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An exact analysis on age-based control policies for perishable inventories

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ABSTRACT

We investigate the impact of effective lifetime of items in an age-based control policy for perishable inventories, a so-called (Q, r, T) policy, with positive lead time and fixed lifetime. The exact analysis of this control policy in the presence of a service level constraint is available in the literature under the restriction that the aging process of a batch begins when it is unpacked for consumption, and that at most one order can be outstanding at any time. In this work, we generalize those results to allow for more than one outstanding order and assume that the aging process of a batch starts since the time that it is ordered. Under this aging process, we derive the effective lifetime distribution of batches at the beginning of embedded cycles in an embedded Markov process. We provide the operating characteristic expressions and construct the cost rate function by the renewal reward theorem approach. We develop an exact algorithm by investigating the cost rate and service level constraint structures. The proposed policy considerably dominates its special two-parameter policies, which are time-dependent (Q, T) and stock-dependent (Q, r) policies. Numerical studies demonstrate that the aging process of items significantly influences the inventory policy performance. Moreover, allowing more than one outstanding order in the system reaps considerable cost savings, especially when the lifetime of items is short and the service level is high.

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Perishable inventories; age-based control policy; effective lifetime; multiple outstanding orders; renewal reward theorem

1. Introduction

The increasing amounts of perishability that are the result of population growth and rising income levels has prompted inventory managers to develop cost-effective ordering policies in order to reduce their operational costs and enhance sustainability. Vegetables, fruits, milk, meat, seafood, blood, chemical materials, pharmaceuticals, and high-tech products are a few examples of perishable products. Perishable inventories constitute a large portion of total stocks held by firms. As reported by the U.S. Department of Agriculture in FirstResearch (2019), the U.S. wholesale food distribution industry includes about 660 000 restaurants that use perishable food. Of all the food produced in the worldwide food supply chain, around 24% is wasted (Kummu *et al.*, 2012). It is estimated that a reduction of 15% in food waste, would make it possible to feed more than 25 million people in the U.S. every year (Gunders & Bloom, 2017). Nowadays, the entities of a supply chain make their decisions interdependently to reduce or even avoid spoilage; for example, Nestlé and Unilever as leading food manufacturers have incorporated waste elimination in their policies (Kirci and Seifert, 2015).

Replenishment policies for perishable products that are based solely on inventory levels are employed in many

industry practices and constitute the majority of works in the literature. Due to the limited lifetime nature of perishable inventories, a created control policy must take into account not only the on-hand stock but also the remaining lifetime of products while making ordering decisions. On the other hand, in many practical situations, perishable inventories start decaying as soon as they are produced/issued by the manufacturer/supplier. With deployment of advanced techniques for recording product information such as Internet-of-Things technologies, the need to created an intelligent inventory control policy that takes into account the remaining lifetime of products when they join inventories is of considerable interest in perishable inventory management.

Due to demand uncertainty and multiple in-transit orders over time, the nature of the aging process of perishable products can significantly affect the dynamics of inventory systems. In view of this, one can analyze a replenishment control policy under two aging processes. Aging process type 1: the aging process of items in a batch begins when the batch becomes unpacked. In this case, all items in the batch have an identical fixed lifetime upon unpacking. Aging process type 2: the aging process of items in a batch begins once the batch is issued from the outside supplier so that an order joining stock has aged for the lead time.

In this article, we consider a single-item single-location inventory system with aging process type 2, consisting of perishable products with fixed lifetimes and Poisson demands. We use a continuous review (Q, r, T) policy for replenishment of perishable products that works as follows. An order of size Q is placed whenever the inventory position (stock on hand plus outstanding orders) reaches r , or T units of time have elapsed since the last instance at which the batch under consumption is depleted by demand or perishes, whichever occurs first. The proposed replenishment policy allows for more than one outstanding order and bases reordering decisions on both the inventory position and the time elapsed to consume the items in a batch. Tracing the remaining lifetime of multiple outstanding orders along with stock on hand and considering a general aging process structure poses substantial technical difficulties in our analysis.

Our study generalizes and improves on previous work in several aspects. First, we investigate the impact of the structure of the aging process on the performance of a (Q, r, T) policy in a perishable inventory system consisting of products with fixed lifetimes. Second, we elicit the effective lifetime distribution by utilizing the concept of the embedded Markov process. Third, we derive explicit expressions for the expected cost rate and the operating characteristics of our model. Fourth, we capture the effect of allowing for multiple outstanding orders in the system. Fifth, we develop an exact algorithm by investigating the cost rate and the service level constraint structures guaranteeing global optimality.

Our numerical experiments reveal that when the lifetime of products is short, the unit perishing cost is large, and the service level is high, the inventory manager needs to place orders with a higher frequency, which results in multiple outstanding orders over the time in the system. Examples of products with short lifetimes are fish, seafood, and platelets, which are highly perishable products whose quality deteriorates very quickly (Ashie *et al.*, 1996). Fresh fruits and vegetables are also highly perishable products with relatively short lifetimes. They are subject to continuous change after harvesting (Yahia, 2019). In the blood and pharmaceutical industries, the inventory system requires a very high service level, and in case of shortages, the system incurs huge shortage costs. Numerical results indicate that allowing for more than one outstanding order in the system can result in up to 43.26% (on average 13.14%) cost savings. To capture the impact of incorporating information related to ages of items in the control policy (i.e., exercising the (Q, r, T) policy), we address two special two-parameter policies. One is the (Q, T) policy, in which orders are placed by tracking time, and the other one is the classic (Q, r) policy, which is solely based on the inventory position. Our results demonstrate that the (Q, r, T) policy always dominates its special policies, and this dominance becomes significant when the lifetime of products is short and the unit perishing cost is high. We observe that the percentage cost deviations between the (Q, r, T) policy and (Q, T) and (Q, r) policies can get as large as 49.26% and 15.68% (on average 10.33% and 4.02%), respectively. Moreover, the (Q, T) policy outperforms the (Q, r)

policy only when the system requires a very high service level and the unit perishing cost is small. Hence, depending on the input parameters of the system, one can decide on which policy between two-parameter policies should be implemented.

The remainder of this article is organized as follows. We briefly review the related literature in Section 2. In Section 3, we provide the basic assumptions of our model and describe the characteristics of the age-based control policy. In this section the effective lifetime distribution of items is also derived. We provide the exact operating characteristic and the expected cost rate expressions in Section 4. The sub-optimal policies elicited from our proposed policy are analyzed in Section 5. We propose our solution methodology for finding the optimal policy parameters in Section 6. In Section 7, we present our numerical study and, finally, we conclude in Section 8 with directions for future research.

2. Literature review

In stochastic perishable inventories with periodic review, when items can be held in stock no more than a period, the replenishment decisions in successive periods are independent and the problem reduces to a sequence of simple news-vendor-type models (Arrow *et al.*, 1958). Van Zyl (1964), Nahmias and Pierskalla (1973), Fries (1975), and Nahmias (1975a) consider a more complicated situation, wherein they analyze the structure of the optimal policy for perishable inventories with the two-period and m -period lifetimes under finite horizon and periodic review settings. Due to the complex structure of the optimal policy for perishable inventories with fixed lifetimes in the presence of non-negligible lead times, researchers have focused on developing heuristic policies; see, for example, Nahmias (1975b), Cohen (1976), Nahmias (1977, 1978), Nahmias and Wang (1979), Nandakumar and Morton (1993), and Williams and Patuwo (2004). Nahmias (1982) and Bakker *et al.* (2012) provide extensive reviews of fixed-lifetime models under a periodic review setting.

Schmidt and Nahmias (1985) analyze a continuous review $(S - 1, S)$ policy for perishable inventory models with fixed lead time. In their proposed policy, known as a lot-for-lot policy, an order is placed immediately whenever a depletion either by demand or perishing occurs. Olsson and Tydesjö (2010) consider the same policy with backorders for perishable inventories. They show that finding the exact optimal solution is not easy when backorders are allowed. Kalpakam and Sapna (1995) also survey the continuous review $(S - 1, S)$ policy, with exponential lifetime and non-Markovian lead times; and Liu and Cheung (1997) consider the same policy with exponential lead time. Kalpakam and Shanthi (2000) analyze a modified continuous review $(S - 1, S)$ policy in which an order is placed when the inventory level depletes by one unit due to the demand, but not perishability. In another research work, Kalpakam and Shanthi (2001) work on the same policy with a general distribution for the lead time.

Weiss (1980) introduces a continuous review (S, r) policy with Poisson demand in an infinite planning horizon, zero lead time, and fixed lifetime setting. Extensions of this policy to different aspects are as follows. Kalpakam and Sapna (1994) and Liu and Yang (1999) extend this policy to the case with both exponential lifetime and lead time. Liu and Shi (1999) address this model with a general renewal demand process. Further, Lian and Liu (1999) extend this policy to a model with perishable inventories, geometric inter-demand times, and batch demands. Liu and Lian (1999) analyze this policy to provide a closed-form solution for the steady-state probability distribution of the stock level by using a Markov renewal approach. Lian and Liu (2001) consider this policy and add a positive lead time to the model and propose a heuristic algorithm. Kalpakam and Shanthi (2006) introduce a new perishing process called discrete point perishability for perishable inventories operating under this control policy. This control policy is extended by Gürler and Özkaya (2008) to a perishable inventory system with a general lifetime distribution. Lian *et al.* (2009) analyze this policy based on the Markovian renewal demand process with a general distribution for inter-demand time. Barron and Baron (2020) analyze a continuous review (S, s) policy in perishable inventory systems with random lead times and times to perishability, and a state-dependent Poisson demand. Under this policy, when the inventory level hits the reorder point s , an order is placed to bring the inventory up to level S . They use the Queueing and Markov Chain Decomposition methodology to derive the stationary distributions of the inventory level, and show that variability of the lead time is more costly than that of perishability time. Barron (2019) revises the work of Barron and Baron (2020) by allowing demand uncertainty, random batch demands, and random perishability.

Chiu (1995) approximates the expected total cost and operating characteristics of a perishable inventory system under a continuous review (Q, r) policy. This policy works as follows. An order of size Q is placed whenever the inventory position drops to r . He shows that this policy does not provide significant improvement compared with the continuous review (S, r) policy. Berk and Gürler (2008) deal with the perishable inventory problem under a continuous review (Q, r) policy with positive lead time and fixed lifetime by introducing a new concept pertaining to the lifetime distribution modeled by the embedded Markov process approach. Kouki *et al.* (2015) consider the continuous review (Q, r) policy when analyzing a model with continuous demand distribution, constant lifetime, and constant lead time. Poormoaid and Atan (2019) analyze this policy under the multi-attribute utility approach for perishable inventories.

The age of items was not considered in policies until Tekin *et al.* (2001) introduced an age-based control policy for perishable inventories, namely a continuous review (Q, r, T) policy, wherein they analyze their proposed policy under at most one order outstanding restriction ($r < Q$), and with a special aging process in which the lifetime of a new order

starts when it is unpacked (referred to as the aging process type 1). Lowalekar and Ravichandran (2017) propose a similar policy, in which the lifetime of items is taken into account when placing orders. In their proposed policy, an order of size Q is placed whenever the inventory level hits r or T units of time have elapsed since the time we receive the batch, whichever occurs first. One of the shortcomings in this policy is that it is not able to analyze inventory systems with more than one outstanding order. They applied a simulation-based optimization to analyze their model. Abouee-Mehrzi *et al.* (2019) present a single-item multi-period stochastic inventory control problem where a firm faces multiple priority classes that require products of different ages. They characterize the structure of the optimal ordering, allocation, and disposal policies, and show that the optimal order quantity is more sensitive to the inventory level of fresher products, as well as the optimal allocation policy is a sequential rationing policy. Interested readers are referred to Karaesmen *et al.* (2011), Janssen *et al.* (2016), Chaudhary *et al.* (2018), and Duong *et al.* (2018) for extensive literature reviews of periodic and continuous review inventory models for perishable inventories.

Our analysis of the continuous review (Q, r, T) policy is different from the one in Tekin *et al.* (2001). We assume that the aging of the products in a batch starts once it is issued by the supplier (referred to as the aging process type 2), whereas in the model of Tekin *et al.* (2001), the lifetime of a new order starts when it is unpacked (i.e., aging process type 1). The system analysis demonstrates that the structure of the aging process influences the dynamics of the system, in the sense that distinct state variables are required to analyze our model. Analogous to Tekin *et al.* (2001), we utilize the renewal reward theorem to derive the operating characteristics of the system. As in our model the aging of items in a batch begins as it is ordered, the lifetime of items at the moment that the batch is unpacked is a random variable (referred to as the *effective lifetime*). Due to the nature of the aging process in our model, we trace both the inventory position and the effective lifetime of items to construct the sequence of regenerative points, which is different from the model of Tekin *et al.* (2001), wherein one can define regenerative points by solely taking into account the inventory level. Due to perishability, the system may observe multiple outstanding orders over time, in order to satisfy the target service level. In this respect, we generalize the model of Tekin *et al.* (2001) by relaxing the restriction on the number of outstanding orders. This also makes our analysis more challenging, due to the multi-dimensional nature of the effective lifetime process. Again, we need to rely on the inventory position instead of the inventory level in order to have tractable analytical results. We show that allowing for multiple outstanding orders in the system influences the effective lifetime of the current batch, which can reap substantial benefits.

Berk and Gürler (2008) is also another study similar to ours, in which the aging process of type 2 in the classic (Q, r) policy is taken into consideration. They use the concept of the embedded Markov process to analyze the system and

elicit the effective lifetime distribution of the current batch under one order outstanding restriction; and extend their study to a multiple outstanding orders setting without the service level constraint in Berk *et al.* (2020). The control policies studied in Berk and Gürlér (2008) and Berk *et al.* (2020) are solely based on the inventory level/inventory position, whereas we incorporate the effective lifetime of the current batch into the policy definition and propose the (Q, r, T) policy. Our analysis reveals that the effective lifetime of the current batch plays an important role in characterizing the system behavior. In the (Q, r, T) policy, the effective lifetime of the current batch appears to be completely different from the one in the (Q, r) policy. This fact can affect the system dynamics over time and result in substantial cost savings. To the best of our knowledge, no study deals with age-based control policy of the (Q, r, T) with aging process type 2 and no restriction on the number of outstanding orders.

3. Model and analysis

In subsection 3.1, we address characteristics of the inventory system under consideration and propose the age-based control policy with multiple outstanding orders at any time. We derive the effective lifetime distribution of items in subsection 3.2.

3.1. System description

We consider a single-item single-location inventory system with perishable products facing unit external demands generated by a Poisson process with mean λ in an infinite time horizon setting. The inventory system receives orders (batches) from an outside supplier, and then items in the batch are picked from the stock based on the First-In-First-Out (FIFO) policy by end-customers. Products have a fixed and finite lifetime τ upon arrival at the inventory system, beyond which they are no longer usable. We assume that lifetimes of all items in the batch are identical. We use the term *perishing* when all the remaining items in a batch perish before complete consumption. Procurement lead time $L > 0$ is assumed to be positive and fixed. The inventory system incurs a fixed replenishment cost of K per replenishment order, an inventory holding cost of h per unit per unit time, and the perishing cost p charged per unit that perishes in stock. Demands are immediately satisfied if the stock level is strictly positive, otherwise they are lost. There is no direct cost for lost sales. Instead, a service level that requires a fraction of unmet demand not to exceed a threshold value α , is imposed on the inventory system. Thus, the expected fixed ordering cost, expected on-hand inventory, and expected number of perishing items are the operating characteristics of the inventory system under consideration.

At a specific time epoch, the batch which is in stock and under consumption is called the *current batch* (or the *oldest batch*), and the batch which is already placed, but has not yet joined the inventory is called the *outstanding batch*. It is assumed that the inventory system allows for an unlimited number of outstanding orders at any time. Among multiple

outstanding orders, we will refer to the most recent one as the *youngest batch*. We use the term *depletion* when all items in a batch are consumed by demands without perishing. We employ the continuous review (Q, r, T) policy which works as follows. *An order of size Q is placed whenever the inventory position hits/crosses r by demand or perishing, or T units of time elapse since the time that the last batch under consumption was depleted or perished, whichever occurs first.* The goal, in our study, is to minimize the expected total cost rate under the service level constraint, where Q and r are two non-negative integer decision variables and T is the only continuous decision variable.

When a batch is under consumption in the system (i.e., the current batch in the system), a new order among outstanding orders may join inventories before complete consumption of the current batch. Based on the FIFO issuing policy, the consumption of the new batch begins when the current batch is completely depleted by demand or perishes. Since demand is stochastic, the time between the arrival time of the new batch and complete consumption of the current batch is random. Therefore, when a new batch is unpacked, all items in the batch have an identical random lifetime (effective lifetime). Hence, under aging process type 2, we need to track both the arrival time of the new batch and the time it becomes unpacked in order to measure its effective lifetime. That is the reason why we record the lifetime of a batch, as it is issued from the supplier. However, under aging process type 1, we do not need to trace the arrival time of the new batch to inventories, since whenever it is unpacked its aging process starts. In what follows, we define the state variables, the regenerative points and address different realizations of the system that works under the (Q, r, T) policy with aging process type 2.

We use $IP(t)$, $IL(t)$, and $O(t)$ to denote the inventory position, inventory level, and number of outstanding orders at time $t \geq 0$, respectively, where $IP(t) = IL(t) + O(t)$ in inventory systems with lost sales. In the inventory system under consideration, the number of outstanding orders, the inventory level (or the inventory position), the age of items, and the remaining delivery time of outstanding orders are the variables changing over time and can be used to define the state of the system. Since analyzing and deriving the operating characteristics of the system are easily obtained by taking the inventory position and the age of items into account, we define the *state of the system* at time t based on the inventory position and the remaining lifetime of items at time t . Our analysis regarding the age of batches in the (Q, r, T) policy reveals that in order to characterize the system behavior with lost sales, we need to track the age of m batches over the time simultaneously, where $m = \lceil r/Q \rceil$, which represents the smallest integer strictly greater than r/Q .

