COMPLETE INTERSECTION MONOMIAL CURVES AND THE COHEN-MACAULAYNESS OF THEIR TANGENT CONES

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Abstract. Let $C(\mathbf{n})$ be a complete intersection monomial curve in the 4-dimensional affine space. In this paper we study the complete intersection property of the monomial curve $C(\mathbf{n}+w\mathbf{v})$, where $w>0$ is an integer and $\mathbf{v}\in\mathbb{N}^4$. Also we investigate the Cohen-Macaulayness of the tangent cone of $C(\mathbf{n}+w\mathbf{v})$.

1. Introduction

Let $\mathbf{n}=(n_1,n_2,\ldots,n_d)$ be a sequence of positive integers with $\gcd(n_1,\ldots,n_d)=1$. Consider the polynomial ring $K[x_1,\ldots,x_d]$ in d variables over a field K. We shall denote by $\mathbf{x}^{\mathbf{u}}$ the monomial $x_1^{u_1}\cdots x_d^{u_d}$ of $K[x_1,\ldots,x_d]$, with $\mathbf{u}=(u_1,\ldots,u_d)\in\mathbb{N}^d$ where $\mathbb{N}$ stands for the set of non-negative integers. The toric ideal $I(\mathbf{n})$ is the kernel of the K-algebra homomorphism $\phi: K[x_1,\ldots,x_d]\to K[t]$ given by

$$\phi(x_i)=t^{n_i} \quad \text{for all} \quad 1\le i\le d.$$

Then $I(\mathbf{n})$ is the defining ideal of the monomial curve $C(\mathbf{n})$ given by the parametrization $x_1=t^{n_1},\ldots,x_d=t^{n_d}$. The ideal $I(\mathbf{n})$ is generated by all the binomials $\mathbf{x}^{\mathbf{u}}-\mathbf{x}^{\mathbf{v}}$, where $\mathbf{u}-\mathbf{v}$ runs over all vectors in the lattice $\ker_{\mathbb{Z}}(n_1,\ldots,n_d)$ see for example, [16, Lemma 4.1]. The height of $I(\mathbf{n})$ is $d-1$ and also equals the rank of $\ker_{\mathbb{Z}}(n_1,\ldots,n_d)$ (see [16]). Given a polynomial $f\in I(\mathbf{n})$, we let f_* be the homogeneous summand of f of least degree. We shall denote by $I(\mathbf{n})_*$ the ideal in $K[x_1,\ldots,x_d]$ generated by the polynomials f_* for $f\in I(\mathbf{n})$.

Deciding whether the associated graded ring of the local ring $K[[t^{n_1},\ldots,t^{n_d}]]$ is Cohen-Macaulay constitutes an important problem studied by many authors, see for instance [1], [6], [14]. The importance of this problem stems partially from the fact that if the associated graded ring is Cohen-Macaulay, then the Hilbert function of $K[[t^{n_1},\ldots,t^{n_d}]]$ is non-decreasing. Since the associated graded ring of $K[[t^{n_1},\ldots,t^{n_d}]]$ is isomorphic to the ring $K[x_1,\ldots,x_d]/I(\mathbf{n})_*$, the Cohen-Macaulayness of the associated graded ring can be studied as the Cohen-Macaulayness of the ring $K[x_1,\ldots,x_d]/I(\mathbf{n})_*$. Recall that $I(\mathbf{n})_*$ is the defining ideal of the tangent cone of $C(\mathbf{n})$ at 0.

The case that $K[[t^{n_1},\ldots,t^{n_d}]]$ is Gorenstein has been particularly studied. This is partly due to the M. Rossi's problem [13] asking whether the Hilbert function of a Gorenstein local ring of dimension one is non-decreasing. Recently, A. Oneto, F. Strazzanti and G. Tamone [12] found many families of monomial curves giving negative answer to the above problem. However M. Rossi's problem is still open for a Gorenstein local ring $K[[t^{n_1},\ldots,t^{n_4}]]$. It is worth to note that, for a complete intersection monomial curve $C(\mathbf{n})$ in the 4-dimensional affine space (i.e. the ideal $I(\mathbf{n})$ is a complete intersection), we have, from [14, Theorem 3.1], that if the minimal number of generators for $I(\mathbf{n})_*$ is either three or four, then $C(\mathbf{n})$ has

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Cohen-Macaulay tangent cone at the origin. The converse is not true in general, see [14, Proposition 3.14].

In recent years there has been a surge of interest in studying properties of the monomial curve $C(\mathbf{n}+w\mathbf{v})$, where $w>0$ is an integer and $\mathbf{v}\in\mathbb{N}^d$, see for instance [4], [7] and [18]. This is particularly true for the case that $\mathbf{v}=(1,\dots,1)$. In fact, J. Herzog and H. Srinivasan conjectured that if $n_1<n_2<\cdots<n_d$ are positive numbers, then the Betti numbers of $I(\mathbf{n}+w\mathbf{v})$ are eventually periodic in w with period n_d-n_1. The conjecture was proved by T. Vu [18]. More precisely, he showed that there exists a positive integer N such that, for all $w>N$, the Betti numbers of $I(\mathbf{n}+w\mathbf{v})$ are periodic in w with period n_d-n_1. The bound N depends on the Castelnuovo-Mumford regularity of the ideal generated by the homogeneous elements in $I(\mathbf{n})$. For $w>(n_d-n_1)^2-n_1$ the minimal number of generators for $I(\mathbf{n}+w(1,\dots,1))$ is periodic in w with period n_d-n_1 (see [4]). Furthermore, for every $w>(n_d-n_1)^2-n_1$ the monomial curve $C(\mathbf{n}+w(1,\dots,1))$ has Cohen-Macaulay tangent cone at the origin, see [15]. The next example provides a monomial curve $C(\mathbf{n}+w(1,\dots,1))$ which is not a complete intersection for every $w>0$.

Example 1.1. Let $\mathbf{n}=(15,25,24,16)$, then $I(\mathbf{n})$ is a complete intersection on the binomials $x_1^5-x_2^3$, $x_3^2-x_4^3$ and $x_1x_2-x_3x_4$. Consider the vector $\mathbf{v}=(1,1,1,1)$. For every $w>85$ the minimal number of generators for $I(\mathbf{n}+w\mathbf{v})$ is either 18, 19 or 20. Using CoCoA ([3]) we find that for every $0<w\leq 85$ the minimal number of generators for $I(\mathbf{n}+w\mathbf{v})$ is greater than or equal to 4. Thus for every $w>0$ the ideal $I(\mathbf{n}+w\mathbf{v})$ is not a complete intersection.

Given a complete intersection monomial curve $C(\mathbf{n})$ in the 4-dimensional affine space, we study (see Theorems 2.6, 3.2) when $C(\mathbf{n}+w\mathbf{v})$ is a complete intersection. We also construct (see Theorems 2.8, 2.9, 3.4) families of complete intersection monomial curves $C(\mathbf{n}+w\mathbf{v})$ with Cohen-Macaulay tangent cone at the origin.

Let a_i be the least positive integer such that $a_in_i\in\sum_{j\neq i}\mathbb{N}n_j$. To study the complete intersection property of $C(\mathbf{n}+w\mathbf{v})$ we use the fact that after permuting variables, if necessary, there exists (see [14, Proposition 3.2] and also Theorems 3.6 and 3.10 in [10]) a minimal system of binomial generators S of $I(\mathbf{n})$ of the following form:

(A) $S=\{x_1^{a_1}-x_2^{a_2},x_3^{a_3}-x_4^{a_4},x_1^{u_1}x_2^{u_2}-x_3^{u_3}x_4^{u_4}\}$.
(B) $S=\{x_1^{a_1}-x_2^{a_2},x_3^{a_3}-x_1^{u_1}x_2^{u_2},x_4^{a_4}-x_1^{v_1}x_2^{v_2}x_3^{v_3}\}$.

In section 2 we focus on case (A). We prove that the monomial curve $C(\mathbf{n})$ has Cohen-Macaulay tangent cone at the origin if and only if the minimal number of generators for $I(\mathbf{n})_*$ is either three or four. Also we explicitly construct vectors $\mathbf{v}_i$, $1\leq i\leq 22$, such that for every $w>0$, the ideal $I(\mathbf{n}+w\mathbf{v}_i)$ is a complete intersection whenever the entries of $\mathbf{n}+w\mathbf{v}_i$ are relatively prime. We show that if $C(\mathbf{n})$ has Cohen-Macaulay tangent cone at the origin, then for every $w>0$ the monomial curve $C(\mathbf{n}+w\mathbf{v}_1)$ has Cohen-Macaulay tangent cone at the origin whenever the entries of $\mathbf{n}+w\mathbf{v}_1$ are relatively prime. Additionally we show that there exists a non-negative integer w_0 such that for all $w\geq w_0$, the monomial curves $C(\mathbf{n}+w\mathbf{v}_9)$ and $C(\mathbf{n}+w\mathbf{v}_{13})$ have Cohen-Macaulay tangent cones at the origin whenever the entries of the corresponding sequence ($\mathbf{n}+w\mathbf{v}_9$ for the first family and $\mathbf{n}+w\mathbf{v}_{13}$ for the second) are relatively prime. Finally we provide an infinite family of complete intersection monomial curves $C_m(\mathbf{n}+w\mathbf{v}_1)$ with corresponding local rings having non-decreasing Hilbert functions, although their tangent cones are not Cohen-Macaulay, thus giving a positive partial answer to M. Rossi's problem.

