# LOCALLY FINITE DERIVATIONS AND MODULAR COINVARIANTS 

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#### Abstract

We consider a finite-dimensional $\mathbb{k} G$-module $V$ of a $p$-group $G$ over a field $\mathbb{k}$ of characteristic $p$. We describe a generating set for the corresponding Hilbert Ideal. In case $G$ is cyclic, this yields that the algebra $\mathbb{k}[V]_{G}$ of coinvariants is a free module over its subalgebra generated by $\mathbb{k} G$-module generators of $V^{*}$. This subalgebra is a quotient of a polynomial ring by pure powers of its variables. The coinvariant ring was known to have this property only when $G$ was cyclic of prime order [M. Sezer, Decomposing modular coinvariants, J. Algebra 423 (2015), 87-92]. In addition, we show that if $G$ is the Klein 4 -group and $V$ does not contain an indecomposable summand isomorphic to the regular module, then the Hilbert Ideal is a complete intersection, extending a result of the second author and Shank [M. Sezer and R. J. Shank, Rings of invariants for modular representations of the Klein four group, Trans. Amer. Math. Soc. 368 (2016), 5655-5673].


## 1. Introduction

Let $\mathbb{k}$ be a field of positive characteristic $p$ and $V$ a finite-dimensional $\mathbb{k}$-vector space, and $G \leq \mathrm{GL}(V)$ a finite group. Then the induced action on $V^{*}$ extends to the symmetric algebra $\mathbb{k}[V]:=S\left(V^{*}\right)$ by the formula $\sigma(f)=f \circ \sigma^{-1}$ for $\sigma \in G$ and $f \in \mathbb{k}[V]$. The ring of fixed points $\mathbb{k}[V]^{G}$ is called the ring of invariants, and is the central object of study in invariant theory. Another object which is often studied is the Hilbert Ideal, $\mathcal{H}$, which is defined to be the ideal of $\mathbb{k}[V]$ generated by invariants of positive degree, in other words

$$
\mathcal{H}=\mathbb{k}[V]_{+}^{G} \mathbb{k}[V] .
$$

In this article, we study the quotient $\mathbb{k}[V]_{G}:=\mathbb{k}[V] / \mathcal{H}$ which is called the algebra of coinvariants. An equivalent definition is $\mathbb{k}[V]_{G}:=\mathbb{k}[V] \otimes_{\mathbb{k}[V]^{G}} \mathbb{k}$, which shows that this object is, in a sense, dual to $\mathbb{k}[V]^{G}$.

As $\mathbb{k}[V]_{G}$ is a finite-dimensional $\mathbb{k} G$-module, it is generally easier to handle than the ring of invariants. On the other hand, much information about $\mathbb{k}[V]^{G}$ is encoded in $\mathbb{k}[V]_{G}$. For example, Steinberg [13] famously showed that $\operatorname{dim}\left(\mathbb{C}[V]_{G}\right)=|G|$ if and only if $(G, V)$ is a complex reflection group. Combined with the theorem of Chevalley [2], Shephard and Todd [11], this shows that

[^0]$\operatorname{dim}\left(\mathbb{C}[V]_{G}\right)=|G|$ if and only if $\mathbb{C}[V]^{G}$ is a polynomial ring. Smith [12] later generalized this by showing that $\operatorname{dim}\left(\mathbb{k}[V]_{G}\right)=|G|$ if and only if $G$ is a (pseudo)-reflection group, where $\mathbb{k}$ is any field. Further, the polynomial property of $\mathbb{C}[V]^{G}$ is equivalent to the Poincaré duality property of $\mathbb{C}[V]_{G}$, by Kane [6] and Steinberg [13].

Before we continue, we fix some terminology. Let $x_{0}, \ldots, x_{n}$ be a basis for $V^{*}$. We will say $x_{i}$ is a terminal variable if the vector space spanned by the other variables is a $\mathbb{k} G$-submodule of $V^{*}$. Note that if $G$ is a $p$-group, then $V^{G} \neq 0$ and there is a choice of a basis for $V$ that contains a fixed point. Then the dual element corresponding to the fixed point is a terminal variable in the basis consisting of dual elements of this basis. For any $f \in \mathbb{k}[V]$, we define the norm

$$
N^{G}(f)=\prod_{h \in G \cdot f} h .
$$

For every terminal variable $x_{i}$, we choose a polynomial $N\left(x_{i}\right)$ in $\mathbb{k}[V]^{G}$ which, when viewed as a polynomial in $x_{i}$ is monic of minimal positive degree. While $N\left(x_{i}\right)$ is not unique in general, its degree is well defined. Since $N^{G}\left(x_{i}\right)$ is monic of degree $\left[G: G_{x_{i}}\right]$, the degree of $N\left(x_{i}\right)$ is bounded above by this number. By 'degree of $x_{i}{ }^{\prime}$, we understand degree of $N\left(x_{i}\right)$ as a polynomial in $x_{i}$ and denote it by $\operatorname{deg}\left(x_{i}\right)$. We will show that the degree of a terminal variable is always a $p$-power.