According to the state definition of the system, regenerative points are instances at which the inventory position hits mQ and the remaining lifetimes of items have a specific value. At these instances, the inventory system renews itself. Therefore, a *cycle* is defined as the time interval between two consecutive regenerative points. Our analysis demonstrates that tracking the operating characteristics of the

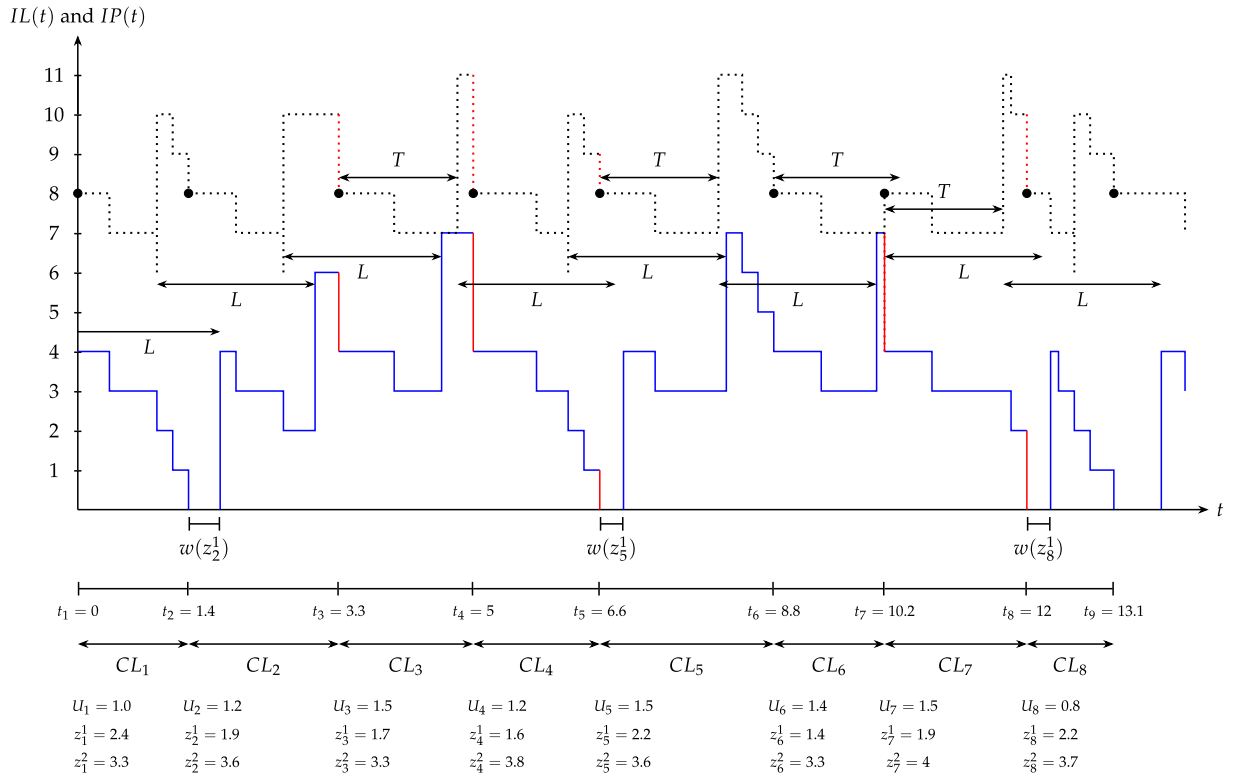


Figure 1. A sample path of the inventory position and the inventory level for $Q=4$, $r=6$ ($m=2$), $T=1.5$, $L=2$, and $\tau=2$ in the (Q, r, T) policy.

systems and deriving the cost function by considering the cycle definition is not an easy task. Hence, we base our analysis on *embedded cycles*. An embedded cycle is defined as the time interval between two consecutive instances (embedded regenerative points) at which the inventory position hits mQ . Thus, a cycle includes a number of embedded cycles. According to the embedded cycle definition, the remaining lifetimes of items at embedded regenerative points yield a vector of random variables.

For illustration purposes, we depict a sample path of the inventory position and the inventory level for the (Q, r, T) policy in Figure 1, where $Q=4$, $r=6$, and $T=1.5$. The blue, black, and red lines represent the inventory level, inventory position, and perishing occurrence, respectively (see the online version). Without loss of generality, we assume that the generated sample path commences with one outstanding order, and the inventory level at time 0 is $Q=4$ (i.e., $IP(t_1) = mQ = 8$); and also assume that the oldest and youngest batch effective lifetimes are 2.4 and 3.3, respectively. The consecutive instances of embedded cycle beginnings (embedded regenerative points) are shown by bold circles (●), at which the inventory position is $mQ=8$. Let CL_n denote the n th embedded cycle length, which is the time interval between two consecutive embedded regenerative points, and let U_n denote a period of time measured from the beginning of embedded cycle n to the instance a new order is placed (reorder point).

At time epoch t_1 (beginning of embedded cycle 1), we have a batch that is under consumption; this batch is the oldest batch and the outstanding order is the youngest

batch. In embedded cycle 1, the inventory position drops from $mQ=8$ to $r=6$ before T units of time elapse since the beginning of the embedded cycle. The remaining $r - (m - 1)Q = 2$ units are depleted by demand before they perish, and embedded cycle 1 ends. Then, embedded cycle 2 begins with a stock-out period. The outstanding order during embedded cycle 1 and the order placed within embedded cycle 1 are the oldest and the youngest batches at the beginning of embedded cycle 2, respectively. A new order is placed in embedded cycle 2 when the inventory position hits r before T units of time elapse since the beginning of this embedded cycle, and some of the Q units (in this case, 2 units) in the current batch perish. The orders that are placed within embedded cycles 1 and 2 are the oldest and the youngest batches at the beginning of embedded cycle 3, respectively. In embedded cycle 3, a new order is issued when T units of time elapse before the inventory position drops to r . This embedded cycle ends when some of the Q units (in this case, 3 units) in the current batch perish. Then, embedded cycle 4 starts. The orders that are placed within embedded cycles 2 and 3 are the oldest and the youngest batches at the beginning of embedded cycle 4, respectively. The dynamics of the system in embedded cycle 4 are the same as those in embedded cycle 2, with the difference that in embedded cycle 4 only one item perishes. The orders that are placed within embedded cycles 3 and 4 are the oldest and the youngest batches at the beginning of embedded cycle 5, respectively. Embedded cycle 5 begins with a stock-out period. In this embedded cycle, a new order is issued when T units of time elapse before the inventory

position drops to r , and the embedded cycle ends when all Q units are depleted by demand before they perish. Then, embedded cycle 6 starts. The orders that are placed within embedded cycles 4 and 5 are the oldest and the youngest batches at the beginning of embedded cycle 6, respectively. In embedded cycle 6, some of the Q items perish before T units of time, which drops the inventory position below r (i.e., the inventory position crosses r). In this case, when a new order is placed the inventory position immediately hits mQ and embedded cycle 6 ends. The orders that are placed within embedded cycles 5 and 6 are the oldest and the youngest batches at the beginning of embedded cycle 7, respectively. The dynamics of the system in embedded cycle 7 are the same as those in embedded cycle 3, with the difference that in embedded cycle 7 two items perish. Embedded cycle 8 begins with a stock-out period. The dynamics of the system in embedded cycle 8 are the same as those in embedded cycle 1. The process continues in this fashion. We will later refer to Figure 1 for more illustration purposes.

We can generalize our observations from the sample path above as follows:

1. At the beginning of embedded cycle n , we need to trace the effective lifetime of the last $m = \lceil \frac{r}{Q} \rceil = \lceil \frac{6}{4} \rceil = 2$ batches ordered in the past.
2. A new order is placed whenever the inventory position crosses $r=6$ by demand or perishing, or $T=1.5$ units of time elapse since the beginning of an embedded cycle, whichever occurs first.
3. An embedded cycle ends when the current batch with size Q is depleted by demand or perishes. For example, embedded cycles 1, 5, and 8 end by depletion, where all Q units are depleted by demand; and embedded cycles 2, 3, 4, 6, and 7 end by perishing, where some of the Q items perish.
4. One can see that the inventory position lies on interval $[r+1, (m+1)Q] = [7, 12]$.
5. During an embedded cycle, we may receive at most $m+1$ outstanding orders.
6. An embedded cycle begins with a stock-out period when the oldest batch at the beginning of the embedded cycle is still an outstanding batch, and the length of the stock-out period is equal to the remaining lead time at the beginning of the embedded cycle.

We define the *effective lifetime* of a batch at the beginning of an embedded cycle as $\tau + L$ minus the time that elapsed since the order time of that particular batch. According to the embedded cycle definition, the effective lifetime of a batch at the beginning of an embedded cycle is the remaining lifetime of the batch if that batch is already in stock and is the remaining lead time plus τ if that batch is still outstanding. Based on the embedded cycle and effective lifetime definitions, one can observe that at the beginning of any embedded cycle the inventory position is mQ , but the remaining lifetimes of the items are random variables. Hence, the system under consideration can be fully characterized by an m -dimensional array of the effective lifetimes

of m batches at the beginning of the embedded cycle. Effective lifetime of batches over embedded cycle beginnings yields a sequence of random variables referred to as the sequence of *effective lifetime vector*. Let Z_n^i denote the effective lifetime of batch i , $i = 1, \dots, m$ at the beginning of embedded cycle n , where $m \geq 1$. Then, $\{Z_1 = (Z_1^1, Z_1^2, \dots, Z_1^m), Z_2 = (Z_2^1, Z_2^2, \dots, Z_2^m), Z_3 = (Z_3^1, Z_3^2, \dots, Z_3^m), \dots\}$ yield a sequence of vectors of random variables; where Z_n^1 and Z_n^m are the oldest and the youngest batches at the beginning of the embedded cycle n , respectively. The oldest batch is the one that is currently under consumption during an embedded cycle (referred to as the current batch). In the next subsection, we derive the effective lifetime distribution of items at the beginning of embedded cycles.

To illustrate the cycle definition, suppose that in Figure 1 the remaining lifetime of items at the beginning of embedded cycle 1 is $Z_1 = (Z_1^1, Z_1^2) = (1.2, 2.3)$, and suppose that for the first time we observe that the remaining lifetime of items at the beginning of embedded cycle 3 is $Z_3 = (Z_3^1, Z_3^2) = (1.2, 2.3)$, and we observe for the second time that the remaining lifetime of items at the beginning of embedded cycle 7 is $Z_7 = (Z_7^1, Z_7^2) = (1.2, 2.3)$. Then, one observes the first cycle with length $\sum_{n=1}^2 CL_n$ including two embedded cycles 1 and 2 and the second cycle with length $\sum_{n=3}^6 CL_n$ including four embedded cycles 3, 4, 5, and 6.

3.2. Effective lifetime distribution

Let $\{t_n, n \geq 1\}$ be the sequence of time epochs at which the inventory position hits mQ for the n th time starting with $t_1 = 0$. Assuming that the inventory system starts with m batches in the system at time 0, $IP(t_n) = mQ$ for all $n \geq 1$. The time interval between t_{n+1} and t_n represents the n th embedded cycle length for $n \geq 1$. One can see the time epochs t_n for $n \geq 1$ in Figure 1.

For each $i \geq 1$, let X_i be the random variable denoting the arrival time of the i th demand since a pre-specified time origin which is taken as the beginning of an embedded cycle (after the completion of a possible stock-out period) unless stated otherwise. Since demand is generated by a Poisson process with rate λ , the arrival time for i th demand which is measured from the beginning of an embedded cycle has an Erlang distribution with parameters λ and i . For an Erlang i variable, its probability density function (p.d.f) is denoted by $f_i(\cdot)$, cumulative distribution function (c.d.f) by $F_i(\cdot)$, and complementary c.d.f by $\bar{F}_i(\cdot)$. Furthermore, let $N(t)$ be the counting process of the arrivals in $(0, t]$. Then, $\{t_n, n \geq 1\}$ is a sequence of stopping times for $(N(t))_{t \geq 0}$.

Let $\{Z_n, n \geq 1\}$ be the sequence of effective lifetimes of m batches in the system at time t_n , where $Z_n = \{Z_n^1, Z_n^2, \dots, Z_n^m\}$; and $z_n = \{z_n^1, z_n^2, \dots, z_n^m\}$ denote a particular realization of Z_n , where $0 \leq z_n^1 \leq z_n^2 \leq \dots \leq z_n^m \leq \tau + L$. Our analysis states that when $z_n^1 > \tau$, the current batch is an outstanding batch at the beginning of the embedded cycle. That particular batch will join the inventory after $z_n^1 - \tau$ units of time and upon its arrival, it is a fresh batch. Hence, we have a *stock-out period* with a length of the remaining

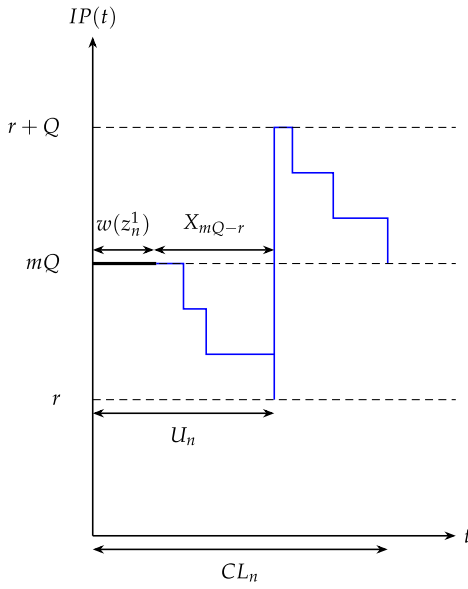


Figure 2. Event 1, $w(z_n^1) + X_{mQ-r} < T$ and $w(z_n^1) + X_Q < z_n^1$; $U_n = w(z_n^1) + X_{mQ-r}$ and $CL_n = w(z_n^1) + X_Q$; $Z_{n+1}^i = Z_n^{i+1} - [w(z_n^1) + X_Q]$, for $i = 1, \dots, m-1$, and $Z_{n+1}^m = \tau + L - [X_Q - X_{mQ-r}]$.

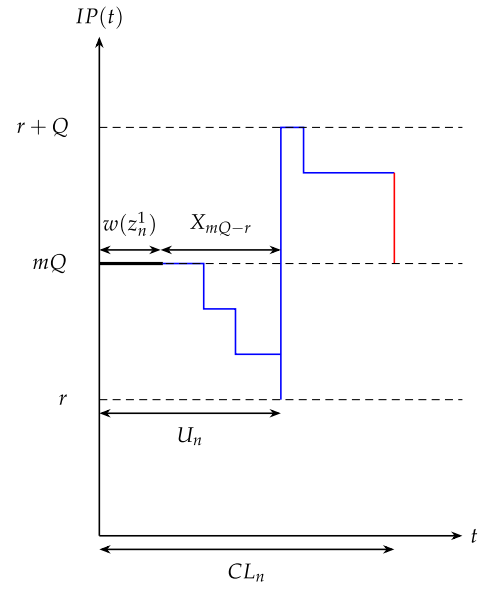


Figure 3. Event 2, $w(z_n^1) + X_{mQ-r} < T$ and $w(z_n^1) + X_Q > z_n^1$; $U_n = w(z_n^1) + X_{mQ-r}$ and $CL_n = z_n^1$; $Z_{n+1}^i = Z_n^{i+1} - z_n^1$, for $i = 1, \dots, m-1$, and $Z_{n+1}^m = \tau + L - [z_n^1 - (w(z_n^1) + X_{mQ-r})]$.

lead time (i.e., $z_n^1 - \tau$) at the beginning of the embedded cycle. Any demand arrival during the stock-out period is lost. For instance, in Figure 1, we have a stock-out period at the beginning of embedded cycles 2, 5, and 8. And when $z_n^1 \leq \tau$, it implies that the current batch is in the stock and under consumption at the beginning of the embedded cycle (i.e., it is not a fresh batch). In this case, we do not have any lost sales at the beginning of the embedded cycle. Hence, the system loses the demands that occur, if any, during the segment $w(Z_n^1)$, where $w(Z_n^1) = \max\{0, Z_n^1 - \tau\}$.

To derive the effective lifetime distribution of the batches, we need to show the Markovian property of $\{Z_n, n \geq 1\}$. To do so, we address different realizations of the system and the effective lifetime expression by which we can show the Markovian property. In Figures 2 to 6, we illustrate possible realizations of the stochastic process under the proposed policy. In these realizations, we assume that we are currently at the beginning of embedded cycle n and the current batch has the effective lifetime of z_n^1 as well as we assume that, without loss of generality, the embedded cycle n begins with a stock-out period $w(z_n^1)$. Thus, the n th embedded cycle length and the effective lifetime of the current batch at the beginning of embedded cycle $(n+1)$ are provided for each realization. We refer to these realizations as Event 1 through Event 5. Recall that U_n is a period of time measured from the beginning of embedded cycle n to the instance a new order is placed (reorder point), and X_{mQ-r} is the time period during which $mQ - r$ items are consumed by demands, which is measured (after the completion of a possible stock-out period) from the beginning of embedded cycle n . A new order with size Q is placed whenever the inventory position hits r or T units of time elapse since the beginning of the embedded cycle n , whichever occurs first. Following

different realizations, one can see that a new order is placed whenever (i) the inventory position hits r by depletion (Events 1 and 2), (ii) T units of time has elapsed since the beginning of the embedded cycle (Events 3 and 4), or (iii) the inventory position crosses r (drops below r) by perishing (Event 5).

In Event 1, as shown in Figure 2, $mQ - r$ demands have arrived before T units of time and the remaining $r - (m-1)Q$ units are also depleted by demand before perishing. This realization can be characterized by the events $w(z_n^1) + X_{mQ-r} < T$ and $w(z_n^1) + X_Q < z_n^1$. In this realization, $U_n = w(z_n^1) + X_{mQ-r}$ and $CL_n = w(z_n^1) + X_Q$. Event 2 (Figure 3) is analogous to Event 1 with a difference that some of Q items perish before complete depletion. This realization can be characterized by the events $w(z_n^1) + X_{mQ-r} < T$ and $w(z_n^1) + X_Q > z_n^1$. In this realization, $U_n = w(z_n^1) + X_{mQ-r}$ and $CL_n = z_n^1$. In Event 3, as shown in Figure 4, we hit T before $mQ - r$ demand arrivals, and the embedded cycle ends by complete depletion. This realization can be characterized by the events $w(z_n^1) + X_{mQ-r} > T$ and $w(z_n^1) + X_Q < z_n^1$. In this realization, $U_n = T$ and $CL_n = w(z_n^1) + X_Q$. Event 4 (Figure 5) is analogous to Event 3 with a difference that some of Q items perish before complete depletion. This realization can be characterized by the events $w(z_n^1) + X_{mQ-r} > T$ and $w(z_n^1) + X_Q > z_n^1$. In this realization, $U_n = T$ and $CL_n = z_n^1$. Finally, in Event 5 (Figure 6), before $mQ - r$ demand arrivals and before we hit T , some of the Q items perish and the inventory position drops below r . This realization can be characterized by the events $w(z_n^1) + X_{mQ-r} > z_n^1$ and $z_n^1 < T$. In this realization, $U_n = z_n^1$ and $CL_n = z_n^1$. It is worthwhile to note that in this event, $Z_{n+1}^m = \tau + L$. In the sample path depicted in Figure 1, embedded cycles 1 and 8 illustrate Event 1; embedded cycles 2 and 4 illustrate Event 2;

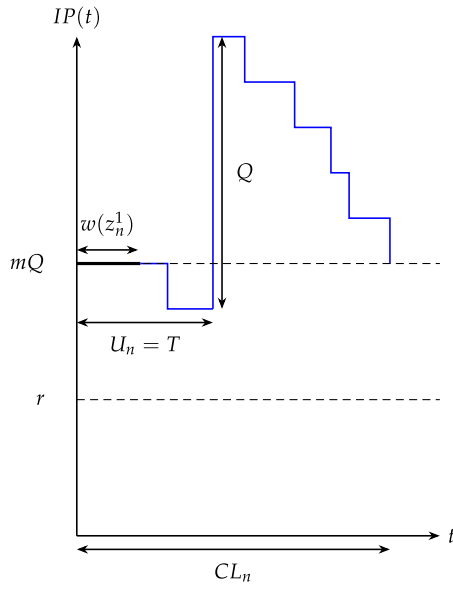


Figure 4. Event 3, $w(z_n^1) + X_{mQ-r} > T$ and $w(z_n^1) + X_Q < z_n^1$; $U_n = T$ and $CL_n = w(z_n^1) + X_Q$; $Z_{n+1}^i = Z_n^{i+1} - [w(z_n^1) + X_Q]$, for $i = 1, \dots, m-1$, and $Z_{n+1}^m = \tau + L - [w(z_n^1) + X_Q - T]$.

