

# Ensuring multidimensional equality in public service

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## ABSTRACT

Service planning problems typically involve decisions that lead to the distribution of multiple benefits to multiple users, and hence include equality and efficiency concerns in a multidimensional way. We develop two mathematical modeling-based approaches that incorporate these concerns in such problems. The first formulation aggregates the multidimensional efficiency and equality (equitability) concerns in a biobjective model. The second formulation defines an objective function for each benefit, which maximizes the total social welfare obtained from that specific benefit distribution; this results in an  $n$ -objective model, where  $n$  is the number of benefits. We illustrate and compare these approaches on an example public service provision problem.

## 1. Introduction

The importance of the role of fairness in real-life decisions has been acknowledged in many operational research (OR) problems studied in recent years [34]. Especially in applications related to social welfare, including fairness in the proposed solution methods is a must. In line with this, there is a notable increase in the reported studies in the OR literature incorporating fairness concerns in various areas, such as supply chain [25,47], logistics [27], allocation problems [32,33], equitable choice [36], network [45] and action planning [24].

A significant challenge that occurs in such applications is the fact that fairness arises as an additional criterion to other, mostly system efficiency-related, criteria (such as total cost or total benefit), and that there a trade-off between these concerns. This calls for the use of multiobjective programming approaches, which enables the decision makers (DMs) to analyze such trade-offs. There are studies in the literature that acknowledge and address the trade-off between efficiency and fairness (equity) concerns (see e.g., Refs. [15,20,22,28,31,42,43]).

Fairness is a broad concept involving many dimensions. The DM's ideally fair allocation may correspond to one where every entity receives the same amount or to another allocation that considers different characteristics of the entities, such as their needs when defining a fair allocation [33]. In the first case, the underlying preference model of the DM satisfies anonymity as the identities of the entities become irrelevant (i.e., an equal allocation is considered as the most fair one by the DM). In the second case, however, anonymity no longer holds as the DM has a

clear desire to prioritize allocation to some of the entities. These two concepts are called equitability (or equality) and balance concerns, respectively [34]. In this study, we consider cases where the entities are indistinguishable; hence, the fairness related concern is an equality (equitability) concern.

If the decision maker considers the distribution of a single benefit to multiple users, each distribution alternative corresponds to a vector. This allocation vector presents the distribution of that single benefit to users/entities. In this case, there are single dimensional efficiency and equality concerns. On the other hand, if the decisions will lead to the allocation of multiple types of benefits to multiple users, there will be a concern for efficiency and equality for each benefit type. We hereby consider cases where the DM has equality (equitability) concerns over the allocation of multiple types of benefits to multiple users. We call such problems multidimensional equitable optimization problems. Most of the current work in the literature focuses on settings where a single benefit is allocated. This study extends these and contributes to the literature by suggesting ways of addressing these concerns in a multidimensional domain and demonstrating their potential use in public service provision.

The rest of the paper is organized as follows: In Section 2, we give a review of the literature and explain how our study extends the current works. In Section 3, we formally define the problem and provide the generic formulations of the two approaches that we consider for the problem. In Section 4, we demonstrate the usage of our approaches on a case study. The specifics of the case study are described, followed by a

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detailed analysis of the results. We conclude our paper in Section 5 and give recommendations for future research.

## 2. Literature review

The problem considered in this paper can be classified as a multi-dimensional equitable optimization problem. We first mention equitable optimization problems in which single benefit distributions are considered. Then we discuss the solution methods used in solving multi-objective resource allocation problems with only efficiency-related concerns.

Fig. 1 demonstrates a categorization of the relevant literature. We refer to problems aiming at equitable allocations of benefits (or resources) to a set of entities as equitable decision-making problems, which can be categorized into two main sets based on whether a choice or an optimization setting is considered. In the choice settings, the options (alternatives) are explicitly given; hence, the problem is choosing the alternative to implement (see e.g. Ref. [35], for a choice problem over alternative allocations of a single benefit and [36] for an extension to settings with multiple benefits).

When alternatives are implicitly defined by constraints, the problem becomes an equitable optimization problem. We can categorize our study under equitable optimization problems in a multibenefit domain (see Fig. 1). Most of the current work focuses on equitable optimization problems with single benefit concern, and there are not many works on the multiple benefit domain. For that reason, we look at the equitable optimization problems in the single benefit domain in the next section.