The algebras of modular coinvariants for cyclic groups of order $p$ were studied by the second author [8], and previously by the second author and Shank [9]. Note that there is a choice of basis such that an indecomposable representation of a $p$-group is afforded by an upper triangular matrix with 1 s on the diagonal and the bottom variable is a terminal variable. In [8], the following was proven.

Proposition 1.1. Let $G$ be a cyclic group of order $p$ and $V a \mathbb{k} G$-module that contains $k+1$ non-trivial summands. Choose a basis $x_{0}, x_{1}, \ldots, x_{n}$ in which the variables $x_{0}, x_{1}, \ldots, x_{k}$ are the bottom variables of the respective Jordan blocks, and let A be the $\mathbb{k} G$-subalgebra of $\mathbb{k}[V]$ generated by $x_{k+1}, \ldots, x_{n}$. Denote the image of $x_{i}$ in $\mathbb{k}[V]_{G}$ by $X_{i}$. Then
(1) The Hilbert Ideal of $\mathbb{k}[V]^{G}$ is generated by $N^{G}\left(x_{0}\right), N^{G}\left(x_{1}\right), \ldots, N^{G}\left(x_{k}\right)$, and polynomials in $A$.
(2) $\mathbb{k}[V]_{G}$ has dimension divisible by $p^{k+1}$.
(3) $\mathbb{k}[V]_{G}$ is free as a module over its subalgebra $\mathcal{T}$ generated by $X_{0}, X_{1}, \ldots, X_{k}$.
(4) $\mathcal{T} \cong \mathbb{k}\left[t_{0}, \ldots, t_{k}\right] /\left(t_{0}^{p}, \ldots, t_{k}^{p}\right)$, where $t_{0}, \ldots, t_{k}$ are independent variables.

The goal of this article is to generalize the above, as far as possible, to the case of all finite $p$-groups. In particular, we show in section two:

Theorem 1.2. Let $G$ be a finite p-group and $V a \mathbb{k} G$-module that contains $k+1$ non-trivial summands. Choose a basis $x_{0}, x_{1}, \ldots, x_{n}$ in which the variables $x_{0}, x_{1}, \ldots, x_{k}$ coming from each summand are terminal variables. Let $d_{i}$ denote $\operatorname{deg}\left(x_{i}\right)$ for $0 \leq i \leq k$. Retain the notation in the proposition above, then
(1) There is a choice for polynomials $N\left(x_{0}\right), N\left(x_{1}\right), \ldots, N\left(x_{k}\right)$ such that the Hilbert Ideal of $\mathbb{k}[V]^{G}$ is generated by $N\left(x_{0}\right), N\left(x_{1}\right), \ldots, N\left(x_{k}\right)$, and polynomials in $A$.
(2) $\mathbb{k}[V]_{G}$ has dimension divisible by $\prod_{i=0}^{k} d_{i}$.

Suppose in addition that one has $d_{i}=\operatorname{deg}\left(N^{G}\left(x_{i}\right)\right)$ for $0 \leq i \leq k$. Then we have:
(3) $\mathbb{k}[V]_{G}$ is free as a module over its subalgebra $\mathcal{T}$ generated by $X_{0}, X_{1}, \ldots, X_{k}$.
(4) $\mathcal{T} \cong \mathbb{k}\left[t_{0}, \ldots, t_{k}\right] /\left(t_{0}^{d_{0}}, \ldots, t_{k}^{d_{k}}\right)$, where $t_{0}, \ldots, t_{k}$ are independent variables.

In Section 3, we describe the situation for a $p$-group, where the complete intersection property of the Hilbert Ideal corresponding to a module is inherited from the Hilbert Ideal of the indecomposable summands of the module. The final section is devoted to applications of our main results to cyclic $p$-groups and the Klein 4 -group. It turns out that for a cyclic $p$-group, the bottom variables $x_{i}$ of Jordan blocks satisfy $\operatorname{deg}\left(x_{i}\right)=\operatorname{deg}\left(N^{G}\left(x_{i}\right)\right)$. Consequently, (3) and (4) above hold for a cyclic $p$-group. Additionally, for the Klein 4 -group, we show that the Hilbert Ideal corresponding to a module is a complete intersection as long as the module does not contain the regular module as a summand. This generalizes a result of the second author and Shank [10], where the complete intersection property was established for indecomposable modules only.

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## 2. Main results

Throughout this section, we let $G$ be a finite $p$-group, $\mathbb{k}$ a field of characteristic $p$ and $V$ a $\mathbb{k} G$-module, which may be decomposable. As trivial summands do not contribute to the coinvariants, we assume no direct summand of $V$ is trivial. Let $x_{0}, x_{1}, x_{2}, \ldots, x_{n}$ be a basis of $V^{*}$ and assume that $x_{0}$ is a terminal variable. Then $x_{1}, x_{2}, \ldots, x_{n}$ generate a $G$-subalgebra which we denote by $A$. We can define a nonlinear action of $(\mathbb{k},+)$ on $\mathbb{k}[V]$ as follows:

$$
\begin{gather*}
t \cdot x_{0}=x_{0}+t  \tag{2.1}\\
t \cdot x_{i}=x_{i} \quad \text { for any } i>0 \tag{2.2}
\end{gather*}
$$

