric by $b(a, r)$. A sequence $\{a_k\}$ in $\mathbb{B}$ is called *separated* if there is a $\rho > 0$ such that $d(a_k, a_m) \geq \rho$ for all $k \neq m$, and we call ρ the separation constant.

The following growth rate estimate turns out to be surprisingly effective for obtaining several inclusion relations.

Lemma 2.5. *If $f \in B_q^p$, then for any s, t satisfying $q + pt > -(1 + N)$, we have*

$$|D_s^t f(z)| \lesssim \frac{\|f\|_{B_q^p}}{(1 - |z|^2)^{(1+N+q+pt)/p}} \qquad (z \in \mathbb{B}).$$

Proof. When f belongs to the Bergman space A_q^p, $q > -1$, and $t = 0$, the result is derived from the subharmonicity of $|f|^p$ using Möbius transformations and is in [3, Corollary 3.5 (ii)], although rediscovered later several times. If f belongs to the general Besov space B_q^p, by Definition 1.1 and Remark 1.3, $D_s^t f \in A_{q+pt}^p$ for any s, t satisfying (1.1). By the same reason, $\|f\|_{B_q^p} = \|I_s^t f\|_{L_q^p} = \|D_s^t f\|_{A_{q+pt}^p}$. The result for t satisfying (1.1) follows by applying the Bergman space case to $D_s^t f$.

What we have so far can be written also in the form $\|f\|_{\mathcal{B}^\alpha_{(1+N+q)/p}} \lesssim \|f\|_{B_q^p}$. But by Remark 1.3, the parameter t used in the norm $\|f\|_{\mathcal{B}^\alpha_{(1+N+q)/p}} < \infty$ can be as low as to satisfy $q + pt > -(1 + N)$. □

Corollary 2.6. *If $f \in B^p_{-(1+N)}$, then $\|f\|_{\mathcal{B}^\infty} \lesssim \|f\|_{B^p_{-(1+N)}}$.*

This corollary appears also in [10, Proposition 3.3] as well as in [3, Corollary 5.5] more generally.

3 | BASIC EXAMPLES

We now develop and collect interesting functions that lie in certain Besov or Bloch spaces but not in certain others. We use them frequently for the strict and the best possible inclusion results.

Example 3.1. The functions in the Hilbert spaces B_q^2 can be characterized by their Taylor series, so it is easy to write a function $h \in B_Q^2 \setminus B_q^2$ if $q < Q$. Let

$$h(z) := \sum_k k^{(2(N-1)+q+Q)/4} z_1^k \qquad (z \in \mathbb{B}).$$

Then by (2.2),

$$\|h\|^2_{B_q^2} \sim \sum_k \frac{1}{k^{1-(Q-q)/2}} = \infty \qquad \text{while} \qquad \|h\|^2_{B_Q^2} \sim \sum_k \frac{1}{k^{1+(Q-q)/2}} < \infty.$$

Example 3.2. An example that is essential for the Bloch–Lipschitz spaces is the family of functions

$$f_\alpha(z) := \sum_{k=0}^{\infty} c_k(\alpha - (1+N)) z_1^k \qquad (z \in \mathbb{B})$$

indexed by $\alpha \in \mathbb{R}$. For any branch of the logarithm, by Definition 2.1,

$$f_\alpha(z) = \frac{1}{(1-z_1)^\alpha} \quad (\alpha > 0) \qquad \text{and} \qquad f_0(z) = \frac{1}{z_1} \log \frac{1}{1-z_1} \qquad (z \in \mathbb{B}). \tag{3.1}$$

Note that, by (2.7),

$$D^t_{\alpha-(1+N)} f_\alpha = f_{\alpha+t}. \tag{3.2}$$

Now Definition 1.2 and Remark 1.3 show immediately that $f_1 \in \mathcal{B}^\infty_1$. The same reasoning shows that also $f_1 \notin \mathcal{B}^\infty_\beta$ if $\beta < 1$. Applying (3.2) and Theorem 2.3 yields

$$f_\alpha \in \mathcal{B}^\infty_\alpha \setminus \mathcal{B}^\infty_{<\alpha} \qquad (\alpha \in \mathbb{R}). \tag{3.3}$$

Further, [28, (16) and (17)] say that

$$f_{<\frac{1+N}{p}} \in A^p_0 \qquad \text{but} \qquad f_{\frac{1+N}{p}} \notin A^p_0. \tag{3.4}$$

Finally, applying (3.2) and Theorem 2.3 yields as before

$$f_{<\frac{1+N+q}{p}} \in B^p_q \qquad \text{but} \qquad f_{\frac{1+N+q}{p}} \notin B^p_q. \tag{3.5}$$

We give a proof of (3.4) simpler than the one in [28]. For $\alpha > 0$, by (3.1), for $w = r\zeta$ with $\zeta \in \mathbb{S}$ and $r \geq 0$, we have

$$\left|f_\alpha(w)\right|^p = \frac{1}{|1-w_1|^{\alpha p}} = \frac{1}{|1-\langle r\zeta, e_1\rangle|^{\alpha p}}.$$

Then by polar coordinates,

$$\int_{\mathbb{B}} |f|^p \, d\nu \sim \int_0^1 \int_{\mathbb{S}} \frac{d\sigma(\zeta)}{|1-\langle re_1, \zeta\rangle|^{\alpha p}} \, r^{2N-1} \, dr.$$

If $\alpha = (1+N)/p$, then $\alpha p = 1+N$, and [21, Proposition 1.4.10] yields

$$\int_{\mathbb{B}} |f|^p \, d\nu \sim \int_0^1 \frac{r^{2N-1}}{1-r^2} \, dr,$$

which diverges. If $\alpha < (1+N)/p$, then $\alpha p < 1+N$, say $\alpha p - N = c < 1$, and [21, Proposition 1.4.10] again yields

$$\int_{\mathbb{B}} |f|^p \, d\nu \lesssim \int_0^1 \frac{r^{2N-1}}{(1-r^2)^c} \, dr,$$

which converges, where we use $\lesssim$ to incorporate the cases $c \leq 0$ too.

Since $c_k(\alpha - (1+N)) \sim k^{\alpha-1}$, it is clear that $f_\alpha \in H^\infty$ if and only if $\alpha < 0$.

