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4-generated pseudo symmetric monomial curves with not Cohen–Macaulay tangent cones

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Abstract: In this article, standard bases of some toric ideals associated to 4-generated pseudo symmetric semigroups with not Cohen-Macaulay tangent cones at the origin are computed. As the tangent cones are not Cohen-Macaulay, nondecreasingness of the Hilbert function of the local ring was not guaranteed. Therefore, using these standard bases, Hilbert functions are explicitly computed as a step towards the characterization of Hilbert function. In addition, when the smallest integer satisfying $k(\alpha_2+1)<(k-1)\alpha_1+(k+1)\alpha_{21}+\alpha_3$ is 1, it is proved that the Hilbert function of the local ring is nondecreasing.

Key words: Hilbert function, tangent cone, monomial curve, numerical semigroup, standard bases

1. Introduction

Let $n_1<n_2<\cdots<n_k$ be positive integers with $\gcd(n_1,\ldots,n_k)=1$ and let S be the numerical semigroup $S=\langle n_1,\ldots,n_k\rangle=\{\sum_{i=1}^{k}u_in_i|u_i\in\mathbb{N}\}$. K being an algebraically closed field, let $K[S]=K[t^{n_1},t^{n_2},\ldots,t^{n_k}]$ be the semigroup ring of S and $A=K[X_1,X_2,\ldots,X_k]$. If $\varphi:A\longrightarrow K[S]$ with $\varphi(X_i)=t^{n_i}$ and $\ker\varphi=I_S$, then $K[S]\simeq A/I_S$. If we denote the affine curve with parametrization

$$X_1=t^{n_1},\quad X_2=t^{n_2},\ \ldots,\ X_k=t^{n_k}$$

corresponding to S with C_S, then I_S is called the defining ideal of C_S. The smallest integer n_1 in the semigroup is called the *multiplicity* of C_S. Denote the corresponding local ring with $R_S=K[[t^{n_1},\ldots,t^{n_k}]]$ and the maximal ideal with $\mathfrak{m}=\langle t^{n_1},\ldots,t^{n_k}\rangle$. Then $gr_{\mathfrak{m}}(R_S)=\bigoplus_{i=0}^{\infty}\mathfrak{m}^i/\mathfrak{m}^{i+1}\cong A/I_S^*$, is the associated graded ring where $I_S^*=\langle f^*|f\in I_S\rangle$ with f^* denoting the least homogeneous summand of f.

The Hilbert function $H_{R_S}(n)$ of the local ring R_S is defined to be the Hilbert function of the associated graded ring $gr_{\mathfrak{m}}(R_S)=\bigoplus_{i=0}^{\infty}\mathfrak{m}^i/\mathfrak{m}^{i+1}$. In other words,

$$H_{R_S}(n)=H_{gr_{\mathfrak{m}}(R_S)}(n)=dim_{R_S/\mathfrak{m}}(\mathfrak{m}^n/\mathfrak{m}^{n+1})\ \ n\geq 0.$$

This function is called nondecreasing if $H_{R_S}(n)\geq H_{R_S}(n-1)$ for all $n\in\mathbb{N}$. If $\exists l\in\mathbb{N}$ such that $H_{R_S}(l)<H_{R_S}(l-1)$ then it is called decreasing at level l. The Hilbert series of R_S is defined to be the generating

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function

$$HS_{R_S}(t) = \sum_{n\in\mathbb{N}} H_{R_S}(n)t^n.$$

By the Hilbert–Serre theorem it can also be written as: $HS_{R_S}(t) = \frac{P(t)}{(1-t)^k} = \frac{Q(t)}{(1-t)^d}$, where $P(t)$ and $Q(t)$ are polynomials with coefficients in $\mathbb{Z}$ and d is the Krull dimension of R_S. $P(t)$ is called first Hilbert series and $Q(t)$ is called second Hilbert series, [6, 14]. It is also known that there is a polynomial $P_{R_S}(n) \in \mathbb{Q}[n]$ called Hilbert polynomial of R_S such that $H_{R_S}(n) = P_{R_S}(n)$ for all $n \geq n_0$, for some $n_0 \in \mathbb{N}$. The smallest n_0 satisfying this condition is the regularity index of the Hilbert function of R_S. A natural question is whether the Hilbert function of the local ring is nondecreasing. In general Cohen-Macaulayness of a one dimensional local ring does not guarantee the nondecreasingness of its Hilbert function for embedding dimensions greater than three. However, it is known that if the tangent cone is Cohen–Macaulay, then the Hilbert function of the local ring is nondecreasing. When the tangent cone is not Cohen–Macaulay, the Hilbert function of the local ring may decrease at some level. Rossi's conjecture states that "The Hilbert function of a Gorenstein local ring of dimension one is nondecreasing". This conjecture is still open in embedding dimension 4 even for monomial curves. It is known that the local ring corresponding to a monomial curve is Gorenstein iff the corresponding numerical semigroup is symmetric, [10]. For symmetric semigroups in affine 4-space, under the condition "$\alpha_2 \leq \alpha_{21} + \alpha_{24}$", Arslan and Mete showed that the tangent cone is Cohen–Macaulay in [1]. They proved that the conjecture is true for 4-generated monomial curves with symmetric semigroup under this condition. The conjecture is open for local rings corresponding to 4-generated symmetric semigroups with $\alpha_2 > \alpha_{21} + \alpha_{24}$. For some recent work on the monotonicity of the Hilbert function, see [1–3, 11–13].

Since symmetric and pseudo-symmetric semigroups are maximal with respect to inclusion with fixed genus, a natural question is whether Rossi's conjecture is even true for local rings corresponding to 4-generated pseudo-symmetric semigroups. In [15], we showed that if $\alpha_2 \leq \alpha_{21} + 1$, then the tangent cone is Cohen–Macaulay, and hence, the Hilbert function of the local ring is nondecreasing. In this paper, we focus on the open case of 4-generated pseudo symmetric monomial curves with $\alpha_2 > \alpha_{21} + 1$. For the computation of the first Hilbert series of the tangent cone, a standard basis computation with the algorithm in [4] will be used.