The terminality of $x_{0}$ ensures this commutes with the action of $G$. It is well known that any action of the additive group of an infinite field of prime characteristic is determined by a locally finite iterative higher derivation. This is a family of $\mathbb{k}$-linear maps $\Delta^{i}: \mathbb{k}[V] \rightarrow \mathbb{k}[V], i \geq 0$ satisfying the following properties:
(1) $\Delta^{0}=\operatorname{id}_{\mathbb{k}[V]}$.
(2) For all $i>0$ and $a, b \in \mathbb{k}[V]$, one has $\Delta^{i}(a b)=\sum_{j+k=i} \Delta^{j}(a) \Delta^{k}(b)$.
(3) For all $b \in \mathbb{k}[V]$, there exists $i \geq 0$ such that $\Delta^{i}(b)=0$.
(4) For all $i, j$, one has $\Delta^{j} \circ \Delta^{i}=\binom{i+j}{j} \Delta^{i+j}$.

The equivalence of the group action and the l.f.i.h.d. is given by the formula

$$
\begin{equation*}
t \cdot b=\sum_{i \geq 0} t^{i} \Delta^{i}(b) \tag{2.3}
\end{equation*}
$$

See [3, 14], for more details on l.f.i.h.d.'s.
Let $f \in \mathbb{k}[V]^{G}$ be homogeneous of degree $d$ in $x_{0}$. We write

$$
f=f_{d} x_{0}^{d}+f_{d-1} x_{0}^{d-1}+\cdots+f_{0},
$$

where $f_{i} \in A$. We have

$$
\begin{equation*}
t \cdot f=f_{d}\left(x_{0}+t\right)^{d}+f_{d-1}\left(x_{0}+t\right)^{d-1}+\cdots+f_{0}=\sum_{i \geq 0} t^{i} \Delta^{i}(f) \tag{2.4}
\end{equation*}
$$

That is to say that $\Delta^{i}(f)$ is the coefficient of $t^{i}$ in the above expression. As the action of $G$ commutes with the action of $\mathbb{k}$, we see that $\Delta^{i}(f) \in \mathbb{k}[V]^{G}$ for all $i \geq 0$.

Remark 2.1.
(1) Clearly $\Delta^{1}=\frac{\partial}{\partial x_{0}}$. So the previous paragraph generalizes [8, Lemma 1].
(2) Equation (2.4) gives that $\Delta^{j}\left(x_{0}^{i}\right)=\binom{i}{j} x_{0}^{i-j}$ provided $i \geq j$. Then, from Lucas's theorem [5] on binomial coefficients in characteristic $p$, we see that we can think of $\Delta p^{j}$ as 'Differentiation by $x_{0}^{p^{j}}$ : if the coefficient of $p^{j}$ in the base $p$ expansion of $m$ is $a$, then we have

$$
\Delta^{p^{j}}\left(x_{0}^{m}\right)= \begin{cases}a x_{0}^{m-p^{j}} & a>0 \\ 0 & a=0\end{cases}
$$

For later use, we also note the following consequence: for a homogeneous $f \in \mathbb{k}[V], \Delta^{j}(f)$ contains a non-zero constant if and only if the monomial $x_{0}^{j}$ appears in $f$.
(3) In [4], a $G$-equivariant map is constructed from polynomials whose $x_{0}$-degree is at most $e p^{r}$ $(0<e<p)$ to polynomials whose $x_{0}$-degree is at most $p^{r}$. This map turns out to be a nonzero scalar multiple of $\Delta^{(e-1) p^{r}}$.

We have the following statement generalizing [8, Lemma 2]:
Lemma 2.2. Let $f \in \mathbb{k}[V]$ be a homogeneous polynomial of degree $d$ in $x_{0}$. Write $f=f_{d} x_{0}^{d}+f_{d-1} x_{0}^{d-1}+\cdots+f_{0}$, where $f_{i} \in A$. Then we have

$$
\sum_{i=0}^{d}(-1)^{i} x_{0}^{i} \Delta^{i}(f)=f_{0}
$$

Proof. Write $f=f\left(x_{0}, x_{1}, x_{2}, \ldots, x_{n}\right)$. For any $t \in \mathbb{k}$, we have

$$
t \cdot f=f\left(t \cdot x_{0}, t \cdot x_{1}, \ldots, t \cdot x_{n}\right)=f\left(x_{0}+t, x_{1}, x_{2}, \ldots, x_{n}\right)
$$

As this holds for all $t$ it also holds when $t$ is replaced by $\left(-x_{0}\right)$, and hence by Equation (2.3), we have $\sum_{i=0}^{d}(-1)^{i} x_{0}^{i} \Delta^{i}(f)=\left(-x_{0}\right) \cdot f=f\left(0, x_{1}, x_{2}, \ldots, x_{n}\right)=f_{0}$ as required.