We say a sequence $\{n_k\}$ in $\mathbb{N}$ has *Hadamard gaps* if there is a $\tau > 1$ such that $n_{k+1} \geq \tau n_k$. For such a sequence, if the homogeneous expansion of an $f \in H(\mathbb{B})$ has the form $f = \sum_k f_{n_k}$, we write $f \in HG$.

Theorem 3.3. *Let $H = \sum_k H_{n_k} \in HG$.*

(i) *$H \in B^p_q$ if and only if $\sum_k n_k^{-(1+q)} \left\| H_{n_k} \right\|^p_{L^p(\sigma)} < \infty$.*

(ii) *$H \in \mathcal{B}^\infty_\alpha$ if and only if $\sup_k n_k^{-\alpha} \left\| H_{n_k} \right\|_{L^\infty(\sigma)} < \infty$.*

Proof. See [15, Lemmas 9.4 and 9.2]. See also [29, Propositions 61 and 63]. For unweighted Bergman spaces A^p_0, see also [28, Proposition 3]. For the spaces $\mathcal{B}^\infty_\alpha$ with $\alpha > -1$, see also [28, Proposition 2]. □

Several of our examples are constructed using the *Ryll–Wojtaszczyk polynomials* W_m, $m = 0, 1, 2, \dots$. Each W_m is a homogeneous polynomial of degree m with the properties

$$\|W_m\|_{L^\infty(\sigma)} = 1 \qquad \text{and} \qquad \|W_m\|_{L^p(\sigma)} \gtrsim 1 \quad (0 < p < \infty). \tag{3.6}$$

These polynomials are invented with the first property and the second with $p = 2$ in [22, Theorem 1.2]. The second property for general p is due to [24, Corollary 1]. Clearly also

$$\|W_m\|_{L^p(\sigma)} \le \|W_m\|_{L^\infty(\sigma)} = 1. \tag{3.7}$$

When $N = 1$, we can simply take $W_m = z^m$. However, taking something like z_1^m for simplicity when $N > 1$ would not be as useful, because it does not satisfy the second property in (3.6).

Example 3.4. The example that is indispensable for the Bergman–Besov spaces is the family of functions

$$G_{qp}(z) := \sum_k 2^{k(1+q)/p}\, W_{2^k}(z) \qquad (z \in \mathbb{B})$$

indexed by q and p. By Theorem 3.3 and (3.6), it is clear that $G_{qp} \notin B^p_q$. But the real question is to determine those spaces B^P_Q that G_{qp} lies in. The answer is that G_{qp} lies in B^P_Q if and only if $(1 + q)/p < (1 + Q)/P$, that is, G_{qp} lies in those B^P_Q whose (P, Q) lies precisely in regions I and IV of Figure 1.

If $(1 + q)/p < (1 + Q)/P$, then by Theorem 3.3 and (3.7),

$$\sum_k 2^{-k(1+Q)}\big(2^{k(1+q)/p}\big)^P \|W_{2^k}\|^P_{L^P(\sigma)} \le \sum_k \frac{1}{2^{kP((1+Q)/P-(1+q)/p)}} < \infty,$$

and $G_{qp} \in B^P_Q$. On the other hand, if $(1 + q)/p \ge (1 + Q)/P$, then by Theorem 3.3 and (3.6),

$$\sum_k 2^{-k(1+Q)}\big(2^{k(1+q)/p}\big)^P \|W_{2^k}\|^P_{L^P(\sigma)} \gtrsim \sum_k \frac{1}{2^{kP((1+Q)/P-(1+q)/p)}} = \infty,$$

and $G_{qp} \notin B^P_Q$.

By Theorem 3.3 and (3.6), it is clear that $G_{qp} \in \mathcal{B}^\infty_\alpha$ if and only if $(1 + q)/p \le \alpha$. Further, $G_{qp} \in H^\infty$ by (3.6) if $q < -1$. On the other hand, if $G_{-1,p} \in H^\infty$ were true, then also $G_{-1,p} \in H^2$. Then we would have $\|G_{-1,p}\|_{H^2} = \sum_k \|W_{2^k}\|_{L^2(\sigma)} < \infty$ by [21, Proposition 1.4.8]. But this is impossible since $\sum_k \|W_{2^k}\|_{L^2(\sigma)} = \infty$ by (3.6).

The use of Ryll–Wojtaszczyk polynomials to show exclusions between pairs of function spaces is advocated in [28] and [19].

Example 3.5. We now construct functions in every Besov space using the atomic decomposition idea. The atomic decomposition of Besov spaces is developed in several places starting with [6, Theorem 2], but most proofs use hypotheses that are too much for our purposes, so we construct our functions from scratch using minimal assumptions. We start with a sequence $\{a_k\}$ in $\mathbb{B}$ that is merely separated with separation constant 2ρ. Given q and p, we also take a sequence $\lambda = \{\lambda_k\}$ in ℓ^p. For $0 < p \le 1$, we take an s satisfying the inequality $1 + N + q < p(1 + N + s)$; for $1 < p < \infty$, we take an s satisfying $1 + q < p(1 + s)$. We set

$$F_{qp}(z) := \sum_k \lambda_k \big(1 - |a_k|^2\big)^{1+N+s-(1+N+q)/p} K_s(z, a_k) \qquad (z \in \mathbb{B}).$$

We start by showing that the series defining F_{qp} converges uniformly on any compact set $M \subset \mathbb{B}$ and hence $F_{qp} \in H(\mathbb{B})$. Let $z \in M$; then $|K_s(z, a_k)| \sim 1$ for any k, and

$$|F_{qp}(z)| \lesssim \sum_k |\lambda_k| \big(1 - |a_k|^2\big)^{1+N+s-(1+N+q)/p}.$$

First for $0 < p \le 1$, by the first choice of s, the power on $1 - |a_k|^2$ is positive and hence

$$|F_{qp}(z)| \lesssim \sum_k |\lambda_k| \le \sum_k |\lambda_k|^p < \infty \qquad (z \in M).$$

Second for $1 < p < \infty$, by the Hölder inequality and that $\lambda \in \ell^p$, we have

$$|F_{qp}(z)| \lesssim \left(\sum_k \left(1 - |a_k|^2\right)^{(p(1+N+s)-(1+N+q))/(p-1)} \right)^{(p-1)/p} \qquad (z \in \boldsymbol{M}).$$

Call the power on $1 - |a_k|^2$ by $\widetilde{p}$. By the second choice of s, we see that $\widetilde{p} > N$. Then since the balls $b(a_k, \rho)$ are disjoint, using [15, Lemmas 2.1 and 2.2], we obtain

$$\begin{aligned} |F_{qp}(z)|^{p/(p-1)} &\lesssim \sum_k \left(1 - |a_k|^2\right)^{\widetilde{p}} \sim \sum_k \int_{b(a_k,\rho)} \left(1 - |w|^2\right)^{\widetilde{p}-(1+N)} d\nu(w) \\ &= \int_{\bigcup_k b(a_k,\rho)} \left(1 - |w|^2\right)^{\widetilde{p}-(1+N)} d\nu(w) \leq \int_{\mathbb{B}} \left(1 - |w|^2\right)^{\widetilde{p}-(1+N)} d\nu(w) \end{aligned}$$

for all $z \in \boldsymbol{M}$. But the last integral is finite.