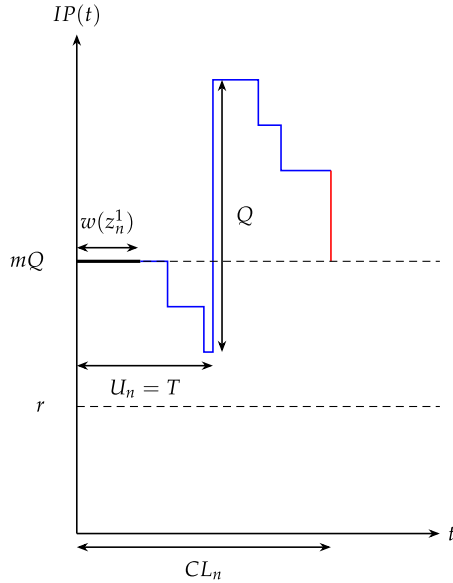


Figure 5. Event 4, $w(z_n^1) + X_{mQ-r} > T$ and $w(z_n^1) + X_Q > z_n^1$; $U_n = T$ and $CL_n = z_n^1$; $Z_{n+1}^i = Z_n^{i+1} - z_n^1$, for $i = 1, \dots, m-1$, and $Z_{n+1}^m = \tau + L - (z_n^1 - T)$.

embedded cycle 5 illustrates Event 3; embedded cycles 3 and 7 illustrate Event 4; and embedded cycle 6 illustrates Event 5.

Following different realizations described above, one can easily verify that:

$$U_n = \min\{Z_n^1, w(z_n^1) + X_{mQ-r}, T\}, \quad (1)$$

and

$$CL_n = w(z_n^1) + \min\{Z_n^1, \tau, X_Q\}. \quad (2)$$

Furthermore, one can find the effective lifetime of items at the beginning of the embedded cycle $n+1$ by

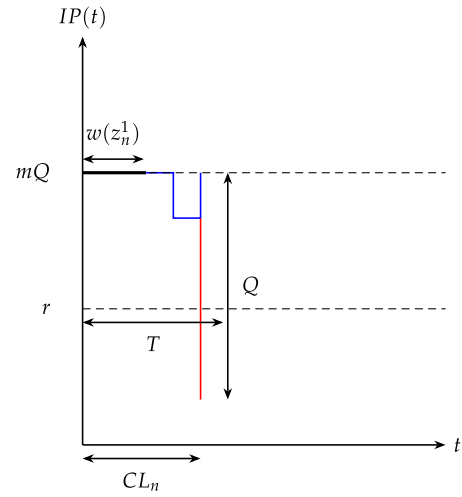


Figure 6. Event 5, $w(z_n^1) + X_{mQ-r} > z_n^1$ and $z_n^1 < T$; $U_n = z_n^1$ and $CL_n = z_n^1$; $Z_{n+1}^i = Z_n^{i+1} - z_n^1$, for $i = 1, \dots, m-1$, and $Z_{n+1}^m = \tau + L$.

$$\begin{aligned} Z_{n+1}^i &= Z_n^{i+1} - CL_n, \text{ for } i = 1, 2, \dots, m-1, \\ Z_{n+1}^m &= \tau + L - (CL_n - U_n). \end{aligned} \quad (3)$$

Equation (3) indicates that the youngest effective lifetime at the beginning of the embedded cycle $n+1$, Z_{n+1}^m , is $\tau + L$ minus the time period that begins from the reorder point till the end of the embedded cycle n , i.e., $CL_n - U_n$. Moreover, the effective lifetime of batch i , for $i = 1, 2, \dots, m-1$, at the beginning of the embedded cycle $n+1$, Z_{n+1}^i , is the effective lifetime of batch $i+1$ at the beginning of the embedded cycle n minus the n th embedded cycle length. One can see z_n^1 and z_n^2 , $n = 1, \dots, 8$, for the sample path depicted in Figure 1. According to (3), we can conclude that the effective lifetime vector \mathbf{Z}_{n+1} at the beginning of the $(n+1)$ th embedded cycle is completely determined by \mathbf{Z}_n and the Poisson demand arrival process after the stopping time t_n . Therefore, the embedded process $\{\mathbf{Z}_n, n \geq 1\}$ has the Markovian property. Moreover, we can easily verify that $0 \leq Z_n^i \leq \tau + L$, for $i = 1, 2, \dots, m-1$, and $L \leq Z_n^m \leq \tau + L$.

Next, we show that the union of Events 1 to 5 contains the entire space. To do so, we consider two cases: (i) When $T \leq z_n^1$, the following events may occur:

$$\begin{aligned} E_{1a} &:= \{X_{mQ-r} \leq T; X_Q \leq z_n^1\}, \\ E_{2a} &:= \{X_{mQ-r} \leq T; X_Q > z_n^1\}, \\ E_3 &:= \{T < X_{mQ-r} < z_n^1; X_Q \leq z_n^1\}, \\ E_{4a} &:= \{T < X_{mQ-r} < z_n^1; X_Q > z_n^1\}, \\ E_{4b} &:= \{X_{mQ-r} > z_n^1\}, \end{aligned} \quad (4)$$

where events E_{1a} and E_{2a} correspond to Event 1 and Event 2, respectively; event E_3 describes Event 3, and both events E_{4a} and E_{4b} represent Event 4. Hence,

$$\begin{aligned}
\mathbb{P}\{\Omega\} &= \mathbb{P}\{\cup E_i\} = \int_0^T \left[\int_0^{z_n^1-t} f_{r-(m-1)Q}(u) du \right. \\
&\quad \left. + \int_{z_n^1-t}^\infty f_{r-(m-1)Q}(u) du \right] f_{mQ-r}(t) dt \\
&\quad + \int_T^{z_n^1} \left[\int_0^{z_n^1-t} f_{r-(m-1)Q}(u) du \right. \\
&\quad \left. + \int_{z_n^1-t}^\infty f_{r-(m-1)Q}(u) du \right] f_{mQ-r}(t) dt \\
&\quad + \int_{z_n^1}^\infty f_{mQ-r}(t) dt \\
&= \int_0^T f_{mQ-r}(t) dt + \int_T^{z_n^1} f_{mQ-r}(t) dt \\
&\quad + \int_{z_n^1}^\infty f_{mQ-r}(t) dt = 1.
\end{aligned} \tag{5}$$

(ii) When $T > z_n^1$, the following events may occur:

$$\begin{aligned}
E_{1b} &:= \{X_{mQ-r} \leq z_n^1; X_Q \leq z_n^1\}, \\
E_{2b} &:= \{X_{mQ-r} \leq z_n^1; X_Q > z_n^1\}, \\
E_5 &:= \{X_{mQ-r} > z_n^1\},
\end{aligned} \tag{6}$$

where events E_{1b} , E_{2b} , and E_5 describe Events 1, 2, and 5, respectively. Hence,

$$\begin{aligned}
\mathbb{P}\{\Omega\} &= \mathbb{P}\{\cup E_i\} = \int_0^{z_n^1} \left[\int_0^{z_n^1-t} f_{r-(m-1)Q}(u) du \right. \\
&\quad \left. + \int_{z_n^1-t}^\infty f_{r-(m-1)Q}(u) du \right] f_{mQ-r}(t) dt \\
&\quad + \int_{z_n^1}^\infty f_{mQ-r}(t) dt = \int_0^{z_n^1} f_{mQ-r}(t) dt \\
&\quad + \int_{z_n^1}^\infty f_{mQ-r}(t) dt = 1.
\end{aligned} \tag{7}$$

The state space of the system is $A = \{(x^1, x^2, \dots, x^m) : 0 \leq x^i \leq \tau + L, i = 1, \dots, m\}$. Let \mathcal{B}^m be the Borel σ -algebra generated by the subsets of A . Without loss of generality, we consider the sets $\mathcal{A} \in \mathcal{B}^m$ which are in the form $\mathcal{A} = (0, z^1] \times (0, z^2] \times \dots \times (0, z^m]$, where $0 \leq z^i \leq \tau + L, i = 1, \dots, m$. Let $\mathbf{x} = (x^1, x^2, \dots, x^m)$ and $\mathbf{z} = (z^1, z^2, \dots, z^m)$; and for the ease of exposition, we define the following notations: $\zeta^x = \max_{1 \leq i \leq m-1} \{x^{i+1} - z^i\}$ and $\zeta_y^x = \min\{x, y\}$; more specifically, $\zeta_\tau^{x^1} = \min\{x^1, \tau\}$ and $\zeta_T^{x^1} = \min\{x^1, T\}$. Let $\ell = \max\{\zeta^x + \zeta_\tau^{x^1} - x^1, \tau + L - z^m + \zeta_\tau^{x^1} + \zeta_T^{x^1} - x^1\}$ and $\mathbb{1}_{\{\cdot\}}$ be the indicator function. Then, considering the realizations of the system, and the Markovian property of the lifetime distribution, we have the following result which finds the transition probability function of effective lifetimes, i.e., $\mathbb{P}\{\mathcal{A}|\mathbf{x}\} \equiv \mathbb{P}\{\mathbf{Z}_{n+1} \leq \mathbf{z} | \mathbf{Z}_n = \mathbf{x}\} \equiv \mathbb{P}\{\mathbf{Z}_{n+1}^i \leq z^i, i = 1, \dots, m | \mathbf{x} = (x^1, x^2, \dots, x^m)\}$. That is, we find the effective lifetime distribution at the beginning of embedded cycle $n+1$, $\mathbf{Z}_{n+1} \leq \mathbf{z}$, given that the effective lifetimes at the beginning of embedded cycle n is $\mathbf{Z}_n = \mathbf{x}$.

Our analysis reveals that when $r = (m-1)Q$, the system behaves differently from the case where $(m-1)Q < r < mQ$ in the sense that some realizations of the system would not occur when $r = (m-1)Q$. For example, if $r = (m-1)Q$, then Events 1 and 2, depicted in Figures 2 and 3 respectively, would not occur. Therefore, we analyze the proposed policy when $r = (m-1)Q$, separately. We call this case as a special region. When $0 \leq r < Q$, then $m=1$ and the special region is $r=0$; when $Q \leq r < 2Q$, then $m=2$ and the special region is $r=Q$, and so on. In what follows, we present the transition probability function for the case where $(m-1)Q < r < mQ$ and postpone the derivation of that of the special region (when $r = (m-1)Q$) to the end of this section.

Theorem 1. (Transition probability function of \mathbf{Z}_n for $(m-1)Q < r < mQ$).

For $x^1 < \zeta^x$, $\mathbb{P}\{\mathcal{A}|\mathbf{x}\} = 0$ and for $x^1 \geq \zeta^x$ we have:

$$\begin{aligned}
\mathbb{P}\{\mathcal{A}|\mathbf{x}\} &= \mathbb{1}_{\{x^1 - \zeta_\tau^{x^1} \geq \tau + L - z^m\}} \\
&\quad \left[\int_{\zeta_\tau^{x^1} + \zeta_T^{x^1} - x^1}^{\zeta_\tau^{x^1}} \bar{F}_{r-(m-1)Q}(\ell - t) dF_{mQ-r}(t) + \bar{F}_{mQ-r}(\zeta_\tau^{x^1}) \right] \\
&\quad + \int_0^{\min\{\zeta_\tau^{x^1} - (\tau + L - z^m), \zeta_\tau^{x^1} + \zeta_T^{x^1} - x^1\}} \bar{F}_{r-(m-1)Q}(\max\{\zeta_\tau^{x^1} \\
&\quad + \zeta_\tau^{x^1} - x^1 - t, \tau + L - z^m\}) dF_{mQ-r}(t).
\end{aligned} \tag{8}$$

All proofs are provided in Appendix A.

Theorem 2. (Ergodicity). The process $\{\mathbf{Z}_n, n \geq 1\}$ is ergodic.

Theorem 2 ensures that the limiting distribution of the effective lifetime process exists. Hence, we let $\lim_{n \rightarrow \infty} G_{n+1} = \lim_{n \rightarrow \infty} G_n \equiv G$ and obtain the limiting distributions in terms of implicit integral equations as follows:

$$\begin{aligned}
G(\mathbf{z}) &= G(z^1, z^2, \dots, z^m) = \int_{x^m=0}^{\tau+L} \dots \int_{x^2=0}^{\tau+L} \int_{x^1=0}^{\tau+L} \\
&\quad \mathbb{P}\{\mathcal{A}|\mathbf{x} = (x^1, x^2, \dots, x^m)\} dG(x^1, x^2, \dots, x^m),
\end{aligned} \tag{9}$$

where $\mathbb{P}\{\mathcal{A}|\mathbf{x} = (x^1, x^2, \dots, x^m)\}$ is given by Equation (8). Our analysis shows that the limiting distribution of the effective lifetime is a continuous function on Z^i for $i = 1, 2, \dots, m-1$, and it is a mixture function on Z^m with only one mass point at $Z^m = \tau + L$. The mass point on Z^m is due to Event 5 (Figure 6), in which the youngest batch has the effective lifetime of $Z^m = \tau + L$. Since the limiting distribution of the effective lifetime is unknown, solving the integral equation in (9) is not an easy task. Thus, to find the limiting distribution of the effective lifetime, we discretize the support of the limiting distribution, $G(\mathbf{z})$, which is in both sides of Equation (9); however, it is a continuous distribution function. To this end, we suppose that the limiting distribution includes several mass points (i.e., (z^1, z^2, \dots, z^m) and (x^1, x^2, \dots, x^m) in Equation (9) are assumed to be mass points) and then for all combinations of mass points we construct a set of linear equations by (9). Given the set of linear equations and knowing the fact that the sum of limiting probabilities is equal to one, we can find the limiting

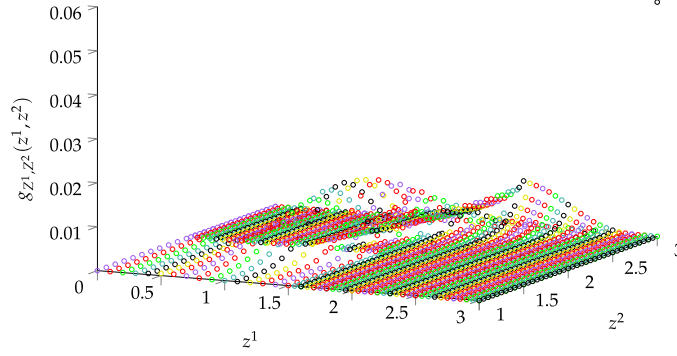


Figure 7. Joint distribution function of effective lifetime for $Q = 5$, $r = 7$, $T = 0.3$, $L = 1$, $\tau = 2$, and $\lambda = 3$.

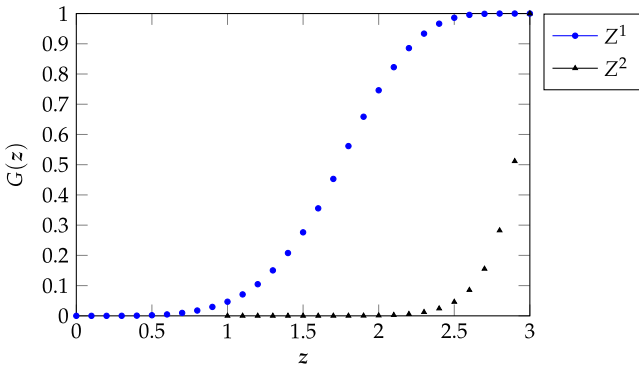


Figure 8. Effective lifetime distribution for $Q = 5$, $r = 7$, $T = 0.3$, $L = 1$, $\tau = 2$, and $\lambda = 3$.

probabilities of mass points. A greater number of mass points requires more time to solve the resultant set of equations, however, it results in more accurate computational results.

For the sake of clarity, we describe our discretization approach in more detail as follows and provide a numerical example in [Appendix B](#) for the illustration purpose. Suppose that vector Z^i is discretized to S^i mass points for $i = 1, 2, \dots, m$. Let k^1, k^2, \dots, k^m represent the mass point indices over vectors Z^1, Z^2, \dots, Z^m , respectively, with its corresponding mass point (x^1, x^2, \dots, x^m) . Thus, $k^i \in \{1, 2, \dots, S^i\}$ for $i = 1, 2, \dots, m$. Denote the corresponding probability of the mass point (x^1, x^2, \dots, x^m) by $\mathbb{P}_{k^1, k^2, \dots, k^m}$, which is a decision variable. Furthermore, suppose that e^1, e^2, \dots, e^m is the corresponding index of a given mass point (z^1, z^2, \dots, z^m) . Then, for any given mass point (z^1, z^2, \dots, z^m) in the discretized vector space, we construct the set of linear equations as follows:

$$\begin{aligned} & \sum_{k^1 \leq e^1} \sum_{k^2 \leq e^2} \dots \sum_{k^m \leq e^m} \mathbb{P}_{k^1, k^2, \dots, k^m} \\ &= \sum_{k^1=1}^{S^1} \sum_{k^2=1}^{S^2} \dots \sum_{k^m=1}^{S^m} \mathbb{P}\{\mathcal{A}|(x^1, x^2, \dots, x^m)\} \times \mathbb{P}_{k^1, k^2, \dots, k^m}, \end{aligned} \quad (10)$$

where $\mathbb{P}\{\mathcal{A}|(x^1, x^2, \dots, x^m)\}$ is computed by [Equation \(8\)](#) for

a given effective lifetime (x^1, x^2, \dots, x^m) . Moreover, we need to include a single equation as below to the set of linear equations in [\(10\)](#), that accounts for the fact that the sum of probabilities is one:

$$\sum_{k^1=1}^{S^1} \sum_{k^2=1}^{S^2} \dots \sum_{k^m=1}^{S^m} \mathbb{P}_{k^1, k^2, \dots, k^m} = 1. \quad (11)$$

The left-hand side of [Equation \(10\)](#) is equivalent to that of [Equation \(9\)](#), indicating the sum of probabilities of mass points which are less than or equal to a given mass point (z^1, z^2, \dots, z^m) . Similarly, the right-hand side of [Equation \(10\)](#) is equivalent to that of [Equation \(9\)](#), expressing the transition probabilities for all set of mass points in the discretized vector space.

[Figure 7](#) demonstrates the joint distribution function, $g_{Z^1, Z^2}(z^1, z^2)$, and [Figure 8](#) depicts its corresponding marginal effective lifetime distributions at the beginning of embedded cycles in the (Q, r, T) policy for a particular data set, where $m = 2$, and Z^1 and Z^2 are discretized by 30 and 20 mass points, respectively.

Next, we derive the transition probability function for the case with at most one outstanding order, $m = 1$. Let $\{Z_n, n \geq 1\}$ be the sequence of effective lifetimes of the current batch in the system at time t_n and z_n denote a particular realization of Z_n where $0 \leq z_n \leq \tau + L$. Considering different realizations for the case where $m = 1$, we can conclude that the $(n + 1)$ th effective lifetime vector can be expressed as

$$Z_{n+1} = \tau + L - (CL_n - U_n), \quad (12)$$

where $U_n = \min\{Z_n, w(Z_n) + X_{Q-r}, T\}$ and $CL_n = w(Z_n) + \min\{Z_n, \tau, X_Q\}$. The transition probability function for the case where $m = 1$ is found by

$$\begin{aligned} \mathbb{P}\{\mathcal{A}|x\} &= \mathbb{P}\{Z_{n+1} < z|Z_n = x\} \\ &= \mathbb{1}_{\{x - \zeta_T^x \geq \tau + L - z\}} \left[\int_{\zeta_T^x + \zeta_T^x - x}^{\zeta_T^x} \bar{F}_r(\ell - t) dF_{Q-r}(t) + \bar{F}_{Q-r}(\zeta_T^x) \right] \\ &\quad + \bar{F}_r(\tau + L - z) F_{Q-r}(\min\{\zeta_T^x - (\tau + L - z), \zeta_T^x + \zeta_T^x - x\}), \end{aligned} \quad (13)$$

where $\ell = \tau + L - z + \zeta_T^x + \zeta_T^x - x$. The proof is similar to [Theorem 1](#), so it is omitted. Note that one can find the transition probability function for the case where $m = 1$ by that

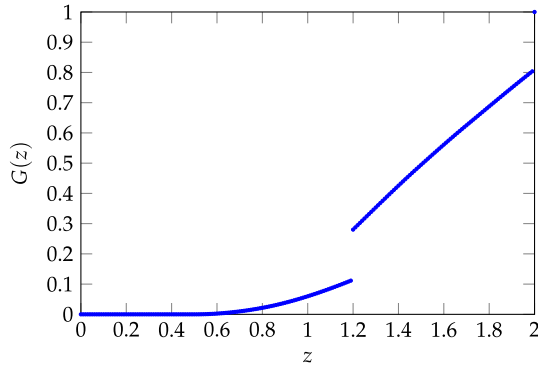


Figure 9. Effective lifetime distribution for $Q=9$, $r=7$, $T=1.2$, $L=0.5$, $\tau=1.5$, and $\lambda=1$.

of the case where $m > 1$ by setting $\zeta^x = 0$, $z^m = z$, $x^1 = x$, and $\ell = \tau + L - z + \zeta_\tau^x + \zeta_T^x - x$ in Equation (8), which reduces to Equation (13).