In section 3 we study the case (B). We construct vectors $\mathbf{b}_i$, $1\leq i\leq 22$, such that for every $w>0$, the ideal $I(\mathbf{n}+w\mathbf{b}_i)$ is a complete intersection whenever the entries of $\mathbf{n}+w\mathbf{b}_i$ are relatively prime. Furthermore we show that there exists a non-negative integer w_1 such that for all $w \geq w_1$, the ideal $I(\mathbf{n}+w\mathbf{b}_{22})_*$ is a complete intersection whenever the entries of $\mathbf{n}+w\mathbf{b}_{22}$ are relatively prime.

2. The case (A)

In this section we assume that after permuting variables, if necessary, $S=\{x_1^{a_1}-x_2^{a_2}, x_3^{a_3}-x_4^{a_4}, x_1^{u_1}x_2^{u_2}-x_3^{u_3}x_4^{u_4}\}$ is a minimal generating set of $I(\mathbf{n})$. First we will show that the converse of [14, Theorem 3.1] is also true in this case.

Let $n_1=\min\{n_1,\ldots,n_4\}$ and also $a_3<a_4$. By [6, Theorem 7] a monomial curve $C(\mathbf{n})$ has Cohen-Macaulay tangent cone if and only if x_1 is not a zero divisor in the ring $K[x_1,\ldots,x_4]/I(\mathbf{n})_*$. Hence if $C(\mathbf{n})$ has Cohen-Macaulay tangent cone at the origin, then $I(\mathbf{n})_*:\langle x_1\rangle = I(\mathbf{n})_*$. Without loss of generality we can assume that $u_2\leq a_2$. In case that $u_2>a_2$ we can write $u_2=ga_2+h$, where $0\leq h<a_2$. Then we can replace the binomial $x_1^{u_1}x_2^{u_2}-x_3^{u_3}x_4^{u_4}$ in S with the binomial $x_1^{u_1+ga_1}x_2^{h}-x_3^{u_3}x_4^{u_4}$. Without loss of generality we can also assume that $u_3\leq a_3$.

Theorem 2.1. *Suppose that $u_3>0$ and $u_4>0$. Then $C(\mathbf{n})$ has Cohen-Macaulay tangent cone at the origin if and only if the ideal $I(\mathbf{n})_*$ is either a complete intersection or an almost complete intersection.*

Proof. ($\Longleftarrow$) If the minimal number of generators of $I(\mathbf{n})_*$ is either three or four, then $C(\mathbf{n})$ has Cohen-Macaulay tangent cone at the origin.

($\Longrightarrow$) Let $f_1=x_1^{a_1}-x_2^{a_2}$, $f_2=x_3^{a_3}-x_4^{a_4}$, $f_3=x_1^{u_1}x_2^{u_2}-x_3^{u_3}x_4^{u_4}$. We distinguish the following cases

(1) $u_2<a_2$. Note that $x_4^{a_4+u_4}-x_1^{u_1}x_2^{u_2}x_3^{a_3-u_3}\in I(\mathbf{n})$. We will show that $a_4+u_4\leq u_1+u_2+a_3-u_3$. Suppose that $u_1+u_2+a_3-u_3<a_4+u_4$, then $x_2^{u_2}x_3^{a_3-u_3}\in I(\mathbf{n})_*:\langle x_1\rangle$ and therefore $x_2^{u_2}x_3^{a_3-u_3}\in I(\mathbf{n})_*$. Since $\{f_1,f_2,f_3\}$ is a generating set of $I(\mathbf{n})$, the monomial $x_2^{u_2}x_3^{a_3-u_3}$ is divided by at least one of the monomials $x_2^{a_2}$ and $x_3^{a_3}$. But $u_2<a_2$ and $a_3-u_3<a_3$, so $a_4+u_4\leq u_1+u_2+a_3-u_3$. Let $G=\{f_1,f_2,f_3,f_4=x_4^{a_4+u_4}-x_1^{u_1}x_2^{u_2}x_3^{a_3-u_3}\}$. We will prove that G is a standard basis for $I(\mathbf{n})$ with respect to the negative degree reverse lexicographical order with $x_3>x_4>x_2>x_1$. Note that $u_3+u_4<u_1+u_2$, since $u_3+u_4\leq u_1+u_2+a_3-a_4$ and also $a_3-a_4<0$. Thus $\mathrm{LM}(f_3)=x_3^{u_3}x_4^{u_4}$. Furthermore $\mathrm{LM}(f_1)=x_2^{a_2}$, $\mathrm{LM}(f_2)=x_3^{a_3}$ and $\mathrm{LM}(f_4)=x_4^{a_4+u_4}$. Therefore $\mathrm{NF}(\mathrm{spoly}(f_i,f_j)|G)=0$ as $\mathrm{LM}(f_i)$ and $\mathrm{LM}(f_j)$ are relatively prime, for $(i,j)\in\{(1,2),(1,3),(1,4),(2,4)\}$. We compute $\mathrm{spoly}(f_2,f_3)=-f_4$, so $\mathrm{NF}(\mathrm{spoly}(f_2,f_3)|G)=0$. Next we compute $\mathrm{spoly}(f_3,f_4)=x_1^{u_1}x_2^{u_2}x_3^{a_3}-x_1^{u_1}x_2^{u_2}x_4^{a_4}$. Then $\mathrm{LM}(\mathrm{spoly}(f_3,f_4))=x_1^{u_1}x_2^{u_2}x_3^{a_3}$ and only $\mathrm{LM}(f_2)$ divides $\mathrm{LM}(\mathrm{spoly}(f_3,f_4))$. Also $\mathrm{ecart}(\mathrm{spoly}(f_3,f_4))=a_4-a_3=\mathrm{ecart}(f_2)$. Then $\mathrm{spoly}(f_2,\mathrm{spoly}(f_3,f_4))=0$ and $\mathrm{NF}(\mathrm{spoly}(f_3,f_4)|G)=0$. By [8, Lemma 5.5.11] $I(\mathbf{n})_*$ is generated by the least homogeneous summands of the elements in the standard basis G. Thus the minimal number of generators for $I(\mathbf{n})_*$ is least than or equal to 4.

(2) $u_2=a_2$. Note that $x_4^{a_4+u_4}-x_1^{u_1+a_1}x_3^{a_3-u_3}\in I(\mathbf{n})$. We will show that $a_4+u_4\leq u_1+a_1+a_3-u_3$. Clearly the above inequality is true when $u_3=a_3$. Suppose that $u_3<a_3$ and $u_1+a_1+a_3-u_3<a_4+u_4$, then $x_3^{a_3-u_3}\in I(\mathbf{n})_*:\langle x_1\rangle$ and therefore $x_3^{a_3-u_3}\in I(\mathbf{n})_*$. Thus $x_3^{a_3-u_3}$ is divided by $x_3^{a_3}$, a contradiction. Consequently $a_4+u_4\leq u_1+a_1+a_3-u_3$. We will prove that $H=\{f_1,f_2,f_5=x_1^{u_1+a_1}-x_3^{u_3}x_4^{u_4}, f_6=x_4^{a_4+u_4}-x_1^{u_1+a_1}x_3^{a_3-u_3}\}$ is a standard basis for $I(\mathbf{n})$ with respect to the negative degree reverse lexicographical order with $x_3>x_4>x_2>x_1$. Here $\mathrm{LM}(f_1)=x_2^{a_2}$, $\mathrm{LM}(f_2)=x_3^{a_3}$, $\mathrm{LM}(f_5)=x_3^{u_3}x_4^{u_4}$ and $\mathrm{LM}(f_6)=x_4^{u_4+a_4}$. Therefore

NF(spoly$(f_i, f_j)|H) = 0$ as LM(f_i) and LM(f_j) are relatively prime, for $(i,j) \in \{(1,2),(1,5),(1,6),(2,6)\}$. We compute spoly$(f_2, f_5) = -f_6$, therefore NF(spoly$(f_2, f_5)|H) = 0$. Furthermore spoly$(f_5, f_6) = x_1^{u_1+a_1}x_3^{a_3} - x_1^{u_1+a_1}x_4^{a_4}$ and also LM(spoly$(f_5, f_6)) = x_1^{u_1+a_1}x_3^{a_3}$. Only LM$(f_2)$ divides LM(spoly(f_5, f_6)) and ecart(spoly$(f_5, f_6)) = a_4 - a_3 =$ ecart(f_2). Then spoly$(f_2$, spoly$(f_5, f_6)) = 0$ and therefore NF(spoly$(f_5, f_6)|H) = 0$. By [8, Lemma 5.5.11] $I(\mathbf{n})_*$ is generated by the least homogeneous summands of the elements in the standard basis H. Thus the minimal number of generators for $I(\mathbf{n})_*$ is least than or equal to 4. □

Corollary 2.2. *Suppose that $u_3 > 0$ and $u_4 > 0$.*

(1) *Assume that $u_2 < a_2$. Then $C(\mathbf{n})$ has Cohen-Macaulay tangent cone at the origin if and only if $a_4 + u_4 \leq u_1 + u_2 + a_3 - u_3$.*
(2) *Assume that $u_2 = a_2$. Then $C(\mathbf{n})$ has Cohen-Macaulay tangent cone at the origin if and only if $a_4 + u_4 \leq u_1 + a_1 + a_3 - u_3$.*