To show that $F_{qp} \in B^p_q$, we define a linear map T by $T\lambda := F_{qp}$ for $\lambda \in \ell^p$. Let t satisfy (1.1); then $s + t > -1$ for any value of p. Then using (2.7),

$$D^t_s T\lambda(z) = \sum_k \lambda_k \frac{\left(1 - |a_k|^2\right)^{1+N+s-(1+N+q)/p}}{(1 - \langle z, a_k\rangle)^{1+N+s+t}} \qquad (z \in \mathbb{B}).$$

First for $0 < p \leq 1$,

$$|I^t_s T\lambda(z)|^p \leq \left(1 - |z|^2\right)^{pt} \sum_k |\lambda_k|^p \frac{\left(1 - |a_k|^2\right)^{p(1+N+s)-(1+N+q)}}{|1 - \langle z, a_k\rangle|^{p(1+N+s+t)}}.$$

Then by [21, Proposition 1.4.10],

$$\begin{aligned} \|T\lambda\|^p_{B^p_q} &\leq \sum_k |\lambda_k|^p \left(1 - |a_k|^2\right)^{p(1+N+s)-(1+N+q)} \int_{\mathbb{B}} \frac{\left(1 - |z|^2\right)^{q+pt}}{|1 - \langle z, a_k\rangle|^{p(1+N+s+t)}} d\nu(z) \\ &\sim \sum_k |\lambda_k|^p = \|\lambda\|^p_{\ell^p}. \end{aligned}$$

Second for $1 < p < \infty$, by [15, Lemma 2.1],

$$|D^t_s T\lambda(z)| \lesssim \sum_k |\lambda_k| \frac{\left(1 - |a_k|^2\right)^{1+N-(1+N+q)/p}}{\nu(b(a_k, \rho))} \int_{b(a_k,\rho)} \frac{\left(1 - |w|^2\right)^s}{|1 - \langle z, w\rangle|^{1+N+s+t}} d\nu(w)$$

and $|I^t_s T\lambda(z)| \leq S\phi(z)$, where

$$\phi(w) = \sum_k |\lambda_k| \frac{\left(1 - |a_k|^2\right)^{1+N-(1+N+q)/p}}{\nu(b(a_k, \rho))} \chi_{\nu(b(a_k,\rho))}(w),$$

S is as in [30, Theorem 2.10], and χ denotes the characteristic function. Now we use that $\{a_k\}$ is separated and [15, Lemma 2.2] to write

$$\begin{aligned} \|\phi\|^p_{L^p_q} &= \sum_k |\lambda_k|^p \frac{\left(1 - |a_k|^2\right)^{p(1+N)-(1+N+q)}}{\nu(b(a_k, \rho))^p} \nu_q(b(a_k, \rho)) \\ &\sim \sum_k |\lambda_k|^p \frac{\left(1 - |a_k|^2\right)^{p(1+N)-(1+N+q)}}{\left(1 - |a_k|^2\right)^{p(1+N)}} (1 - |a_k|^2)^{1+N+q} \\ &= \sum_k |\lambda_k|^p = \|\lambda\|^p_{\ell^p}. \end{aligned}$$

Applying [30, Theorem 2.10] implies that $S : L^p_q \to L^p_q$ is bounded by the conditions imposed above on s, t. It follows that $\|T\lambda\|_{B^p_q} \leq \|S\phi\|_{L^p_q} \lesssim \|\phi\|_{L^p_q} \lesssim \|\lambda\|_{\ell^p}$.

Thus $T : \ell^p \to B^p_q$ is bounded and $F_{qp} = T\lambda \in B^p_q$ for any value of p. Note that we need the separation property of $\{a_k\}$ only for $p > 1$.

4 | SOME INITIAL INCLUSIONS

Here we take care of a few better-known and straightforward inclusions that are part of the full picture although not listed among our main results.

Theorem 4.1.

(i) *If $q < Q$, then $B^p_q \subset B^p_Q$.*
(ii) *If $\alpha < \beta$, then $\mathcal{B}^\infty_\alpha \subset \mathcal{B}^\infty_\beta$.*

Proof. For both (i) and (ii), the inclusion follows directly from Definitions 1.1 and 1.2 and Remark 1.3.

The inclusion in (ii) is strict because of (3.3). This part appears earlier in [28, (12) and (13)] with f_α for $\alpha > 0$ and in [16, Example 2.2].

The inclusion in (i) is strict too, because $G_{qp} \in B^p_Q \setminus B^p_q$ by Example 3.4, since (p, Q) lies in region I with respect to B^p_q. A similar example when $N = 1$ appears earlier in [17, Example 4.5]. □

Theorem 4.2. $\mathcal{B}^\infty_{<0} \subset H^\infty \subset \mathcal{B}^\infty$.

Proof. The left hand inclusion on $\mathbb{B}$ is in [21, Theorem 6.4.10]. There is a different proof in [16, Theorem 5.4] using Bergman projections. Both proofs show also $\mathcal{B}^\infty_{<0} \subset A(\mathbb{B})$.