### 2.1. Equitable optimization problems (single benefit)

In these problems, if the decisions are associated with the distribution of a single benefit across multiple entities, then efficiency and equality concerns are single dimensional, and the allocation alternatives are vectors, showing how the benefit is distributed across entities. Most of the current literature is concerned with such settings and mainly relies on these three methods: using an (in)equality-related function in addition to an efficiency-related one; combining these two functions in a welfare function; or treating the problem as a multiobjective optimization problem and finding the equitable nondominated solutions (see Ref. [37]).

The first group of studies attempts to quantify the inequality degree of an allocation using specific indices, some of which are borrowed from the income inequality measurement literature. Our first approach follows this line of thought and is an index based approach. In this method, various measures of inequality are used to assess the equality level of a given distribution. In the mathematical models, this index is either optimized as an objective function or restricted through constraints in an efficiency maximizing setting. In the latter approach, the inequality measure is ensured to be less than a certain threshold. The most popular approach is using the amount that the worst-off entity gets as an indicator of inequality, which is called a Rawlsian approach [49]. Examples of that type of approach are seen in the hospital location problems, in which the total distance is minimized while limiting the longest distance to any neighborhood by a certain threshold. Further examples can be given from various applications, including (but not limited to) location [39,46], scheduling [26,55], logistics [48], resource allocation [13], network [14] and project selection [41].

Rather than using separate equality and efficiency-related functions in a model, some studies use special types of functions that incorporate both efficiency and equality (or, as called in Ref. [38], equity) concerns [17,21,40]. Such functions are called *social welfare* or *equitable aggregation* functions [38]. Maximizing a function which incorporates both efficiency and equity is analogous to the multicriteria decision-making methods that assume that the decision maker has a known utility function and maximizes this utility function. The welfare function must be an increasing function (to encourage efficiency) and must be

symmetric and satisfy the Pigou-Dalton transfers principle<sup>1</sup> (see e.g., Ref. [50]) to promote egalitarian allocations. Such functions are selected from the set of Schur-concave functions,<sup>2</sup> which are symmetric by definition [51]. Ordered weighted averaging (OWA) functions, in which weights are ordered such that relatively worse-off entities receive relatively higher weights, are typical examples of this type of social-welfare function [57].

[44] studies OWA aggregation for multicriteria problems. They introduce two linear programming formulations to linearize OWA-type objective functions [40]. uses ordered median functions, which are symmetric concave to address equity concerns [52]. tries to allocate indivisible tasks while using the Minmax share approach [30]. proposes a social welfare function, which combines equity and efficiency concerns for a two-person problem and a many-person problem [21]. works on a nurse rostering model with a welfare function that determines shifts of nurses according to their skills [15]. utilizes OWA functions and introduces a parametric welfare dominance concept so as to parameterize the degree of inequality aversion in resource allocation settings.

Our second method is based on the same idea of using welfare functions, but since there are multiple benefit allocations the resulting optimization problems are multiobjective.

The third group of methods formulates a single benefit distribution problem as a multi-objective optimization problem and suggests finding solutions that are equitably nondominated [18].<sup>3</sup>

In this paper, we focus on problems where decisions result in the distribution of multiple benefits to multiple entities. Examples of this setting could be seen in many public and private sector decision-making problems, such as location, task assignment, scheduling, bandwidth allocation, health investment, health-care systems and course design (see e.g. Ref. [52], for an equitable task allocation problem with multidimensional cost). In these problems, the DMs have efficiency-related concerns and try to maximize the total benefits. Moreover, in most of these settings they also have equality concerns, and hence, would like to distribute the benefits to users as equally as possible. Since multiple benefits exist, the concerns of efficiency and equality are multidimensional. A typical example occurs in public education course allocation settings, where a DM decides which courses to will be offered in different neighborhoods. The DM wants to open courses so that the population can benefit as much as possible (subject to a given budget), while ensuring that equitable service is offered to different population groups.

In such settings, the DM is faced with the problem of evaluating alternative distributions of multiple benefits to multiple users (see Fig. 1). This problem can be considered as an extension of two different problem types in the literature: The first one is the equitable optimization problem, whose applications in the literature have focused on the distribution of a single good or bad as equally and efficiently as possible. We mentioned this type above. The problem we define extends such problems as it concerns the distributions of multiple goods (bundles) across entities; hence, alternatives are matrices. The second one is the multiobjective resource allocation problem with only efficiency concern, which is discussed below.