Theorem 2.3. *Suppose that either $u_3 = 0$ or $u_4 = 0$. Then $C(\mathbf{n})$ has Cohen-Macaulay tangent cone at the origin if and only if the ideal $I(\mathbf{n})_*$ is a complete intersection.*

Proof. It is enough to show that if $C(\mathbf{n})$ has Cohen-Macaulay tangent cone at the origin, then the ideal $I(\mathbf{n})_*$ is a complete intersection. Suppose first that $u_3 = 0$. Then $\{f_1 = x_1^{a_1} - x_2^{a_2}, f_2 = x_3^{a_3} - x_4^{a_4}, f_3 = x_4^{u_4} - x_1^{u_1}x_2^{u_2}\}$ is a minimal generating set of $I(\mathbf{n})$. If $u_2 = a_2$, then $\{f_1, f_2, x_4^{u_4} - x_1^{u_1+a_1}\}$ is a standard basis for $I(\mathbf{n})$ with respect to the negative degree reverse lexicographical order with $x_3 > x_4 > x_2 > x_1$. By [8, Lemma 5.5.11] $I(\mathbf{n})_*$ is a complete intersection. Assume that $u_2 < a_2$. We will show that $u_4 \leq u_1 + u_2$. Suppose that $u_4 > u_1 + u_2$, then $x_2^{u_2} \in I(\mathbf{n})_* : \langle x_1 \rangle$ and therefore $x_2^{u_2} \in I(\mathbf{n})_*$. Thus $x_2^{u_2}$ is divided by $x_2^{a_2}$, a contradiction. Then $\{f_1, f_2, f_3\}$ is a standard basis for $I(\mathbf{n})$ with respect to the negative degree reverse lexicographical order with $x_3 > x_4 > x_2 > x_1$. Note that LM$(f_1) = x_2^{a_2}$, LM$(f_2) = x_3^{a_3}$ and LM$(f_3) = x_4^{u_4}$. By [8, Lemma 5.5.11] $I(\mathbf{n})_*$ is a complete intersection. Suppose now that $u_4 = 0$, so necessarily $u_3 = a_3$. Then $\{f_1, f_2, f_4 = x_4^{a_4} - x_1^{u_1}x_2^{u_2}\}$ is a minimal generating set of $I(\mathbf{n})$. If $u_2 = a_2$, then $\{f_1, f_2, x_4^{a_4} - x_1^{a_1+u_1}\}$ is a standard basis for $I(\mathbf{n})$ with respect to the negative degree reverse lexicographical order with $x_3 > x_4 > x_2 > x_1$. Thus, from [8, Lemma 5.5.11], $I(\mathbf{n})_*$ is a complete intersection. Assume that $u_2 < a_2$, then $a_4 \leq u_1 + u_2$ and also $\{f_1, f_2, f_4\}$ is a standard basis for $I(\mathbf{n})$ with respect to the negative degree reverse lexicographical order with $x_3 > x_4 > x_2 > x_1$. From [8, Lemma 5.5.11] we deduce that $I(\mathbf{n})_*$ is a complete intersection. □

Remark 2.4. In case (B) the minimal number of generators of $I(\mathbf{n})_*$ can be arbitrarily large even if the tangent cone of $C(\mathbf{n})$ is Cohen-Macaulay, see [14, Proposition 3.14].

Given a complete intersection monomial curve $C(\mathbf{n})$, we next study the complete intersection property of $C(\mathbf{n}+w\mathbf{v})$. Let M be a non-zero $r \times s$ integer matrix, then there exist an $r \times r$ invertible integer matrix U and an $s \times s$ invertible integer matrix V such that $UMV = \text{diag}(\delta_1, \ldots, \delta_m, 0, \ldots, 0)$ is the diagonal matrix, where δ_j for all $j = 1, 2, \ldots, m$ are positive integers such that $\delta_i | \delta_{i+1}$, $1 \leq i \leq m-1$, and m is the rank of M. The elements $\delta_1, \ldots, \delta_m$ are the invariant factors of M. By [9, Theorem 3.9] the product $\delta_1\delta_2 \cdots \delta_m$ equals the greatest common divisor of all non-zero $m \times m$ minors of M.

The following proposition will be useful in the proof of Theorem 2.6.

Proposition 2.5. *Let $B = \{f_1 = x_1^{b_1} - x_2^{b_2}, f_2 = x_3^{b_3} - x_4^{b_4}, f_3 = x_1^{v_1}x_2^{v_2} - x_3^{v_3}x_4^{v_4}\}$ be a set of binomials in $K[x_1, \ldots, x_4]$, where $b_i \geq 1$ for all $1 \leq i \leq 4$, at least one of v_1,*

v_2 is non-zero and at least one of v_3, v_4 is non-zero. Let $n_1 = b_2(b_3v_4+v_3b_4)$, $n_2 = b_1(b_3v_4+v_3b_4)$, $n_3 = b_4(b_1v_2+v_1b_2)$, $n_4 = b_3(b_1v_2+v_1b_2)$. If $\gcd(n_1,\dots,n_4)=1$, then $I(\mathbf{n})$ is a complete intersection ideal generated by the binomials f_1, f_2 and f_3.

Proof. Consider the vectors $\mathbf{d}_1 = (b_1,-b_2,0,0)$, $\mathbf{d}_2 = (0,0,b_3,-b_4)$ and $\mathbf{d}_3 = (v_1,v_2,-v_3,-v_4)$. Clearly $\mathbf{d}_i \in \ker_{\mathbb{Z}}(n_1,\dots,n_4)$ for $1 \le i \le 3$, so the lattice $L = \sum_{i=1}^{3} \mathbb{Z}\mathbf{d}_i$ is a subset of $\ker_{\mathbb{Z}}(n_1,\dots,n_4)$. Consider the matrix

$$M = \begin{pmatrix} b_1 & 0 & v_1 \\ -b_2 & 0 & v_2 \\ 0 & b_3 & -v_3 \\ 0 & -b_4 & -v_4 \end{pmatrix}.$$

It is not hard to show that the rank of M equals 3. We will prove that L is saturated, namely the invariant factors δ_1, δ_2 and δ_3 of M are all equal to 1. The greatest common divisor of all non-zero 3×3 minors of M equals the greatest common divisor of the integers n_1, n_2, n_3 and n_4. But $\gcd(n_1,\dots,n_4)=1$, so $\delta_1\delta_2\delta_3 = 1$ and therefore $\delta_1=\delta_2=\delta_3=1$. Note that the rank of the lattice $\ker_{\mathbb{Z}}(n_1,\dots,n_4)$ is 3 and also equals the rank of L. By [17, Lemma 8.2.5] we have that $L = \ker_{\mathbb{Z}}(n_1,\dots,n_4)$. Now the transpose M^t of M is mixed dominating. Recall that a matrix P is mixed dominating if every row of P has a positive and negative entry and P contains no square submatrix with this property. By [5, Theorem 2.9] $I(\mathbf{n})$ is a complete intersection on the binomials f_1, f_2 and f_3. $\square$

Theorem 2.6. *Let $I(\mathbf{n})$ be a complete intersection ideal generated by the binomials $f_1 = x_1^{a_1} - x_2^{a_2}$, $f_2 = x_3^{a_3} - x_4^{a_4}$ and $f_3 = x_1^{u_1}x_2^{u_2} - x_3^{u_3}x_4^{u_4}$. Then there exist vectors $\mathbf{v}_i$, $1 \le i \le 22$, in $\mathbb{N}^4$ such that for all $w > 0$, the toric ideal $I(\mathbf{n}+w\mathbf{v}_i)$ is a complete intersection whenever the entries of $\mathbf{n}+w\mathbf{v}_i$ are relatively prime.*