The right hand inclusion for $N = 1$ is commonly proved using the Schwarz–Pick lemma, but the proof for $N > 1$ is nowhere to be found, so we provide one. Let $f \in H^\infty$, and without loss of generality, assume $f : \mathbb{B} \to \mathbb{D}$ so that $\|f\|_\infty \leq 1$. Let also $w = f(z)$, and set $g = \varphi_w \circ f \circ \varphi_z$, where φ_w on $\mathbb{D}$ and φ_z on $\mathbb{B}$ are as in (2.8). Then $g \in H(\mathbb{B})$, $g(0) = 0$, and $|g'(0)\zeta| \leq 1$ for all $\zeta \in \mathbb{B}$, where $g'(0)$ is called the *hyperbolic derivative* of f at z; see [12, p. 651]. Applying the chain rule yields that $g'(0) = \left(1 - |w|^2\right)^{-1} \nabla f(z) \varphi'_z(0)$, where $\varphi'_z(0) = -\left(1 - |z|^2\right) P_z + \sqrt{1 - |z|^2}(I - P_z)$ by [21, Theorem 2.2.2 (ii)]. Since $|w| \leq 1$, using the special value $\zeta = z$, we obtain that $|\nabla f(z)|(1 - |z|^2)|z| \leq 1$ for all $z \in \mathbb{B}$. This proves that $f \in \mathcal{B}^\infty$.

By Example 3.2, the function $f_0 \in \mathcal{B}^\infty \setminus H^\infty$ shows that the right hand inclusion is strict. If $\alpha < 0$, choose β such that $\alpha < \beta < 0$. Then $f_\beta \in H^\infty \setminus \mathcal{B}^\infty_\alpha$ by (3.3), and the left hand inclusion is also strict. □

5 | A BERGMAN–BESOV AND A BLOCH–LIPSCHITZ SPACE

In this section, we prove Theorem 1.5. This theorem already appears in [29, Theorem 66] with a proof that follows the same long path as the proof of the case $q = 0$ of unweighted Bergman spaces in [28, Theorem]. However, once the result is established for this case, we can use the idea in Example 3.2 to differentiate and obtain the full result in all Besov spaces. We follow a different path though. We prove the right hand inclusion for a different value of q, because it has a much more direct proof. For the left hand inclusion, we simplify the proof given in [28].

The right hand inclusion for general q appears in [3, Corollary 5.5]. It is probably known for some time that the Lipschitz spaces and the usual Bloch space $\mathcal{B}^\infty$ lie in all Bergman spaces, which are special cases of the left hand inclusion and are direct consequences of Remark 1.3.

Proof of Theorem 1.5. Corollary 2.6 supplies us with a sufficiently general instance of the right hand inclusion, which is

$$B^p_{-(1+N)} \subset \mathcal{B}^\infty. \tag{5.1}$$

We take from [28, Theorem (a)] the case $q = 0$ of the left hand inclusion, which is

$$\mathcal{B}^\infty_{<\frac{1}{p}} \subset A^p_0. \tag{5.2}$$

Here's a proof of this that depends on Remark 1.3 and is slightly simpler than the one given in [28]. It is sufficient to take $0 < \alpha < 1/p$ and show that $\mathcal{B}^\infty_\alpha \subset A^p_0$; for such an α, the norm $\|\cdot\|_{\mathcal{B}^\infty_\alpha}$ does not require any derivative. If $f \in \mathcal{B}^\infty_\alpha$, it holds that $|f(z)|^p \lesssim (1-|z|^2)^{-\alpha p}$ for all $z \in \mathbb{B}$. Then by polar coordinates,

$$\int_{\mathbb{B}} |f|^p \, d\nu \lesssim \int_0^1 \frac{r^{2N-1}}{(1-r^2)^{\alpha p}} \int_{\mathbb{S}} d\sigma \, dr \lesssim \int_0^1 \frac{r}{(1-r^2)^{\alpha p}} \, dr,$$

which is finite since $\alpha p < 1$.

To boost all these to B^p_q with arbitrary q, we simply apply $D_s^{(1+N+q)/p}$ to the spaces in (5.1), and we apply $D_s^{q/p}$ to the spaces in (5.2). These yield all the inclusions claimed in the statement of the theorem by Theorem 2.3.

We follow the proof of [28, Theorem (b)] and exhibit functions that show the inclusions just obtained are strict and the best possible. We use (3.3) and (3.5) three times. The right hand inclusion is strict because we have $f_{\frac{1+N+q}{p}} \in \mathcal{B}^\infty_{\frac{1+N+q}{p}} \setminus B^p_q$, and the left hand inclusion is strict because we have $f_{\frac{1+q}{p}} \in B^p_q \setminus \mathcal{B}^\infty_{<\frac{1+q}{p}}$. Next, if $\beta < \eta < (1+N+q)/p$, then $f_\eta \in B^p_q \setminus \mathcal{B}^\infty_\beta$, and in view of Theorem 4.1, this shows the right hand inclusion is the best possible. Finally, $G_{qp} \in \mathcal{B}^\infty_{\frac{1+q}{p}} \setminus B^p_q$ by Example 3.4, and this shows the left hand inclusion is the best possible. □

6 | TWO BERGMAN–BESOV SPACES

This section is devoted to a new proof of Theorem 1.6. Its both parts appear also in [29, Theorems 69 and 70], but their proofs rely on difficult Carleson measure results. We give totally different direct proofs of the two parts based on two representative cases, Theorem 1.5, and the differentiation idea in Example 3.2. Our proof also sheds more light on the relationships among the B^p_q spaces.

There are several earlier partial results. [11, p. 703] and [18, Theorem 1] have a mixture of the two cases in certain Bergman spaces. Again for Bergman spaces with $0 < p < P = 1$ and $(2+q)/p = 2+Q$, the inclusion in (i) is obtained in [23, Theorem 3]. With $0 < p < P \le 1$ and $(1+N+q)/p = (1+N+Q)/P$, the inclusion in (i) is shown in [6, Proposition 4.2] still for Bergman spaces. The full inclusion in (i) for Besov spaces is in [3, Theorem 5.13]. With $1 < p < P < \infty$ and $q = Q = -(1+N)$, the inclusion in (i) is developed in [10, Proposition 3.3] using Möbius invariance, and [20, Theorem 2.5] adds the $p = 1$ case to the previous inclusions. None of these results contain the only if parts.