Our results show that in the case where $m=1$, we have two mass points at T and $\tau + L$. In Event 5, the effective lifetime of a new embedded cycle turns out to be $\tau + L$. On the other hand, if an embedded cycle begins with effective lifetime of $\tau + L$, a new batch is ordered by hitting T , and the embedded cycle ends by perishing (i.e., Event 4 occurs), then $Z_{n+1} = \tau + L - (CL_n - U_n) = Z_{n+1} = \tau + L - (\tau + L - T) = T$, implying that we have a mass point at T . Therefore, we have two mass points at T and $\tau + L$ in the case where $m=1$. Figure 9 depicts a sample of effective lifetime distribution for $Q=9$, $r=7$, $T=1.2$, $L=0.5$, $\tau=1.5$, and $\lambda=1$, where we have two mass points at $T=1.2$ and $\tau + L = 2$.

Now, we investigate the system characteristics by considering the special region of $r = (m-1)Q$, i.e., when r is an integer multiple of Q . In the following, we provide some main results for the special region $r = (m-1)Q$ analogous to the case where $(m-1)Q < r < mQ$. Given that the state spaces for the cases $r = (m-1)Q$ and $(m-1)Q < r < mQ$ are identical, we have the following result.

Theorem 3. (Transition probability function of Z_n for $r = (m-1)Q$).

For $x^1 < \zeta^x$, $\mathbb{P}\{A|x\} = 0$, and for $x^1 \geq \zeta^x$ we have:

$$\begin{aligned} \mathbb{P}\{A|x\} = & F_Q(\zeta_\tau^{x^1}) - F_Q(\max\{\zeta_\tau^x + \zeta_\tau^{x^1} - x^1, \tau + L - z^m \\ & + \zeta_\tau^{x^1} + \zeta_T^{x^1} - x^1\}) \\ & + \mathbb{1}_{\{x^1 - \zeta_\tau^{x^1} \geq \tau + L - z^m\}} \bar{F}_Q(\zeta_\tau^{x^1}) \\ & + \mathbb{1}_{\{z^m = \tau + L\}} \left[F_Q(\zeta_\tau^{x^1} + \zeta_T^{x^1} - x^1) - F_Q(\zeta_\tau^x + \zeta_\tau^{x^1} - x^1) \right]. \end{aligned} \quad (14)$$

Proof is similar to Theorem 1, so it is omitted. And, for the case where $m=1$ (i.e., the $(Q, 0, T)$ policy), we have

$$\begin{aligned} \mathbb{P}\{A|x\} = & F_Q(\zeta_\tau^x) - F_Q(\tau + L - z + \zeta_\tau^x + \zeta_T^x - x) \\ & + \mathbb{1}_{\{x - \zeta_\tau^x \geq \tau + L - z\}} \bar{F}_Q(\zeta_\tau^x) \\ & + \mathbb{1}_{\{z = \tau + L\}} F_Q(\zeta_\tau^x + \zeta_T^x - x). \end{aligned} \quad (15)$$

Similar to the case where $(m-1)Q < r < mQ$, we use the discretization approach for finding the limit distribution of the effective lifetime in case where $r = (m-1)Q$.

4. Operating characteristics and the objective function

Having the steady-state effective lifetime distribution, we next derive the operating characteristics of the system, including the expected embedded cycle length, number of lost sales, number of perishing items, as well as on-hand inventory for a given lifetime $Z = z = (z^1, z^2, \dots, z^m)$. Our analysis reveals that only the current batch lifetime affects the operating characteristics of the system. Therefore, in the following, we derive the operating characteristics for a given current batch lifetime $Z^1 = z^1$.

Considering the events depicted in Figures 2 to 6, we can write the conditional embedded cycle length for a given current batch lifetime $Z^1 = z^1$:

$$[CL|z^1] = \begin{cases} \min\{X_Q, z^1\} & \text{if } z^1 < \tau, \\ z^1 - \tau + \min\{X_Q, \tau\} & \text{if } z^1 \geq \tau. \end{cases} \quad (16)$$

After some modification, we can rewrite Equation (16) as follows:

$$[CL|z^1] = \begin{cases} X_Q + w(z^1) & \text{if } X_Q < z^1 - w(z^1), \\ z^1 & \text{if } X_Q \geq z^1 - w(z^1). \end{cases} \quad (17)$$

Therefore, the conditional expected embedded cycle length is obtained as follows:

$$\begin{aligned} \mathbb{E}[CL|z^1] = & \int_0^{z^1 - w(z^1)} (y + w(z^1)) dF_Q(y) + \int_{z^1 - w(z^1)}^\infty z^1 dF_Q(y) \\ = & \frac{Q}{\lambda} F_{Q+1}(z^1 - w(z^1)) + w(z^1) F_Q(z^1 - w(z^1)) \\ & + z^1 \bar{F}_Q(z^1 - w(z^1)). \end{aligned} \quad (18)$$

Note that $z^1 - w(z^1) = \min\{z^1, \tau\} = \zeta_\tau^{z^1}$. Thus, we can simplify Equation (18) as follows:

$$\mathbb{E}[CL|z^1] = \frac{Q}{\lambda} F_{Q+1}(\zeta_\tau^{z^1}) - \zeta_\tau^{z^1} F_Q(\zeta_\tau^{z^1}) + z^1. \quad (19)$$

We have a stock-out period at the beginning of the embedded cycle with length $z^1 - \tau$ if $z^1 > \tau$. Then, the conditional expected number of lost sales in an embedded cycle is obtained as follows:

$$\mathbb{E}[LS|z^1] = \lambda w(z^1). \quad (20)$$

The conditional number of perishing items can be calculated as:

$$[P|z^1] = \begin{cases} 0 & \text{if } X_Q < z^1 - w(z^1), \\ Q - N(z^1 - w(z^1)) & \text{if } X_Q \geq z^1 - w(z^1), \end{cases} \quad (21)$$

where $N(z^1 - w(z^1))$ represents the number of demand arrivals during $z^1 - w(z^1)$ units of time. Then, the conditional expected number of perishing items is obtained as follows:

$$\mathbb{E}[P|z^1] = \sum_{n=0}^{Q-1} (Q-n) e^{-\lambda \zeta_\tau^1} \frac{(\lambda \zeta_\tau^1)^n}{n!}. \quad (22)$$

Without loss of generality, we charge the holding cost as an item is withdrawn from stock either through demand occurrence or perishing. We can do this because Little's Law holds for the ergodic system at hand. To find the conditional expectation of the time over which inventory is held, we find the average time that the items are retained in stock before they are either consumed by demand or perish. To this end, we consider two cases:

1. If $z^1 > \tau$ and n demands have arrived at the system after the end of the stock-out period during an embedded cycle, where $n \leq Q$, the waiting time of the item consumed by the n th demand is X_n and $Q - n$ items will be left at the end of the embedded cycle and they will be retained τ units of time in the stock within the embedded cycle. Therefore, the expected waiting time of products is given by $(x_1 + x_2 + \dots + x_n) + (Q - n)\tau$.
2. If $z^1 < \tau$ and n demands arrive at the system during an embedded cycle, where $n \leq Q$, all of the n items wait for $\tau - z^1$ units of time before the embedded cycle starts; and item n waits X_n units of time during the embedded cycle till it is depleted by demand. Moreover, the waiting time of the $Q - n$ items is τ . Therefore, the expected waiting time of products is given by $n(\tau - z^1) + (x_1 + x_2 + \dots + x_n) + (Q - n)\tau = (x_1 + x_2 + \dots + x_n) + Q\tau - nz^1$.

Considering the two cases above and having the joint distribution of arrival times of Q demands, $f(x_1, x_2, \dots, x_Q) = \lambda^Q e^{-\lambda x_Q}$ where $0 < x_1 < x_2 < \dots < x_Q < \infty$ (Ross *et al.*, 1996), we can obtain the expected waiting time of products for a given effective lifetime vector $\mathbf{Z} = \mathbf{z} = (z^1, z^2, \dots, z^m)$ and given n as follows:

$$\begin{aligned} \mathbb{E}[OH_n|z^1] &= \int_{0 < x_1 < \dots < x_n < \zeta_\tau^1 < x_{n+1} < \dots < x_Q} \\ &\quad \left[(x_1 + x_2 + \dots + x_n) + (Q\tau - n\zeta_\tau^1) \right] \lambda^Q e^{-\lambda x_Q} dx_1 \dots dx_Q, \end{aligned} \quad (23)$$

and the expected on-hand inventory for a given z^1 is obtained as

$$\begin{aligned} \mathbb{E}[OH|z^1] &= \sum_{n=0}^Q \mathbb{E}[OH_n|z^1] \\ &= Q(\tau - \zeta_\tau^1) + \frac{1}{\lambda} \sum_{k=0}^{Q-1} \frac{(Q-k)}{k!} \gamma(k+1, \lambda \zeta_\tau^1), \end{aligned} \quad (24)$$

where

$$\gamma(k+1, \lambda \zeta_\tau^1) = \int_0^{\lambda \zeta_\tau^1} x^k e^{-x} dx. \quad (25)$$

The derivation of (24) is provided in Appendix C.

Finally, we can find the expected value of operating characteristics as

$$\begin{aligned} \mathbb{E}[CL] &= \int_{z^1} \mathbb{E}[CL|z^1] dG(z^1), \\ \mathbb{E}[LS] &= \int_{z^1} \mathbb{E}[LS|z^1] dG(z^1), \\ \mathbb{E}[P] &= \int_{z^1} \mathbb{E}[P|z^1] dG(z^1), \\ \mathbb{E}[OH] &= \int_{z^1} \mathbb{E}[OH|z^1] dG(z^1). \end{aligned} \quad (26)$$

The approach to construct the objective function is similar to Berk and Gürlér (2008), Berk *et al.* (2020) and is motivated by the results of Ross (1970) and Tijms and Tijms (1994). After them, we have the following argument. Let $C_i \equiv C_i(\mathbf{Z}_i, \mathbf{X})$ and $L_i \equiv L_i(\mathbf{Z}_i, \mathbf{X})$ be the cost and the length of the i th embedded cycle for $i \geq 1$, where \mathbf{X} denotes the array of inter-arrival times of Poisson demands within the i th embedded cycle, independent of $(\mathbf{Z}_1, \dots, \mathbf{Z}_i)$. Also, let $\mathcal{C}(\mathbf{z}) = \mathbb{E}[C_i(\mathbf{Z}_i, \mathbf{X})|\mathbf{Z}_i = \mathbf{z}]$ and $\mathcal{L}(\mathbf{z}) = \mathbb{E}[L_i(\mathbf{Z}_i, \mathbf{X})|\mathbf{Z}_i = \mathbf{z}]$ for $i \geq 1$. The expectations are independent of the index i when $\mathbf{Z}_i = \mathbf{z}$ is given and are calculated with respect to the inter-arrival times of Poisson demands \mathbf{X} . Then, we use the following objective function:

$$\phi = \frac{\int_{\mathbf{z}} \mathcal{C}(\mathbf{z}) dG(\mathbf{z})}{\int_{\mathbf{z}} \mathcal{L}(\mathbf{z}) dG(\mathbf{z})}. \quad (27)$$

That is, we use the ratio of the expected cost of an embedded cycle in the limit divided by the length of such an embedded cycle in the limit. We do not provide a rigorous proof here, but the use of this objective function is motivated by:

1. The results of Ross (1970) where it is shown that under a mild regularity condition which states that transitions in the embedded Markov process do not occur too quickly, it holds that:

$$\begin{aligned} \phi_1(\mathbf{x}) &= \lim_{t \rightarrow \infty} \mathbb{E} \left[\frac{C(t)}{t} | \mathbf{Z}_1 = \mathbf{x} \right] = \phi_2(\mathbf{x}) \\ &= \lim_{n \rightarrow \infty} \frac{\mathbb{E} \left[\sum_{i=1}^n C_i(\mathbf{Z}_i, \mathbf{X}) | \mathbf{Z}_1 = \mathbf{x} \right]}{\mathbb{E} \left[\sum_{i=1}^n L_i(\mathbf{Z}_i, \mathbf{X}) | \mathbf{Z}_1 = \mathbf{x} \right]}, \end{aligned} \quad (28)$$

where $C(t)$ is the cost incurred in interval $[0, t]$.

2. As we have shown $\{Z_n, n \geq 1\}$ converges in distribution, and we have $Z_n \rightarrow Z$ and $G_n \rightarrow G$. Since $\{Z_n, n \geq 1\}$ is a bounded sequence, we have $\mathbb{E}[Z_n] \rightarrow \mathbb{E}[Z]$ by bounded convergence theorem.
3. Finally, since both the cost and the length of the embedded cycles are continuous and bounded, we assume that $\mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n C_i(\mathbf{Z}_i, \mathbf{X}) | \mathbf{Z}_1 = \mathbf{z} \right] \rightarrow \mathbb{E}[C(\mathbf{Z}, \mathbf{X})] = \mathbb{E}[\mathbb{E}[C(\mathbf{Z}, \mathbf{X})|\mathbf{Z}]] = \mathbb{E}[\mathcal{C}(\mathbf{Z})] = \int_{\mathbf{z}} \mathcal{C}(\mathbf{z}) dG(\mathbf{z})$.

The objective function given in Equation (27) implicitly involves the policy parameters (Q, r, T) . Hence, we explicitly write the objective function of our policy as

$$\phi = \mathbb{E}[TC(Q, r, T)] = \frac{K + h\mathbb{E}[OH] + p\mathbb{E}[P]}{\mathbb{E}[CL]}, \quad (29)$$

and the optimization problem is

$$\begin{aligned} (P) : \min_{Q, r, T} \mathbb{E}[TC(Q, r, T)], \\ \text{subject to :} \\ \frac{\mathbb{E}[LS]}{\lambda \mathbb{E}[CL]} \leq \alpha, \end{aligned}$$

where $\mathbb{E}[LS]/\lambda \mathbb{E}[CL]$ represents the fraction of demands lost over a long time, which is denoted by $FL(Q, r, T)$. For the ease of exposition, we use $\mathbb{E}[TC(Q, r, T)]$ and $\mathbb{E}[TC]$ interchangeably.

Since in the case where $m=1$ we have only one batch on hand, the effective lifetime of the current batch $Z = z$ affects the operating characteristics. Therefore, z^1 and z are equivalent and the operating characteristics expressions are the same in both cases $m=1$ and $m>1$ as expressed in Equations (19), (20), (22), and (24). It should be noted that both operating characteristic expressions for a given $Z = z$ and the objective function structures for the cases $(m-1)Q < r < mQ$ and $r = (m-1)Q$ are identical.

5. Special policies

In this section, we address two special policies which are either a stock-based or an age-based policy. The (Q, r, T) policy has a special case, namely the (Q, r) policy, which works as follows: *an order of size Q is placed whenever the inventory position hits r* . The (Q, T) policy is another special case of the (Q, r, T) policy which works as follows: *an order of size Q is placed whenever T units of time elapse since the beginning of the embedded cycle*. For the general case $m \geq 1$, the (Q, r) and (Q, T) policies can be achieved by setting $T = \tau + L$ and $r = -\infty$ in the (Q, r, T) policy, respectively. The exact analysis of the (Q, r) policy with the effective lifetime consideration is studied by Berk *et al.* (2020). In what follows, we provide the exact analysis of the (Q, T) policy. As shown later in Section 6, the (Q, T) policy performs well for inventory systems that hold perishable items with a short lifetime (such as flowers, bread, fruits, milk products, vegetables, fish, and foodstuffs). This means that when the lifetime of products is short, tracking only the age of items for triggering a new order is sufficient. Furthermore, our analysis reveals that the computation time for calculating the operating characteristics under this policy is negligible, so that when a large number of outstanding orders is required, this policy is applicable.

In the (Q, T) policy, we only consider the case where $m=1$. There exist three realizations in this policy. In Event 1, after reordering, Q items are depleted. In Event 2, after reordering, a number of Q items perish. In Event 3, Q items are either depleted or perish before reordering. Considering the different events above, we can conclude that the $(n+1)$ th effective lifetime can be expressed as

$$Z_{n+1} = \tau + L - (CL_n - U_n), \quad (30)$$

where $U_n = T$ and $CL_n = \max\{w(Z_n) + \min\{Z_n, \tau, X_Q\}, T\}$; and the transition probability function is given by the following result.

Theorem 4. (Transition Probability Function of Z_n).

$$\begin{aligned} \mathbb{P}\{\mathcal{A}|x\} = \mathbb{1}_{\{z=\tau+L\}} F_Q(\zeta_\tau^x + \zeta_T^x - x) + \mathbb{1}_{\{x-\zeta_T^x \geq \tau+L-z\}} \bar{F}_Q(\zeta_\tau^x) \\ + F_Q(\zeta_\tau^x) - F_Q(\tau + L - z + \zeta_\tau^x + \zeta_T^x - x). \end{aligned} \quad (31)$$

The limiting distributions in terms of implicit integral equations is found by

$$G(z) = \int_0^{\tau+L} \mathbb{P}\{\mathcal{A}|x\} dG(x). \quad (32)$$

Similar to the (Q, r, T) policy, we use the discretization approach to find the limiting distributions. We observe that in the (Q, T) policy we have at most two mass points at T and $\tau + L$.