<sup>1</sup> Pigou-Dalton transfer principle states that, for an allocation, any new allocation created by taking some benefit from a relatively better off entity and transferring it to the other(s), should be a more preferred alternative.

<sup>2</sup> A function  $f(\cdot) : \mathbb{R}^m \rightarrow \mathbb{R}$  is Schur-concave if and only if for all doubly stochastic matrices  $Q$ ,  $f(Qz) \geq f(z)$ .

<sup>3</sup> An allocation vector is equitably nondominated if there is no other allocation vector that equitably dominates it, i.e. is preferred to it with respect to all rational preference relations that satisfy the additional axioms of symmetry and Pigou-Dalton Principle of transfers. Symmetry (anonymity) ensures that the decisions are not affected by the identities of the entities [37].

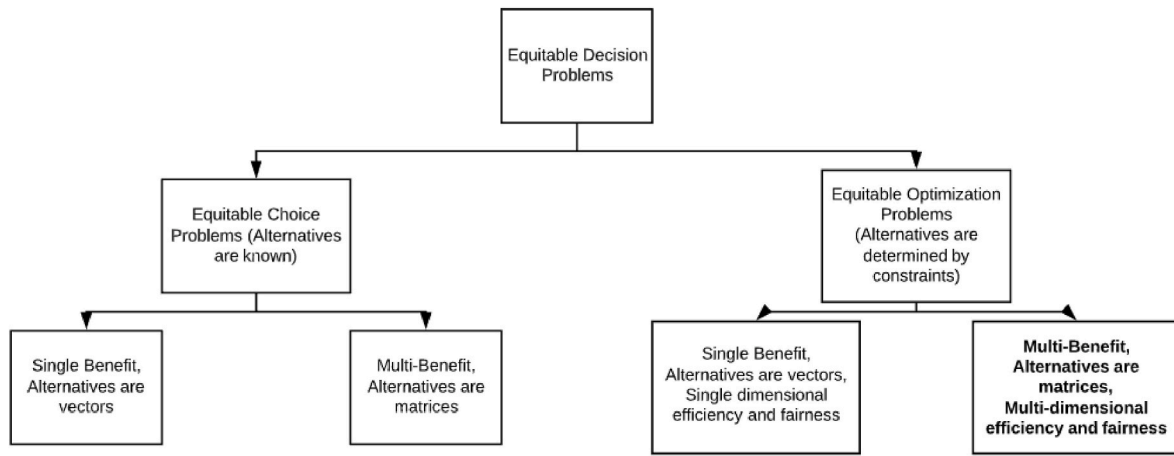


Fig. 1. Categorization of equitable decision-making problems in the literature.

## 2.2. Multiobjective resource allocation problems with only efficiency concern

Most of the optimization settings in the operations research literature formulated multiobjective resource allocation problems so as to maximize the total amount for each type of benefit, and, therefore, considered only efficiency.

In the optimization field, problems with more than one outcome have been studied for multiobjective knapsack problems, though this has been done without considering equality [56]. works on a branch and cut algorithm, and implements a two-phase algorithm to generate efficient solutions for biobjective knapsack problems [19]. provides a new dynamic programming algorithm and experimentally compares the method with other exact solution methods that have been proposed; it shows that their algorithm works faster than the best algorithm ([23]) known to date.

These problems generalize the single benefit knapsack problems to multibenefit knapsack problems. However, in these problems, only the total output is maximized and distribution is not addressed. In many real-life decision-making processes, however, maximizing efficiency may not be convenient in terms of equity. Our problem extends these problems as well, as it incorporates equality concerns into these settings and concerns the distributions of the benefits across entities, defining alternatives as matrices.

The contributions of the current work can be summarized as follows: i) We introduce and study the multidimensional equitable optimization problems, which extend equitable optimization problems by considering allocation of bundles; ii) These problems also extend multiobjective resource allocation problems with only efficiency concerns by incorporating equality; iii) We provide generic models to address the multidimensional equitable optimization problems. (To the best of our knowledge, our work is the first attempt to handle multidimensional efficiency and equality concerns in optimization settings by proposing generic structures that allow trading equality off-against efficiency or trading welfare with respect to one benefit off-against welfare with respect to other benefits); and iv) The proposed methodology is generic in the sense that it can be adapted to any setting in which decisions lead to the allocation of bundles to a number of entities who are considered indistinguishable (having equal rights for sharing the benefits). We demonstrate the use of the proposed methodology through a public sector resource allocation example. Our analysis reveals the benefits of considering equality in public service planning.