Recall from [9] that a 4-generated semigroup $S = \langle n_1, n_2, n_3, n_4\rangle$ is pseudo-symmetric if and only if there are integers $\alpha_i > 1$, for $1 \leq i \leq 4$, and $\alpha_{21} > 0$ with $0 < \alpha_{21} < \alpha_1 - 1$, such that

$$\begin{aligned}
n_1 &= \alpha_2\alpha_3(\alpha_4 - 1) + 1,\\
n_2 &= \alpha_{21}\alpha_3\alpha_4 + (\alpha_1 - \alpha_{21} - 1)(\alpha_3 - 1) + \alpha_3,\\
n_3 &= \alpha_1\alpha_4 + (\alpha_1 - \alpha_{21} - 1)(\alpha_2 - 1)(\alpha_4 - 1) - \alpha_4 + 1,\\
n_4 &= \alpha_1\alpha_2(\alpha_3 - 1) + \alpha_{21}(\alpha_2 - 1) + \alpha_2.
\end{aligned}$$

Then, the toric ideal is $I_S = \langle f_1, f_2, f_3, f_4, f_5\rangle$ with

$$\begin{aligned}
f_1 &= X_1^{\alpha_1} - X_3X_4^{\alpha_4-1}, \qquad f_2 = X_2^{\alpha_2} - X_1^{\alpha_{21}}X_4, \quad f_3 = X_3^{\alpha_3} - X_1^{\alpha_1-\alpha_{21}-1}X_2,\\
f_4 &= X_4^{\alpha_4} - X_1X_2^{\alpha_2-1}X_3^{\alpha_3-1}, \quad f_5 = X_1^{\alpha_{21}+1}X_3^{\alpha_3-1} - X_2X_4^{\alpha_4-1}.
\end{aligned}$$

If $n_1 < n_2 < n_3 < n_4$ then it is known from [15] that

(1) $\alpha_1 > \alpha_4$

(2) $\alpha_3 < \alpha_1 - \alpha_{21}$

(3) $\alpha_4 < \alpha_2 + \alpha_3 - 1$

and these conditions completely determine the leading monomials of f_1, f_3 and f_4. Indeed, $\mathrm{LM}(f_1) = X_3X_4^{\alpha_4-1}$ by (1), $\mathrm{LM}(f_3) = X_3^{\alpha_3}$ by (2), $\mathrm{LM}(f_4) = X_4^{\alpha_4}$ by (3) If we also let

(4) $\alpha_2 > \alpha_{21} + 1$

then $\mathrm{LM}(f_2) = X_1^{\alpha_{21}}X_4$ by (4). To determine the leading monomial of f_5 we need the following remark.

Remark 1.1 *Let $n_1 < n_2 < n_3 < n_4$. Then (4) implies*

(5) $\alpha_{21} + \alpha_3 > \alpha_4$

Proof Assume to the contrary $\alpha_{21} + \alpha_3 \le \alpha_4$. Then, $\alpha_4 = \alpha_{21} + \alpha_3 + n$, $(*)$, for some nonnegative n. Then from (1), $\alpha_1 = \alpha_{21} + \alpha_3 + n + m$ $(**)$, for some positive m and also $\alpha_2 = \alpha_{21} + 1 + k$ $(***)$, for some positive k by (4). Then from $n_1 < n_2$ we have:
$\alpha_2\alpha_3(\alpha_4 - 1) + 1 < \alpha_{21}\alpha_3\alpha_4 + (\alpha_1 - \alpha_{21} - 1)(\alpha_3 - 1) + \alpha_3$. Using $(***)$, we have:
$\alpha_{21}\alpha_3\alpha_4 + (1 + k)\alpha_3\alpha_4 - \alpha_3(\alpha_{21} + 1 + k) + 1 < \alpha_{21}\alpha_3\alpha_4 + (\alpha_1 - \alpha_{21} - 1)(\alpha_3 - 1)\alpha_3$,
$\alpha_3((1 + k)\alpha_4 - \alpha_{21} - 1 - k - \alpha_1 + \alpha_{21} + 1 - 1) + 1 < 1 + \alpha_{21} - \alpha_1$. Then from $(**)$:
$\alpha_3(\alpha_4 + k\alpha_4 - 1 - k - \alpha_1) < -\alpha_3 - n - m$. Using $(*)$ and $(**)$ again, we get:
$\alpha_3(m + k\alpha_4 - k) < -n - m < 0$. As $\alpha_3 > 0$, we have $m + k(\alpha_4 - 1) < 0$ which is a contradiction as each term in the sum is positive. □

Remark 1.2 *Let $n_1 < n_2 < n_3 < n_4$. Then (1) and (4) implies*

(6) $\alpha_1 + \alpha_{21} + 1 \ge \alpha_2 + \alpha_4$

Proof Assume to the contrary $\alpha_1 + \alpha_{21} + 1 < \alpha_2 + \alpha_4$. We know from (1) and (4) that $\alpha_1 = \alpha_4 + m$ and $\alpha_2 = \alpha_{21} + 1 + k$, for some positive m and k. Hence

$$\begin{aligned} \alpha_1 + \alpha_{21} + 1 < \alpha_2 + \alpha_4 \quad &\Rightarrow \quad \alpha_4 + m + \alpha_{21} + 1 < \alpha_{21} + 1 + k + \alpha_4 \\ &\Rightarrow \quad m < k \end{aligned}$$

On the other hand, from $n_1 < n_2$ we have:
$\alpha_2\alpha_3(\alpha_4 - 1) + 1 < \alpha_{21}\alpha_3\alpha_4 + (\alpha_1 - \alpha_{21} - 1)(\alpha_3 - 1) + \alpha_3$. From (1) and (4):
$(\alpha_{21} + 1 + k)\alpha_3(\alpha_4 - 1) + 1 < \alpha_{21}\alpha_3\alpha_4 + (\alpha_4 - \alpha_{21} + m - 1)(\alpha_3 - 1) + \alpha_3$. Expanding this, we obtain
$k\alpha_3\alpha_4 + \alpha_4 + m < \alpha_3(m + 1 + k) + \alpha_{21}$. Now as $\alpha_4 + m = \alpha_1$ and $\alpha_{21} < \alpha_1$:
$k\alpha_3\alpha_4 + \alpha_1 < \alpha_3(m + 1 + k) + \alpha_{21} < \alpha_3(m + 1 + k) + \alpha_1 \Rightarrow k\alpha_3\alpha_4 < \alpha_3(m + k + 1) \Rightarrow k\alpha_4 < m + k + 1$
$\Rightarrow k(\alpha_4 - 1) < m + 1 < k + 1$ which is a contradiction as $\alpha_4 - 1 \ge 1$.Hence, $\alpha_1 + \alpha_{21} + 1 \ge \alpha_2 + \alpha_4$.

□

Being able to determine the leading monomials of the generators of I_S in the open case, lots of different possibilities must be considered for the leading monomials of the s-polynomials appearing in standard bases computation. Unfortunately, there is not a general form for the standard basis if $\alpha_2 > \alpha_{21} + 1$ contrary to the case $\alpha_2 \le \alpha_{21} + 1$. Though there are five elements in minimal standard basis of I_S if $\alpha_2 \le \alpha_{21} + 1$ (see [15]), the number of elements in the standard basis increases as α_4 increases if $\alpha_2 > \alpha_{21} + 1$.

Example 1.3 *The following examples are done using SINGULAR*[*].