Proof. By [11, Theorem 6] $n_1 = a_2(a_3u_4+u_3a_4)$, $n_2 = a_1(a_3u_4+u_3a_4)$, $n_3 = a_4(a_1u_2+u_1a_2)$, $n_4 = a_3(a_1u_2+u_1a_2)$. Let $\mathbf{v}_1 = (a_2a_3, a_1a_3, a_2a_4, a_2a_3)$ and $B = \{f_1, f_2, f_4 = x_1^{u_1+w}x_2^{u_2} - x_3^{u_3}x_4^{u_4+w}\}$. Then $n_1+wa_2a_3 = a_2(a_3(u_4+w)+u_3a_4)$, $n_2+wa_1a_3 = a_1(a_3(u_4+w)+u_3a_4)$, $n_3+wa_2a_4 = a_4(a_1u_2+(u_1+w)a_2)$ and $n_4+wa_2a_3 = a_3(a_1u_2+(u_1+w)a_2)$. By Proposition 2.5 for every $w>0$, the ideal $I(\mathbf{n}+w\mathbf{v}_1)$ is a complete intersection on f_1, f_2 and f_4 whenever $\gcd(n_1+wa_2a_3, n_2+wa_1a_3, n_3+wa_2a_4, n_4+wa_2a_3)=1$. Consider the vectors $\mathbf{v}_2 = (a_2a_3, a_1a_3, a_1a_4, a_1a_3)$, $\mathbf{v}_3 = (a_2a_4, a_1a_4, a_2a_4, a_2a_3)$, $\mathbf{v}_4 = (a_2a_4, a_1a_4, a_1a_4, a_1a_3)$, $\mathbf{v}_5 = (a_2(a_3+a_4), a_1(a_3+a_4), 0, 0)$ and $\mathbf{v}_6 = (0,0,a_4(a_1+a_2), a_3(a_1+a_2))$. By Proposition 2.5 for every $w>0$, $I(\mathbf{n}+w\mathbf{v}_2)$ is a complete intersection on f_1, f_2 and $x_1^{u_1}x_2^{u_2+w} - x_3^{u_3}x_4^{u_4+w}$ whenever the entries of $\mathbf{n}+w\mathbf{v}_2$ are relatively prime, $I(\mathbf{n}+w\mathbf{v}_3)$ is a complete intersection on f_1, f_2 and $x_1^{u_1+w}x_2^{u_2} - x_3^{u_3+w}x_4^{u_4}$ whenever the entries of $\mathbf{n}+w\mathbf{v}_3$ are relatively prime, and $I(\mathbf{n}+w\mathbf{v}_4)$ is a complete intersection on f_1, f_2 and $x_1^{u_1}x_2^{u_2+w} - x_3^{u_3+w}x_4^{u_4}$ whenever the entries of $\mathbf{n}+w\mathbf{v}_4$ are relatively prime. Furthermore for all $w>0$, $I(\mathbf{n}+w\mathbf{v}_5)$ is a complete intersection on f_1, f_2 and $x_1^{u_1}x_2^{u_2} - x_3^{u_3+w}x_4^{u_4+w}$ whenever the entries of $\mathbf{n}+w\mathbf{v}_5$ are relatively prime, and $I(\mathbf{n}+w\mathbf{v}_6)$ is a complete intersection on f_1, f_2 and $x_1^{u_1+w}x_2^{u_2+w} - x_3^{u_3}x_4^{u_4}$ whenever the entries of $\mathbf{n}+w\mathbf{v}_6$ are relatively prime. Consider the vectors $\mathbf{v}_7 = (a_2(a_3+a_4), a_1(a_3+a_4), a_2a_4, a_2a_3)$, $\mathbf{v}_8 = (a_2(a_3+a_4), a_1(a_3+a_4), a_4(a_1+a_2), a_3(a_1+a_2))$, $\mathbf{v}_9 = (0,0,a_2a_4,a_2a_3)$, $\mathbf{v}_{10} = (a_2a_4, a_1a_4, a_4(a_1+a_2), a_3(a_1+a_2))$, $\mathbf{v}_{11} = (a_2a_3, a_1a_3, a_4(a_1+a_2), a_3(a_1+a_2))$, $\mathbf{v}_{12} = (a_2(a_3+a_4), a_1(a_3+a_4), a_1a_4, a_1a_3)$, $\mathbf{v}_{13} = (0,0,a_1a_4,a_1a_3)$, $\mathbf{v}_{14} = (a_2a_4, a_1a_4, 0, 0)$ and $\mathbf{v}_{15} = (a_2a_3, a_1a_3, 0, 0)$. Using Proposition 2.5 we have that for all $w>0$, $I(\mathbf{n}+w\mathbf{v}_i)$, $7 \le i \le 15$, is a complete intersection whenever the entries of $\mathbf{n}+w\mathbf{v}_i$ are relatively prime. For instance $I(\mathbf{n}+w\mathbf{v}_9)$ is a complete intersection on the binomials f_1, f_2 and $x_1^{u_1+w}x_2^{u_2} - x_3^{u_3}x_4^{u_4}$. Consider the vectors $\mathbf{v}_{16} = (a_3u_4+u_3a_4, a_3u_4+u_3a_4, a_4(u_1+u_2), a_3(u_1+u_2))$, $\mathbf{v}_{17} = (0, a_3u_4+$

$u_3a_4, u_2a_4, u_2a_3)$, $\mathbf{v}_{18} = (a_3u_4 + u_3a_4, 0, u_1a_4, u_1a_3)$, $\mathbf{v}_{19} = (a_2u_4, a_1u_4, 0, a_1u_2 + u_1a_2)$, $\mathbf{v}_{20} = (a_2u_3, a_1u_3, a_1u_2 + u_1a_2, 0)$, $\mathbf{v}_{21} = (a_2(a_4 + u_4), a_1(a_4 + u_4), 0, a_1u_2 + u_1a_2)$ and $\mathbf{v}_{22} = (a_2(u_3 + u_4), a_1(u_3 + u_4), a_1u_2 + u_1a_2, a_1u_2 + u_1a_2)$. It is easy to see that for all $w > 0$, the ideal $I(\mathbf{n} + w\mathbf{v}_i)$, $16 \le i \le 22$, is a complete intersection whenever the entries of $\mathbf{n} + w\mathbf{v}_i$ are relatively prime. For instance $I(\mathbf{n} + w\mathbf{v}_{16})$ is a complete intersection on the binomials f_2, f_3 and $x_1^{a_1+w} - x_2^{a_2+w}$. □

Example 2.7. Let $\mathbf{n} = (93, 124, 195, 117)$, then $I(\mathbf{n})$ is a complete intersection on the binomials $x_1^4 - x_2^3$, $x_3^3 - x_4^5$ and $x_1^9x_2^3 - x_3^2x_4^7$. Here $a_1 = 4$, $a_2 = 3$, $a_3 = 3$, $a_4 = 5$, $u_1 = 9$, $u_2 = 3$, $u_3 = 2$ and $u_4 = 7$. Consider the vector $\mathbf{v}_1 = (9, 12, 15, 9)$. For all $w \ge 0$ the ideal $I(\mathbf{n}+w\mathbf{v}_1)$ is a complete intersection on $x_1^4-x_2^3$, $x_3^3-x_4^5$ and $x_1^{9+w}x_2^3 - x_3^2x_4^{w+7}$ whenever $\gcd(93 + 9w, 124 + 12w, 195 + 15w, 117 + 9w) = 1$. By Corollary 2.2 the monomial curve $C(\mathbf{n} + w\mathbf{v}_1)$ has Cohen-Macaulay tangent cone at the origin. Consider the vector $\mathbf{v}_4 = (15, 20, 20, 12)$ and the sequence $\mathbf{n} + 9\mathbf{v}_4 = (228, 304, 375, 225)$. The toric ideal $I(\mathbf{n} + 9\mathbf{v}_4)$ is a complete intersection on the binomials $x_1^4 - x_2^3$, $x_3^3 - x_4^5$ and $x_1^{21}x_2^3 - x_3^2x_4^{22}$. Note that $x_1^{25} - x_3^2x_4^{22} \in I(\mathbf{n} + 9\mathbf{v}_4)$, so $x_3^2x_4^{22} \in I(\mathbf{n} + 9\mathbf{v}_4)_*$ and also $x_3^2 \in I(\mathbf{n} + 9\mathbf{v}_4)_* : \langle x_4\rangle$. If $C(\mathbf{n} + 9\mathbf{v}_4)$ has Cohen-Macaulay tangent cone at the origin, then $x_3^2 \in I(\mathbf{n} + 9\mathbf{v}_4)_*$ a contradiction. Thus $C(\mathbf{n} + 9\mathbf{v}_4)$ does not have a Cohen-Macaulay tangent cone at the origin.

Theorem 2.8. *Let $I(\mathbf{n})$ be a complete intersection ideal generated by the binomials $f_1 = x_1^{a_1} - x_2^{a_2}$, $f_2 = x_3^{a_3} - x_4^{a_4}$ and $f_3 = x_1^{u_1}x_2^{u_2} - x_3^{u_3}x_4^{u_4}$. Consider the vector $\mathbf{d} = (a_2a_3, a_1a_3, a_2a_4, a_2a_3)$. If $C(\mathbf{n})$ has Cohen-Macaulay tangent cone at the origin, then for every $w > 0$ the monomial curve $C(\mathbf{n} + w\mathbf{d})$ has Cohen-Macaulay tangent cone at the origin whenever the entries of $\mathbf{n} + w\mathbf{d}$ are relatively prime.*

Proof. Let $n_1 = \min\{n_1, \dots, n_4\}$ and also $a_3 < a_4$. Without loss of generality we can assume that $u_2 \le a_2$ and $u_3 \le a_3$. By Theorem 2.6 for every $w > 0$, the ideal $I(\mathbf{n}+w\mathbf{d})$ is a complete intersection on f_1, f_2 and $f_4 = x_1^{u_1+w}x_2^{u_2} - x_3^{u_3}x_4^{u_4+w}$ whenever the entries of $\mathbf{n} + w\mathbf{d}$ are relatively prime. Note that $n_1 + wa_2a_3 = \min\{n_1+wa_2a_3, n_2+wa_1a_3, n_3+wa_2a_4, n_4+wa_2a_3\}$. Suppose that $u_3 > 0$ and $u_4 > 0$. Assume that $u_2 < a_2$. By Corollary 2.2 it holds that $a_4 + u_4 \le u_1 + u_2 + a_3 - u_3$ and therefore

$$a_4 + (u_4 + w) \le (u_1 + w) + u_2 + a_3 - u_3.$$

Thus, from Corollary 2.2 again $C(\mathbf{n} + w\mathbf{d})$ has Cohen-Macaulay tangent cone at the origin. Assume that $u_2 = a_2$. Then, from Corollary 2.2, we have that $a_4 + u_4 \le u_1 + a_1 + a_3 - u_3$ and therefore $a_4 + (u_4 + w) \le (u_1 + w) + a_1 + a_3 - u_3$. By Corollary 2.2 $C(\mathbf{n} + w\mathbf{d})$ has Cohen-Macaulay tangent cone at the origin.