## 3. Problem definition and solution approaches

Multidimensional optimization problems are multiobjective

optimization problems by nature due to the existence of multiple efficiency and equality concerns. However they are different from classical multiobjective optimization problems due to additional properties assumed for the underlying preference relation. Therefore handling such problems requires customized approaches.

To be able to incorporate these (multidimensional) efficiency and equality concerns in the decision making process, we propose multi-objective mathematical modeling based approaches in this paper. These approaches are generic in the sense that they can be adapted to various settings in which the decisions result in allocations of multiple goods to multiple users. To keep the cognitive burden of the DM at a reasonable level, we first formulate biobjective programming problems, the solving of which would provide the DM with a set of Pareto solutions. Hence we allow the DM to analyze the trade-off between efficiency and equality and choose the solution that she will implement. Our second approach maximizes welfare functions, each of which is associated with one of the benefits distributed. These functions are concave (Schur-concave); hence, they encourage efficiency and equality in the allocations.

Consider an optimization setting where any decision results in allocations of a set of benefits over a set of entities and the decision maker(s) has efficiency and equality concerns for all benefits. We discuss two alternative approaches that could be used in these settings, *aggregate efficiency-equality framework* and *concave welfare framework*, respectively. In the first approach, we aggregate the multidimensional efficiency concerns, that is the concern for maximizing the sums of all types of benefits, using an efficiency-related aggregation function. Similarly, the multidimensional equality concerns, that is the concern for distributing each type of benefit as equally as possible, are aggregated using an equality-related aggregation function, resulting in a biobjective programming problem as follows:

$$\begin{aligned} & \max \text{ "Efficiency, Equality" } \\ & \text{Subject to :} \\ & z_{ik} = g_{ik}(x) \quad \forall i \in I, \forall k \in K \end{aligned} \quad (1)$$

$$x \in X \quad (2)$$

In this generic formulation  $x$  is the decision variable vector and  $X$  is the set of feasible decisions. Each decision  $x$  results in a distribution of multiple benefits across a set of entities.  $z_{ik}$  is the amount of benefit type  $i$  enjoyed by entity  $k$ .  $g_{ik}(x)$  is a function that determines the value of the enjoyed benefit of  $z_{ik}$ . "Efficiency" and "Equality" refer to the efficiency and equality aggregation functions, the explicit forms of which will be provided in the upcoming sections. This approach explicitly focuses on the trade-off between these concerns.

In the second approach, we investigate the case where the aggregation is performed over the efficiency and equality concerns of each benefit allocation, resulting in the following  $n$ -objective programming

problem:

$$\begin{aligned} \max \quad & \text{"Welfare}_1, \text{ Welfare}_2, \dots, \text{ Welfare}_n \\ \text{Subject to :} \quad & \\ z_{ik} = g_{ik}(x) \quad & \forall i \in I, \forall k \in K \end{aligned} \quad (3)$$

$$x \in X \quad (4)$$

( $\text{Welfare}_1, \text{Welfare}_2, \dots, \text{Welfare}_n$ ) is the vector of  $n$  objective functions where  $\text{Welfare}_i$  is the aggregation function used for benefit  $i$ , the exact form of which will be given later.

We now provide the detailed descriptions of these two approaches.

### 3.1. Aggregate efficiency-equality framework (AEE)

This approach aggregates the multidimensional efficiency concerns in one objective and the multidimensional equality concerns in the other objective. The overall equality and efficiency levels of a decision are calculated as the sum of equality and efficiency scores assigned to each benefit allocation.

The efficiency score function is a function of *total benefits* distributed from each benefit type. Since the benefits would typically take values on different ranges and are measured in different units, a scalarization would be needed to aggregate the total amounts of different benefits. For scalarization purposes, we define lower and upper bounds on the total amount of benefit  $i$  that could be enjoyed by the entities (the upper bound is determined by solving the problem as if the only concern were maximizing the total amount of that specific benefit; the lower bounds are simply taken as 0) and denote these as  $L_i$  and  $H_i$ , respectively. Moreover, we assume that the DM would like to avoid cases with very high levels of total efficiency score coming only from a small subset of the benefit types, which indicates very low totals in other benefit types.