In the following, we provide the operating characteristics of the (Q, T) policy (see Appendix D for their derivations). Our analysis reveals that the operating characteristics for $T < L$ and $T \geq L$ are different. Then, we need to consider $\mathbb{E}[CL]$ and $\mathbb{E}[LS]$ in two cases:

(i) If $T < L$,

$$\begin{aligned} \mathbb{E}[CL] = \int_0^T T dG(z) + \int_T^\tau [TF_Q(T) \\ + \frac{Q}{\lambda} [F_{Q+1}(z) - F_{Q+1}(T)] + z\bar{F}_Q(z)] dG(z) \\ + \int_\tau^{\tau+T} [TF_Q(T - z + \tau) \\ + (z - \tau)[F_Q(\tau) - F_Q(T - z + \tau)] \\ + \frac{Q}{\lambda} [F_{Q+1}(\tau) - F_{Q+1}(T - z + \tau)] + z\bar{F}_Q(\tau)] dG(z) \\ + \int_{\tau+T}^{\tau+L} \left[z - \tau F_Q(\tau) + \frac{Q}{\lambda} F_{Q+1}(\tau) \right] dG(z). \end{aligned} \quad (33)$$

$$\begin{aligned} \mathbb{E}[LS] = \int_0^T [\lambda T - QF_{Q+1}(z) - \lambda z\bar{F}_Q(z)] dG(z) \\ + \int_T^\tau [\lambda TF_Q(T) - QF_{Q+1}(T)] dG(z) \\ + \int_\tau^{\tau+T} [\lambda TF_Q(T - z + \tau) - QF_{Q+1}(T - z + \tau) \\ + \lambda(z - \tau)\bar{F}_Q(T - z + \tau)] dG(z) \\ + \int_{\tau+T}^{\tau+L} \lambda(z - \tau) dG(z). \end{aligned} \quad (34)$$

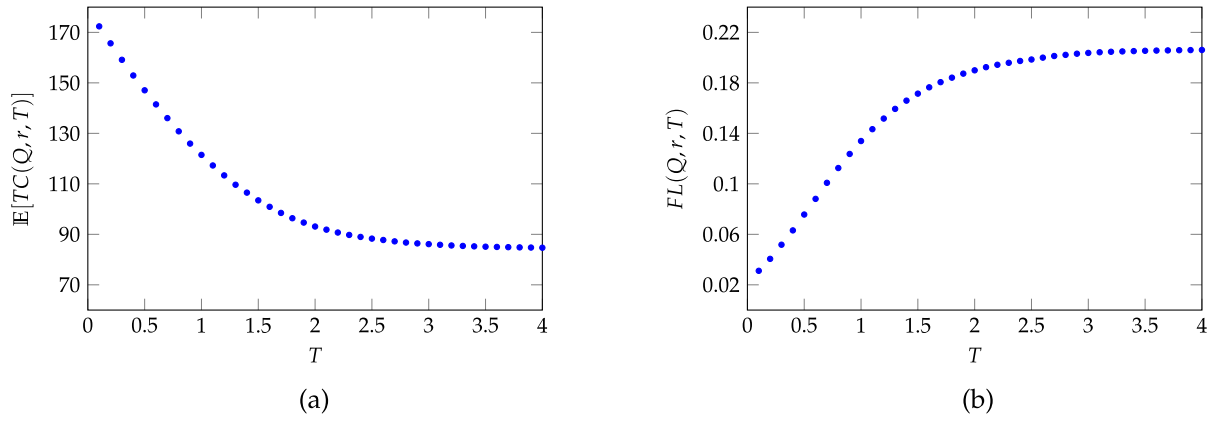


Figure 10. The behaviors of (a) $\mathbb{E}[TC(Q, r, T)]$ and (b) $FL(Q, r, T)$ w.r.t T .

(ii) If $T \geq L$,

$$\begin{aligned} \mathbb{E}[CL] = & \int_0^T T dG(z) + \int_T^\tau [TF_Q(T) \\ & + \frac{Q}{\lambda} [F_{Q+1}(z) - F_{Q+1}(T)] + z\bar{F}_Q(z)] dG(z) \\ & + \int_\tau^{\tau+L} [TF_Q(T - z + \tau) \\ & + (z - \tau)[F_Q(\tau) - F_Q(T - z + \tau)] \\ & + \frac{Q}{\lambda} [F_{Q+1}(\tau) - F_{Q+1}(T - z + \tau)] + z\bar{F}_Q(\tau)] dG(z). \end{aligned} \quad (35)$$

$$\begin{aligned} \mathbb{E}[LS] = & \int_0^T [\lambda T - QF_{Q+1}(z) - \lambda z\bar{F}_Q(z)] dG(z) \\ & + \int_T^\tau [\lambda TF_Q(T) - QF_{Q+1}(T)] dG(z) \\ & + \int_\tau^{\tau+L} [\lambda TF_Q(T - z + \tau) - QF_{Q+1}(T - z + \tau) \\ & + \lambda(z - \tau)\bar{F}_Q(T - z + \tau)] dG(z). \end{aligned} \quad (36)$$

Note that $\mathbb{E}[P]$ and $\mathbb{E}[OH]$ in the (Q, T) policy are the same as those in the (Q, r, T) policy. Finally, it is worthwhile to note that the structures of the objective function and the service level constraint for all proposed policies are the same; and in all policies, the discretization approach is used to find the limit distribution of the effective lifetime.

6. Solution approach

In this section, we present our solution methodology for finding the global optimal solution of the nonlinear optimization problem (P). Since the effective lifetime distribution is not explicitly attainable, it is difficult to provide an explicit expression for the optimal solution. We develop an exact algorithm that is based on the structures of the objective function and the service level constraint.

As Q and r are integer decision variables, we can execute an exhaustive search on different values of Q and r over a specific search space; however, enumerating the continuous variable T is not an easy task, which makes the optimization

problem (P) sophisticated. Thus, we try to capture the behaviors of the expected cost rate function and the service level constraint with respect to T . Now, we present our main result in the following proposition.

Proposition 1. $\mathbb{E}[TC(Q, r, T)]$ and $FL(Q, r, T)$ are decreasing and increasing in T , respectively.

Since the explicit expressions of $\mathbb{E}[TC(Q, r, T)]$ and $FL(Q, r, T)$ are not attainable, we use the sample-path analysis for the proof (see Appendix A). One can argue the proposition above as follows. By fixing Q and r and increasing T , we expect to have a lower expected on-hand inventory because we postpone reordering. Therefore, by increasing T we expect to have less holding costs and consequently fewer operating costs over time. On the other hand, late reordering leads to late delivery of ordered batches which results in more unmet demands during the lead time, and consequently a lower service level (higher $FL(Q, r, T)$) in the system. Figure 10 illustrates the behaviors of the expected total cost rate and the fraction of lost sales with respect to (w.r.t) T for a particular data set: $\lambda=5$, $L=1$, $\tau=2$, $h=1$, $p=50$, and $K=100$. From Figure 10, we can see that after some point, an increase in T does not have an impact on the expected total cost rate and the fraction of lost sales. The reason for this behavior is that when T becomes large, a new batch is always placed when the inventory position hits r and any increase in T does not affect the reorder point.

To achieve the minimum objective value, we should increase T as much as possible, since the expected total cost rate is decreasing in T (by Proposition 1). Adversely, by increasing T , $FL(Q, r, T)$ increases and the service level constraint may be violated accordingly. Therefore, for given Q and r , the optimal T binds the service level constraint. Thus, we conclude the following result.

Corollary 1. With given Q and r , the optimal value of T is obtained by solving $FL(Q, r, T) = \alpha$.

We utilize the result of Corollary 1 to propose an exact algorithm for finding the optimal threshold time in the (Q, r, T) policy given Q and r as follows. Let Q^* , r^* , and T^* denote the optimal values of the control policy parameters. Since the explicit expression of the $FL(Q, r, T)$ function w.r.t

T is unavailable, we apply the iterative interpolation algorithm over the interval $[0, \tau + L]$ to find the desired feasible region for the optimal T , $T^* \in [T_L, T_U]$. In the first iteration of the interpolation method, we set the new value of T to the midpoint of the interval $[0, \tau + L]$, i.e., $T = (\tau + L)/2$. If $T = (\tau + L)/2$ satisfies the service level constraint, then the new interval for search will be $[T_L, T_U] = [(\tau + L)/2, \tau + L]$, otherwise $[T_L, T_U] = [0, (\tau + L)/2]$. Then, we select T as the midpoint of the new interval and continue this procedure while the desired range including the lower bound T_L and the upper bound T_U is attained. Then, by doing an enumeration search over the feasible region $[T_L, T_U]$ we can find T^* . We iterate the interpolation algorithm till the interval $[T_L, T_U]$ contains 10 mass points. That is, $T_U - T_L = 10 \times \delta$. The interval $[T_L, T_U]$ might contain a higher number of mass points. With a more extensive feasible region, the algorithm converges faster, but it needs to search a higher number of solutions on the interval $[T_L, T_U]$ after convergence. Note also that in any iteration, T is set to the nearest multiple of δ .

Now, we can develop our exact algorithm as Algorithm 1, which is based on an exhaustive search over Q and r and finding the corresponding optimal time threshold T^* by the iterative interpolation algorithm described above. We search over the range $[1, Q_U]$ for Q and the range $[0, r_U]$ for r , where Q_U and r_U are set to arbitrary values and the search is conducted on an increment size of one unit. Noting that the effective lifetime distribution is different in the special region $r = (m - 1)Q$, we find the optimal solution of the special region, separately. Then, the global optimal solution is obtained by comparing the optimal solutions of two cases $r = (m - 1)Q$ and $(m - 1)Q < r < mQ$.

Algorithm 1. The pseudo code of the exact algorithm.

Initialization: Set $\lambda, L, \tau, h, p, K, \alpha, \delta, \mathbb{E}[TC^*] \leftarrow \infty$;
for $Q = 1 : Q_U$ **do**
 for $r = 0 : r_U$ **do**
 $T_L \leftarrow 0$;
 $T_U \leftarrow \tau + L$;
 while $T_U - T_L \geq 10 \times \delta$ **do**
 $T = \lceil \frac{T_L + T_U}{2\delta} \rceil \times \delta$;
 if $FL(Q, r, T) < \alpha$ **then**
 $T_L \leftarrow T$;
 else
 $T_U \leftarrow T$;
 end if
 end while
 for $T = T_L : \delta : T_U$ **do**
 if $FL(Q, r, T) \leq \alpha$ **then**
 $T_1^* \leftarrow T$;
 end if
 end for
 Calculate $\mathbb{E}[TC(Q, r, T_1^*)]$;
 if $\mathbb{E}[TC(Q, r, T_1^*)] < \mathbb{E}[TC^*]$ **then**
 $\mathbb{E}[TC^*] \leftarrow \mathbb{E}[TC(Q, r, T_1^*)]$;
 $Q^* \leftarrow Q$;
 $r^* \leftarrow r$;

$T^* \leftarrow T_1^*$;
end if
end for
Return: Q^*, r^*, T^* , and $\mathbb{E}[TC^*]$;

The computation time of the exact algorithm depends on the shape of $FL(Q, r, T)$, especially its shape close to the optimal T . There are some cases in which the exact algorithm returns the optimal T for given Q and r after five iterations and in some cases after 20 iterations. The proposed exact algorithm is fast enough for finding the optimal solution, as for given Q and r , we do not need to search over all possible values of the continuous variable T .

It is worthwhile to note that by adding a direct cost for lost sales, instead of the service level constraint, the expected total cost per unit time is no longer decreasing in T (i.e., Proposition 1 does not hold). We can use the sample-path analysis to demonstrate this argument. According to Figure 11, in the appendix, by increasing the time threshold T by ΔT time units the probability of experiencing a stock-out situation increases (see embedded Cycle 3 in Figure 11); however, as shown in the proof of Proposition 1, the expected on-hand inventory decreases. Hence, by involving the lost sale costs in the objective function, the expected cost per unit time is not decreasing in T any more. Our numerical experiences reveal that by adding a direct cost for lost sales, the cost rate function is neither convex nor concave in T .

7. Numerical study

We conduct our numerical study to gain insight into when our model is economically worthwhile and to address the following research questions:

1. How does the age-based policy of (Q, r, T) work under the aging process type 2 compared to its special policies?
2. How much and when does the relaxation on number of outstanding orders bring benefits for the inventory system?

For this purpose, we use the data set proposed by Tekin *et al.* (2001) as it is provided in Table 1.

We do not impose any restriction on Q and r values. That is, r and Q can take any non-negative integer value resulting in m outstanding orders, where $m \geq 1$. We use the exact algorithm to find the optimal solution of control policies. We set $\delta = 0.01$ in this algorithm. In the (Q, r, T) policy, to find the effective lifetime distribution in cases where $m=1$, we use 200 mass points in the discretization approach; and in cases where $m=2$, we use 120 and 80 mass points over the oldest and youngest batch effective lifetime distributions. Our numerical results reveal that we have at most two outstanding orders in the (Q, r, T) policy. In the (Q, r) policy, we have up to three outstanding orders in the system. We use the discretization profile provided in Berk *et al.* (2020) for the (Q, r) policy. Finally, in the (Q, T)

Table 1. Test parameters.

Notation	Values
L	1
h	1
λ	5
K	50, 100
p	1, 10, 50
τ	2, 4, 6
α	0.005, 0.01, 0.02, 0.05, 0.1

policy, we use 200 mass points for discretization. We specify the number of mass points mentioned above in such a way that convergence in the effective lifetime distribution is observed.

7.1. The (Q, r, T) policy performance

To capture the efficiency of the (Q, r, T) policy, we compare it with two special policies of (Q, T) and (Q, r) under different settings. We base our analysis by varying the input parameters K , p , τ , and α . In Tables 2 and 3, we report the optimal policy parameters of different policies for $K=50$ and $K=100$, respectively, where $\mathbb{E}[TC_i]$ denotes the expected total cost per unit time for policy i . The index of $i=1, 2$, and 3 are reserved to represent the optimal solution of the (Q, r, T) , (Q, T) , and (Q, r) policies, respectively. Δ_{ij} represents the percentage deviation between two policies i and j and it is computed as

$$\Delta_{ij} = \frac{\mathbb{E}[TC_j] - \mathbb{E}[TC_i]}{\mathbb{E}[TC_i]} \times 100.$$

In these tables, the bold numbers are the average gaps for different τ , and the bold numbers in parentheses are the average gaps for different p .

Recall that $m = \lceil r/Q \rceil$. In cases with a short lifetime, high service level and high unit perishing cost, the system requires more frequent orders, which leads to a higher number of outstanding orders over time. Moreover, as the fixed ordering cost increases, the batch size increases, which results in a fewer number of outstanding orders (compare the results in Tables 2 and 3).

From our numerical results in Table 2, one can see that we have maximum number of outstanding orders when $K=50$, $p=50$, $\tau=2$, and $\alpha=0.005$. In this case, we have a $m=2$ outstanding orders in the optimal (Q, r, T) policy, where $Q_1^*=6$, $r_1^*=10$, and $T_1^*=0.42$, and $m=3$ outstanding orders in the optimal (Q, r) policy, where $Q_3^*=5$, $r_3^*=11$. As expected, the variable T in the (Q, r, T) policy can result in a reduction in the optimal reorder point r_1^* . Therefore, we have fewer number of outstanding orders in the (Q, r, T) policy compared with the (Q, r) policy.

As in the (Q, T) policy at most one order can be outstanding, the maximum gap between (Q, r, T) and (Q, T) policies occurs when the inventory system requires multiple outstanding orders. The reason for this behavior is that in cases where we need multiple outstanding orders, the optimal batch size in the (Q, T) policy is high, and in the case of perishing, large perishing costs are incurred by the system. From our numerical results in Table 2, one can see

that when $p=50$, $\tau=2$, and $\alpha=0.005$ the optimal batch size in the (Q, r, T) policy is six, whereas it is 11 in the (Q, T) policy. That is why we observe a huge gap in the optimal costs of these two policies ($\Delta_{12}=49.26\%$ when $K=50$ and $\Delta_{12}=22.77\%$ when $K=100$). As a result, the (Q, r, T) policy significantly outperforms the (Q, T) policy when multiple outstanding orders are in the system, and this dominance becomes more considerable as the unit perishing cost increases.

Our numerical experiment shows that the (Q, r) policy performs well when the lifetime of items is long so that in this case, the (Q, r, T) policy converges to the (Q, r) policy. Therefore the maximum gap between (Q, r, T) and (Q, r) policies occurs when the lifetime of products is short ($\Delta_{13}=14.07\%$ when $K=50$, and $\Delta_{13}=15.68\%$ when $K=100$).

The (Q, T) policy dominates the (Q, r) policy (Δ_{23} is positive) when the unit perishing cost is low, the lifetime of items is short and the service level is high. As the unit perishing cost increases, the performance of the (Q, T) policy declines and it cannot dominate the (Q, r) policy even if the lifetime of items is short and the service level is high.

Regarding the computation time, as the number of outstanding orders increases, finding the vector of effective lifetime distribution takes more time. Overall, the computation time for finding the operating characteristic values (the expected cost rate values) in the (Q, T) policy is less than that of the (Q, r) policy. Due to the complexity of the transition probability function of the (Q, r, T) policy, the computation time in this policy is more than the other policies. Suppose that in the discretization approach, the vector Z^i is discretized to S^i mass points for $i=1, 2, \dots, m$. Then, we need to solve a set of $S^1 \times S^2 \times \dots \times S^m + 1$ linear equations in order to find the effective lifetime distribution. Thus, the computational time for finding the effective lifetime distribution increases exponentially with the value of m . Moreover, the computation time in the (Q, r, T) policy is less than that of the (Q, r) policy, as we observe a lower number of outstanding orders m in the (Q, r, T) policy compared with the (Q, r) policy. Our numerical study shows that even in extreme cases, it is a rare event to experience more than four outstanding orders at the same time in the inventory system (i.e., $m \leq 4$), and also reveals that for $m \leq 4$, the computation time for finding the effective lifetime distribution is reasonable. For instance, it takes 38 seconds on average (programmed with Gurobi C++ interface on a personal computer with speed of 2.71 GHz) to find the effective lifetime for $m=4$ with 50 mass points on each vector in the (Q, r, T) policy. Since in the (Q, T) policy we need to track only the current batch effective lifetime, deriving the effective lifetime distribution even for a large number of mass points is fast (on average 1.04 seconds for 500 mass points). Thus, this policy can be considered as a heuristic if we observe $m > 4$ in the (Q, r, T) policy.

As a result, the inventory manager always prefers implementing the (Q, r, T) policy in order to achieve the minimum expected cost per unit time, especially if one holds perishable items with a short lifetime and requires a high service level. Further, if one relies on only two-parameter policies, one needs to compare the expected cost rate

Table 2. Comparison of policies for $K = 50$ ($\lambda = 5$, $L = 1$, $h = 1$).