We also note that the degree of a terminal variable is a $p$-power.
Lemma 2.3. For any terminal variable $x_{0} \in V^{*}, \operatorname{deg}\left(x_{0}\right)$ is a power of $p$.
Proof. Let $d$ denote the degree of $x_{0}$ and suppose $f \in \mathbb{k}[V]^{G}$ is monic as a polynomial in $x_{0}$ of degree $d=d_{r} p^{r}+d_{r-1} p^{r-1}+\cdots+d_{0}$ with $0 \leq d_{i}<p$ and $d_{r} \neq 0$. If $d_{j} \neq 0$ for some $j<r$, then $\Delta^{p^{j}}(f) \in \mathbb{k}[V]^{G}$ has degree $d-p^{j}>0$ as a polynomial in $x_{0}$ and its leading coefficient is
in $\mathbb{k}$. Similarly, if $d_{j}=0$ for $j<r$ and $d_{r}>1$, then $\Delta^{p^{r}}(f) \in \mathbb{k}[V]^{G}$ has degree $d-p^{r}>0$ in $x_{0}$ and its leading coefficient is in $\mathbb{k}$. Both cases violate the minimality of $d$.

Lemma 2.4. Let d denote the degree of $x_{0}$. Then $\Delta^{j}(\mathcal{H}) \subseteq \mathcal{H}$ for $j<d$.
Proof. Let $f \in \mathbb{k}[V]$. From the second assertion of Remark 2.1, we get that $\Delta^{j}(f)$ contains a nonzero constant if and only if the monomial $x_{0}^{j}$ appears in $f$. Therefore, by the minimality of $d$, we have $\Delta^{j}\left(\mathbb{k}[V]_{+}^{G}\right) \subseteq \mathcal{H}$ for $j<d$. Now the result follows from property (2) of 1.f.i.h.d.'s.

From this point on, we adopt the notation of the introduction. This means that $x_{0}, x_{1}, x_{2}, \ldots, x_{k}$ are terminal variables coming from different summands, and $A=\mathbb{k}\left[x_{k+1}, x_{k+2}, \ldots, x_{n}\right]$. For each $i=0, \ldots, k$ let $d_{i}=p^{r_{i}}$ be the degree of $x_{i}$. Since setting variables outside of a summand to zero sends invariants to invariants of the summand, we may also assume that $N\left(x_{i}\right)$ depends only on variables that come from the summand that contains $x_{i}$. We denote by $\Delta_{i}$ the l.f.i.h.d. associated to $x_{i}$. We use reverse lexicographic order with $x_{i}>x_{j}$ whenever $0 \leq i \leq k$ and $k+1 \leq j \leq n$.

Theorem 2.5. $\mathcal{H}$ is generated by $N\left(x_{0}\right), \ldots, N\left(x_{k}\right)$ and polynomials in $A$. Moreover, the lead term ideal of $\mathcal{H}$ is generated by $x_{0}^{p^{p^{\prime 0}}}, x_{1}^{p^{p_{1}}}, \ldots, x_{k}^{p^{r_{k}}}$ and monomials in $A$.

Proof. Let $f \in \mathbb{k}[V]^{G}$. Since $N\left(x_{0}\right)$ is monic in $x_{0}$, we may perform polynomial division and write $f=q N\left(x_{0}\right)+r$ where $r$ has $x_{0}$-degree $<p^{r_{0}}$, and it is easily shown that $q, r \in \mathbb{k}[V]^{G}$. Then dividing $r$ by $N\left(x_{1}\right)$ yields another invariant remainder $r^{\prime}$ that has $x_{1}$-degree $<p^{r_{1}}$. Since $x_{0}$-degree of $N\left(x_{1}\right)$ is zero, it follows that $x_{0}$-degree of $r^{\prime}$ is still $<p^{r_{0}}$. Thus, by repeating the process with each terminal variable, and replacing $f$ with the final remainder we assume that $x_{i}$-degree of $f$ is $<p^{r_{i}}$ for $0 \leq i \leq k$.

Let $i$ be minimal such that $f$ has non-zero degree $d<p^{r_{i}}$ in the terminal variable $x_{i}$. We apply Lemma 2.2 with $\Delta=\Delta_{i}$ to see that

$$
f=f_{0}-\left(\sum_{j=1}^{d}(-1)^{j} x_{i}^{j} \Delta_{i}^{j}(f)\right),
$$

where $f_{0}$ is the 'constant term' of $f$, that is, $f_{0} \in \mathbb{k}\left[x_{i+1}, \ldots, x_{n}\right]$. So from the previous lemma, we get that $f_{0} \in \mathcal{H}$ since $d<p^{r_{i}}$. Moreover, since $\Delta_{i}$ decreases $x_{i}$-degrees and does not increase degrees in any other variable, the $x_{i}$-degree of each $\Delta_{i}^{j}(f)$ in the expression above is strictly less than $d$, and the $x_{l}$-degree for every $i<l \leq k$ remains strictly less than $p^{{ }^{h}}$. Thus, by induction on degree, $f$ can be expressed as a $\mathbb{k}[V]$-combination of elements of $\mathcal{H}$ whose degrees in the terminal variables $x_{0}, \ldots, x_{i}$ are all zero, and degrees in the remaining terminal variables $x_{l}$ for $i<l \leq k$ are strictly less than $p^{r_{i}}$, respectively. Repeating the same argument with the remaining terminal variables gives us that $f$ can be written as a $\mathbb{k}[V]$-combination of elements of $\mathcal{H} \cap A$ together with $N\left(x_{1}\right), \ldots, N\left(x_{k}\right)$ as required. The first assertion of the theorem follows.