Suppose now that $u_3 = 0$. Then $\{f_1, f_2, f_5 = x_4^{u_4+w} - x_1^{u_1+w}x_2^{u_2}\}$ is a minimal generating set of $I(\mathbf{n} + w\mathbf{d})$. If $u_2 = a_2$, then $\{f_1, f_2, x_4^{u_4+w} - x_1^{u_1+a_1+w}\}$ is a standard basis for $I(\mathbf{n}+w\mathbf{d})$ with respect to the negative degree reverse lexicographical order with $x_3 > x_4 > x_2 > x_1$. Thus $I(\mathbf{n} + w\mathbf{d})_*$ is a complete intersection and therefore $C(\mathbf{n}+w\mathbf{d})$ has Cohen-Macaulay tangent cone at the origin. Assume that $u_2 < a_2$, then $u_4 \le u_1+u_2$ and therefore $u_4+w \le (u_1+w)+u_2$. The set $\{f_1, f_2, f_5\}$ is a standard basis for $I(\mathbf{n}+w\mathbf{d})$ with respect to the negative degree reverse lexicographical order with $x_3 > x_4 > x_2 > x_1$. Thus $I(\mathbf{n}+w\mathbf{d})_*$ is a complete intersection and therefore $C(\mathbf{n} + w\mathbf{d})$ has Cohen-Macaulay tangent cone at the origin.

Suppose that $u_4 = 0$, so necessarily $u_3 = a_3$. Then $\{f_1, f_2, x_4^{a_4+w} - x_1^{u_1+w}x_2^{u_2}\}$ is a minimal generating set of $I(\mathbf{n} + w\mathbf{d})$. If $u_2 = a_2$, then $\{f_1, f_2, x_4^{a_4+w} - x_1^{u_1+a_1+w}\}$ is a standard basis for $I(\mathbf{n}+w\mathbf{d})$ with respect to the negative degree reverse lexicographical order with $x_3 > x_4 > x_2 > x_1$. Thus $I(\mathbf{n}+w\mathbf{d})_*$ is a complete intersection and therefore $C(\mathbf{n} + w\mathbf{d})$ has Cohen-Macaulay tangent cone at the origin. Assume that $u_2 < a_2$, then $a_4 \le u_1 + u_2$ and therefore $a_4 + w \le (u_1 + w) + u_2$. The set $\{f_1, f_2, x_4^{a_4+w} - x_1^{u_1+w}x_2^{u_2}\}$ is a standard basis for $I(\mathbf{n} + w\mathbf{d})$ with respect to the negative degree reverse lexicographical order with $x_3 > x_4 > x_2 > x_1$. Thus $I(\mathbf{n}+w\mathbf{d})_*$ is a complete intersection and therefore $C(\mathbf{n}+w\mathbf{d})$ has Cohen-Macaulay tangent cone at the origin. $\square$

Theorem 2.9. *Let $I(\mathbf{n})$ be a complete intersection ideal generated by the binomials $f_1 = x_1^{a_1} - x_2^{a_2}$, $f_2 = x_3^{a_3} - x_4^{a_4}$ and $f_3 = x_1^{u_1}x_2^{u_2} - x_3^{u_3}x_4^{u_4}$. Consider the vectors $\mathbf{d}_1 = (0, 0, a_2a_4, a_2a_3)$ and $\mathbf{d}_2 = (0, 0, a_1a_4, a_1a_3)$. Then there exists a non-negative integer w_0 such that for all $w \geq w_0$, the monomial curves $C(\mathbf{n} + w\mathbf{d}_1)$ and $C(\mathbf{n} + w\mathbf{d}_2)$ have Cohen-Macaulay tangent cones at the origin whenever the entries of the corresponding sequence ($\mathbf{n} + w\mathbf{d}_1$ for the first family and $\mathbf{n} + w\mathbf{d}_2$ for the second) are relatively prime.*

Proof. Let $n_1 = \min\{n_1, \dots, n_4\}$ and $a_3 < a_4$. Suppose that $u_2 \leq a_2$ and $u_3 \leq a_3$. By Theorem 2.6 for all $w \geq 0$, $I(\mathbf{n} + w\mathbf{d}_1)$ is a complete intersection on f_1, f_2 and $f_4 = x_1^{u_1+w}x_2^{u_2} - x_3^{u_3}x_4^{u_4}$ whenever the entries of $\mathbf{n} + w\mathbf{d_1}$ are relatively prime. Remark that $n_1 = \min\{n_1, n_2, n_3 + wa_2a_4, n_4 + wa_2a_3\}$. Let w_0 be the smallest non-negative integer greater than or equal to $u_3 + u_4 - u_1 - u_2 + a_4 - a_3$. Then for every $w \geq w_0$ we have that $a_4 + u_4 \leq u_1 + w + u_2 + a_3 - u_3$, so $u_3 + u_4 < u_1 + w + u_2$. Let $G = \{f_1, f_2, f_4, f_5 = x_4^{a_4+u_4} - x_1^{u_1+w}x_2^{u_2}x_3^{a_3-u_3}\}$. We will prove that for every $w \geq w_0$, G is a standard basis for $I(\mathbf{n} + w\mathbf{d}_1)$ with respect to the negative degree reverse lexicographical order with $x_3 > x_4 > x_2 > x_1$. Note that $\mathrm{LM}(f_1) = x_2^{a_2}$, $\mathrm{LM}(f_2) = x_3^{a_3}$, $\mathrm{LM}(f_4) = x_3^{u_3}x_4^{u_4}$ and $\mathrm{LM}(f_5) = x_4^{a_4+u_4}$. Therefore $\mathrm{NF}(\mathrm{spoly}(f_i, f_j)|G) = 0$ as $\mathrm{LM}(f_i)$ and $\mathrm{LM}(f_j)$ are relatively prime, for $(i, j) \in \{(1, 2), (1, 4), (1, 5), (2, 5)\}$. We compute $\mathrm{spoly}(f_2, f_4) = -f_5$, so $\mathrm{NF}(\mathrm{spoly}(f_2, f_4)|G) = 0$. Next we compute $\mathrm{spoly}(f_4, f_5) = x_1^{u_1+w}x_2^{u_2}x_3^{a_3} - x_1^{u_1+w}x_2^{u_2}x_4^{a_4}$. Then $\mathrm{LM}(\mathrm{spoly}(f_4, f_5)) = x_1^{u_1+w}x_2^{u_2}x_3^{a_3}$ and only $\mathrm{LM}(f_2)$ divides $\mathrm{LM}(\mathrm{spoly}(f_4, f_5))$. Also $\mathrm{ecart}(\mathrm{spoly}(f_4, f_5)) = a_4 - a_3 = \mathrm{ecart}(f_2)$. Then $\mathrm{spoly}(f_2, \mathrm{spoly}(f_4, f_5)) = 0$ and $\mathrm{NF}(\mathrm{spoly}(f_4, f_5)|G) = 0$. Thus the minimal number of generators for $I(\mathbf{n} + w\mathbf{d}_1)_*$ is either three or four, so from [14, Theorem 3.1] for every $w \geq w_0$, $C(\mathbf{n} + w\mathbf{d}_1)$ has Cohen-Macaulay tangent cone at the origin whenever the entries of $\mathbf{n} + w\mathbf{d}_1$ are relatively prime.

By Theorem 2.6 for all $w \geq 0$, $I(\mathbf{n} + w\mathbf{d}_2)$ is a complete intersection on f_1, f_2 and $f_6 = x_1^{u_1}x_2^{u_2+w} - x_3^{u_3}x_4^{u_4}$ whenever the entries of $\mathbf{n} + w\mathbf{d_2}$ are relatively prime. Remark that $n_1 = \min\{n_1, n_2, n_3 + wa_1a_4, n_4 + wa_1a_3\}$. For every $w \geq w_0$ the set $H = \{f_1, f_2, f_6, x_4^{a_4+u_4} - x_1^{u_1}x_2^{u_2+w}x_3^{a_3-u_3}\}$ is a standard basis for $I(\mathbf{n} + w\mathbf{d}_2)$ with respect to the negative degree reverse lexicographical order with $x_3 > x_4 > x_2 > x_1$. Thus the minimal number of generators for $I(\mathbf{n} + w\mathbf{d}_2)_*$ is either three or four, so from [14, Theorem 3.1] for every $w \geq w_0$, $C(\mathbf{n}+w\mathbf{d}_2)$ has Cohen-Macaulay tangent cone at the origin whenever the entries of $\mathbf{n} + w\mathbf{d}_2$ are relatively prime. $\square$

Example 2.10. Let $\mathbf{n} = (15, 25, 24, 16)$, then $I(\mathbf{n})$ is a complete intersection on the binomials $x_1^5 - x_2^3$, $x_3^2 - x_4^3$ and $x_1x_2 - x_3x_4$. Here $a_1 = 5$, $a_2 = 3$, $a_3 = 2$, $a_4 = 3$, $u_i = 1$, $1 \leq i \leq 4$. Note that $x_4^4 - x_1x_2x_3 \in I(\mathbf{n})$, so, from Corollary 2.2, $C(\mathbf{n})$ does not have a Cohen-Macaulay tangent cone at the origin. Consider the vector $\mathbf{d}_1 = (0, 0, 9, 6)$. For every $w > 0$ the ideal $I(\mathbf{n} + w\mathbf{d}_1)$ is a complete intersection on the binomials $x_1^5 - x_2^3$, $x_3^2 - x_4^3$ and $x_1^{w+1}x_2 - x_3x_4$ whenever $\gcd(15, 25, 24+9w, 16+6w) = 1$. By Theorem 2.9 for every $w \geq 1$, the monomial curve $C(\mathbf{n} + w\mathbf{d}_1)$ has Cohen-Macaulay tangent cone at the origin whenever $\gcd(15, 25, 24 + 9w, 16 + 6w) = 1$.