			(Q, r, T) Policy		(Q, T) Policy		(Q, r) Policy					
			(Q ₁ , r ₁ , T ₁)	ℰ[TC ₁]	(Q ₂ , T ₂)	ℰ[TC ₂]	(Q ₃ , r ₃)	ℰ[TC ₃]	Δ ₁₂	Δ ₁₃	Δ ₂₃	
p = 1	τ = 2	0.005	(11, 10, 0.17)	45.18	(11, 0.06)	46.30	(11, 11)	47.26	2.48	4.60	2.07	
		0.01	(11, 9, 0.29)	43.13	(11, 0.17)	44.35	(10, 10)	45.90	2.83	6.42	3.49	
		0.02	(11, 8, 0.48)	40.57	(12, 0.39)	42.36	(10, 9)	42.21	4.41	4.04	−0.35	
		0.05	(11, 7, 0.91)	36.55	(11, 0.55)	39.06	(10, 8)	38.89	6.87	6.40	−0.44	
		0.1	(11, 5, 1.29)	32.49	(11, 0.85)	35.65	(11, 7)	34.17	9.73	5.17	−4.15	
	τ = 4	0.005	(16, 9, 1.68)	30.10	(15, 0.86)	32.34	(14, 10)	31.73	7.44	5.42	−1.89	
		0.01	(16, 8, 1.98)	28.61	(17, 1.34)	31.04	(15, 9)	29.71	8.49	3.84	−4.28	
		0.02	(17, 7, 2.44)	27.00	(16, 1.42)	29.50	(16, 8)	27.86	9.26	3.19	−5.56	
		0.05	(18, 5, 3.06)	24.57	(17, 2.00)	27.12	(16, 6)	25.23	10.38	2.69	−6.97	
		0.1	(18, 3, 3.56)	22.53	(17, 2.56)	24.39	(16, 4)	22.72	8.26	0.84	−6.85	
	τ = 6	0.005	(22, 8, 3.27)	26.70	(19, 1.58)	29.71	(21, 9)	27.07	11.27	1.39	−8.89	
		0.01	(21, 7, 3.36)	25.56	(19, 1.80)	28.54	(22, 8)	25.97	11.66	1.60	−9.00	
		0.02	(21, 6, 3.75)	24.40	(20, 2.26)	27.20	(22, 7)	24.91	11.48	2.09	−8.42	
		0.05	(22, 4, 4.26)	22.79	(20, 2.80)	24.85	(21, 5)	22.85	9.04	0.26	−8.05	
		0.1	(20, 3, 6.00)	20.92	(21, 3.66)	22.41	(20, 3)	20.92	7.12	0.00	−6.65	
	p = 10	τ = 2	0.005	(9, 10, 1.46)	60.92	(11, 0.06)	67.52	(11, 11)	69.49	10.83	14.07	2.92
			0.01	(8, 9, 1.44)	56.93	(10, 0.06)	61.27	(10, 10)	63.04	7.62	10.73	2.89
			0.02	(9, 8, 0.30)	52.50	(9, 0.10)	54.85	(10, 9)	56.03	4.48	6.72	2.15
			0.05	(9, 7, 1.15)	45.72	(8, 0.20)	48.80	(8, 7)	46.68	6.74	2.10	−4.34
			0.1	(9, 5, 1.11)	39.24	(8, 0.46)	44.15	(9, 6)	40.87	12.51	4.15	−7.43
τ = 4		0.005	(13, 9, 1.46)	33.33	(12, 0.42)	35.65	(14, 10)	35.26	6.96	5.79	−1.09	
		0.01	(14, 8, 1.84)	31.26	(12, 0.58)	34.17	(14, 9)	32.99	9.31	5.53	−3.45	
		0.02	(14, 7, 2.26)	29.27	(13, 0.94)	32.46	(15, 8)	30.90	10.90	5.57	−4.81	
		0.05	(15, 5, 2.54)	26.40	(14, 1.50)	29.62	(15, 6)	27.09	12.20	2.61	−8.54	
		0.1	(15, 4, 4.00)	23.90	(14, 1.98)	26.53	(15, 4)	23.90	11.00	0.00	−9.91	
τ = 6		0.005	(19, 8, 2.61)	27.40	(17, 1.26)	30.46	(19, 9)	27.86	11.17	1.68	−8.54	
		0.01	(21, 7, 3.36)	26.24	(17, 1.46)	29.26	(19, 8)	26.61	11.51	1.41	−9.06	
		0.02	(21, 6, 3.75)	24.89	(18, 1.90)	27.86	(19, 7)	25.39	11.93	2.01	−8.87	
		0.05	(20, 5, 6.00)	23.09	(18, 2.38)	25.49	(20, 5)	23.09	10.37	0.00	−9.40	
		0.1	(20, 3, 6.00)	21.02	(19, 3.20)	22.90	(20, 3)	21.02	8.94	0.00	−8.20	
p = 50		τ = 2	0.005	(6, 10, 0.42)	107.90	(11, 0.06)	161.05	(5, 11)	115.26	49.26	6.82	−28.43
			0.01	(7, 9, 0.98)	98.73	(10, 0.06)	133.08	(5, 10)	103.87	34.79	5.21	−21.95
			0.02	(7, 8, 1.40)	83.64	(9, 0.10)	105.22	(5, 9)	90.14	25.80	7.77	−14.33
			0.05	(6, 6, 1.20)	67.91	(7, 0.04)	72.45	(6, 7)	69.65	6.69	2.56	−3.86
			0.1	(7, 5, 0.56)	55.43	(6, 0.12)	59.10	(7, 6)	58.25	6.62	5.09	−1.44
	τ = 4	0.005	(11, 9, 1.06)	39.61	(10, 0.12)	41.14	(11, 10)	43.83	3.86	10.65	6.54	
		0.01	(12, 8, 1.48)	37.00	(10, 0.25)	39.24	(10, 9)	39.35	6.05	6.35	0.28	
		0.02	(12, 7, 1.66)	33.72	(10, 0.41)	37.24	(11, 8)	36.12	10.44	7.12	−3.01	
		0.05	(12, 6, 3.34)	30.46	(11, 0.92)	33.96	(12, 6)	30.47	11.49	0.03	−10.28	
		0.1	(13, 4, 3.48)	26.06	(11, 1.32)	30.31	(14, 4)	26.41	16.32	1.35	−12.87	
	τ = 6	0.005	(16, 9, 3.51)	29.25	(14, 0.76)	31.82	(16, 9)	29.30	8.79	0.17	−7.92	
		0.01	(16, 8, 3.96)	27.72	(14, 0.94)	30.59	(16, 8)	27.78	10.35	0.22	−9.19	
		0.02	(17, 7, 5.01)	26.25	(15, 1.34)	29.14	(17, 7)	26.28	11.01	0.12	−9.81	
		0.05	(18, 5, 5.94)	23.55	(16, 1.98)	26.57	(18, 5)	23.55	12.82	0.00	−11.37	
		0.1	(19, 3, 5.34)	21.39	(17, 2.76)	23.69	(20, 3)	21.46	10.75	0.33	−9.41	
									Max:	49.26	14.07	6.54
									Min:	2.48	0.00	−28.43
									Ave:	10.94	3.65	−6.19

associated with each policy and select the best one. However, one can exercise the (Q, T) policy when the life-time of items is short and the unit perishing cost is low. Finally, one can adopt the classic (Q, r) policy when the life-time of items is long enough (such as conserved products and beans).

7.2. Relaxation on number of outstanding orders

From numerical results in Tables 2 and 3, we select the cases with multiple outstanding orders and compare those results with the case with one outstanding order restriction. In Tables 4 and 5, we report the optimal policy parameters and the optimal expected cost rates for the cases $m > 1$ and $m = 1$, where

Table 3. Comparison of policies for $K = 100$ ($\lambda = 5$, $L = 1$, $h = 1$).

		α	(Q, r, T) Policy		(Q, T) Policy		(Q, r) Policy		Δ_{12}	Δ_{13}	Δ_{23}
			(Q_1, r_1, T_1)	$\mathbb{E}[TC_1]$	(Q_2, T_2)	$\mathbb{E}[TC_2]$	(Q_3, r_3)	$\mathbb{E}[TC_3]$			
$p = 1$	$\tau = 2$	0.005	(15, 10, 0.39)	76.58	(15, 0.36)	77.27	(11, 11)	81.14	0.90	5.95	5.01
		0.01	(13, 9, 0.41)	73.29	(15, 0.44)	75.23	(10, 10)	80.27	2.65	9.52	6.70
		0.02	(13, 8, 0.57)	69.71	(13, 0.44)	72.28	(10, 9)	74.38	3.69	6.70	2.91
		0.05	(13, 7, 0.94)	63.13	(13, 0.72)	66.42	(10, 8)	69.03	5.21	9.35	3.93
		0.1	(13, 5, 1.34)	56.78	(13, 1.02)	61.18	(11, 7)	60.22	7.75	6.06	-1.57
									4.04	7.52	3.39
	$\tau = 4$	0.005	(20, 8, 1.90)	46.26	(19, 1.39)	49.11	(14, 10)	50.90	6.16	10.03	3.64
		0.01	(20, 8, 2.22)	44.12	(20, 1.70)	47.30	(15, 9)	47.44	7.21	7.52	0.30
		0.02	(20, 7, 2.56)	41.91	(19, 1.82)	45.29	(16, 8)	44.28	8.06	5.65	-2.23
		0.05	(20, 5, 3.16)	38.40	(20, 2.40)	41.85	(19, 7)	39.65	8.98	3.26	-5.26
		0.1	(20, 3, 3.74)	35.35	(21, 3.22)	37.54	(20, 5)	35.96	6.20	1.73	-4.21
									7.32	5.64	-1.55
	$\tau = 6$	0.005	(24, 8, 3.60)	37.80	(23, 2.24)	42.04	(21, 9)	39.20	11.22	3.70	-6.76
		0.01	(25, 7, 4.05)	36.37	(24, 2.64)	40.50	(22, 8)	37.49	11.36	3.08	-7.43
		0.02	(25, 6, 4.50)	34.90	(24, 2.94)	38.79	(24, 7)	35.53	11.15	1.81	-8.40
		0.05	(26, 4, 5.25)	32.53	(25, 3.72)	35.62	(26, 5)	32.79	9.50	0.80	-7.94
		0.1	(26, 2, 5.97)	30.32	(25, 4.52)	32.12	(26, 3)	30.66	5.94	1.14	-4.53
									9.83	2.10	-7.01
$p = 10$	$\tau = 2$	0.005	(10, 10, 1.34)	96.98	(11, 0.06)	100.56	(11, 11)	103.36	3.69	6.58	2.78
		0.01	(10, 9, 0.26)	90.90	(10, 0.07)	94.45	(10, 10)	97.40	3.91	7.15	3.12
		0.02	(9, 8, 0.39)	85.29	(9, 0.10)	88.98	(10, 9)	88.20	4.33	3.41	-0.88
		0.05	(10, 7, 0.93)	75.66	(10, 0.44)	81.72	(10, 8)	80.16	8.01	5.95	-1.91
		0.1	(10, 5, 1.24)	66.07	(10, 0.75)	73.68	(9, 6)	69.74	11.52	5.55	-5.35
									6.29	5.73	-0.45
	$\tau = 4$	0.005	(15, 9, 1.62)	51.63	(14, 0.72)	55.87	(14, 10)	54.43	8.21	5.42	-2.58
		0.01	(16, 8, 1.98)	48.80	(15, 1.04)	53.52	(15, 9)	50.79	9.67	4.08	-5.10
		0.02	(16, 7, 2.40)	45.96	(16, 1.40)	51.09	(16, 8)	47.50	11.16	3.35	-7.03
		0.05	(17, 5, 2.92)	41.64	(17, 1.98)	46.84	(16, 6)	42.74	12.49	2.64	-8.75
		0.1	(17, 3, 3.40)	38.02	(17, 2.56)	41.72	(16, 4)	38.44	9.73	1.10	-7.86
									10.25	3.32	-6.26
	$\tau = 6$	0.005	(23, 8, 3.51)	39.46	(20, 1.74)	44.66	(21, 9)	40.28	13.18	2.08	-9.81
		0.01	(23, 7, 3.81)	37.81	(20, 1.98)	42.83	(22, 8)	38.54	13.28	1.93	-10.02
		0.02	(23, 6, 4.26)	36.11	(21, 2.44)	40.83	(24, 7)	36.86	13.07	2.08	-9.72
		0.05	(24, 4, 4.86)	33.63	(21, 2.98)	37.51	(24, 5)	34.00	11.54	1.10	-9.36
		0.1	(24, 2, 5.58)	31.12	(22, 3.88)	33.48	(24, 3)	31.47	7.58	1.12	-6.00
									11.73	1.66	-8.98
$p = 50$	$\tau = 2$	0.005	(7, 10, 0.72)	158.28	(11, 0.06)	194.32	(5, 11)	174.62	22.77	10.32	-10.14
		0.01	(8, 9, 1.44)	139.50	(10, 0.06)	166.86	(5, 10)	161.37	19.61	15.68	-3.29
		0.02	(8, 8, 2.00)	127.02	(9, 0.11)	138.63	(7, 9)	133.29	9.14	4.94	-3.85
		0.05	(8, 7, 1.80)	107.12	(7, 0.04)	110.57	(8, 7)	107.12	3.22	0.00	-3.12
		0.1	(8, 5, 0.91)	87.23	(7, 0.30)	98.03	(8, 6)	93.32	12.38	6.98	-4.80
									13.42	7.58	-5.04
	$\tau = 4$	0.005	(12, 9, 1.42)	62.29	(11, 0.26)	66.57	(12, 10)	67.13	6.87	7.77	0.84
		0.01	(13, 8, 1.74)	57.96	(11, 0.42)	63.45	(12, 9)	62.14	9.47	7.21	-2.06
		0.02	(13, 7, 2.00)	53.74	(12, 0.76)	60.58	(13, 8)	57.53	12.73	7.05	-5.03
		0.05	(14, 5, 2.26)	48.12	(12, 1.12)	54.99	(14, 6)	49.38	14.28	2.62	-10.20
		0.1	(14, 4, 4.00)	42.72	(13, 1.76)	49.06	(14, 4)	42.69	14.84	-0.08	-12.99
									11.64	4.91	-5.89
	$\tau = 6$	0.005	(20, 8, 2.91)	43.09	(17, 1.26)	48.76	(18, 9)	43.75	13.16	1.53	-10.27
		0.01	(21, 7, 3.36)	41.22	(17, 1.46)	46.69	(19, 8)	41.65	13.27	1.04	-10.79
		0.02	(21, 6, 3.75)	38.84	(18, 1.90)	44.50	(19, 7)	39.62	14.57	2.01	-10.97
		0.05	(21, 5, 5.91)	35.89	(18, 2.38)	40.61	(21, 5)	35.89	13.15	0.00	-11.62
		0.1	(21, 3, 6.00)	32.65	(20, 3.44)	36.04	(21, 3)	32.65	10.38	0.00	-9.40
									12.91	0.92	-10.61
									(12.66)	(4.47)	(-7.18)
							Max:	22.77	15.68	6.70	
							Min:	0.90	-0.08	-12.99	
							Ave:	9.71	4.38	-4.71	

(Q_5^*, r_5^*, T_5^*) is the optimal solution of the (Q, r, T) policy under one order outstanding restriction. Numerical results indicate that relaxation on the number of outstanding orders can bring up to 43.26% cost savings for the system. As expected, we observe significant cost savings in cases with high unit perishing cost, short lifetime, and high service level.

The interesting result is that the optimal reorder points in both cases $m > 1$ and $m = 1$ are the same, i.e., $r_5^* = r_1^*$. Despite

the equality of reorder points for the instances analyzed here, it is not necessarily the case in general. The optimal time threshold for reordering in case where $m = 1$, T_5^* , is smaller than that in case where $m > 1$, T_1^* (i.e., $T_5^* < T_1^*$). The reason is that when the system requires multiple outstanding orders over time, but we are allowed to have at most one outstanding order, we need to decrease T in order to increase the inventory on-hand and satisfy the service level target.

Table 4. Comparison of policies for $K = 50$ ($\lambda = 5$, $L = 1$, $h = 1$).

			(Q, r, T) Policy when $m > 1$		(Q, r, T) Policy when $m = 1$		Δ_{15}
			(Q_1, r_1, T_1)	$\mathbb{E}[TC_1]$	(Q_5, r_5, T_5)	$\mathbb{E}[TC_5]$	
$p = 10$	$\tau = 2$	0.005	(9, 10, 1.46)	60.92	(11, 10, 0.16)	65.34	7.26
		0.01	(8, 9, 1.44)	56.93	(10, 9, 0.26)	58.02	1.91
$p = 50$	$\tau = 2$	0.005	(6, 10, 0.42)	107.90	(11, 10, 0.16)	154.58	43.26
		0.01	(7, 9, 0.98)	98.73	(10, 9, 0.26)	123.19	24.77
		0.02	(7, 8, 1.40)	83.64	(9, 8, 0.38)	96.42	15.28
		0.05	(6, 6, 1.20)	67.91	(7, 6, 0.06)	71.53	5.33

Table 5. Comparison of policies for $K = 100$ ($\lambda = 5$, $L = 1$, $h = 1$).

			(Q, r, T) Policy when $m > 1$		(Q, r, T) Policy when $m = 1$		Δ_{15}
			(Q_1, r_1, T_1)	$\mathbb{E}[TC_1]$	(Q_5, r_5, T_5)	$\mathbb{E}[TC_5]$	
$p = 10$	$\tau = 2$	0.005	(10, 10, 1.34)	96.98	(11, 10, 0.16)	98.08	1.13
$p = 50$	$\tau = 2$	0.005	(7, 10, 0.72)	158.28	(11, 10, 0.16)	187.33	18.35
		0.01	(8, 9, 1.44)	139.50	(10, 9, 0.26)	156.07	11.88
		0.02	(8, 8, 2.00)	127.02	(9, 8, 0.38)	129.80	2.19

7.3. Observations

The optimal r in the (Q, r) policy is always greater than or equal to that of the (Q, r, T) policy (i.e., $r_3^* \geq r_1^*$). Since r is an integer decision variable, reordering at point $r^* - 1$ may violate the service level constraint, and at point $r^* + 1$ may increase the holding cost. Hence, by incorporating the variable T into the (Q, r, T) policy, we can have reordering at any time epoch. By doing so, we can bind the service level constraint and have an opportunity to decrease r in order to have a lower holding cost. Therefore, we can conclude that the optimal r in the (Q, r, T) policy is an upper bound for the one in the (Q, r, T) policy.

From our numerical results, one can see that $T_1^* \geq T_2^*$. The reason is that in the (Q, r, T) policy we have two control variables r and T for triggering orders, while in (Q, T) policy we have only the variable T . Regarding the batch sizes in different policies, we do not observe any monotonic behavior. Moreover, non-monotonic behavior in gap values is due to the integrality of decision variables Q and r .

8. Concluding remarks

In this article, we study the age-based control policy of (Q, r, T) which is proposed by Tekin *et al.* (2001). We utilize the embedded Markov process to analyze this policy by considering the effective lifetime distribution of items. We derive the effective lifetime distribution of items at the beginning of embedded cycles by which we find the exact explicit expressions for operating characteristics and the expected cost rate function. Investigating the structures of the expected cost rate and the fraction of lost sales with respect to the time threshold T , we develop an exact algorithm guaranteeing the global optimality. Our numerical results show that the effective lifetime of items significantly affects the optimal policy parameters and the expected cost rate, especially when we have products with short lifetimes and a high target service level. We consider two special policies of (Q, r) and (Q, T) which are sub-optimal policies. We observe that as the lifetime of items increases the (Q, r, T) policy performance declines and it converges to the classic stock-based policy of (Q, r) , and it converges to the (Q, T)

policy under no condition. The (Q, T) policy dominates the (Q, r) policy when the lifetime of items is very short and the target service level is high. An interesting property of the (Q, T) policy is that we only need to track the effective lifetime of the current batch; however, in (Q, r, T) and (Q, r) policies, we need to track multiple orders' effective lifetime under specific circumstances. Therefore, under the (Q, r, T) policy, when the inventory system requires a high number of outstanding orders ($m > 4$), the (Q, T) policy can be considered as a heuristic. Moreover, we relax the one order outstanding restriction and analyze the system under multiple outstanding orders setting. In cases with a high unit perishing cost, short product lifetime, and high target service level, we observe multiple outstanding orders in the system. As the unit perishing cost and the service level increase, the higher number of outstanding orders we would observe in the system. In conclusion, the (Q, r, T) policy is preferred over other control policies since it provides the minimum expected total cost per unit time; and if the inventory manager decides to apply a two-parameter control policy, the (Q, r) and (Q, T) policies are proposed for inventory systems with long and short lifetimes, respectively. Having the effective lifetime of items at the beginning of embedded cycles, we can develop a remaining lifetime-based control policy, which is left for future study.

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Appendices

Appendix A: Proofs

Proof of Theorem 1

We can rewrite Z_{n+1}^i for $i = 1, \dots, m-1$ and Z_{n+1}^m as follows:

$$\begin{aligned}
 \mathbb{P}\{\mathcal{A}|\mathbf{x}\} &= \mathbb{P}\{Z_{n+1}^1 \leq z^1; Z_{n+1}^2 \leq z^2; \dots; Z_{n+1}^m \leq z^m | \mathbf{Z}_n = (x^1, x^2, \dots, x^m)\} \\
 &= \mathbb{P}\{x^2 - CL_n \leq z^1; x^3 - CL_n \leq z^2; \dots; \tau + L - (CL_n - U_n) \leq z^m\} \\
 &= \mathbb{P}\{CL_n \geq \max_{1 \leq i \leq m-1} \{x^{i+1} - z^i\}; CL_n - U_n \geq \tau + L - z^m\} \\
 &= \mathbb{P}\{CL_n \geq \zeta^x; CL_n - U_n \geq \tau + L - z^m\}.
 \end{aligned} \tag{37}$$

Using Equation (37) and considering different realizations of CL_n and U_n for different regions A, B, and C expressed below, we can derive the transition probability function. Let us denote X_{mQ-r} by t and $X_{r-(m-1)Q}$ by u .