Note that the leading monomial of $N\left(x_{i}\right)$ is $x_{i}^{p_{i}}$ for $0 \leq i \leq k$. So it remains to show that all other monomials in the lead term ideal of $\mathcal{H}$ lie in $A$. Recall that by Buchberger's algorithm a Gröbner basis is obtained by reduction of $S$-polynomials of a generating set by polynomial division, see [1, Section 1.7]. By the first part, $\mathcal{H}$ has a generating set consisting of $N\left(x_{i}\right)$ for $0 \leq i \leq k$ and polynomials in $A$. But the $S$-polynomial of two polynomials in $A$ is also in $A$, and via polynomials in $A$, it also reduces to a polynomial in $A$. Finally, the $S$-polynomial of $N\left(x_{i}\right)$ and
a polynomial in $A$ and the $S$-polynomial of a pair $N\left(x_{i}\right)$ and $N\left(x_{j}\right)$ with $0 \leq i \neq j \leq k$ reduce to zero since their leading monomials are pairwise relatively prime.
Corollary 2.6. The vector space dimension of $\mathbb{k}[V]_{G}$ is divisible by $\prod_{0 \leq i \leq k} d_{i}=p^{\sum_{i=0}^{k} r_{i}}$.
Proof. The set of monomials that are not in the lead term ideal of $\mathcal{H}$ form a vector space basis for $\mathbb{k}[V]_{G}$. Let $\Lambda$ denote this set of monomials. By the previous theorem, a monomial $M \in A$ lies in $\Lambda$ if and only if $M x_{0}^{a_{0}}, \ldots, x_{k}^{a_{k}}$ lies in $\Lambda$ for $0 \leq a_{i}<p^{r_{i}}$ and $0 \leq i \leq k$. It follows that the size of the set $\Lambda$ is divisible by $p^{\sum_{i=0}^{k} r_{i}}$.

The following generalizes the content of [8, Theorem 5] partially for a $p$-group.
Theorem 2.7. Let $x_{i}$ be a terminal variable of degree $d$, and write $N\left(x_{i}\right)=x_{i}^{d}+\sum_{j=0}^{d-1} f_{j} x_{i}^{j}$, where $x_{i}$-degree of $f_{j}$ is zero for $0 \leq j \leq d-1$. Then $x_{i}^{d}+f_{0} \in \mathcal{H}$.

Proof. Consider $\bar{N}=N\left(x_{i}\right)-x_{i}^{d}$. This is a polynomial of degree $e<d$ in $x_{i}$. By Lemma 2.2,

$$
\sum_{j=0}^{e}(-1)^{j} x_{i}^{j} \Delta_{i}^{j}(\bar{N})=f_{0}
$$

since $f_{0}$ is the constant term of $\bar{N}$. Now recall that $\Delta_{i}^{j}\left(x_{i}^{d}\right)$ is the coefficient of $t^{j}$ in $\left(x_{i}+t\right)^{d}=$ $x_{i}^{d}+t^{d}$ (note that $d$ is a $p$-power by Lemma 2.3). Thus, $\Delta^{j}\left(x_{i}^{d}\right)=0$ for all $0<j<d$. As $\Delta_{i}^{j}$ is a linear map for all $j$ it follows that $\Delta_{i}^{j}\left(N\left(x_{i}\right)\right)=\Delta_{i}^{j}(\bar{N})$ for all $0<j<d$. Therefore,

$$
\sum_{j=1}^{e}(-1)^{j} x_{i}^{j} \Delta^{j}\left(N\left(x_{i}\right)\right)=f_{0}-\bar{N}
$$

As $\Delta_{i}^{j}\left(N\left(x_{i}\right)\right) \in \mathcal{H}$ for all $j<d$ by Lemma 2.4, we get that $f_{0}-\bar{N} \in \mathcal{H}$. Therefore $x_{i}^{d}+f_{0}=N\left(x_{i}\right)-\bar{N}+f_{0} \in \mathcal{H}$ as required.
Lemma 2.8. Suppose that for each $i=0, \ldots, k$ we have $x_{i}^{d_{i}} \in \mathcal{H}$. Then $\mathbb{k}[V]_{G}$ is free as a module over its subalgebra $\mathcal{T}$ generated by $X_{0}, X_{1}, \ldots, X_{k}$, and $\mathcal{T} \cong \mathbb{k}\left[t_{0}, \ldots, t_{k}\right] /\left(t_{0}^{d_{0}}, \ldots, t_{k}^{d_{k}}\right)$, where $t_{0}, \ldots, t_{k}$ are independent variables.

Proof. The hypothesis on the $x_{i}$ is equivalent to $X_{i}^{d_{i}}=0$ in $\mathbb{k}[V]_{G}$. Let $t_{0}, \ldots, t_{k}$ be independent variables and consider the natural surjective ring homomorphism from $\mathbb{k}\left[t_{0}, \ldots, t_{k}\right]$ to $\mathbb{k}\left[X_{0}, \ldots, X_{k}\right]$. Since $X_{i}^{d_{i}}=0$, the kernel of this map contains $\left(t_{0}^{d_{0}}, \ldots, t_{k}^{d_{k}}\right)$. If this ideal is not all the kernel, then $\mathcal{H}$ must contain a polynomial in $x_{0}, \ldots, x_{k}$ such that no monomial in this polynomial is divisible by $x_{i}^{d_{i}}$ for $0 \leq i \leq k$. This is a contradiction with the description of the lead term ideal in Theorem 2.5.