- *Standard basis for* $\alpha_{21}=8, \alpha_1=16, \alpha_2=20, \alpha_3=7, \alpha_4=2$ *is* $\{X_1^{16}-X_3X_4, X_2^{20}-X_1^8X_4, X_3^7-X_1^7X_2, X_4^2-X_1X_2^{19}X_3^6, X_1^9X_3^6-X_2X_4, X_1^{24}-X_2^{20}X_3, X_1^{17}X_3^6-X_2^{21}\}$

- *Standard basis for* $\alpha_{21}=2, \alpha_1=9, \alpha_2=5, \alpha_3=3, \alpha_4=3$ *is* $\{X_3^3-X_1^6X_2, X_1^2X_4-X_2^5, X_2X_4^2-X_1^3X_3^2, X_3X_4^2-X_1^9, X_4^3-X_1X_2^4X_3^2, X_1^5X_3^2-X_2^6X_4, X_2^5X_3X_4-X_1^{11}, X_2^{10}X_3-X_1^{13}, X_2^{16}X_4-X_1^{18}X_3, X_2^{21}-X_1^{20}X_3\}$

- *Standard basis for* $\alpha_{21}=8, \alpha_1=16, \alpha_2=11, \alpha_3=3, \alpha_4=5$ *is* $\{X_3^3-X_1^7X_2, X_2X_4^4-X_1^9X_3^2, X_3X_4^4-X_1^{16}, X_4^5-X_1X_2^{10}X_3^2, X_1^8X_4-X_2^{11}, X_2^{12}X_4^3-X_1^{17}X_3^2, X_2^{11}X_3X_4^3-X_1^{24}, X_2^{23}X_4^2-X_1^{25}X_3^2, X_2^{22}X_3X_4^2-X_1^{32}, X_1^{33}X_3^2-X_2^{34}X_4, X_2^{33}X_3X_4-X_1^{40}, X_2^{44}X_3-X_1^{48}, X_2^{78}X_4-X_1^{81}X_3, X_2^{89}-X_1^{89}X_3\}$

We focus on the case $\alpha_4=2$.

2. Standard bases

Theorem 2.1 *Let* $S=\langle n_1,n_2,n_3,n_4\rangle$ *be a 4-generated pseudosymmetric numerical semigroup with* $n_1<n_2<n_3<n_4$ *and* $\alpha_2>\alpha_{21}+1$. *If* $\alpha_4=2$ *and* k *is the smallest positive integer such that* $k(\alpha_2+1)<(k-1)\alpha_1+(k+1)\alpha_{21}+\alpha_3$ *then the standard basis for* I_S *is*

$$\{f_1,f_2,f_3,f_4,f_5,f_6,...,f_{6+k}\}$$

where $f_6=X_1^{\alpha_1+\alpha_{21}}-X_2^{\alpha_2}X_3$ *and* $f_{6+j}=X_1^{(j-1)\alpha_1+(j+1)\alpha_{21}+1}X_3^{\alpha_3-j}-X_2^{j\alpha_2+1}$ *for* $j=1,2,...,k$

Proof We will prove the theorem by using induction on k and applying standard basis algorithm with NFMora as the normal form algorithm, see [6]. Here $G=\{f_1,f_2,f_3,f_4,f_5,f_6,...,f_{6+k}\}$ and T_h denotes the set $\{g\in G:\mathrm{LM}(g)\mid \mathrm{LM}(h)\}$ and ecart(h) is $deg(h)-deg(\mathrm{LM}(h))$. Note that $\mathrm{LM}(f_6)=X_2^{\alpha_2}X_3$ by (6).
Before we start the basis step of the induction, note that, we have

$$k(\alpha_2+1)<(k-1)\alpha_1+(k+1)\alpha_{21}+\alpha_3 \Leftrightarrow \underbrace{k(\alpha_1+\alpha_{21}-\alpha_2-1)>\alpha_1-\alpha_{21}-\alpha_3}_{(A)}.$$

We know from (6) that $(\alpha_1+\alpha_{21}-\alpha_2-1)\geq 0$. When $(\alpha_1+\alpha_{21}-\alpha_2-1)=0$, then it follows from (A) that $\alpha_1-\alpha_{21}-\alpha_3<0$. However, this contradicts with (2), which says that $\alpha_3<\alpha_1-\alpha_{21}$. Therefore, we conclude that after defining k as in the statement of Theorem 2.1, (A) and $\alpha_1+\alpha_{21}-\alpha_2-1=0$ can not hold at the same time under the assumptions that we have done.

<u>For $k=1$:</u>

In this case $f_7=X_1^{2\alpha_{21}+1}X_3^{\alpha_3-1}-X_2^{\alpha_2+1}$ and $\alpha_2+1<2\alpha_{21}+\alpha_3$ which implies that $\mathrm{LM}(f_7)=X_2^{\alpha_2+1}$. We need to show that $NF(\mathrm{spoly}(f_m,f_n)|G)=0$ for all m,n with $1\leq m<n\leq 7$.

- spoly$(f_1,f_2)=f_6$ and hence $NF(\mathrm{spoly}(f_1,f_2)|G)=0$

[*]Singular 2.0 (Year). A Computer Algebra System for Polynomial Computations [online]. Website http://www.singular.uni-kl.de [accessed 00 Month Year].