We propose using an increasing concave function as an aggregation function to ensure balance in efficiency score values across multiple benefits.

The equality score function is a function of the *benefit proportions*, showing the proportion of the total benefit distributed to each entity. Similar to the efficiency case, we assume that the DM would like to avoid cases in which some benefits are equitably distributed while there is extreme inequity in other benefit distributions. Hence, we use an increasing concave function as an aggregation function to ensure equity in equality score values across multiple benefits.

Note that the aggregated equality function considers proportions and hence does not incorporate any efficiency concerns (i.e., there is no difference between alternative allocations with different totals as long as the users enjoy the total with the same proportions). To illustrate, the following two benefit distributions over three entities would have the same equality score, although their efficiency levels are different: (10, 10, 20) and (25, 25, 50). We incorporate the (multidimensional) efficiency preferences only in the first objective. In this sense, we use an approach that is similar to the index-based approaches discussed in Section 2 and capture the degree of equity using our equality score. However, unlike the existing methods, our approach aggregates scores over multiple benefits; hence, we propose an indicator for *multidimensional inequality measurement*.

We linearize the concave aggregation functions using piecewise linearization in our mathematical models. The formulation is as follows:

#### Problem parameters

$K$ : the set of entities,  $K = \{1, 2, \dots, l\}$ , index  $k$

$I$ : the set of benefits,  $I = \{1, 2, \dots, n\}$ , index  $i$

$M$ : number of thresholds used for piecewise linearization of the concave functions

$L_i$ : a lower bound on the total amount of benefit  $i$  enjoyed by the entities

$H_i$ : an upper bound on the total amount of benefit  $i$  enjoyed by the entities

$\Delta T_m^f$ : difference between two consecutive normalized benefit thresholds defining interval  $m$  in equality aggregation function,  $m = 1, \dots, (M - 1)$

$\Delta T_m^e$ : difference between two consecutive normalized benefit thresholds defining interval  $m$  in efficiency aggregation function,  $m = 1, \dots, (M - 1)$

$\Delta U_m^f$ : difference between equality scores of benefit thresholds defining interval  $m$ , in equality aggregation function,  $m = 1, \dots, (M - 1)$

$\Delta U_m^e$ : difference between efficiency scores of benefit thresholds defining interval  $m$  in efficiency aggregation function,  $m = 1, \dots, (M - 1)$

$W_m^f$ :  $\Delta U_m^f / \Delta T_m^f$ ,  $m = 1, \dots, (M - 1)$ .

$W_m^e$ :  $\Delta U_m^e / \Delta T_m^e$ ,  $m = 1, \dots, (M - 1)$ .

#### Decision variables

$z_{ik}$ : amount of benefit  $i$  enjoyed by entity  $k$

$z_{ik}^N$ : normalized  $z_{ik}$  ( $\frac{z_{ik}}{\sum_{k \in K} z_{ik}}$ )

$f_{ik}$ : equality score contribution of  $z_{ik}^N$

$t_i$ : scalarized value of total amount of benefit  $i$

$e_i$ : efficiency score contribution of  $t_i$

$\bar{X}_{ikm}^f$ : amount of normalized benefit obtained within interval  $m$  for entity  $k$  from benefit type  $i$  in equality aggregation function

$\bar{X}_{im}^e$ : amount of normalized total benefit obtained within interval  $m$  from benefit type  $i$  in efficiency aggregation function

#### Aggregate efficiency-equality model (AEE-C)

$$\text{Max} \sum_{i \in I} e_i, \quad \text{Max} \sum_{k \in K} \sum_{i \in I} f_{ik} \quad (5)$$

$$\begin{aligned} \text{Subject to :} \quad & \\ x \in X \end{aligned} \quad (6)$$

$$z_{ik} = g_{ik}(x) \quad \forall i \in I, \forall k \in K \quad (7)$$

$$z_{ik}^N = \left( \frac{z_{ik}}{\sum_{k \in K} z_{ik}} \right) \quad \forall i \in I, \forall k \in K \quad (8)$$

$$z_{ik}^N = \sum_{m=1}^{M-1} \bar{X}_{ikm}^f \quad \forall i \in I, \forall k \in K \quad (9)$$

$$f_{ik} = \sum_{m=1}^{M-1} W_m^f \bar{X}_{ikm}^f \quad \forall i \in I, \forall k \in K \quad (10)$$