Secondly, let $\Lambda$ denote the set of monomials in $\mathbb{k}[V]$ that are not in the lead term ideal of $\mathcal{H}$. Then the set of images of monomials in $\Lambda^{\prime}=\Lambda \cup A$ generate $\mathbb{k}[V]_{G}$ over $\mathcal{T}$. Further, they generate freely because $M x_{0}^{a_{0}}, \ldots, x_{k}^{a_{k}} \in \Lambda$ for all $M \in \Lambda^{\prime}$ and $0 \leq a_{i}<d_{i}$ and $0 \leq i \leq k$, and the images of monomials in $\Lambda$ form a vector space basis for $\mathbb{k}[V]_{G}$.

Proof of Theorem 1.2. The first two assertions of the theorem are contained in Theorem 2.5 and its corollary. Next assume that $d_{i}=\operatorname{deg}\left(N^{G}\left(x_{i}\right)\right)$ for $0 \leq i \leq k$. So we can take $N\left(x_{i}\right)=N^{G}\left(x_{i}\right)$. Then from Theorem 2.7, it follows that $x_{i}^{d_{i}} \in \mathcal{H}$ for $0 \leq i \leq k$ since the constant term of $N^{G}\left(x_{i}\right)$ (as a polynomial in $x_{i}$ ) is zero. Now the third and the fourth assertions follow from Lemma 2.8.

## 3. Complete intersection property of $\mathcal{H}$

In this section, we show that if the Hilbert Ideals of two modules are generated by fixed points and powers of terminal variables, then so is the Hilbert Ideal of the direct sum. As an incidental result, we prove that the degree of a terminal variable does not change after taking direct sums. We continue with the notation and the convention of the previous section. Let $V_{1}$ and $V_{2}$ be arbitrary $\mathbb{k} G$-modules. We choose a basis $x_{1,1}, \ldots, x_{n_{1}, 1}, y_{1,1}, \ldots, y_{m_{1}, 1}$ for $V_{1}^{*}$ and $x_{1,2}, \ldots, x_{n_{2}, 2}, y_{1,2}, \ldots, y_{m_{2}, 2}$ for $V_{2}^{*}$ such that $x_{1,1}, \ldots, x_{n_{1}, 1}, x_{1,2}, \ldots, x_{n_{2}, 2}$ are fixed points. Note that both $\mathbb{k}\left[V_{1}\right]$ and $\mathbb{k}\left[V_{2}\right]$ are subrings of $\mathbb{k}\left[V_{1} \oplus V_{2}\right]$, and we identify

$$
\mathbb{k}\left[V_{1} \oplus V_{2}\right]=\mathbb{k}\left[x_{1,1}, \ldots, x_{n_{1}, 1}, x_{1,2}, \ldots, x_{n_{2}, 2}, y_{1,1}, \ldots, y_{m_{1}, 1}, y_{1,2}, \ldots, y_{m_{2}, 2}\right]
$$

Note that if $y_{i, j}$ is a terminal variable in $V_{j}^{*}$ for some $1 \leq i \leq m_{j}, 1 \leq j \leq 2$, then it is also a terminal variable in $V_{1}^{*} \oplus V_{2}^{*}$.

Lemma 3.1. Assume the notation of the previous paragraph. Let $y_{i, j} \in V_{j}^{*}$ be a terminal variable. Then the degrees of $y_{i, j}$ in $V_{j}^{*}$ and $V_{1}^{*} \oplus V_{2}^{*}$ are equal.

Proof. Since $\mathbb{k}\left[V_{j}\right]^{G} \subseteq \mathbb{k}\left[V_{1} \oplus V_{2}\right]^{G}$, we have that the degree of $y_{i, j}$ in $V_{j}^{*}$ is bigger than its degree in $V_{1}^{*} \oplus V_{2}^{*}$. On the other hand, the restriction map $\mathbb{k}\left[V_{1} \oplus V_{2}\right]^{G} \rightarrow \mathbb{k}\left[V_{j}\right]^{G}$ given $f \rightarrow f_{\mid V_{j}}$ preserves any power of the form $y_{i, j}^{d}$. This gives the reverse inequality.

We denote the Hilbert Ideals $\mathbb{k}\left[V_{1} \oplus V_{2}\right]_{+}^{G} \mathbb{k}\left[V_{1} \oplus V_{2}\right], \mathbb{k}^{[ }\left[V_{1}\right]_{+}^{G} \mathbb{k}\left[V_{1}\right]$ and $\mathbb{k}\left[V_{2}\right]_{+}^{G} \mathbb{k}\left[V_{2}\right]$ with $\mathcal{H}, \mathcal{H}_{1}$ and $\mathcal{H}_{2}$, respectively.