Proof of Theorem 1.6. (i) Assume $p \le P$ and (1.3). We follow the method of [9, Proposition 13.3] to obtain the desired inclusion. First let s, t satisfy (1.1) with $q = -(1+N)$ and p; then s, t clearly satisfy (1.2) with $\alpha = 0$. Let also $f \in B^p_{-(1+N)}$. Then by Corollary 2.6,

$$\begin{aligned}\|f\|^P_{B^P_{-(1+N)}} &= \int_{\mathbb{B}} |I^t_s f|^P \, d\nu_{-(1+N)} = \int_{\mathbb{B}} |I^t_s f|^{P-p} \, |I^t_s f|^p \, d\nu_{-(1+N)} \\ &\le \|f\|^{P-p}_{\mathcal{B}^\infty} \int_{\mathbb{B}} |I^t_s f|^p \, d\nu_{-(1+N)} \lesssim \|f\|^{P-p}_{B^p_{-(1+N)}} \, \|f\|^p_{B^p_{-(1+N)}} = \|f\|^P_{B^p_{-(1+N)}},\end{aligned}$$

which gives us one of the two fundamental inclusions $B^p_{-(1+N)} \subset B^P_{-(1+N)}$.

To pass to the remaining Bergman–Besov spaces, we use (1.3) and call its two fractions $-t$ and $-T$; then we have the equalities $q + pt = -(1+N) = Q + PT$. We apply D_s^{-T} to both sides of the fundamental inclusion in the previous paragraph. Theorem 2.3 implies $B^p_{-(1+N)-pT} \subset B^P_Q$, that is, $B^p_{q+p(t-T)} \subset B^P_Q$. Since $t - T \ge 0$, we obtain $B^p_q \subset B^P_Q$ by Theorem 4.1.

Conversely, suppose that $p \le P$ and $B^p_q \subset B^P_Q$. Since $B^P_Q \subset \mathcal{B}^\infty_{\frac{1+N+Q}{P}}$ by Theorem 1.5, we also have $B^p_q \subset \mathcal{B}^\infty_{\frac{1+N+Q}{P}}$. By the best possible assertion of that theorem, we conclude that (1.3) holds.

(ii) Assume now $P < p$ and (1.4). The proof is a variant of those of [9, Proposition 13.2] and part (i). For any $r > -1$, the finiteness of the measure ν_r gives us the other fundamental inclusion $A^p_r \subset A^P_r$.

Next we pass to the remaining Bergman–Besov spaces. Let

$$r = \frac{Qp - qP}{p - P}, \qquad \text{that is,} \qquad \frac{-r + Q}{P} = \frac{-r + q}{p}.$$

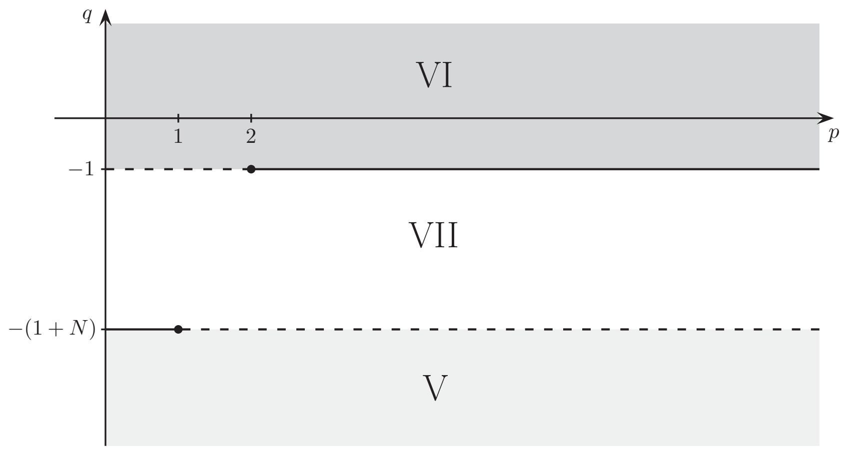

FIGURE 2 If $(p,q) \in$ V, then $B^p_q \subset H^\infty$; if $(p,q) \in$ VI, then $H^\infty \subset B^p_q$; if $(p,q) \in$ VII, then neither B^p_q nor H^∞ contains the other, but $B^p_q \cap H^\infty \neq \emptyset$

Then $r > -1$ by (1.4). Now call the common value of the two fractions on the right t; then $r + Pt = Q$ and $r + pt = q$. We apply D^t_s to the fundamental inclusion in the previous paragraph. Theorem 2.3 implies that $B^p_q \subset B^P_Q$.

Conversely, suppose that $P < p$ and $B^p_q \subset B^P_Q$. Since $\mathcal{B}^\infty_{<\frac{1+q}{p}} \subset B^p_q$ by Theorem 1.5, also $\mathcal{B}^\infty_{<\frac{1+q}{p}} \subset B^P_Q$. By the best possible assertion of that theorem, we can only conclude that (1.4) holds with $\leq$. We prove that equality cannot occur by assuming $(1+q)/p = (1+Q)/P$ and showing that the claimed inclusion does not hold. We do it through a gap series of Ryll–Wojtaszczyk polynomials once again. Let

$$\widetilde{G}_{QP} = \sum_k \frac{2^{k(1+Q)/P}}{k^{1/P}} W_{2^k}. \tag{6.1}$$

By Theorem 3.3, (3.6), and (3.7), we see that $\widetilde{G}_{QP} \in B^p_q \setminus B^P_Q$. This proves (1.4). In the last part, G_{QP} in place of $\widetilde{G}_{QP}$ would not work, because the G_{QP} are designed for the regions in Figure 1, and a point (P,Q) satisfying $P < p$ and $(1+q)/p = (1+Q)/P$ lies not in region I but on its boundary with respect to B^p_q.

(i), (ii) The inclusions are the best possible since they are given by if and only if conditions. They are also strict because Example 3.4 says exactly that. □

7 | A BERGMAN–BESOV SPACE AND H^∞

We are now ready to prove Theorem 1.7. We have nothing new to say on the inclusion part, but for certain cases in the strict and the best possible parts, the functions developed in Section 3 are not good enough, and we have to attempt even more elaborate constructions.

The inclusions of Theorem 1.7 are shown graphically in Figure 2. The space H^∞ includes all the B^p_q spaces in region V and is included all the B^p_q in region VI. A B^p_q in region VII has nonempty intersection with H^∞ without one including the other.

Proof of Theorem 1.7. (i) The inclusions are already proved in [29, Theorems 21 and 22] in their entirety as well as $B^p_q \subset A(\mathbb{B})$. What is left is to show that they are strict and the best possible. Strictness is easy. Example 3.4 explains that $G_{qp} \notin B^p_q$. But (3.6) shows that $G_{qp} \in H^\infty$ for $q \leq -(1+N)$ and any p.