The next example gives a family of complete intersection monomial curves supporting M. Rossi's problem, although their tangent cones are not Cohen-Macaulay. To prove it we will use the following proposition.

Proposition 2.11. [2, Proposition 2.2] *Let $I \subset K[x_1, x_2, \dots, x_d]$ be a monomial ideal and $I = \langle J, \mathbf{x}^{\mathbf{u}} \rangle$ for a monomial ideal J and a monomial $\mathbf{x}^{\mathbf{u}}$. Let $p(I)$ denote* *the numerator $g(t)$ of the Hilbert Series for $K[x_1,x_2,\dots,x_d]/I$. Then $p(I)=p(J)-t^{\deg(\mathbf{x^u})}p(J:\langle \mathbf{x^u}\rangle)$.*

Example 2.12. Consider the family $n_1=8m^2+6$, $n_2=20m^2+15$, $n_3=12m^2+15$ and $n_4=8m^2+10$, where $m\geq 1$ is an integer. The toric ideal $I(\mathbf{n})$ is minimally generated by the binomials

$$x_1^5-x_2^2, x_3^2-x_4^3, x_1^{2m^2}x_2-x_3x_4^{2m^2}.$$

Consider the vector $\mathbf{v}_1=(4,10,6,4)$ and the family $n_1'=n_1+4w$, $n_2'=n_2+10w$, $n_3'=n_3+6w$, $n_4'=n_4+4w$ where $w\geq 0$ is an integer. Let $\mathbf{n}'=(n_1',n_2',n_3',n_4')$, then for all $w\geq 0$ the toric ideal $I(\mathbf{n}')$ is minimally generated by the binomials

$$x_1^5-x_2^2, x_3^2-x_4^3, x_1^{2m^2+w}x_2-x_3x_4^{2m^2+w}$$

whenever $\gcd(n_1',n_2',n_3',n_4')=1$. Let $C_m(\mathbf{n}')$ be the corresponding monomial curve. By Corollary 2.2 for all $w\geq 0$, the monomial curve $C_m(\mathbf{n}')$ does not have Cohen-Macaulay tangent cone at the origin whenever $\gcd(n_1',n_2',n_3',n_4')=1$. We will show that for every $w\geq 0$, the Hilbert function of the ring $K[[t^{n_1'},\dots,t^{n_4'}]]$ is non-decreasing whenever $\gcd(n_1',n_2',n_3',n_4')=1$. It suffices to prove that for every $w\geq 0$, the Hilbert function of $K[x_1,x_2,x_3,x_4]/I(\mathbf{n}')_*$ is non-decreasing whenever $\gcd(n_1',n_2',n_3',n_4')=1$. The set

$$G=\{x_1^5-x_2^2, x_3^2-x_4^3, x_1^{2m^2+w}x_2-x_3x_4^{2m^2+w}, x_4^{2m^2+w+3}-x_1^{2m^2+w}x_2x_3,$$
$$x_1^{2m^2+w+5}x_3-x_2x_4^{2m^2+w+3}, x_1^{4m^2+2w+5}-x_4^{4m^2+2w+3}\}$$

is a standard basis for $I(\mathbf{n}')$ with respect to the negative degree reverse lexicographical order with $x_4>x_3>x_2>x_1$. Thus $I(\mathbf{n}')_*$ is generated by the set

$$\{x_2^2, x_3^2, x_4^{4m^2+2w+3}, x_1^{2m^2+w}x_2x_3, x_1^{2m^2+w}x_2-x_3x_4^{2m^2+w}, x_2x_4^{2m^2+w+3}\}.$$

Also $\langle \mathrm{LT}(I(\mathbf{n}')_*)\rangle$ with respect to the aforementioned order can be written as,

$$\langle \mathrm{LT}(I(\mathbf{n}')_*)\rangle=\langle x_2^2, x_3^2, x_4^{4m^2+2w+3}, x_2x_4^{2m^2+w+3}, x_3x_4^{2m^2+w}, x_1^{2m^2+w}x_2x_3\rangle.$$

Since the Hilbert function of $K[x_1,x_2,x_3,x_4]/I(\mathbf{n}')_*$ is equal to the Hilbert function of $K[x_1,x_2,x_3,x_4]/\langle \mathrm{LT}(I(\mathbf{n}')_*)\rangle$, it is sufficient to compute the Hilbert function of the latter. Let

$$J_0=\langle \mathrm{LT}(I(\mathbf{n}')_*)\rangle, J_1=\langle x_2^2, x_3^2, x_4^{4m^2+2w+3}, x_2x_4^{2m^2+w+3}, x_3x_4^{2m^2+w}\rangle,$$
$$J_2=\langle x_2^2, x_3^2, x_4^{4m^2+2w+3}, x_2x_4^{2m^2+w+3}\rangle, J_3=\langle x_2^2, x_3^2, x_4^{4m^2+2w+3}\rangle.$$

Remark that $J_i=\langle J_{i+1},q_i\rangle$, where $q_0=x_1^{2m^2+w}x_2x_3$, $q_1=x_3x_4^{2m^2+w}$ and $q_2=x_2x_4^{2m^2+w+3}$. We apply Proposition 2.11 to the ideal J_i for $0\leq i\leq 2$, so

$$p(J_i)=p(J_{i+1})-t^{\deg(q_i)}p(J_{i+1}:\langle q_i\rangle). \tag{2.1}$$

Note that $\deg(q_0)=2m^2+w+2$, $\deg(q_1)=2m^2+w+1$ and $\deg(q_2)=2m^2+w+4$. In this case, it holds that $J_1:\langle q_0\rangle=\langle x_2,x_3,x_4^{2m^2+w}\rangle$, $J_2:\langle q_1\rangle=\langle x_2^2,x_3,x_4^{2m^2+w+3},x_2x_4^3\rangle$ and $J_3:\langle q_2\rangle=\langle x_2,x_3^2,x_4^{2m^2+w}\rangle$. We have that

$$p(J_3)=(1-t)^3(1+3t+4t^2+\cdots+4t^{4m^2+2w+2}+3t^{4m^2+2w+3}+t^{4m^2+2w+4}).$$

Substituting all these recursively in Equation (2.1), we obtain that the Hilbert series of $K[x_1,x_2,x_3,x_4]/J_0$ is

$$\frac{1+3t+4t^2+\cdots+4t^{2m^2+w}+3t^{2m^2+w+1}+t^{2m^2+w+2}+t^{2m^2+w+3}+t^{4m^2+2w+2}}{1-t}.$$

Since the numerator does not have any negative coefficients, the Hilbert function of $K[x_1,x_2,x_3,x_4]/J_0$ is non-decreasing whenever $\gcd(n_1',n_2',n_3',n_4')=1$.

3. The case (B)

In this section we assume that after permuting variables, if necessary, $S = \{x_1^{a_1} - x_2^{a_2}, x_3^{a_3} - x_1^{u_1}x_2^{u_2}, x_4^{a_4} - x_1^{v_1}x_2^{v_2}x_3^{v_3}\}$ is a minimal generating set of $I(\mathbf{n})$. Proposition 3.1 will be useful in the proof of Theorem 3.2.

Proposition 3.1. *Let $B = \{f_1 = x_1^{b_1} - x_2^{b_2}, f_2 = x_3^{b_3} - x_1^{c_1}x_2^{c_2}, f_3 = x_4^{b_4} - x_1^{m_1}x_2^{m_2}x_3^{m_3}\}$ be a set of binomials in $K[x_1, \dots, x_4]$, where $b_i \geq 1$ for all $1 \leq i \leq 4$, at least one of c_1, c_2 is non-zero and at least one of m_1, m_2 and m_3 is non-zero. Let $n_1 = b_2b_3b_4$, $n_2 = b_1b_3b_4$, $n_3 = b_4(b_1c_2 + c_1b_2)$, $n_4 = m_3(b_1c_2 + b_2c_1) + b_3(b_1m_2 + m_1b_2)$. If $\gcd(n_1, \dots, n_4) = 1$, then $I(\mathbf{n})$ is a complete intersection ideal generated by the binomials f_1, f_2, f_3.*