Theorem 3.2. Assume that $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$ are generated by the powers of the variables in $V_{1}^{*}$ and $V_{2}^{*}$, respectively, and that the variables $y_{1,1}, \ldots, y_{m_{1}, 1}, y_{1,2}, \ldots, y_{m_{2}, 2}$ are terminal variables. Then $\mathcal{H}$ is generated by the union of the generating sets for $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$.

Proof. Assume that $\mathcal{H}_{1}$ is generated by $x_{1,1}, \ldots, x_{n_{1}, 1}, y_{1,1}^{d_{1,1}}, \ldots, y_{m_{1}, 1}^{d_{n, 1}}$ and $\mathcal{H}_{2}$ is generated by $x_{1,2}, \ldots, x_{n_{2}, 2}, y_{1,2}^{d_{1,2}}, \ldots, y_{m_{2}, 2}^{d_{m, 2}}$. We show that $d_{i, j}$ is equal to the degree of the variable $y_{i, j}$ for $1 \leq i \leq m_{j}$ and $1 \leq j \leq 2$. For simplicity, we set $i=j=1$ and denote the degree of $y_{1,1}$ with $d$. Since $\mathcal{H}_{1}$ is generated by monomials, each monomial in a polynomial in $\mathcal{H}_{1}$ is divisible by one of its monomial generators. So we get $d_{1,1} \leq d$. On the other hand, since $y_{1,1}^{d_{1,1}}$ is a member of $\mathcal{H}_{1}$ there is a positive degree invariant with a monomial that divides $y_{1,1}^{d_{1,1}}$. So by the minimality of $d$, we get $d \leq d_{1,1}$ as well. By Lemma 3.1, $d_{i, j}$ is also equal to the degree of $y_{i, j}$ in $\mathbb{k}\left[V_{1} \oplus V_{2}\right]^{G}$. We claim that the union of the generating sets for $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$ generate $\mathcal{H}$. Otherwise, there exists a polynomial $f$ in $\mathcal{H}$ that contains a non-constant monomial $\prod_{1 \leq i \leq m_{j}, 1 \leq j \leq 2} y_{i, j}^{e_{i j}}$ with $0 \leq e_{i, j}<d_{i, j}$. Let $\Delta_{i, j}$ denote the derivation with respect to the terminal variable $y_{i, j}$. Then applying $\Delta_{i, j}^{e_{i, j}}$ successively to $f$ for $1 \leq i \leq m_{j}, 1 \leq j \leq 2$ yields an invariant with a non-zero constant. This is a contradiction by Lemma 2.4 since $e_{i, j}<d_{i, j}$.

We end this section with an example which shows that the degree of a terminal variable may be strictly less than the degree of its norm:

Example 3.3. Let $H=\langle\sigma, \tau\rangle$ be the Klein 4-group, $\mathbb{k}$ a field of characteristic 2 and $m \geq 2$. Let $\Omega^{-m}(\mathbb{k})$ be a vector space of dimension $m=2 n+1$ over $\mathbb{k}$. Choose a basis $\left\{x_{1}, x_{2}, \ldots, x_{m}, y_{1}\right.$, $\left.y_{2}, \ldots, y_{m+1}\right\}$ of $V^{*}$. One can define an action of $H$ on $V$ in such a way that its action on $V^{*}$ is given
by $\sigma\left(y_{j}\right)=y_{j}+x_{j}, \sigma\left(x_{j}\right)=x_{j}, \tau\left(y_{j}\right)=y_{j}+x_{j-1}, \tau\left(x_{j}\right)=x_{j}$ using the convention that $x_{0}=$ $x_{m+1}=0$.

The variables $y_{1}, y_{2}, \ldots, y_{m+1}$ are terminal. One can readily check that

$$
y_{2}^{2}+x_{2} y_{2}+x_{1} y_{2}+x_{2} y_{1}+x_{1} y_{3}+y_{1} x_{3}
$$

is invariant under $H$ (note the last term is zero if $m=2$ ), so $y_{2}$ has degree 2 . On the other hand, $y_{2}$ is not fixed by either $\sigma$ or $\tau$, which means $N^{H}\left(y_{2}\right)$ has degree 4 . It is interesting to note that $x_{1} y_{2}+x_{2} y_{1}+x_{1} y_{3}+y_{1} x_{3} \in \mathcal{H}$, so we still have $y_{2}^{2} \in \mathcal{H}$.

## 4. Cyclic $\boldsymbol{p}$-groups and the Klein 4-group

In this section, we apply the results of the previous sections to cyclic $p$-groups and the Klein 4group. Let $G=Z_{p^{r}}$ denote a cyclic group of order $p^{r}$. Fix a generator $\sigma$ of $G$. There are $p^{r}$ indecomposable $\mathbb{k} G$-modules $V_{1}, \ldots, V_{p^{r}}$ over $\mathbb{k}$, and each indecomposable module $V_{i}$ is afforded by $\sigma^{-1}$ acting via a Jordan block of dimension $i$ with ones on the diagonal. For an arbitrary $\mathbb{k} G$-module $V$, we write

$$
V=\bigoplus_{i=0}^{k} V_{n_{i}} \quad\left(\text { with } 1 \leq n_{i} \leq p^{r} \text { for all } i\right)
$$