For the best possible claim, we consider the two cases

$$q > -(1+N) \quad \text{and} \quad 0 < p \leq 1, \qquad \text{or} \qquad q = -(1+N) \quad \text{and} \quad 1 < p < \infty.$$

Example 3.5 furnishes us with $F_{qp} \in B^p_q$, and we have to pick suitable $\{a_k\}$ and $\{\lambda_k\}$ to force $F_{qp} \notin H^\infty$ in these two cases. We take $b_k = 1 - 2^{-k}$ and $a_k = b_k e_1$. Then $P_{a_k}(a_m) = a_m$, and for $m < k$, we compute that $|a_k - a_m| = \left(2^k - 2^m\right)2^{-(k+m)}$ and

$|1-\langle a_k, a_m\rangle| = 2^{-k} + 2^{-m} - 2^{-(k+m)} < \left(2^k + 2^m\right)2^{-(k+m)}$. Hence

$$|\varphi_{a_k}(a_m)| \geq \frac{2^k - 2^m}{2^k + 2^m} \geq \frac{1}{4} \qquad \text{and} \qquad d(a_k, a_m) \geq \tanh^{-1}\frac{1}{4} =: 2\rho;$$

in other words, $\{a_k\}$ is separated.

For $c \in \mathbb{C}$, we write $\operatorname{sgn}(c) = c/|c|$ if $c \neq 0$ and $\operatorname{sgn}(0) = 0$. Next we pick $\lambda_k = k^{-1-\operatorname{sgn}(1+N+q)/p}$; explicitly, $\lambda_k = 1/k$ for $q = -(1+N)$ and $\lambda_k = 1/k^{1+1/p}$ for $q \neq -(1+N)$; but for both cases under consideration, $\{\lambda_k\} \in \ell^p$. Also by the choice of s in Example 3.5, in both cases $1+N+s > 0$, which means that the kernel used in the definition of F_{qp} is binomial and not hypergeometric. If $z = re_1$, then $\langle z, a_k\rangle = rb_k$. Then

$$F_{qp}(re_1) = \sum_k \frac{1}{k^{1+\operatorname{sgn}(1+N+q)/p}} \frac{\left(1-b_k^2\right)^{1+N+s-(1+N+q)/p}}{(1-rb_k)^{1+N+s}}.$$

Since $1-b_k^2 \sim 1-b_k$, by the monotone convergence theorem,

$$\lim_{r\to 1^-} F_{qp}(re_1) \sim \sum_k \frac{(1-b_k)^{-(1+N+q)/p}}{k^{1+\operatorname{sgn}(1+N+q)/p}} = \sum_k \frac{2^{k(1+N+q)/p}}{k^{1+\operatorname{sgn}(1+N+q)/p}} = \infty$$

in both cases, and thus $F_{qp} \notin H^\infty$.

(ii) Evidently H^∞ lies in all weighted Bergman spaces A_q^p simply by the finiteness of the measures ν_q for $q > -1$. The next question is whether the Besov spaces B_{-1}^p at the boundary of the Bergman zone are large enough to include H^∞. Here a Littlewood–Paley inequality helps us. For $p \geq 2$, [5, Theorem] states that

$$\int_{\mathbb{B}} \left(|\nabla f(z)|^2 - |Rf(z)|^2\right)^{p/2} \left(1-|z|^2\right)^{p/2-1} d\nu(z) \lesssim \|f\|_{H^p}^p.$$

But $|Rf(z)| \leq |z|\,|\nabla f(z)|$. Substituting this into the Littlewood–Paley inequality, we obtain

$$\int_{\mathbb{B}} |\nabla f(z)|^p \left(1-|z|^2\right)^{p-1} d\nu(z) \lesssim \|f\|_{H^p}^p \qquad (2 \leq p < \infty).$$

By virtue of Remark 2.4, this says nothing but $H^p \subset B_{-1}^p$ for $p \geq 2$. In fact, because both are Hilbert spaces, $B_{-1}^2 = H^2$. Thus $H^\infty \subset B_{-1}^p$ for $p \geq 2$. Note that Littlewood–Paley inequalities are in general reversed for $0 < p < 2$.

With the same $\{a_k\}$ in the proof of part (i) and $\lambda_k = k^{-1-\operatorname{sgn}(1+q)/p}$, we have $F_{qp} \in B_q^p \setminus H^\infty$ with the same proof as above in all q, p combinations mentioned in the statement of this part since now $1+N+q \geq N > 0$. This shows strictness.

By construction, $G_{qp} \notin B_q^p$, and for $q < -1$ and any p, also $G_{qp} \in H^\infty$ by Example 3.4; this shows that the inclusions of this part are the best possible for $p \geq 2$. Here's another proof of this fact. If $H^\infty \subset B_q^p$ for some $q < -1$, let $(1+q)/p < \beta < 0$. Then by Theorem 4.2, also $\mathcal{B}_\beta^\infty \subset B_q^p$. But this is impossible by the best possible conclusion of Theorem 1.5.

When $q = -1$ and $0 < p < 1$, then from (6.1), $\widetilde{G}_{-1,p} \in A(\mathbb{B}) \setminus B_{-1}^p$ by (3.6), and this shows that the inclusion is the best possible for the q, p at hand.

This leaves us with proving that the inclusion is the best possible for the case $q = -1$ and $1 \leq p < 2$. This is the most involved part of the proof, so we isolate it as the next theorem, which is also of independent interest. □

Theorem 7.1. *(i) If $\{c_k\} \in \ell^2$, there is a $G \in A(\mathbb{B})$ such that $[G, W_{2^k}]_{L^2(\sigma)} = c_k$. (ii) For every $1 \leq p < 2$, there is a $\breve{G}_p \in A(\mathbb{B}) \setminus B_{-1}^p$.*

Proof. (i) This is just [27, Proposition] written in different words. There is a constructive proof for $N = 1$ in [8, Theorem]. Let's write carefully what this part says. If $G = \sum_m G_m$ is the homogeneous expansion of G, then we have no control on G_m if $m \neq 2^k$ for some k. When $m = 2^k$, both G_{2^k} and W_{2^k} are finite sums of δ_{2^k} monomials possibly with different coefficients and possibly with different sets of coefficients equal to zero; yet $\int_{\mathbb{S}} G_{2^k}(\zeta)\overline{W_{2^k}(\zeta)}\, d\sigma(\zeta) = c_k$ by [21, Proposition 1.4.8].