For Region A:

A1.

$$\mathbf{A}: \max\{T, \tau\} < x^1 \leq \tau + L$$

$$\text{A1. If } t < T - (x^1 - \tau),$$

$$CL_n = x^1 - \tau + \min\{\tau, X_Q\} \text{ and } U_n = x^1 - \tau + t.$$

$$\text{A2. If } T - (x^1 - \tau) < t < \tau,$$

$$CL_n = x^1 - \tau + \min\{\tau, X_Q\} \text{ and } U_n = T.$$

$$\text{A3. If } \tau < t,$$

$$CL_n = x^1 \text{ and } U_n = T.$$

$$\mathbf{B}: \min\{T, \tau\} < x^1 \leq \max\{T, \tau\}$$

$$\text{B1. If } t < \min\{T, \tau\},$$

$$CL_n = x^1 - \zeta_\tau^{x^1} + \min\{\zeta_\tau^{x^1}, X_Q\} \text{ and } U_n = x^1 - \zeta_\tau^{x^1} + t.$$

$$\text{B2. If } \min\{T, \tau\} < t < \zeta_\tau^{x^1},$$

$$CL_n = \min\{x^1, X_Q\} \text{ and } U_n = T.$$

$$\text{B3. If } \zeta_\tau^{x^1} < t,$$

$$CL_n = x^1 \text{ and } U_n = \zeta_\tau^{x^1}.$$

$$\mathbf{C}: 0 \leq x^1 \leq \min\{T, \tau\}$$

$$\text{C1. If } t < x^1,$$

$$CL_n = \min\{x^1, X_Q\} \text{ and } U_n = t.$$

$$\text{C2. If } t > x^1,$$

$$CL_n = x^1 \text{ and } U_n = x^1.$$

$$\mathbb{P}\{\mathcal{A}|\mathbf{x}\} = \mathbb{P}\{X_{mQ-r} < T - (x^1 - \tau); x^1 - \tau$$

$$+ \min\{\tau, X_Q\} \geq \zeta^x; \min\{\tau, X_Q\} - t \geq \tau + L - z^m\}$$

$$= \mathbb{P}\{X_{mQ-r} < T - (x^1 - \tau); x^1 \geq \zeta^x; \tau \geq \tau + L - z^m + t;$$

$$X_Q \geq \zeta^x + \zeta_\tau^{x^1} - x^1; X_Q \geq \tau + L - z^m + t\}$$

$$= \int_0^{\min\{z^m - L, T - (x^1 - \tau)\}} \int_{\max\{\zeta^x + \zeta_\tau^{x^1} - x^1 - t, \tau + L - z^m\}}^\infty$$

$$dF_{r-(m-1)Q}(u) dF_{mQ-r}(t).$$

A2.

$$\mathbb{P}\{\mathcal{A}|\mathbf{x}\} = \mathbb{P}\{T - (x^1 - \tau) < X_{mQ-r} < \tau; x^1 - \tau + \min\{\tau, X_Q\} \geq \zeta^x;$$

$$x^1 - \tau + \min\{\tau, X_Q\} - T \geq \tau + L - z^m\}$$

$$= \mathbb{P}\{T - (x^1 - \tau) < X_{mQ-r} < \tau; x^1 \geq \zeta^x;$$

$$\tau \geq \tau + L - z^m + \zeta_\tau^{x^1} - x^1 + T;$$

$$X_Q \geq \zeta^x + \zeta_\tau^{x^1} - x^1; X_Q \geq \tau + L - z^m + \zeta_\tau^{x^1} - x^1 + T\}$$

$$= \mathbb{1}_{\{x^1 - T \geq \tau + L - z^m\}} \int_{T - (x^1 - \tau)}^\tau \bar{F}_{r-(m-1)Q}(\ell - t) dF_{mQ-r}(t),$$

where $\ell = \max\{\zeta^x + \zeta_\tau^{x^1} - x^1, \tau + L - z^m + \zeta_\tau^{x^1} - x^1 + T\}$.

A3.

$$\mathbb{P}\{\mathcal{A}|\mathbf{x}\} = \mathbb{P}\{X_{mQ-r} > \tau; x^1 \geq \zeta^x; x^1 - \tau \geq \tau + L - z^m\}$$

$$= \mathbb{1}_{\{x^1 - T \geq \tau + L - z^m\}} \int_\tau^\infty dF_{mQ-r}(t).$$

Then, we can simply express the transition probability function for Region A as follows:

$$\begin{aligned}
 \mathbb{P}\{\mathcal{A}|\mathbf{x}\} &= \mathbb{1}_{\{x^1 - T \geq \tau + L - z^m\}} \left[\int_{T - (x^1 - \tau)}^\tau \bar{F}_{r-(m-1)Q}(\ell - t) dF_{mQ-r}(t) + \bar{F}_{mQ-r}(\tau) \right] \\
 &+ \int_0^{\min\{z^m - L, T - (x^1 - \tau)\}} \int_{\max\{\zeta^x + \zeta_\tau^{x^1} - x^1 - t, \tau + L - z^m\}}^\infty dF_{r-(m-1)Q}(u) dF_{mQ-r}(t).
 \end{aligned}$$

Similarly, we can derive the transition probability function for Regions B and C. Finally, we can express all regions by a single term as Equation (8). Also, note that any negative term, for example the term $T - (x^1 - \tau)$, should be equalized to zero, if there is any. \square

Proof of Theorem 2

Recall that if there are no closed classes in a Markov chain, except for the entire class of states, the chain is irreducible (Cinlar (2013), p.127).

Inspecting (8), we see that for any $\mathcal{A} \in \mathcal{B}^m$, there exists some $\mathbf{x} \in \mathcal{A}$ such that $\mathbb{P}\{\mathcal{A}|\mathbf{x}\} = 0$. In particular, for any \mathbf{x} that satisfies $x^i \leq z^i, i = 1, \dots, m$ and $x^1 < \zeta^x$, $\mathbb{P}\{\mathcal{A}|\mathbf{x}\} = 0$. That is, $\mathbb{P}\{\mathbf{Z}_{n+1} \in \mathcal{A}|\mathbf{Z}_n = \mathbf{x}\} = 0$ and \mathbf{Z}_{n+1} gets out of the subset \mathcal{A} . This implies that a set of the form \mathcal{A} given above is not closed. Similarly, it is easy to check that $\mathbb{P}\{\mathcal{A}|\mathbf{x}\} = 1$ if $z^i = \tau + L, i = 1, \dots, m$, in which case, \mathcal{A} is the entire state space. Hence, irreducibility follows.

We use μ_τ to denote the mean hitting time, which is defined as the mean time between two consecutive embedded cycles at the beginning of which the vectors of effective lifetimes are the same. To confirm the ergodicity of the process, we need to show that the mean hitting time is bounded. First, consider the case where $m = 1$. Denote the probability that the embedded cycle $n + 1$ starts with a fresh batch if the embedded cycle n starts with a fresh batch ($Z_n = \tau$) by p_g . This event occurs if:

1. we hit r before $\tau - L$ time units and the remaining r units are depleted during the lead time L ; or
2. we hit r after $\tau - L$ time units and the remaining r units will perish before complete depletion. Hence, $p_g = F_{Q-r}(\tau - L)F_r(L) + \bar{F}_{Q-r}(\tau - L) = 1 - \bar{F}_r(L)F_{Q-r}(\tau - L) < \infty$. Now, suppose that an embedded cycle starts with a fresh batch; the number of embedded cycles required to observe for the first time an embedded cycle with a fresh batch has a geometric distribution with probability of success p_g . Then, the expected number of embedded cycles required to reach, hitting time is $1/p_g$ (i.e., mean of geometric distribution with parameter p_g). On the other hand, from Equation (16), one can see that the maximum length of an embedded cycle is $\tau + L$. Thus, to derive an upper bound for the mean hitting time, we can assume that all the embedded cycles occurring during the hitting time take maximum value of $\tau + L$. Then, we have:

$$\mu_\tau \leq \frac{\tau + L}{p_g} < \infty, \quad (38)$$

which implies that the mean hitting time is bounded above. Analogously, we can apply this approach to find an upper bound for the mean hitting time in the case where $m > 1$. The only difference is the derivation of p_g . In what follows, we approximate p_g by using the discretization approach. Following Equation (3), one can see that a neighborhood \mathbf{w} can be defined such that, if the embedded cycle n starts with effective lifetime vector \mathbf{w} , it is possible for the effective lifetime vector to visit the same neighborhood after two transitions (at the beginning of embedded cycle $n + 2$) with the following probability:

$$p_g = \sum_{\mathbf{x}} \mathbb{P}\{\mathbf{Z}_{n+2} = \mathbf{w}|\mathbf{Z}_{n+1} = \mathbf{x}, \mathbf{Z}_n = \mathbf{w}\} \times \mathbb{P}\{\mathbf{Z}_{n+1} = \mathbf{x}|\mathbf{Z}_n = \mathbf{w}\}. \quad \square$$

Proof of Theorem 4

Similar to the (Q, r, T) policy, we can derive different regions:

- A: $\max\{T, \tau\} < \mathbf{x} \leq \tau + L$**
- A1. If $X_Q < T - (x - \tau)$, $CL_n = T$ and $U_n = T$.
 - A2. If $T - (x - \tau) < X_Q < \tau$, $CL_n = x - \tau + X_Q$ and $U_n = T$.
 - A3. If $\tau < X_Q$, $CL_n = x$ and $U_n = T$.
- B: $\min\{T, \tau\} < \mathbf{x} \leq \max\{T, \tau\}$**
- B1. If $X_Q < \min\{T, \tau\}$, $CL_n = T$ and $U_n = T$.
 - B2. If $\min\{T, \tau\} < X_Q < \zeta_\tau^x$, $CL_n = X_Q$ and $U_n = T$.
 - B3. If $\zeta_\tau^x < X_Q$, $CL_n = T + x - \zeta_\tau^x$ and $U_n = T$.
- C: $0 \leq \mathbf{x} \leq \min\{T, \tau\}$**
- C1. If $X_Q < x$, $CL_n = T$ and $U_n = T$.
 - C2. If $X_Q > x$, $CL_n = T$ and $U_n = T$.

For Region A:

A1.

$$\begin{aligned} \mathbb{P}\{\mathcal{A}|\mathbf{x}\} &= \mathbb{P}\{X_Q < T - (x - \tau); CL_n - U_n \geq \tau + L - z\} \\ &= \mathbb{P}\{X_Q < T - (x - \tau); 0 \geq \tau + L - z\} \\ &= \mathbb{1}_{\{z=\tau+L\}} \int_0^{T-(x-\tau)} dF_Q(y). \end{aligned}$$

A2.

$$\begin{aligned} \mathbb{P}\{\mathcal{A}|\mathbf{x}\} &= \mathbb{P}\{T - (x - \tau) < X_Q < \tau; CL_n - U_n \geq \tau + L - z\} \\ &= \mathbb{P}\{T - (x - \tau) < X_Q < \tau; x - \tau + X_Q - T \geq \tau + L - z\} \\ &= \mathbb{P}\{T - (x - \tau) < X_Q < \tau; X_Q \geq \tau + L - z + \zeta_\tau^x - x + T\} \\ &= \int_{\tau+L-z+\zeta_\tau^x-x+T}^{\tau} dF_Q(y). \end{aligned}$$

A3.

$$\mathbb{P}\{\mathcal{A}|\mathbf{x}\} = \mathbb{P}\{X_Q > \tau; x - T \geq \tau + L - z\} = \mathbb{1}_{\{x-T \geq \tau+L-z\}} \int_{\tau}^{\infty} dF_Q(y).$$

Then, we can simply express the transition probability function for Region A as follows:

$$\begin{aligned} \mathbb{P}\{\mathcal{A}|\mathbf{x}\} &= \mathbb{1}_{\{z=\tau+L\}} F_Q(T - (x - \tau)) + \mathbb{1}_{\{x-T \geq \tau+L-z\}} \bar{F}_Q(\tau) + F_Q(\tau) \\ &\quad - F_Q(\tau + L - z + \zeta_\tau^x - x + T). \end{aligned}$$

Similarly, we can derive the transition probability function for Regions B and C. Finally, we can express all regions by a single term as Equation (31). \square

Proof of Proposition 1.

We provide a sketch of the proof for this proposition by using the sample-path analysis, which is based on imposing a perturbation into the sample path and tracing its effect (Ho and Cao, 1983; Fu, 1994). Under this approach, we increase the time threshold T by ΔT time units and investigate the effect of this change on the expected total cost rate and the fraction of lost sales.

Figure 11 depicts a particular sample path, where dotted and solid black lines represent the inventory position and the inventory level, respectively. Blue lines depicted below the time threshold T and lead time L are the perturbed ones resulting from an increase in T by ΔT time units. Moreover, the hatched blue areas are the reduction in on-hand inventories due to the perturbation. Perishing occurrences are shown by red lines. In this sample path, we depict all possibilities that may occur due to the perturbation in T .

Without loss of generality, we assume that the sample path begins with one outstanding order. Suppose that the outstanding order in embedded cycle 1 is triggered by the time threshold T and increasing the time threshold T by ΔT does not affect the reorder point. Thus, the arrival time of outstanding order in embedded cycle 1 is postponed up to ΔT time units which results in less inventory on-hand. The hatched blue area in embedded cycle 1 illustrates this reduction in on-hand inventory.

Within embedded cycle 1, a new order is placed by hitting the time threshold T , which joins inventories during embedded cycle 3. When T is increased by ΔT in embedded cycle 1, a new order is placed by crossing the inventory position r . This change in reorder point postpones the arrival time of the new batch during embedded cycle 3 and shifts the inventory level, which causes less on-hand inventories. The hatched blue area in embedded cycle 3 represents the decrease in on-hand inventory due to the increase in T in embedded cycle 1. Note that the shift in the inventory level is less than the perturbation amount ΔT . Moreover, a perturbation in T in embedded cycle 1 results in a longer stock-out period at the beginning of embedded cycle 3 (compare the black stock-out period with the blue stock-out period).

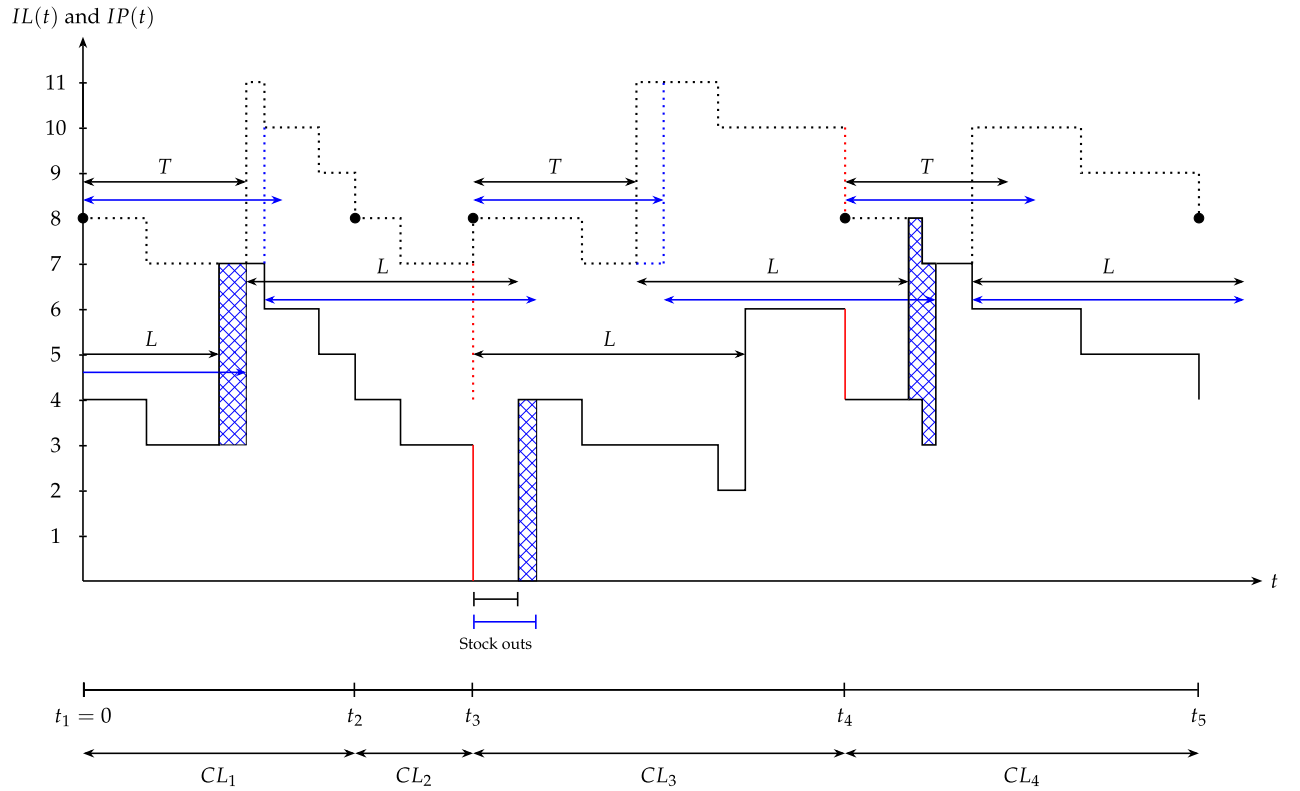


Figure 11. Effect on sample path of the inventory position and the inventory level when T is perturbed for $Q = 4$, $r = 6$, $T = 1.5$, and $L = 3$.

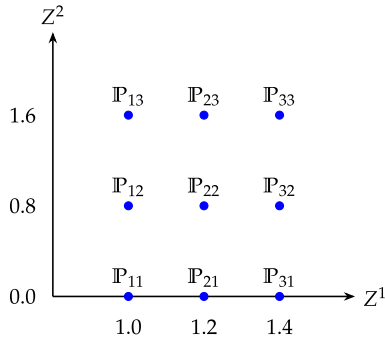


Figure 12. Discretized vector space.

Hence, a perturbation in T in embedded cycle 1 causes more lost sales and less stock on-hand in embedded cycle 3.

In embedded cycle 2, before reordering the current batch perishes and the embedded cycle ends. Thus, T does not play any role in triggering an order in this embedded cycle and the increase in T is not influential. In embedded cycle 3, a new batch is ordered when T time units elapse since the beginning of this embedded cycle and it arrives during embedded cycle 4. Perturbing T by ΔT in this embedded cycle postpones the reorder point by ΔT as well. Thus, in this case, the arrival time of the batch is postponed by ΔT time units, which leads to a decrease in on-hand inventories in embedded cycle 4 (see the hatched blue area in embedded cycle 4). The shift in the inventory level is the same as the perturbation amount ΔT . Finally, in embedded cycle 4, since a new order is placed by hitting the inventory position r , perturbing T does not influence the reorder point and consequently the dynamics of the system.