Proof. Consider the vectors $\mathbf{d}_1 = (b_1, -b_2, 0, 0)$, $\mathbf{d}_2 = (-c_1, -c_2, b_3, 0)$ and $\mathbf{d}_3 = (-m_1, -m_2, -m_3, b_4)$. Clearly $\mathbf{d}_i \in \ker_{\mathbb{Z}}(n_1, \dots, n_4)$ for $1 \leq i \leq 3$, so the lattice $L = \sum_{i=1}^{3} \mathbb{Z}\mathbf{d}_i$ is a subset of $\ker_{\mathbb{Z}}(n_1, \dots, n_4)$. Let

$$M = \begin{pmatrix} b_1 & -c_1 & -m_1 \\ -b_2 & -c_2 & -m_2 \\ 0 & b_3 & -m_3 \\ 0 & 0 & b_4 \end{pmatrix},$$

then the rank of M equals 3. We will prove that the invariant factors δ_1, δ_2 and δ_3 of M are all equal to 1. The greatest common divisor of all non-zero 3×3 minors of M equals the greatest common divisor of the integers n_1, n_2, n_3 and n_4. But $\gcd(n_1, \dots, n_4) = 1$, so $\delta_1\delta_2\delta_3 = 1$ and therefore $\delta_1 = \delta_2 = \delta_3 = 1$. Note that the rank of the lattice $\ker_{\mathbb{Z}}(n_1, \dots, n_4)$ is 3 and also equals the rank of L. By [17, Lemma 8.2.5] we have that $L = \ker_{\mathbb{Z}}(n_1, \dots, n_4)$. Now the transpose M^t of M is mixed dominating. By [5, Theorem 2.9] the ideal $I(\mathbf{n})$ is a complete intersection on f_1, f_2 and f_3. □

Theorem 3.2. *Let $I(\mathbf{n})$ be a complete intersection ideal generated by the binomials $f_1 = x_1^{a_1} - x_2^{a_2}$, $f_2 = x_3^{a_3} - x_1^{u_1}x_2^{u_2}$ and $f_3 = x_4^{a_4} - x_1^{v_1}x_2^{v_2}x_3^{v_3}$. Then there exist vectors $\mathbf{b}_i$, $1 \leq i \leq 22$, in $\mathbb{N}^4$ such that for all $w > 0$, the toric ideal $I(\mathbf{n} + w\mathbf{b}_i)$ is a complete intersection whenever the entries of $\mathbf{n} + w\mathbf{b}_i$ are relatively prime.*

Proof. By [11, Theorem 6] $n_1 = a_2a_3a_4$, $n_2 = a_1a_3a_4$, $n_3 = a_4(a_1u_2 + u_1a_2)$, $n_4 = v_3(a_1u_2 + a_2u_1) + a_3(a_1v_2 + v_1a_2)$. Let $\mathbf{b}_1 = (a_2a_3, a_1a_3, a_1u_2 + u_1a_2, a_2a_3)$ and consider the set $B = \{f_1, f_2, f_4 = x_4^{a_4+w} - x_1^{v_1+w}x_2^{v_2}x_3^{v_3}\}$. Then $n_1 + wa_2a_3 = a_2a_3(a_4+w)$, $n_2+wa_1a_3 = a_1a_3(a_4+w)$, $n_3+w(a_1u_2+u_1a_2) = (a_4+w)(a_1u_2+u_1a_2)$ and $n_4 + wa_2a_3 = v_3(a_1u_2 + a_2u_1) + a_3(a_1v_2 + (v_1 + w)a_2)$. By Proposition 3.1 for every $w > 0$, the ideal $I(\mathbf{n} + w\mathbf{b}_1)$ is a complete intersection on f_1, f_2 and f_4 whenever the entries of $\mathbf{n} + w\mathbf{b}_1$ are relatively prime. Consider the vectors $\mathbf{b}_2 = (a_2a_3, a_1a_3, a_1u_2 + u_1a_2, a_1a_3)$, $\mathbf{b}_3 = (a_2a_3, a_1a_3, a_1u_2 + u_1a_2, a_1u_2 + u_1a_2)$, $\mathbf{b}_4 = (0, 0, 0, a_3(a_1 + a_2))$, $\mathbf{b}_5 = (0, 0, 0, a_1u_2 + a_2u_1 + a_2a_3)$ and $\mathbf{b}_6 = (0, 0, 0, a_1u_2 + a_2u_1 + a_1a_3)$. By Proposition 3.1 for every $w > 0$, $I(\mathbf{n} + w\mathbf{b}_2)$ is a complete intersection on f_1, f_2 and $x_4^{a_4+w} - x_1^{v_1}x_2^{v_2+w}x_3^{v_3}$ whenever the entries of $\mathbf{n} + w\mathbf{b}_2$ are relatively prime, $I(\mathbf{n} + w\mathbf{b}_3)$ is a complete intersection on f_1, f_2 and $x_4^{a_4+w} - x_1^{v_1}x_2^{v_2}x_3^{v_3+w}$ whenever the entries of $\mathbf{n} + w\mathbf{b}_3$ are relatively prime, and $I(\mathbf{n} + w\mathbf{b}_4)$ is a complete intersection on f_1, f_2 and $x_4^{a_4} - x_1^{v_1+w}x_2^{v_2+w}x_3^{v_3}$ whenever the entries of $\mathbf{n} + w\mathbf{b}_4$ are relatively prime. Furthermore for every $w > 0$, $I(\mathbf{n} + w\mathbf{b}_5)$ is a complete intersection on f_1, f_2 and $x_4^{a_4} - x_1^{v_1+w}x_2^{v_2}x_3^{v_3+w}$ whenever the entries of $\mathbf{n} + w\mathbf{b}_5$ are relatively prime, and $I(\mathbf{n} + w\mathbf{b}_6)$ is a complete intersection on f_1, f_2 and $x_4^{a_4} - x_1^{v_1}x_2^{v_2+w}x_3^{v_3+w}$ whenever the entries of $\mathbf{n} + w\mathbf{b}_6$ are relatively prime. Consider the vectors $\mathbf{b}_7 = (a_2a_3, a_1a_3, a_1u_2 + u_1a_2, a_3(a_1 + a_2))$, $\mathbf{b}_8 = (a_2a_3, a_1a_3, a_1u_2 + u_1a_2, a_1u_2 + u_1a_2 + a_2a_3)$, $\mathbf{b}_9 = (a_2a_3, a_1a_3, a_1u_2 + u_1a_2, a_1u_2 +$

$u_1a_2+a_1a_3)$, $\mathbf{b}_{10}=(0,0,0,a_1u_2+a_2u_1+a_3(a_1+a_2))$, $\mathbf{b}_{11}=(a_2a_3,a_1a_3,a_1u_2+u_1a_2,0)$, $\mathbf{b}_{12}=(0,0,0,a_2a_3)$, $\mathbf{b}_{13}=(0,0,0,a_1a_3)$, $\mathbf{b}_{14}=(0,0,0,a_1u_2+a_2u_1)$ and $\mathbf{b}_{15}=(a_2a_3,a_1a_3,a_1u_2+u_1a_2,a_1u_2+u_1a_2+a_3(a_1+a_2))$. Using Proposition 3.1 we have that for all $w>0$, the ideal $I(\mathbf{n}+w\mathbf{b}_i)$, $7\leq i\leq 15$, is a complete intersection whenever the entries of $\mathbf{n}+w\mathbf{b}_i$ are relatively prime. Finally consider the vectors $\mathbf{b}_{16}=(a_3a_4,a_3a_4,a_4(u_1+u_2),v_3(u_1+u_2)+a_3(v_1+v_2))$, $\mathbf{b}_{17}=(0,a_3a_4,a_4u_2,u_2v_3+a_3v_2)$, $\mathbf{b}_{18}=(a_3a_4,0,a_4u_1,u_1v_3+v_1a_3)$, $\mathbf{b}_{19}=(a_2a_4,a_1a_4,a_2a_4,a_2v_3+a_1v_2+v_1a_2)$, $\mathbf{b}_{20}=(a_2a_4,a_1a_4,a_1a_4,a_1v_3+a_1v_2+v_1a_2)$, $\mathbf{b}_{21}=(a_2a_4,a_1a_4,a_4(a_1+a_2),v_3(a_1+a_2)+a_1v_2+v_1a_2)$ and $\mathbf{b}_{22}=(0,0,a_4(a_1+a_2),v_3(a_1+a_2)+a_3(a_1+a_2))$. It is easy to see that for all $w>0$, the ideal $I(\mathbf{n}+w\mathbf{b}_i)$, $16\leq i\leq 22$, is a complete intersection whenever the entries of $\mathbf{n}+w\mathbf{b}_i$ are relatively prime. For instance $I(\mathbf{n}+w\mathbf{b}_{22})$ is a complete intersection on the binomials f_1, $x_3^{a_3}-x_1^{u_1+w}x_2^{u_2+w}$ and $x_4^{a_4}-x_1^{v_1+w}x_2^{v_2+w}x_3^{v_3}$. $\square$