(ii) We follow the sketch of the proof of [26, Lemma 1.6], which is for $N = 1$, and fill in all its details. Let $1 \leq p < 2$ and suppose $G \in A(\mathbb{B})$ of part (i) lies in B_{-1}^p. Let s, t satisfy (1.1) for $q = -1$ and such p. So $D_s^t G(r\zeta) = \sum_m d_m(s,t) r^m G_m(\zeta)$ for $0 < r < 1$. By [21, Proposition 1.4.8] and (2.3),

$$\int_{\mathbb{S}} D_s^t G(r\zeta)\overline{W_{2^k}(\zeta)}\, d\sigma(\zeta) = d_{2^k} r^{2^k} \int_{\mathbb{S}} G_{2^k}(\zeta)\overline{W_{2^k}(\zeta)}\, d\sigma(\zeta) = d_{2^k} r^{2^k} c_k \sim 2^{tk} c_k r^{2^k}.$$

Then by the convexity of the pth power function and (3.6),

$$2^{ptk}|c_k|^p\, r^{p\,2^k} \lesssim \int_{\mathbb{S}} |D_s^t G(r\zeta)|^p\, d\sigma(\zeta) \qquad (0 < r < 1).$$

Now by the polar coordinates formula,

$$\sum_k 2^{ptk}|c_k|^p \int_{1-2^{-k}}^{1-2^{-(1+k)}} r^{2N-1+p\,2^k}\left(1-r^2\right)^{-1+pt} dr \lesssim \int_{\mathbb{B}} |D_s^t G(z)|^p \left(1-|z|^2\right)^{-1+pt} d\nu(z).$$

On the interval $\left[1-2^{-k}, 1-2^{-(1+k)}\right]$ of length 2^{-k}, we have $1-r^2 \sim 1-r \sim 2^{-k}$, and

$$r^{2N-1+p\,2^k} \ge \left(1-2^{-k}\right)^{2N+p\,2^k} \ge \left(e^{-2^{1-k}}\right)^{2N+p\,2^k} \ge e^{-N\,2^{2-k}-2p} \ge e^{-4N-2p} > 0$$

for large enough k since $1-x \ge e^{-2x}$ for small enough $x>0$. Hence the integral on this interval is $\sim 2^{-ptk}$. We therefore obtain $\sum_k |c_k|^p \lesssim \|G\|^p_{B^p_{-1}}$. This implies that $\{c_k\} \in \ell^p$ if $G \in B^p_{-1}$. If we take a $\{c_k\} \in \ell^2 \setminus \ell^p$, then the function $G \in A(\mathbb{B})$ obtained in part (i) is the desired $\breve{G}_p \notin B^p_{-1}$. □

The fact that there are functions in the ball algebra that do not lie in certain Bergman–Besov spaces has an interesting consequence: Essentially, the derivative on the function in the integral norm (2.5) of B^p_q cannot be dispensed with when $(p,q) \in \mathrm{V} \cup \mathrm{VII}$. This fact is first noticed in [2, p. 180] for the Drury–Arveson space B^2_{-N}.

Corollary 7.2. *Let $\kappa : [0,\infty) \to \mathbb{R}$ be an increasing function with $\kappa(0)=0$, and let μ be a positive Borel measure with support in $\overline{\mathbb{B}}$. Define $E_{\kappa\mu}$ as the set of all $f \in H(\mathbb{B})$ for which*

$$\limsup_{r\to 1^-} \int_{\overline{\mathbb{B}}} \kappa(|f(rz)|)\, d\mu(z) < \infty.$$

Then $E_{\kappa\mu} \ne B^p_q$ if $q < -1$, or if $q=-1$ and $0<p<2$.

Proof. We imitate the proof of [4, Theorem 4.3] that takes care of the Hardy–Sobolev-space counterpart. Let q,p be as in the statement of the corollary, and suppose $B^p_q = E_{\kappa\mu}$ for some κ and μ. Applying the definition of $E_{\kappa\mu}$ to $f = 1 \in B^p_q$, we obtain $\kappa(1)\mu\left(\overline{\mathbb{B}}\right) < \infty$; so μ must be finite. If $f \in A(\mathbb{B})$, then

$$\limsup_{r\to 1^-} \int_{\overline{\mathbb{B}}} \kappa(|f(rz)|)\, d\mu(z) \le \kappa\left(\|f\|_{L^\infty(\sigma)}\right)\mu\left(\overline{\mathbb{B}}\right) < \infty,$$

which yields that $f \in B^p_q$ too. This contradicts the fact that there are functions in $A(\mathbb{B}) \setminus B^p_q$ for the values of q,p considered. For $q<-1$, one such function is G_{qp} of Example 3.4 by (3.6); for $q=-1$ and $0<p<1$, one such function is $\widetilde{G}_{-1,p}$ as indicated in the proof of Theorem 1.7 (ii); for $q=-1$ and $1 \le p < 2$, one such function is $\breve{G}_p$ as indicated in Theorem 7.1 (ii). □

Remark 7.3. We do not know whether or not the norm of B^p_{-1} with $p>2$ can be written as an integral without using a derivative on the function. On the other hand, B^2_{-1} is the Hardy space H^2 and its norm is the same as that of $L^2(\sigma)$. If $q>-1$, then the B^p_q are the Bergman spaces and clearly have integral norms without derivatives.

The following can be considered the $p=\infty$ version of Corollary 7.2 and concerns the derivative in (2.6).

Corollary 7.4. *Let $\kappa : [0,\infty) \to \mathbb{R}$ be an increasing function with $\kappa(0)=0$, and let $\omega : \mathbb{B} \to [0,\infty)$. Define $\mathcal{E}_{\kappa\omega}$ as the set of all $f \in H(\mathbb{B})$ for which*

$$\sup_{z\in\mathbb{B}} \kappa(|f(z)|)\,\omega(z) < \infty.$$

Then $\mathcal{E}_{\kappa\omega} \ne \mathcal{B}^\infty_\alpha$ if $\alpha < 0$.