- spoly$(f_1, f_3) = X_1^{\alpha_1} X_3^{\alpha_3 - 1} - X_1^{\alpha_1 - \alpha_{21} - 1} X_2 X_4$ and LM(spoly$(f_1, f_3)) = X_1^{\alpha_1 - \alpha_{21} - 1} X_2 X_4$ by (5). Let $h_1 =$ spoly(f_1, f_3). If $\alpha_1 < 2\alpha_{21} + 1$ or $2\alpha_{21} + \alpha_3 \leq \alpha_2 + 1$ then $T_{h_1} = \{f_5\}$ and since spoly$(h_1, g) = 0$, $NF($spoly$(f_1, f_3)|G) = 0$. Otherwise $T_{h_1} = \{f_2\}$ and spoly$(h_1, f_2) = X_1^{\alpha_1 - 2\alpha_{21} - 1} X_2^{\alpha_2 + 1} - X_1^{\alpha_1} X_3^{\alpha_3 - 1}$. Set $h_2 =$ spoly(h_1, f_2), LM$(h_2) = X_1^{\alpha_1 - 2\alpha_{21} - 1} X_2^{\alpha_2 + 1}$ and $T_{h_2} = \{f_7\}$ and spoly$(h_2, f_7) = 0$, hence $NF($spoly$(f_1, f_3)|G) = 0$.
- spoly$(f_1, f_4) = X_1^{\alpha_1} X_4 - X_1 X_2^{\alpha_2 - 1} X_3^{\alpha_3}$. Set $h_1 =$ spoly(f_1, f_4). If LM$(h_1) = X_1^{\alpha_1} X_4$ then $T_{h_1} = \{f_2\}$ and spoly$(h_1, f_2) = X_1 X_2^{\alpha_2 - 1} f_3$. If LM$(h_1) = X_1 X_2^{\alpha_2 - 1} X_3^{\alpha_3}$ then $T_{h_1} = \{f_3\}$ and spoly$(h_1, f_3) = X_1^{\alpha_1 - \alpha_{21}} f_2$. Hence in both cases, $NF($spoly$(f_1, f_4)|G) = 0$
- spoly$(f_1, f_5) = X_1^{\alpha_{21}+1} X_3^{\alpha_3} - X_1^{\alpha_1} X_2 = X_1^{\alpha_{21}+1} f_3$ hence $NF($spoly$(f_1, f_5)|G) = 0$
- spoly$(f_1, f_6) = X_1^{\alpha_1} f_2$ and hence $NF($spoly$(f_1, f_6)|G) = 0$
- spoly$(f_1, f_7) = X_1^{\alpha_1 + 2\alpha_{21} + 1} X_3^{\alpha_3 - 2} - X_2^{\alpha_2 + 1} X_4$ if $\alpha_3 + 2\alpha_{21} < \alpha_2 + 1$. Set $h_1 =$ spoly(f_1, f_7). Using (6) and the fact that $\alpha_{21} + \alpha_3 > 2$, we can conclude that LM$(h_1) = X_2^{\alpha_2 + 1} X_4$ and $T_{h_1} = \{f_5\}$ then spoly$(h_1, f_5) = X_1^{\alpha_{21}+1} X_3^{\alpha_3 - 2} f_6$ and hence $NF($spoly$(f_1, f_7)|G) = 0$.
  $NF($spoly$(f_1, f_7)|G) = 0$ if $\alpha_3 + 2\alpha_{21} > \alpha_2 + 1$, as LM$(f_1)$ and LM(f_7) are relatively prime.
- $NF($spoly$(f_2, f_3)|G) = 0$ as LM(f_2) and LM(f_3) are relatively prime.
- spoly$(f_2, f_4) = X_2^{\alpha_2 - 1} f_5$ and hence $NF($spoly$(f_2, f_4)|G) = 0$
- spoly$(f_2, f_5) = f_7$ and hence $NF($spoly$(f_2, f_5)|G) = 0$
- $NF($spoly$(f_2, f_6)|G) = 0$ as LM(f_2) and LM(f_6) are relatively prime.
- spoly$(f_2, f_7) = X_2^{\alpha_2} f_5$ if $2\alpha_{21} + \alpha_3 < \alpha_2 + 1$. Otherwise LM$(f_2)$ and LM(f_7) are relatively prime. Hence, in both cases $NF($spoly$(f_2, f_7)|G) = 0$.
- $NF($spoly$(f_3, f_4)|G) = 0$ as LM(f_3) and LM(f_4) are relatively prime.
- $NF($spoly$(f_3, f_5)|G) = 0$ as LM(f_3) and LM(f_4) are relatively prime.
- spoly$(f_3, f_6) = X_1^{\alpha_1 - \alpha_{21} - 1} f_6$ and hence $NF($spoly$(f_3, f_6)|G) = 0$
- spoly$(f_3, f_7) = X_2 f_6$ if $2\alpha_{21} + \alpha_3 < \alpha_2 + 1$. Otherwise LM$(f_3)$ and LM(f_7) are relatively prime. Hence, in both cases $NF($spoly$(f_3, f_7)|G) = 0$
- spoly$(f_4, f_5) = X_1 X_3^{\alpha_3 - 1} f_2$ and hence $NF($spoly$(f_4, f_5)|G) = 0$
- $NF($spoly$(f_4, f_6)|G) = 0$ as LM(f_4) and LM(f_6) are relatively prime.
- $NF($spoly$(f_4, f_7)|G) = 0$ as LM(f_4) and LM(f_7) are relatively prime.
- spoly$(f_5, f_6) = X_1^{\alpha_{21}+1} X_2^{\alpha_2 - 1} X_3^{\alpha_3} - X_1^{\alpha_1 + \alpha_{21}} X_4$ and let $h_1 =$ spoly(f_5, f_6). If LM$(h_1) = X_1^{\alpha_{21}+1} X_2^{\alpha_2 - 1} X_3^{\alpha_3}$ then $T_{h_1} = \{f_3\}$ and spoly$(h_1, f_3) = X_1^{\alpha_1 - \alpha_{21} - 1} f_2$ and hence $NF($spoly$(f_5, f_6)|G) = 0$. If LM$(h_1) = X_1^{\alpha_1 + \alpha_{21}} X_4$ then $T_{h_1} = \{f_2\}$ and spoly$(h_1, f_2) = X_2^{\alpha_2 - 1} f_3$ and hence $NF($spoly$(f_5, f_6)|G) = 0$

- $NF(\text{spoly}(f_5, f_7)|G) = 0$ if $\alpha_3 + 2\alpha_{21} < \alpha_2 + 1$, as $\text{LM}(f_5)$ and $\text{LM}(f_7)$ are relatively prime. Otherwise $\text{spoly}(f_5, f_7) = X_1^{\alpha_{21}+1}X_3^{\alpha_3-1}f_2$ and hence $NF(\text{spoly}(f_5, f_7)|G) = 0$

- $\text{spoly}(f_6, f_7) = f_8$ if $\alpha_3 + 2\alpha_{21} < \alpha_2 + 1$ and hence $NF(\text{spoly}(f_6, f_7)|G) = 0$. Otherwise $\text{spoly}(f_6, f_7) = X_1^{2\alpha_{21}+1}f_3$ and hence $NF(\text{spoly}(f_6, f_7)|G) = 0$

<u>Assume the statement is true for $k < l$:</u>

If k is the smallest positive integer such that $k(\alpha_2 + 1) < (k-1)\alpha_1 + (k+1)\alpha_{21} + \alpha_3$ then the standard basis for I_S is

$$\{f_1, f_2, f_3, f_4, f_5, f_6, ..., f_{6+k}\}$$

where $f_6 = X_1^{\alpha_1+\alpha_{21}} - X_2^{\alpha_2}X_3$ and $f_{6+j} = X_1^{(j-1)\alpha_1+(j+1)\alpha_{21}+1}X_3^{\alpha_3-j} - X_2^{j\alpha_2+1}$ for $j = 1, 2, ..., k$.