As a result, we conclude that perturbing T can have three types of effects:

1. The reorder point changes so that a new order is placed when the inventory position hits r (like embedded cycle 1). In this case, the shift in inventory level is less than the perturbation amount.
2. Reordering occurs by hitting T and by perturbing T the new batch is yet triggered by crossing T (like embedded cycle 3). In this case, the inventory level is shifted by the perturbation amount.
3. The perturbation in T does not influence the reorder point (like embedded cycles 2 and 4), and no shift in the inventory level is observed. Therefore, a perturbation in T decreases the expected on-hand inventory and hence the expected total costs per unit time. Moreover, in some instances, the shift in the inventory level may result in longer stock-out periods (like embedded cycle 3), which increases the fraction of lost sales over time.

It should be noted that an increase in T causes an increase in the current batch effective lifetime Z^1 . One can easily verify this fact by following the recursive equations in (3). As the effective lifetimes get larger, the expected embedded cycle length increases and consequently the fixed ordering cost incurred to the system per unit time reduces. Our analysis reveals that changing the effective lifetime at the beginning of embedded cycles can only change the time epochs of shifts in the inventory level; however, we yet observe these shifts in the sample path. Therefore, the cost savings obtained by a perturbation in T is greater than the one depicted in Figure 11. \square

Proof of Corollary 1

As shown in Proposition 1, the expected total cost rate is decreasing in T and the fraction of unmet demands is increasing in T . Therefore, the maximum value of T satisfying the service level constraint is the optimal T . This maximum value binds the service level constraint. \square

Appendix B: An illustrative example for discretization approach

In the following, we provide an illustrative example to clarify our discretization approach. We consider the case with $m=2$, where we have two vectors of remaining lifetimes. Without loss of generality, suppose that the support of vectors Z^1 and Z^2 are $[1.0, 1.4]$ and $[0.0, 1.6]$, respectively. Vector Z^1 is discretized to three mass points 1.0, 1.2, and 1.4; and vector Z^2 is discretized to three mass points 0.0, 0.8, and 1.6. Thus, we have nine mass points in total, as depicted in Figure 12. The blue bullets represent the set of mass points with corresponding probability values, which are indeed the set of decision variables.

Now, we select two mass points from the set of mass points and write their corresponding linear equations.

Example 1: Given $(z^1, z^2) = (1.2, 1.6)$, we have

$$\begin{aligned} & \mathbb{P}_{11} + \mathbb{P}_{12} + \mathbb{P}_{13} + \mathbb{P}_{21} + \mathbb{P}_{22} + \mathbb{P}_{23} \\ &= \mathbb{P}\{\mathcal{A}|(1.0, 0)\}\mathbb{P}_{11} + \mathbb{P}\{\mathcal{A}|(1.0, 0.8)\}\mathbb{P}_{12} + \mathbb{P}\{\mathcal{A}|(1.0, 1.6)\}\mathbb{P}_{13} \\ &+ \mathbb{P}\{\mathcal{A}|(1.2, 0)\}\mathbb{P}_{21} + \mathbb{P}\{\mathcal{A}|(1.2, 0.8)\}\mathbb{P}_{22} + \mathbb{P}\{\mathcal{A}|(1.2, 1.6)\}\mathbb{P}_{23} \\ &+ \mathbb{P}\{\mathcal{A}|(1.4, 0)\}\mathbb{P}_{31} + \mathbb{P}\{\mathcal{A}|(1.4, 0.8)\}\mathbb{P}_{32} + \mathbb{P}\{\mathcal{A}|(1.4, 1.6)\}\mathbb{P}_{33}. \end{aligned} \quad (39)$$

Example 2: Given $(z^1, z^2) = (1.4, 0.0)$, we have

$$\begin{aligned} & \mathbb{P}_{11} + \mathbb{P}_{21} + \mathbb{P}_{31} \\ &= \mathbb{P}\{\mathcal{A}|(1.0, 0)\}\mathbb{P}_{11} + \mathbb{P}\{\mathcal{A}|(1.0, 0.8)\}\mathbb{P}_{12} + \mathbb{P}\{\mathcal{A}|(1.0, 1.6)\}\mathbb{P}_{13} \\ &+ \mathbb{P}\{\mathcal{A}|(1.2, 0)\}\mathbb{P}_{21} + \mathbb{P}\{\mathcal{A}|(1.2, 0.8)\}\mathbb{P}_{22} + \mathbb{P}\{\mathcal{A}|(1.2, 1.6)\}\mathbb{P}_{23} \\ &+ \mathbb{P}\{\mathcal{A}|(1.4, 0)\}\mathbb{P}_{31} + \mathbb{P}\{\mathcal{A}|(1.4, 0.8)\}\mathbb{P}_{32} + \mathbb{P}\{\mathcal{A}|(1.4, 1.6)\}\mathbb{P}_{33}. \end{aligned} \quad (40)$$

Note that $\mathbb{P}\{\mathcal{A}|(x^1, x^2)\}$ with given (x^1, x^2) is a function of (z^1, z^2) and is computed by Equation (8). We should write these sets of linear equations for all nine mass points, which eventually results in nine linear equations. Then, given that the sum of probabilities is one, we can find the mass point probabilities by solving linear equations.

Appendix C: Derivation of expected on-hand inventory, Equation (24)

The expected waiting time of products for a given effective lifetime vector $\mathbf{Z} = \mathbf{z} = (z^1, z^2, \dots, z^m)$ and given n are found as follows:

$$\begin{aligned} \mathbb{E}[OH_n|z^1] &= \int_{0 < x_1 < \dots < x_n < \zeta_\tau^{z^1} < x_{n+1} < \dots < x_Q} \left[(Q\tau - n\zeta_\tau^{z^1}) + (x_1 + x_2 + \dots + x_n) \right] \\ &\quad \times \lambda^Q e^{-\lambda x_Q} dx_1 \dots dx_Q \\ &= \int_{0 < x_1 < \dots < x_n < \zeta_\tau^{z^1} < x_{n+1} < \dots < x_Q} (Q\tau - n\zeta_\tau^{z^1}) \lambda^Q e^{-\lambda x_Q} dx_1 \dots dx_Q \\ &\quad + \int_{0 < x_1 < \dots < x_n < \zeta_\tau^{z^1} < x_{n+1} < \dots < x_Q} (x_1 + x_2 + \dots + x_n) \lambda^Q e^{-\lambda x_Q} dx_1 \dots dx_Q. \end{aligned} \quad (41)$$

Let \mathcal{R} be the region $0 < x_1 < \dots < x_n < \zeta_\tau^{z^1} < x_{n+1} < \dots < x_Q < \infty$, and denote the first and second term of Equation (41) by T_n^1 and T_n^2 , respectively. Then, $\mathbb{E}[OH|z^1] = \sum_{n=0}^Q \mathbb{E}[OH_n|z^1]$, where $\mathbb{E}[OH_n|z^1] = T_n^1 + T_n^2$. In what follows, we derive T_n^1 and T_n^2 expressions, separately.

We first provide the following lemma for deriving T_n^1 .

Lemma 1. For $n = 0, 1, \dots, Q-1$:

$$\begin{aligned} \mathcal{I} &= \int_{\mathcal{R}} \lambda^Q e^{-\lambda x_Q} dx_1 dx_2 \dots dx_Q \\ &= \int_{x_n=0}^{\zeta_\tau^{z^1}} \dots \int_{x_2=0}^{x_3} \int_{x_1=0}^{x_2} \int_{x_{n+1}=\zeta_\tau^{z^1}}^{\infty} \int_{x_{n+2}=x_{n+1}}^{\infty} \dots \int_{x_Q=x_{Q-1}}^{\infty} \lambda^Q e^{-\lambda x_Q} dx_Q \dots dx_{n+1} dx_1 \dots dx_n \\ &= e^{-\lambda \zeta_\tau^{z^1}} \frac{(\lambda \zeta_\tau^{z^1})^n}{n!}. \end{aligned} \quad (42)$$

And for $n = Q$:

$$\begin{aligned} \mathcal{I} &= \int_{\mathcal{R}} \lambda^Q e^{-\lambda x_Q} dx_1 \dots dx_Q = \int_{x_Q=0}^{\zeta_\tau^{z^1}} \int_{x_{Q-1}=0}^{x_Q} \dots \int_{x_2=0}^{x_3} \int_{x_1=0}^{x_2} \lambda^Q e^{-\lambda x_Q} dx_1 dx_2 \dots dx_Q \\ &= 1 - \sum_{i=0}^{Q-1} e^{-\lambda \zeta_\tau^{z^1}} \frac{(\lambda \zeta_\tau^{z^1})^i}{i!} = F_Q(\zeta_\tau^{z^1}). \end{aligned} \quad (43)$$

Then, using the result of Lemma 1, T_n^1 is given by

$$T_n^1 = \begin{cases} (Q\tau - n\zeta_\tau^{z^1}) e^{-\lambda \zeta_\tau^{z^1}} \frac{(\lambda \zeta_\tau^{z^1})^n}{n!} & \text{for } n = 0, 1, \dots, Q-1, \\ Q(\tau - \zeta_\tau^{z^1}) F_Q(\zeta_\tau^{z^1}) & \text{for } n = Q. \end{cases} \quad (44)$$

In the following, we derive the T_n^2 expression.

For $n = 1, 2, \dots, Q-1$:

$$\begin{aligned} T_n^2 &= \sum_{i=1}^n \int_{x_n=0}^{\zeta_\tau^{z^1}} \int_{x_{n-1}=0}^{x_n} \dots \int_{x_2=0}^{x_3} \int_{x_1=0}^{x_2} x_i \lambda^n e^{-\lambda \zeta_\tau^{z^1}} dx_1 dx_2 \dots dx_n \\ &= e^{-\lambda \zeta_\tau^{z^1}} \lambda^n \sum_{i=1}^n \int_{x_n=0}^{\zeta_\tau^{z^1}} \dots \int_{x_i=0}^{x_{i+1}} x_i \frac{x_i^{i-1}}{(i-1)!} dx_i dx_{i+1} \dots dx_n \\ &= e^{-\lambda \zeta_\tau^{z^1}} \lambda^n \sum_{i=1}^n \frac{i(\zeta_\tau^{z^1})^{n+1}}{(n+1)!} = \frac{n\zeta_\tau^{z^1}}{2} e^{-\lambda \zeta_\tau^{z^1}} \frac{(\lambda \zeta_\tau^{z^1})^n}{n!}. \end{aligned} \quad (45)$$

And for $n = Q$:

$$\begin{aligned} T_Q^2 &= \int_{x_Q=0}^{\zeta_\tau^{z^1}} \int_{x_{Q-1}=0}^{x_Q} \dots \int_{x_2=0}^{x_3} \int_{x_1=0}^{x_2} (x_1 + x_2 + \dots + x_Q) \lambda^Q e^{-\lambda x_Q} dx_1 dx_2 \dots dx_Q \\ &= \sum_{i=1}^Q \int_{x_Q=0}^{\zeta_\tau^{z^1}} \int_{x_{Q-1}=0}^{x_Q} \dots \int_{x_i=0}^{x_{i+1}} \dots \int_{x_1=0}^{x_2} x_i \lambda^Q e^{-\lambda x_Q} dx_1 \dots dx_i \dots dx_Q \\ &= \sum_{i=1}^Q \frac{i}{\lambda} F_{Q+1}(\zeta_\tau^{z^1}) = \frac{Q(Q+1)}{2\lambda} F_{Q+1}(\zeta_\tau^{z^1}). \end{aligned} \quad (46)$$

Then,

$$\begin{aligned} \mathbb{E}[OH|z^1] &= \sum_{n=0}^Q \mathbb{E}[OH_n|z^1] = \sum_{n=0}^Q (T_n^1 + T_n^2) \\ &= \sum_{n=0}^{Q-1} \left(Q\tau - n\zeta_\tau^{z^1} \right) e^{-\lambda \zeta_\tau^{z^1}} \frac{(\lambda \zeta_\tau^{z^1})^n}{n!} + Q(\tau - \zeta_\tau^{z^1}) F_Q(\zeta_\tau^{z^1}) \\ &\quad + \frac{Q(Q+1)}{2\lambda} F_{Q+1}(\zeta_\tau^{z^1}). \end{aligned} \quad (47)$$

Substituting $F_Q(\zeta_\tau^{z^1})$ by $1 - \sum_{n=0}^{Q-1} e^{-\lambda \zeta_\tau^{z^1}} \frac{(\lambda \zeta_\tau^{z^1})^n}{n!}$ and $F_{Q+1}(\zeta_\tau^{z^1})$ by $1 - \sum_{n=0}^Q e^{-\lambda \zeta_\tau^{z^1}} \frac{(\lambda \zeta_\tau^{z^1})^n}{n!}$, after some simplifications we have

$$\mathbb{E}[OH|z^1] = \sum_{n=0}^Q \mathbb{E}[OH_n|z^1] = Q(\tau - \zeta_\tau^{z^1}) + \frac{1}{\lambda} \sum_{k=0}^{Q-1} \frac{(Q-k)}{k!} \gamma(k+1, \lambda \zeta_\tau^{z^1}), \quad (48)$$

where

$$\gamma(k+1, \lambda \zeta_\tau^{z^1}) = \int_0^{\lambda \zeta_\tau^{z^1}} x^k e^{-x} dx. \quad (49)$$

Appendix D: Derivation of operating characteristics of the (Q, T) policy

Considering different regions presented in proof of [Theorem 4](#), we can derive the $\mathbb{E}[CL]$ as following.

$$\begin{aligned} \mathbb{E}[CL] = & \int_0^{\min\{T, \tau\}} T dG(z) + \int_{\min\{T, \tau\}}^{\max\{T, \tau\}} \left[\int_0^{\min\{T, \tau\}} T dF_Q(y) \right. \\ & + \int_{\min\{T, \tau\}}^{\zeta_\tau^z} y dF_Q(y) + \int_{\zeta_\tau^z}^{\infty} (T + z - \zeta_\tau^z) dF_Q(y) \Big] dG(z) \\ & + \int_{\max\{T, \tau\}}^{\tau+L} \left[\int_0^{T+\zeta_\tau^z-z} T dF_Q(y) + \int_{T+\zeta_\tau^z-z}^{\tau} (z - \zeta_\tau^z + y) dF_Q(y) \right. \\ & \left. + \int_{\tau}^{\tau+L} z dF_Q(y) \right] dG(z). \end{aligned} \quad (50)$$

Similarly for $\mathbb{E}[LS]$, we can consider the same regions:

A: $\max\{T, \tau\} < x \leq \tau + L$

- | | |
|--|--|
| A1. If $X_Q < T - (x - \tau)$, | $\mathbb{E}[LS x] = \lambda(T - X_Q)$. |
| A2. If $T - (x - \tau) < X_Q < \tau$, | $\mathbb{E}[LS x] = \lambda(x - \tau)$. |
| A3. If $\tau < X_Q$, | $\mathbb{E}[LS x] = \lambda(x - \tau)$. |

B: $\min\{T, \tau\} < x \leq \max\{T, \tau\}$

- | | |
|---|---|
| B1. If $X_Q < \min\{T, \tau\}$, | $\mathbb{E}[LS x] = \lambda(T - X_Q)$. |
| B2. If $\min\{T, \tau\} < X_Q < \zeta_\tau^x$, | $\mathbb{E}[LS x] = 0$. |
| B3. If $\zeta_\tau^x < X_Q$, | $\mathbb{E}[LS x] = \lambda(T - \min\{T, \tau\})$. |

C: $0 \leq x \leq \min\{T, \tau\}$

- | | |
|--------------------|---|
| C1. If $X_Q < x$, | $\mathbb{E}[LS x] = \lambda(T - X_Q)$. |
| C2. If $X_Q > x$, | $\mathbb{E}[LS x] = \lambda(T - x)$. |

Then, we have:

$$\begin{aligned} \mathbb{E}[LS] = & \int_0^{\min\{T, \tau\}} \left[\int_0^x \lambda(T - y) dF_Q(y) + \int_x^\infty \lambda(T - z) dF_Q(y) \right] dG(z) \\ & + \int_{\min\{T, \tau\}}^{\max\{T, \tau\}} \left[\int_0^{\min\{T, \tau\}} \lambda(T - y) dF_Q(y) + \int_{\zeta_\tau^x}^\infty \lambda(T - \min\{T, \tau\}) dF_Q(y) \right] dG(z) \\ & + \int_{\max\{T, \tau\}}^{\tau+L} \left[\int_0^{T+\zeta_\tau^x-z} \lambda(T - y) dF_Q(y) + \int_{T+\zeta_\tau^x-z}^\infty \lambda(z - \tau) dF_Q(y) \right] dG(z). \end{aligned} \quad (51)$$

Note that in the (Q, T) policy, $T + \zeta_\tau^z - z$ can take negative value. In this case, we set it to zero.

Our analysis indicates that when $T \leq \tau$ the operating characteristics are different from the case where $T > \tau$. In what follows, we provide

the operating characteristic when $T \leq \tau$. Then, one can analyze the case where $T > \tau$ similarly.

Suppose that $T \leq \tau$ in the (Q, T) policy. Then, we need to consider $\mathbb{E}[CL]$ and $\mathbb{E}[LS]$ in two cases $T < L$ and $T \geq L$ as follows.

If $T < L$:

$$\begin{aligned} \mathbb{E}[CL] = & \int_0^T T dG(z) + \int_T^\tau \left[TF_Q(T) + \frac{Q}{\lambda} [F_{Q+1}(z) - F_{Q+1}(T)] + z\bar{F}_Q(z) \right] dG(z) \\ & + \int_\tau^{\tau+T} [TF_Q(T - z + \tau) + (z - \tau)[F_Q(\tau) - F_Q(T - z + \tau)] \\ & + \frac{Q}{\lambda} [F_{Q+1}(\tau) - F_{Q+1}(T - z + \tau)] + z\bar{F}_Q(\tau)] dG(z) \\ & + \int_{\tau+T}^{\tau+L} \left[z - \tau F_Q(\tau) + \frac{Q}{\lambda} F_{Q+1}(\tau) \right] dG(z). \end{aligned} \quad (52)$$

$$\begin{aligned} \mathbb{E}[LS] = & \int_0^T [\lambda T - QF_{Q+1}(z) - \lambda z\bar{F}_Q(z)] dG(z) \\ & + \int_T^\tau [\lambda TF_Q(T) - QF_{Q+1}(T)] dG(z) \\ & + \int_\tau^{\tau+T} [\lambda TF_Q(T - z + \tau) - QF_{Q+1}(T - z + \tau) \\ & + \lambda(z - \tau)\bar{F}_Q(T - z + \tau)] dG(z) \\ & + \int_{\tau+T}^{\tau+L} \lambda(z - \tau) dG(z). \end{aligned} \quad (53)$$

If $T \geq L$:

$$\begin{aligned} \mathbb{E}[CL] = & \int_0^T T dG(z) + \int_T^\tau \left[TF_Q(T) + \frac{Q}{\lambda} [F_{Q+1}(z) - F_{Q+1}(T)] + z\bar{F}_Q(z) \right] dG(z) \\ & + \int_\tau^{\tau+L} [TF_Q(T - z + \tau) + (z - \tau)[F_Q(\tau) - F_Q(T - z + \tau)] \\ & + \frac{Q}{\lambda} [F_{Q+1}(\tau) - F_{Q+1}(T - z + \tau)] + z\bar{F}_Q(\tau)] dG(z). \end{aligned} \quad (54)$$

$$\begin{aligned} \mathbb{E}[LS] = & \int_0^T [\lambda T - QF_{Q+1}(z) - \lambda z\bar{F}_Q(z)] dG(z) \\ & + \int_T^\tau [\lambda TF_Q(T) - QF_{Q+1}(T)] dG(z) \\ & + \int_\tau^{\tau+L} [\lambda TF_Q(T - z + \tau) - QF_{Q+1}(T - z + \tau) \\ & + \lambda(z - \tau)\bar{F}_Q(T - z + \tau)] dG(z). \end{aligned} \quad (55)$$