where each $V_{n_{i}}$ is spanned as a vector space by $e_{1, i}, \ldots, e_{n_{i} i}$. Then the action of $\sigma^{-1}$ is given by $\sigma^{-1}\left(e_{j, i}\right)=e_{j, i}+e_{j+1, i}$ for $1 \leq j<n_{i}$ and $\sigma^{-1}\left(e_{n_{i}, i}\right)=e_{n_{i}, i}$. Note that the fixed point space $V^{G}$ is $\mathbb{k}$-linearly spanned by $e_{n_{1}, 0}, \ldots, e_{n_{k}, k}$. The dual $V_{n_{i}}^{*}$ is isomorphic to $V_{n_{i}}$. Let $x_{1, i}, \ldots, x_{n_{i}, i}$ denote the corresponding dual basis, then we have

$$
\mathbb{k}[V]=\mathbb{k}\left[x_{j, i} \mid 1 \leq j \leq n_{i}, \quad 0 \leq i \leq k\right],
$$

and the action of $\sigma$ is given by $\sigma\left(x_{j, i}\right)=x_{j, i}+x_{j-1, i}$ for $1<j \leq n_{i}$ and $\sigma\left(x_{1, i}\right)=x_{1, i}$ for $0 \leq i \leq k$. Notice that the variables $x_{n_{i}, i}$ for $0 \leq i \leq k$ are terminal variables. We follow the notation of Section 2 and denote $x_{n_{i}, i}$ with $x_{i}$. We show that Theorem 1.2 applies completely to $G$ by computing $\operatorname{deg}\left(x_{i}\right)$ explicitly for $0 \leq i \leq k$. For each $0 \leq i \leq k$, let $a_{i}$ denote the largest integer such that $n_{i}>p^{a_{i}-1}$.

Lemma 4.1. We have $\operatorname{deg}\left(x_{i}\right)=p^{a_{i}}$. In particular, we may take $N\left(x_{i}\right)=N^{G}\left(x_{i}\right)$.
Proof. From [7, Lemma 3], we get that $\operatorname{deg}\left(x_{i}\right)$ is at least $p^{a_{i}}$. On the other hand, since $p^{a_{i}} \geq n_{i}>p^{a_{i}-1}$, a Jordan block of size $n_{i}$ has order $p^{a_{i}}$. That is, this block affords a faithful module of the subgroup of $G$ of size $p^{a_{i}}$. It follows that the orbit of $x_{i}$ has $p^{a_{i}}$ elements and so that the orbit product $N^{G}\left(x_{i}\right)$ is a monic polynomial that is of degree $p^{a_{i}}$ in $x_{i}$.

Applying Theorem 1.2, we obtain the following.

Proposition 4.2. Assume the notation of Theorem 1.2 with specialization $G=Z_{p^{r}}$. We have an isomorphism

$$
\mathbb{k}\left[X_{0}, \ldots, X_{k}\right] \cong \mathbb{k}\left[t_{0}, \ldots, t_{k}\right] /\left(t_{0}^{p^{a_{0}}}, \ldots, t_{k}^{p^{a_{k}}}\right)
$$

Moreover, $\mathbb{k}[V]_{G}$ is free as a module over $\mathbb{k}\left[X_{0}, \ldots, X_{k}\right]$.
Now let $H$ denote the Klein 4 -group and $p=2$. For each indecomposable $\mathbb{k} H$-module $V$, there exists a basis of $V^{*}$ with one of the terminal variables $x_{i}$ satisfying $\operatorname{deg}\left(x_{i}\right)=\left[H: H_{x_{i}}\right]$, see [10]. In this source, it is also proven that with the exception of the regular module, each basis consists of fixed points and the terminal variables, and the Hilbert Ideal of every such module is generated by fixed points and the powers of the terminal variables. So we have by Theorems 1.2 and 3.2:

Proposition 4.3. Let $V$ be $a \mathbb{k} H$-module containing $k+1$ indecomposable summands. There is a basis $\left\{x_{0}, x_{1}, \ldots, x_{n}\right\}$ of $V^{*}$ in which $x_{0}, x_{1}, \ldots, x_{k}$ are terminal variables, each coming from one summand, such that $\mathbb{k}[V]_{H}$ is free as a module over its subalgebra $\mathcal{T}$ generated by the images $X_{0}, X_{1}, \ldots, X_{k}$ of the terminal variables. Moreover, $\mathcal{T} \cong \mathbb{k}\left[t_{0}, \ldots, t_{k}\right] /\left(t_{0}^{a_{0}}, \ldots, t_{k}^{a_{k}}\right)$, where $t_{0}, \ldots, t_{k}$ are independent variables, and for each $i$, we have $a_{i}=2$ or 4 .

Proposition 4.4. Let $V$ be $a \mathbb{k} H$-module such that $V$ does not contain the regular module $\mathbb{k} H$ as a summand. Then there exists a basis of $V^{*}$ such that $\mathbb{k}[V]_{+}^{H} \mathbb{k}[V]$ is generated by powers of basis elements. In particular, $\mathbb{k}[V]_{+}^{H} \mathbb{k}[V]$ is a complete intersection.

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