<u>For $k = l$:</u>

Now let l be the smallest positive integer such that $l(\alpha_2 + 1) < (l-1)\alpha_1 + (l+1)\alpha_{21} + \alpha_3$. Note that for any $j < l$, $j(\alpha_2 + 1) \geq (j-1)\alpha_1 + (j+1)\alpha_{21} + \alpha_3$ and $\text{LM}(f_{6+j}) = X_1^{(j-1)\alpha_1+(j+1)\alpha_{21}+1}X_3^{\alpha_3-j}$ for $j = 1, 2, ..., l-1$ and $\text{LM}(f_{6+l}) = X_2^{l\alpha_2+1}$. Note also that $NF(\text{spoly}(f_m, f_n)|G) = 0$ for $1 \leq m < n \leq 6$ from the basis step since $\text{LM}(f_7)$ is not involved in calculations. Hence it is enough to prove $NF(\text{spoly}(f_m, f_n)|G) = 0$ for all other $1 \leq m < n \leq 6 + l$

- $\text{spoly}(f_1, f_{6+j}) = X_1^{j\alpha_1+(j+1)\alpha_{21}}X_3^{\alpha_3-j-1} - X_2^{j\alpha_2+1}X_4 = h_1$. $\text{LM}(h_1) = X_2^{j\alpha_2+1}X_4$ by (5) and (6) and $T_{h_1} = \{f_5\}$. $\text{spoly}(h_1, f_5) = X_1^{\alpha_{21}+1}X_2^{j\alpha_2}X_3^{\alpha_3-1} - X_1^{j\alpha_1+(j+1)\alpha_{21}+1}X_3^{\alpha_3-j-1} = h_2$ and $\text{LM}(h_2) = X_1^{\alpha_{21}+1}X_2^{j\alpha_2}X_3^{\alpha_3-1}$ by (6). As a result, $T_{h_2} = \{f_6\}$ and $\text{spoly}(h_2, f_6) = X_1^{\alpha_1+2\alpha_{21}}X_2^{(j-1)\alpha_2}X_3^{\alpha_3-2} - X_1^{j\alpha_1+(j+1)\alpha_{21}+1}X_3^{\alpha_3-j-1} = h_3$, $\text{LM}(h_3) = X_1^{\alpha_1+2\alpha_{21}}X_2^{(j-1)\alpha_2}X_3^{\alpha_3-2}$ by (6). $T_{h_3} = \{f_6\}$ and continuing inductively, $h_{j+1} = \text{spoly}(h_j, f_6) = X_1^{j\alpha_1+(j+1)\alpha_{21}}X_3^{\alpha_3-3}f_6$. Hence, $NF(\text{spoly}(f_1, f_{6+j})|G) = 0$

- $\text{spoly}(f_2, f_{6+j}) = X_1^{(j-1)\alpha_1+j\alpha_{21}+1}X_2^{\alpha_2}X_3^{\alpha_3-j} - X_2^{j\alpha_2+1}X_4 = h_1$. $\text{LM}(h_1) = X_2^{j\alpha_2+1}X_4$ by (5) and (6) and $T_{h_1} = \{f_5\}$. $\text{spoly}(h_1, f_5) = X_1^{\alpha_{21}+1}X_2^{j\alpha_2}X_3^{\alpha_3-1} - X_1^{(j-1)\alpha_1+j\alpha_{21}+1}X_2^{\alpha_2}X_3^{\alpha_3-j} = h_2$. $\text{LM}(h_2) = X_1^{\alpha_{21}+1}X_2^{j\alpha_2}X_3^{\alpha_3-1}$ by (6). As a result, $T_{h_2} = \{f_6\}$ and $\text{spoly}(h_2, f_6) = X_1^{\alpha_1+2\alpha_{21}+1}X_2^{(j-1)\alpha_2}X_3^{\alpha_3-2} - X_1^{(j-1)\alpha_1+j\alpha_{21}+1}X_2^{\alpha_2}X_3^{\alpha_3-j} = h_3$. $\text{LM}(h_3) = X_1^{\alpha_1+2\alpha_{21}+1}X_2^{(j-1)\alpha_2}X_3^{\alpha_3-2}$ by (6). $T_{h_3} = \{f_6\}$ and continuing inductively, we obtain $h_j = \text{spoly}(h_{j-1}, f_6) = X_1^{(j-2)\alpha_1+(j-1)\alpha_{21}+1}X_2^{\alpha_2}X_3^{\alpha_3-j}f_6$. Hence, $NF(\text{spoly}(f_2, f_{6+j})|G) = 0$

- $\text{spoly}(f_3, f_{6+j}) = X_1^{j\alpha_1+j\alpha_2}X_2 - X_2^{j\alpha_2+1}X_3^j = h_1$. $\text{LM}(h_1) = X_2^{j\alpha_2+1}X_3^j$ by (6) and $T_{h_1} = \{f_6\}$. $\text{spoly}(h_1, f_6) = X_1^{\alpha_1+\alpha_{21}}X_2^{(j-1)\alpha_2+1}X_3^{j-1} - X_1^{j\alpha_1+j\alpha_{21}}X_2 = h_2$. $\text{LM}(h_2) = X_2^{(j-1)\alpha_2+1}X_3^{j-1}$ by (6) and $T_{h_2} = \{f_6\}$. Continuing inductively $\text{spoly}(h_{j-1}, f_6) = X_1^{(j-1)(\alpha_1+\alpha_{21})}X_2f_6 = h_j$ and hence $NF(\text{spoly}(f_3, f_{6+j})|G) = 0$

- $NF(\text{spoly}(f_m, f_{6+j})|G) = 0$ for $m = 4, 5$ as $\text{LM}(f_m)$ and $\text{LM}(f_{6+j})$ are relatively prime for all $1 \leq j < l$

- $\text{spoly}(f_6, f_{6+j}) = f_{6+(j+1)}$ and hence $NF(\text{spoly}(f_6, f_{6+j})|G) = 0$ for all $1 \leq j < l$

- $\text{spoly}(f_{6+i}, f_{6+j}) = X_1^{(j-i)\alpha_1+(j-i)\alpha_{21}} X_2^{i\alpha_2+1} - X_2^{j\alpha_2+1} X_3^{j-i} = h_1$. $\text{LM}(h_1) = X_2^{j\alpha_2+1} X_3^{j-i}$ by (6). $T_{h_1} = \{f_6\}$, $\text{spoly}(h_1, f_6) = X_1^{\alpha_1+\alpha_{21}} X_2^{(j-1)\alpha_2+1} X_3^{j-i-1} - X_1^{(j-i)(\alpha_1+\alpha_{21})} X_2^{i\alpha_2+1} = h_2$ and $\text{LM}(h_2) = X_1^{\alpha_1+\alpha_{21}} X_2^{(j-1)\alpha_2+1} X_3^{j-i-1}$ by (6) and $T_{h_2} = \{f_6\}$. Continuing inductively $h_{j-i} = \text{spoly}(h_{j-i-1}, f_6) = X_1^{(j-i-1)(\alpha_1+\alpha_{21})} X_2^{i\alpha_2+1} f_6$ and hence $NF(\text{spoly}(f_{6+i}, f_{6+j})|G) = 0$ for all $1 \leq i < j < l$.

- $NF(\text{spoly}(f_m, f_{6+l})|G) = 0$ for $m = 1, 2, 3, 4, 7, ..., 5+l$ as $\text{LM}(f_m)$ and $\text{LM}(f_{6+l})$ are relatively prime.