Example 3.3. Let $\mathbf{n}=(231,770,1023,674)$, then $I(\mathbf{n})$ is a complete intersection on the binomials $x_1^{10}-x_2^3$, $x_3^7-x_1^{11}x_2^6$ and $x_4^{11}-x_1x_2^8x_3$. Here $a_1=10$, $a_2=3$, $a_3=7$, $a_4=11$, $u_1=11$, $u_2=6$, $v_1=1$, $v_2=8$ and $v_3=1$. Consider the vector $\mathbf{b}_{22}=(0,0,143,104)$, then for all $w\geq 0$ the ideal $I(\mathbf{n}+w\mathbf{b}_{22})$ is a complete intersection on $x_1^{10}-x_2^3$, $x_3^7-x_1^{11+w}x_2^{6+w}$ and $x_4^{11}-x_1^{1+w}x_2^{8+w}x_3$ whenever $\gcd(231,770,1023+143w,674+104w)=1$. In fact, $I(\mathbf{n}+w\mathbf{b}_{22})$ is minimally generated by $x_1^{10}-x_2^3$, $x_3^7-x_1^{11+w}x_2^{6+w}$ and $x_4^{11}-x_1^{11+w}x_2^{5+w}x_3$. Remark that $231=\min\{231,770,1023+143w,674+104w\}$. The set $\{x_1^{10}-x_2^3,x_3^7-x_1^{11+w}x_2^{6+w},x_4^{11}-x_1^{11+w}x_2^{5+w}x_3\}$ is a standard basis for $I(\mathbf{n}+w\mathbf{b}_{22})$ with respect to the negative degree reverse lexicographical order with $x_4>x_3>x_2>x_1$. So $I(\mathbf{n}+w\mathbf{b}_{22})_*$ is a complete intersection on x_2^3, x_3^7 and x_4^{11}, and therefore for every $w\geq 0$ the monomial curve $C(\mathbf{n}+w\mathbf{b}_{22})$ has Cohen-Macaulay tangent cone at the origin whenever $\gcd(231,770,1023+143w,674+104w)=1$. Let $\mathbf{b}_{16}=(77,77,187,80)$. For every $w\geq 0$, $I(\mathbf{n}+w\mathbf{b}_{16})$ is a complete intersection on $x_1^{10+w}-x_2^{3+w}$, $x_3^7-x_1^{11}x_2^6$ and $x_4^{11}-x_1x_2^8x_3$ whenever $\gcd(231+77w,770+77w,1023+187w,674+80w)=1$. Note that $231+77w=\min\{231+77w,770+77w,1023+187w,674+80w\}$. For $0\leq w\leq 5$ the set $\{x_1^{10+w}-x_2^{3+w},x_3^7-x_1^{11}x_2^6,x_4^{11}-x_1^{11+w}x_2^{5-w}x_3\}$ is a standard basis for $I(\mathbf{n}+w\mathbf{b}_{16})$ with respect to the negative degree reverse lexicographical order with $x_4>x_3>x_2>x_1$. Thus $I(\mathbf{n}+w\mathbf{b}_{16})_*$ is minimally generated by $\{x_2^{3+w},x_3^7,x_4^{11}\}$, so for $0\leq w\leq 5$ the monomial curve $C(\mathbf{n}+w\mathbf{b}_{16})$ has Cohen-Macaulay tangent cone at the origin whenever $\gcd(231+77w,770+77w,1023+187w,674+80w)=1$. Suppose that there is $w\geq 6$ such that $C(\mathbf{n}+w\mathbf{b}_{16})$ has Cohen-Macaulay tangent cone at the origin. Then $x_2^8x_3\in I(\mathbf{n}+w\mathbf{b}_{16})_*:\langle x_1\rangle$ and therefore $x_2^8x_3\in I(\mathbf{n}+w\mathbf{b}_{16})_*$. Thus $x_2^8x_3$ is divided by x_2^{3+w}, a contradiction. Consequently for every $w\geq 6$ the monomial curve $C(\mathbf{n}+w\mathbf{b}_{16})$ does not have Cohen-Macaulay tangent cone at the origin whenever $\gcd(231+77w,770+77w,1023+187w,674+80w)=1$.

Theorem 3.4. *Let $I(\mathbf{n})$ be a complete intersection ideal generated by the binomials $f_1=x_1^{a_1}-x_2^{a_2}$, $f_2=x_3^{a_3}-x_1^{u_1}x_2^{u_2}$ and $f_3=x_4^{a_4}-x_1^{v_1}x_2^{v_2}x_3^{v_3}$. Consider the vector $\mathbf{d}=(0,0,a_4(a_1+a_2),v_3(a_1+a_2)+a_3(a_1+a_2))$. Then there exists a non-negative integer w_1 such that for all $w\geq w_1$, the ideal $I(\mathbf{n}+w\mathbf{d})_*$ is a complete intersection whenever the entries of $\mathbf{n}+w\mathbf{d}$ are relatively prime.*

Proof. By Theorem 3.2 for all $w\geq 0$, the ideal $I(\mathbf{n}+w\mathbf{d})$ is minimally generated by $G=\{f_1,f_4=x_3^{a_3}-x_1^{u_1+w}x_2^{u_2+w},f_5=x_4^{a_4}-x_1^{v_1+w}x_2^{v_2+w}x_3^{v_3}\}$ whenever the entries of $\mathbf{n}+w\mathbf{d}$ are relatively prime. Let w_1 be the smallest non-negative integer greater than or equal to $\max\{\frac{a_3-u_1-u_2}{2},\frac{a_4-v_1-v_2-v_3}{2}\}$. Then $a_3\leq u_1+u_2+2w_1$ and $a_4\leq v_1+v_2+v_3+2w_1$. It is easy to prove that for every $w\geq w_1$ the set G is a standard basis for $I(\mathbf{n}+w\mathbf{d})$ with respect to the negative degree reverse lexicographical order with $x_4>x_3>x_2>x_1$. Note that $\mathrm{LM}(f_1)$ is either $x_1^{a_1}$

or $x_2^{a_2}$, $\mathrm{LM}(f_4) = x_3^{a_3}$ and $\mathrm{LM}(f_5) = x_4^{a_4}$. By [8, Lemma 5.5.11] $I(\mathbf{n} + w\mathbf{d})_*$ is generated by the least homogeneous summands of the elements in the standard basis G. Thus for all $w \geq w_1$, the ideal $I(\mathbf{n} + w\mathbf{d})_*$ is a complete intersection whenever the entries of $\mathbf{n} + w\mathbf{d}$ are relatively prime. $\square$

Proposition 3.5. *Let $I(\mathbf{n})$ be a complete intersection ideal generated by the binomials $f_1 = x_1^{a_1} - x_2^{a_2}$, $f_2 = x_3^{a_3} - x_1^{u_1}x_2^{u_2}$ and $f_3 = x_4^{a_4} - x_1^{v_1}x_2^{v_2}$, where $v_1 > 0$ and $v_2 > 0$. Assume that $a_2 < a_1$, $a_3 < u_1 + u_2$, $v_2 < a_2$ and $a_1 + v_1 \leq a_2 - v_2 + a_4$. Then there exists a vector $\mathbf{b}$ in $\mathbb{N}^4$ such that for all $w \geq 0$, the ideal $I(\mathbf{n} + w\mathbf{b})_*$ is almost complete intersection whenever the entries of $\mathbf{n} + w\mathbf{b}$ are relatively prime.*

Proof. From the assumptions we deduce that $v_1 + v_2 < a_4$. Consider the vector $\mathbf{b} = (a_2a_3, a_1a_3, a_1u_2 + u_1a_2, a_2a_3)$. For every $w \geq 0$ the ideal $I(\mathbf{n} + w\mathbf{b})$ is a complete intersection on f_1, f_2 and $f_4 = x_4^{a_4+w} - x_1^{v_1+w}x_2^{v_2}$ whenever the entries of $\mathbf{n} + w\mathbf{b}$ are relatively prime. We claim that the set $G = \{f_1, f_2, f_4, f_5 = x_1^{a_1+v_1+w} - x_2^{a_2-v_2}x_4^{a_4+w}\}$ is a standard basis for $I(\mathbf{n} + w\mathbf{b})$ with respect to the negative degree reverse lexicographical order with $x_3 > x_2 > x_1 > x_4$. Note that $\mathrm{LM}(f_1) = x_2^{a_2}$, $\mathrm{LM}(f_2) = x_3^{a_3}$, $\mathrm{LM}(f_4) = x_1^{v_1+w}x_2^{v_2}$ and $\mathrm{LM}(f_5) = x_1^{a_1+v_1+w}$. Also $\mathrm{spoly}(f_1, f_4) = -f_5$. It suffices to show that $\mathrm{NF}(\mathrm{spoly}(f_4, f_5)|G) = 0$. We compute $\mathrm{spoly}(f_4, f_5) = x_2^{a_2}x_4^{a_4+w} - x_1^{a_1}x_4^{a_4+w}$. Then $\mathrm{LM}(\mathrm{spoly}(f_4, f_5)) = x_2^{a_2}x_4^{a_4+w}$ and only $\mathrm{LM}(f_1)$ divides $\mathrm{LM}(\mathrm{spoly}(f_4, f_5))$. Moreover $\mathrm{ecart}(\mathrm{spoly}(f_4, f_5)) = a_1 - a_2 = \mathrm{ecart}(f_1)$. So $\mathrm{spoly}(f_1, \mathrm{spoly}(f_4, f_5)) = 0$ and also $\mathrm{NF}(\mathrm{spoly}(f_4, f_5)|G) = 0$. Thus

(1) If $a_1 + v_1 < a_2 - v_2 + a_4$, then $I(\mathbf{n} + w\mathbf{b})_*$ is minimally generated by $\{x_2^{a_2}, x_3^{a_3}, x_1^{v_1+w}x_2^{v_2}, x_1^{a_1+v_1+w}\}$.
(2) If $a_1 + v_1 = a_2 - v_2 + a_4$, then $I(\mathbf{n} + w\mathbf{b})_*$ is minimally generated by $\{x_2^{a_2}, x_3^{a_3}, x_1^{v_1+w}x_2^{v_2}, f_5\}$. $\square$

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