Proof. Let $\alpha<0$ and suppose $\mathcal{B}^\infty_\alpha = \mathcal{E}_{\kappa\omega}$ for some κ and ω. Applying the definition of $\mathcal{E}_{\kappa\omega}$ to $f = 1 \in \mathcal{B}^\infty_\alpha$, we obtain $\kappa(1)\sup_{z\in\mathbb{B}} \omega(z) < \infty$; so ω is bounded. If $f \in H^\infty$, then

$$\sup_{z\in\mathbb{B}} \kappa(|f(z)|)\,\omega(z) \le \kappa(\|f\|_\infty) \sup_{z\in\mathbb{B}} \omega(z) < \infty,$$

which yields that $f \in \mathcal{B}^\infty_\alpha$ too. This contradicts the fact that there are functions in $H^\infty \setminus \mathcal{B}^\infty_\alpha$ for $\alpha < 0$. One such function is f_α of Example 3.2. □

Remark 7.5. We do not know whether or not the norm of $\mathcal{B}^\infty$ can be written without using a derivative on the function. On the other hand, the $\mathcal{B}^\infty_\alpha$ with $\alpha > 0$ have norms without derivatives by (1.2).

Remark 7.6. There are characterizations of Bergman–Besov and Bloch–Lipschitz spaces that do not use a derivative directly but use a difference quotient of some sort; see, for example, [25] and the references therein. But a difference quotient behaves very much like a derivative. When we say "without using a derivative" in Remarks 7.3 and 7.5, we exclude such characterizations too.

8 | SOBOLEV IMBEDDINGS

Our final intention is to compare the inclusions in Theorems 1.5 and 1.6 with those predicted by the holomorphic versions of the Sobolev imbedding theorems.

Following [1, Chapter 3], for $1 \le p < \infty$, we let the *Sobolev space* $W^{m,p}$ be the space of all locally integrable functions on $\mathbb{B}$ all of whose generalized partial derivatives of order up to and including $m = 1, 2, \dots$ belong to L^p. The subspace of $W^{m,p}$ consisting of holomorphic functions on $\mathbb{B}$ can be regarded as the Besov space B^p_{-mp} in which always $q = -mp \le -1$.

The Sobolev imbedding theorem we are interested in is [1, Theorem 4.12] and concerns the continuous inclusion of $W^{m,p}$ in Lebesgue or Lipschitz spaces. It is for Sobolev spaces defined on some types of domains in $\mathbb{R}^n$ for all of which $\mathbb{B}$ is a standard example. We read this theorem by setting $n = 2N$ and taking the intersections of all spaces with $H(\mathbb{B})$. We analyze our findings in five regions of the parameters.

(a) If $mp < 2N$ (that is, $-2N < q \le -1$), the Sobolev imbedding is $W^{m,p} \subset L^{p^*}$, which translates to the holomorphic setting as $B^p_{-mp} \subset B^{p^*}_0$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{m}{2N}$. The sharpest (that is, into smallest space) inclusion that Theorem 1.6 (i) gives is $B^p_{-mp} \subset B^{p_2}_0$, where $\frac{1}{p_2} = \frac{1}{p} - \frac{m}{1+N}$. Since this $p_2 > p^*$, our inclusion is sharper than that of the Sobolev imbedding theorem for $N > 1$. The two results say the same for $N = 1$.

(b) If $mp = 2N$ (that is, $q = -2N$), the Sobolev imbedding is $W^{m,p} \subset L^{p^*}$ for any $p^* \ge 1$, which yields $B^p_{-2N} \subset B^{p^*}_0$. The inclusions that Theorem 1.6 (i) gives are $B^p_{-2N} \subset B^{p_2}_0$, where $\frac{1}{p_2} \ge \frac{1-N}{1+N}\,\frac{1}{p}$, which says $0 < p_2 < \infty$ for all N. So the two imbeddings are equivalent for $q = -2N$.

(c) If $2N < mp < 2N + p$ (that is, $-2N - p < q < -2N$), the Sobolev imbedding is $W^{m,p} \subset \Lambda_\beta$, where $\beta = m - 2N/p$ and $0 < \beta < 1$, which yields $B^p_{-mp} \subset \mathcal{B}^\infty_{-\beta}$. The inclusion that Theorem 1.5 gives is $B^p_{-mp} \subset \mathcal{B}^\infty_\alpha$, where $\alpha = (1 + N)/p - m$. Since this $\alpha < -\beta$, our inclusion is sharper than that of the Sobolev imbedding theorem for $N > 1$. The two results say the same for $N = 1$.

(d) If $mp = 2N + p$ (that is, $q = -2N - p$), the Sobolev imbedding is $W^{m,p} \subset \Lambda_\beta$ for any $0 < \beta < 1$, yielding $B^p_{-2N-p} \subset \mathcal{B}^\infty_{-\beta}$. If also $p = 1$, then $W^{m,1} \subset \Lambda_1$, that is, $B^1_{-2N-1} \subset \mathcal{B}^\infty_{-1}$. The inclusion that Theorem 1.5 gives is $B^p_{-2N-p} \subset \mathcal{B}^\infty_\alpha$ for any $1 \le p < \infty$, where $\alpha = (1 - N)/p - 1$. Since this $\alpha < -\beta$, our inclusion is sharper than that of the Sobolev imbedding theorem for $N > 1$ or $p > 1$. The two results say the same when both $N = 1$ and $p = 1$.

(e) If $mp > 2N + p$ (that is, $q < -2N - p$), the Sobolev imbedding is $W^{m,p} \subset \Lambda_1$, which yields $B^p_{-mp} \subset \mathcal{B}^\infty_{-1}$. The inclusion that Theorem 1.5 gives is $B^p_{-mp} \subset \mathcal{B}^\infty_\alpha$, where $\alpha = (1 + N)/p - m$. Since now $\alpha < -1$, our inclusion is sharper than that of the Sobolev imbedding theorem.

We can compare the number of derivatives lost in the imbeddings in (c), (d), and (e), which are all in the form $W^{m,p} \subset \Lambda_\beta$. The number of derivatives we lose is indicated by the difference $m - \beta$ and is $2N/p$. On the other hand, the derivatives needed for B^p_{-mp} is given by $-mp + pt > -1$ and is $t > m - 1/p$, while for $\Lambda_{m-2N/p}$ it is given as $t > m - 2N/p$; hence the number of derivatives we lose is $m - 1/p - (m - 2N/p) = (2N - 1)/p$. So in our imbeddings, we lose derivatives of order $1/p$ less than those lost in the Sobolev imbeddings. This difference has already been noted in [3, pp. 39–40].

The fact that our inclusions are stronger and our loss of derivatives is less in general than those predicted by the Sobolev imbedding theorem should come as no surprise, because our spaces consist of holomorphic functions that are very smooth on a very nice domain.

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