- $\text{spoly}(f_5, f_{6+l}) = X_1^{\alpha_{21}+1} X_2^{l\alpha_2} X_3^{\alpha_3-1} - X_1^{(l-1)\alpha_1+(l+1)\alpha_{21}+1} X_3^{\alpha_3-l} X_4$. Set $h_1 = \text{spoly}(f_5, f_{6+l})$. If $\text{LM}(h_1) = X_1^{\alpha_{21}+1} X_2^{l\alpha_2} X_3^{\alpha_3-1}$. Then $T_{h_1} = \{f_6\}$ and $\text{spoly}(h_1, f_6) = X_1^{\alpha_1+2\alpha_{21}+1} X_2^{(l-1)\alpha_2} X_3^{\alpha_3-2} - X_1^{(l-1)\alpha_1+(l+1)\alpha_{21}+1} X_3^{\alpha_3-l} X_4 = h_2$. $\text{LM}(h_2) = X_1^{(l-1)\alpha_1+(l+1)\alpha_{21}+1} X_3^{\alpha_3-l} X_4$ since $(l-1)(\alpha_2+1) \geq (l-2)\alpha_1 + l\alpha_{21} + \alpha_3$ and (5). Then $T_{h_2} = \{f_1, f_2\}$ and since $\text{ecart}(f_2)$ is minimal by (6), $\text{spoly}(h_2, f_2) = X_1^{\alpha_1+2\alpha_{21}+1} X_2^{(l-1)\alpha_2} X_3^{\alpha_3-2} - X_1^{(l-1)\alpha_1+l\alpha_{21}+1} X_2^{\alpha_2} X_3^{\alpha_3-l} = h_3$ and $\text{LM}(h_3) = X_1^{\alpha_1+2\alpha_{21}+1} X_2^{(l-1)\alpha_2} X_3^{\alpha_3-2}$ by (6). $T_{h_3} = \{f_6\}$ and $\text{spoly}(h_3, f_6) = X_1^{2\alpha_1+3\alpha_{21}+1} X_2^{(l-2)\alpha_2} X_3^{\alpha_3-3} - X_1^{(l-1)\alpha_1+l\alpha_{21}+1} X_2^{\alpha_2} X_3^{\alpha_3-l} = h_4$ and $\text{LM}(h_4) = X_1^{2\alpha_1+3\alpha_{21}+1} X_2^{(l-2)\alpha_2} X_3^{\alpha_3-3}$ by (6). $T_{h_4} = \{f_6\}$ and $\text{spoly}(h_4, f_6) = X_1^{3\alpha_1+4\alpha_{21}+1} X_2^{(l-3)\alpha_2} X_3^{\alpha_3-4} - X_1^{(l-1)\alpha_1+l\alpha_{21}+1} X_2^{\alpha_2} X_3^{\alpha_3-l} = h_5$ and $\text{LM}(h_5) = X_1^{3\alpha_1+4\alpha_{21}+1} X_2^{(l-3)\alpha_2} X_3^{\alpha_3-4}$ by (6) and $T_{h_5} = \{f_6\}$. Continuing inductively, $h_l = \text{spoly}(h_{l-1}, f_6) = X_1^{(l-2)\alpha_1+(l-1)\alpha_{21}+1} X_2^{\alpha_2} X_3^{\alpha_3-l} f_6$ and this implies that $NF(\text{spoly}(f_5, f_{6+l})|G) = 0$
  Otherwise $T_{h_1} = \{f_1, f_2\}$ and $\text{ecart}(f_2)$ is minimal by (6). $\text{spoly}(h_1, f_2) = X_1^{(l-1)\alpha_1+l\alpha_{21}+1} X_2^{\alpha_2} X_3^{\alpha_3-l} - X_1^{\alpha_{21}+1} X_2^{l\alpha_2} X_3^{\alpha_3-1} = h_2$ and $\text{LM}(h_2) = X_1^{\alpha_{21}+1} X_2^{l\alpha_2} X_3^{\alpha_3-1}$ by (6). Since $T_{h_2} = \{f_6\}$, we compute $\text{spoly}(h_2, f_6) = X_1^{(l-1)\alpha_1+l\alpha_{21}+1} X_2^{\alpha_2} X_3^{\alpha_3-l} - X_1^{\alpha_1+2\alpha_{21}+1} X_2^{(l-1)\alpha_2} X_3^{\alpha_3-2} = h_3$, $T_{h_3} = \{f_6\}$. Continuing inductively, $h_l = \text{spoly}(h_{l-1}, f_6) = X_1^{(l-1)\alpha_1+l\alpha_{21}+1} X_2^{\alpha_2} X_3^{\alpha_3-l} - X_1^{(l-2)\alpha_1+(l-1)\alpha_{21}+1} X_2^{2\alpha_2} X_3^{\alpha_3-l+1} = X_1^{(l-2)\alpha_1+(l-1)\alpha_{21}+1} X_2^{\alpha_2} X_3^{\alpha_3-l} f_6$ and hence $NF(\text{spoly}(f_5, f_{6+l})|G) = 0$

- $\text{spoly}(f_6, f_{6+l}) = X_1^{\alpha_1+\alpha_{21}} f_{6+(l-1)}$ hence, $NF(\text{spoly}(f_6, f_{6+l})|G) = 0$.

Since all normal forms reduce to zero, $\{f_1, f_2, f_3, f_4, f_5, f_6, ..., f_{6+k}\}$ is a standard basis for I_C □

Corollary 2.2 *$\{{f_1}^*, {f_2}^*, ..., {f_6}^*, ..., {f_{6+k}}^*\}$ is a standard basis for I_C^* where ${f_1}^* = X_3X_4$, ${f_2}^* = X_1^{\alpha_{21}} X_4$, ${f_3}^* = X_3^{\alpha_3}$, ${f_4}^* = X_4^2$, ${f_5}^* = X_2X_4$, ${f_6}^* = X_2^{\alpha_2} X_3$, $f_{6+k}^* = X_2^{k\alpha_2+1}$ and ${f_{6+j}}^* = X_1^{(j-1)\alpha_1+(j+1)\alpha_{21}+1} X_3^{\alpha_3-j}$ for $j = 1, 2, ..., k-1$. Since $X_1 | {f_2}^*$, the tangent cone is not Cohen–Macaulay by the criterion given in [2] as expected.*

3. Hilbert function

Theorem 3.1 *The numerator of the Hilbert series of the local ring R_S is*

$$P(I_S^*) = 1 - 3t^2 + 3t^3 - t^4 - t^{\alpha_{21}+1}(1-t)^3 - t^{\alpha_3}(1-t) - t^{\alpha_2+1}(1-t)(1-t^{\alpha_3-1}) - t^{k\alpha_2+1}(1-t)^2 - r(t)$$

where $r(t) = 0$ if $k = 1$ and $r(t) = \sum_{j=2}^{k} t^{(k-j)\alpha_1+(k-j+2)\alpha_{21}+\alpha_3+j-k}(1-t)^2(1-t^{\alpha_2})$ otherwise.

Proof To compute the Hilbert series, we will use Algorithm 2.6 of [4] that is formed by continuous use of the proposition

"If I is a monomial ideal with $I =< J, w >$, then the numerator of the Hilbert series of A/I is $P(I) = h(J) - t^{\deg w} h(J : w)$ where w is a monomial and $\deg w$ is the total degree of w."

Let $P(I_S^*)$ denote the numerator of the Hilbert series of A/I_S^*

- Let $w_1 = X_2^{k\alpha_2+1}$, then

$$J_1 =< X_3X_4, X_1^{\alpha_{21}}X_4, X_3^{\alpha_3}, X_4^2, X_2X_4, X_2^{\alpha_2}X_3, X_1^{2\alpha_{21}+1}X_3^{\alpha_3-1}, ..., X_1^{(k-2)\alpha_1+(k)\alpha_{21}+1}X_3^{\alpha_3-k+1} >,$$

$$P(I_S^*) = P(J_1) - t^{k\alpha_2+1}P(< X_4, X_3 >) = P(J_1) - t^{k\alpha_2+1}(1-t)^2$$

If $k = 1$ then $w_2 = w_{k+1}$, otherwise:

- Let $w_2 = X_1^{(k-2)\alpha_1+(k)\alpha_{21}+1}X_3^{\alpha_3-k+1}$, then

$$J_2 =< X_3X_4, X_1^{\alpha_{21}}X_4, X_3^{\alpha_3}, X_4^2, X_2X_4, X_2^{\alpha_2}X_3, X_1^{2\alpha_{21}+1}X_3^{\alpha_3-1}, ..., X_1^{(k-3)\alpha_1+(k-1)\alpha_{21}+1}X_3^{\alpha_3-k+2} >,$$

$$\begin{aligned} P(J_1) &= P(J_2) - t^{(k-2)\alpha_1+k(\alpha_{21}-1)+\alpha_3+2}P(< X_4, X_2^{\alpha_2}, X_3 >) \\ &= P(J_2) - t^{(k-2)\alpha_1+k(\alpha_{21}-1)+\alpha_3+2}(1-t)^2(1-t^{\alpha_2}) \end{aligned}$$

- Continue inductively and let $w_k = X_1^{2\alpha_{21}+1}X_3^{\alpha_3-1}$, then

$$J_k =< X_3X_4, X_1^{\alpha_{21}}X_4, X_3^{\alpha_3}, X_4^2, X_2X_4, X_2^{\alpha_2}X_3 >,$$

$$P(J_{k-1}) = P(J_k) - t^{2\alpha_{21}+\alpha_3}P(< X_4, X_2^{\alpha_2}, X_3 >) = P(J_k) - t^{2\alpha_{21}+\alpha_3}(1-t)^2(1-t^{\alpha_2})$$

- Let $w_{k+1} = X_2^{\alpha_2}X_3$, then

$$J_{k+1} =< X_3X_4, X_1^{\alpha_{21}}X_4, X_3^{\alpha_3}, X_4^2, X_2X_4 >,$$

$$P(J_k) = P(J_{k+1}) - t^{\alpha_2+1}P(< X_4, X_3^{\alpha_3-1} >) = P(J_{k+1}) - t^{\alpha_2+1}(1-t)(1-t^{\alpha_3-1})$$

- Let $w_{k+2} = X_3^{\alpha_3}$, then

$$J_{k+2} =< X_3X_4, X_1^{\alpha_{21}}X_4, X_4^2, X_2X_4 >,$$

$$P(J_{k+1}) = P(J_{k+2}) - t^{\alpha_3}P(< X_4 >) = P(J_{k+2}) - t^{\alpha_3}(1-t)$$

- Let $w_{k+3} = X_1^{\alpha_{21}}X_4$, then

$$J_{k+3} =< X_3X_4, X_4^2, X_2X_4 >$$

$$P(J_{k+2}) = P(J_{k+3}) - t^{\alpha_{21}+1}h(< X_3, X_4, X_2 >) = P(J_{k+3}) - t^{\alpha_{21}+1}(1-t)^3$$

- Let $w_{k+4} = X_2X_4$, then

$$J_{k+4} =< X_3X_4, X_4^2 >$$

$$P(J_{k+3}) = P(J_{k+4}) - t^2P(< X_3, X_4 >) = P(J_{k+4}) - t^2(1-t)^2$$

- Let $w_{k+5} = X_4^2$, then

$$J_{k+5} =< X_3X_4 >$$

$$P(J_{k+4}) = P(J_{k+5}) - t^2P(< X_3 >) = (1-t^2) - t^2(1-t)$$

Hence, $P(I_S^*) = (1-t^2)-t^2(1-t)-t^2(1-t)^2-t^{\alpha_{21}+1}(1-t)^3-t^{\alpha_3}(1-t)-t^{\alpha_2+1}(1-t)(1-t^{\alpha_3-1})-t^{k\alpha_2+1}(1-t)^2-r(t)$ where $r(t) = 0$ if $k = 1$ and $r(t) = \sum_{j=2}^{k} t^{(k-j)\alpha_1+(k-j+2)\alpha_{21}+\alpha_3+j-k}(1-t)^2(1-t^{\alpha_2})$ if $k > 1$ □

Clearly, since the krull dimension is one, if there are no negative terms in the second Hilbert series, then the Hilbert function will be nondecreasing. We can state and prove the next theorem.

Theorem 3.2 *The local ring R_S has a nondecreasing Hilbert function if $k = 1$.*

Proof Observe that

$$P(I_S^*) = (1-t)P_1(t)$$

with $P_1(t) = 1+t-t^2-t^2(1-t)-t^{\alpha_{21}+1}(1-t)^2-t^{\alpha_3}-t^{\alpha_2+1}(1-t^{\alpha_3-1})-t^{k\alpha_2+1}(1-t)-r_1(t)$ where $r_1(t) = 0$ if $k = 1$ and $r_1(t) = \sum_{j=2}^{k} t^{(k-j)\alpha_1+(k-j+2)\alpha_{21}+\alpha_3+j-k}(1-t)(1-t^{\alpha_2})$ if $k > 1$.

$$P_1(t) = (1-t)P_2(t)$$

with $P_2(t) = 1+t+t(1+t+...+t^{\alpha_3-2})-t^2-t^{\alpha_{21}+1}(1-t)-t^{\alpha_2+1}(1+t+...+t^{\alpha_3-2})-t^{k\alpha_2+1}-r_2(t)$ where $r_2(t) = 0$ if $k = 1$ and $r_2(t) = \sum_{j=2}^{k} t^{(k-j)\alpha_1+(k-j+2)\alpha_{21}+\alpha_3+j-k}(1-t^{\alpha_2})$ if $k > 1$. Combining some terms, $P_2(t) = 1-t^2+t(1-t^{k\alpha_2})+t(1-t^{\alpha_2})(1+t+...+t^{\alpha_3-2})-t^{\alpha_{21}+1}(1-t)-r_2(t)$ which shows that,

$$P_2(t) = (1-t)Q(t)$$

with

$$Q(t) = 1+t+t(1+t+...+t^{k\alpha_2-1})+t(1+t+...+t^{\alpha_2-1})(1+t+...+t^{\alpha_3-2})-t^{\alpha_{21}+1}+r_3(t)$$

where $r_3(t) = 0$ if $k = 1$ and $r_3(t) = \sum_{j=2}^{k} t^{(k-j)\alpha_1+(k-j+2)\alpha_{21}+\alpha_3+j-k}(1+t+...+t^{\alpha_2-1})$ if $k > 1$.Then $HS_R(t) = \frac{P(t)}{(1-t)^4} = \frac{Q(t)}{(1-t)}$, and $Q(t) = 1+t+t(1+t+...+t^{k\alpha_2-1})+t(1+t+...+t^{\alpha_2-1})(1+t+...+t^{\alpha_3-2})-t^{\alpha_{21}+1}+r_3(t)$ is the second Hilbert series of the local ring. Note that $\alpha_2 > \alpha_{21}+1$ and $-t^{\alpha_{21}+1}$ disappear when we expand $Q(t)$. Hence for $k = 1$, there are no negative coefficients in the second Hilbert series, the Hilbert function is nondecreasing. □

Remark 3.3 *For $k > 1$,*

$$Q(t) = 1+t-t^{\alpha_{21}+1}+(1+t+...+t^{\alpha_2-1})\left[t(2+t+...+t^{\alpha_3-2})+\sum_{j=1}^{k-1} S_j(t)\right]$$

where $S_j(t) = t^{(j)\alpha_2+1} - t^{(j-1)\alpha_1+(j+1)\alpha_{21}+\alpha_3+1-j}$. Recall that k is the smallest positive integer such that $k(\alpha_2+1) < (k-1)\alpha_1+(k+1)\alpha_{21}+\alpha_3$ hence for any $1 \le j \le k-1$, we have $j(\alpha_2+1) \ge (j-1)\alpha_1+(j+1)\alpha_{21}+\alpha_3$, which means $S_j(t) \ge 0$ for every $t \in \mathbb{N}$.

4. Examples

The following examples are done via the computer algebra system SINGULAR.

Example 4.1 *For* $\alpha_{21} = 8, \alpha_1 = 16, \alpha_2 = 20, \alpha_3 = 7, \alpha_4 = 2$*,* $k = 1$ *and the corresponding standard basis is* $\{f_1 = X_1^{16} - X_3X_4, f_2 = X_2^{20} - X_1^8X_4, f_3 = X_3^7 - X_1^7X_2, f_4 = X_4^2 - X_1X_2^{19}X_3^6, f_5 = X_1^9X_3^6 - X_2X_4, f_6 = X_1^{24} - X_2^{20}X_3, f_7 = X_1^{17}X_3^6 - X_2^{21}\}$*. Numerator of the Hilbert series of the tangent cone is* $P(I_S^*) = 1 - 3t^2 + 3t^3 - t^4 - t^7 + t^8 - t^9 + 3t^{10} - 3t^{11} + t^{12} - 2t^{21} + 3t^{22} - t^{23} + t^{27} - t^{28}$*. Direct computation shows that the Hilbert function is nondecreasing.*

Example 4.2 *For* $\alpha_{21} = 4$*,* $\alpha_1 = 22, \alpha_2 = 13$*,* $\alpha_3 = 5$*,* $\alpha_4 = 2$*, we have* $k = 2$ *and the corresponding standard basis is* $\{f_1 = X_1^{22} - X_3X_4, f_2 = X_2^{13} - X_1^4X_4, f_3 = X_3^5 - X_1^{17}X_2, f_4 = X_4^2 - X_1X_2^{12}X_3^4, f_5 = X_1^5X_3^4 - X_2X_4, f_6 = X_1^{26} - X_2^{13}X_3, f_7 = X_1^9X_3^4 - X_2^{14}, f_8 = X_1^{35}X_3^3 - X_2^{27}\}$*. Numerator of the Hilbert series of the tangent cone is* $P(I_S^*) = 1 - 3t^2 + 3t^3 - t^4 - 2t^5 + 4t^6 - 3t^7 + t^8 - t^{13} + t^{14} + t^{18} - t^{19} + t^{26} - 3t^{27} + 3t^{28} - t^{29}$*. Direct computation shows that the Hilbert function is nondecreasing.*

Example 4.3 *For* $\alpha_{21} = 10$*,* $\alpha_1 = 17, \alpha_2 = 25$*,* $\alpha_3 = 4$*,* $\alpha_4 = 2$*, we have* $k = 3$ *and the corresponding standard basis is* $\{f_1 = X_1^{17} - X_3X_4, f_2 = X_2^{25} - X_1^{10}X_4, f_3 = X_3^4 - X_1^6X_2, f_4 = X_4^2 - X_1X_2^{24}X_3^3, f_5 = X_1^{11}X_3^3 - X_2X_4, f_6 = X_1^{27} - X_2^{25}X_3, f_7 = X_2^{26} - X_1^{21}X_3^3, f_8 = X_1^{48}X_3^2 - X_2^{51}, f_9 = X_1^{75}X_3 - X_2^{76}\}$*. Numerator of the Hilbert series of the tangent cone is* $P(I_S^*) = 1 - 3t^2 + 3t^3 - 2t^4 + t^5 - t^{11} + 3t^{12} - 3t^{13} + t^{14} - t^{24} + 2t^{25} - 2t^{26} + t^{27} + t^{29} - t^{30} + t^{49} - 3t^{50} + 3t^{51} - t^{52} + t^{75} - 3t^{76} + 3t^{77} - t^{78}$*. Direct computation shows that the Hilbert function is nondecreasing.*

Example 4.4 *For* $\alpha_{21} = 3$*,* $\alpha_1 = 13, \alpha_2 = 14$*,* $\alpha_3 = 6$*,* $\alpha_4 = 2$*, we have* $k = 4$ *and the corresponding standard basis becomes* $\{f_1 = X_1^{13} - X_3X_4, f_2 = X_2^{14} - X_1^3X_4, f_3 = X_3^6 - X_1^9X_2, f_4 = X_4^2 - X_1X_2^{13}X_3^5, f_5 = X_1^4X_3^5 - X_2X_4, f_6 = X_1^{16} - X_2^{14}X_3, f_7 = X_1^7X_3^5 - X_2^{15}, f_8 = X_1^{23}X_3^4 - X_2^{29}, f_9 = X_1^{39}X_3^3 - X_2^{43}, f_{10} = X_1^{55}X_3^2 - X_2^{57}\}$*. Numerator of the Hilbert series of the tangent cone is* $P(I_S^*) = 1 - 3t^2 + 3t^3 - 2t^4 + 3t^5 - 4t^6 + 2t^7 - t^{12} + 2t^{13} - t^{14} - t^{15} + t^{16} + t^{20} - t^{21} + t^{26} - 3t^{27} + 3t^{28} - t^{29} + t^{41} - 3t^{42} + 3t^{43} - t^{44} + t^{56} - 3t^{57} + 3t^{58} - t^{59}$*. Direct computation shows that the Hilbert function is nondecreasing.*

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