$$t_i = \left( \frac{\sum_{k \in K} z_{ik}^N}{H_i - L_i} \right) \quad \forall i \in I \quad (11)$$

$$t_i = \sum_{m=1}^{M-1} \bar{X}_{im}^e \quad \forall i \in I \quad (12)$$

$$e_i = \sum_{m=1}^{M-1} W_m^e \bar{X}_{im}^e \quad \forall i \in I \quad (13)$$

$$0 \leq \bar{X}_{ikm}^f \leq \Delta T_m^f \quad \forall i \in I, \forall k \in K, m = 1, \dots, (M - 1) \quad (14)$$

$$0 \leq \bar{X}_{im}^e \leq \Delta T_m^e \quad \forall i \in I, m = 1, \dots, (M - 1) \quad (15)$$

Constraint set (6) ensures that  $x$ , the decision vector, is an element of  $X$ , which is the feasible set in the decision space. The decisions and the feasible decision space are problem specific, hence we give them in



generic form. Constraint set (7) is used to calculate total amount of benefit  $i$  enjoyed by entity  $k$  as a result of decision  $x$ . Constraint set (8) scales the total amount of benefit  $i$  enjoyed by entity  $k$  to a  $[0,1]$  interval by converting  $z_{ik}$  to the proportion of the total amount,  $z_{ik}^N$ . Note that a concave equality function is used to ensure that these equality scores are allocated evenly across entities. This function is linearized using piecewise linearization. For that purpose, the range of normalized benefit values,  $[0,1]$ , is divided into  $M-1$  intervals as seen in Fig. 2, which shows an example equality score function with 10 intervals. Constraint sets (9) and (10) are used to calculate the equality score value of the entity  $k$ 's share from benefit type  $i$  ( $z_{ik}^N$ ).

We perform a similar linearization for the concave efficiency function. Constraint set (11) is used to calculate the scalarized total value for each benefit, and (12)–(13) are used to calculate the efficiency score. Finally, constraint sets (14) and (15) ensure that  $\bar{X}_{ikm}^f$  and  $\bar{X}_{im}^e$  values are all in  $[0, \Delta T_m^f]$  and  $[0, \Delta T_m^e]$ , respectively.

We call the above model AEE-C (Aggregate efficiency-equality model with concave efficiency score function). For comparative purposes we also consider another variant (AEE-L), where the efficiency aggregation function is a linear function instead of a concave one. This implies that the DM is only concerned with the total amounts, hence decisions with very high totals in one benefit and low totals in others are also deemed acceptable. Parameters and decision variables of AEE-L are the same as in AEE-C. We delete constraints (12), (13), (15) and replace the first objective function with  $\sum_{i \in I} t_i$ .

### 3.2. Concave welfare model

This approach defines the objective functions based on the benefit types and separately maximizes social welfare obtained from each benefit distribution.

When allocation alternatives are vectors and equality concerns exist, the DM's preference relation is assumed to be an equitable preference relation, which satisfies axioms of symmetry and Pigou-Dalton principle of transfers. If the DM's equitable preference relation can be represented by a function, the function should be an equitable aggregation function, which satisfies Pigou-Dalton transfer, anonymity and monotonicity properties. It is well known that equitable aggregation functions should be Schur-concave [37]. Hence, in this approach, we use Schur-concave welfare functions for single benefit allocations to incorporate efficiency and equality concerns. We define the objective functions based on the benefit types and separately maximize social welfare obtained from each benefit distribution.

Each such function is defined as a Schur-concave function of the form  $SW_i = \sum_k u_i(z_{ik})$ , where  $u_i(\cdot)$  shows the welfare gained by providing  $z_{ik}$  units of benefit  $i$  to entity  $k$ . Using concave welfare ( $u_i(\cdot)$ ) functions encourages equitable distribution of the benefits. We use piecewise linearization to calculate the  $u_i(\cdot)$  values for the benefits from concave welfare function.

In addition to the decision variables and the problem specific parameters of the AEE-C model, we introduce the following:

#### Additional parameters for concave welfare model

$\Delta T_m$ : difference between two consecutive benefit thresholds defining interval  $m$ ,  $m = 1, \dots, (M-1)$ .

$\Delta U_m$ : difference between welfare scores of benefit thresholds defining interval  $m$ ,  $m = 1, \dots, (M-1)$ .

$W_m$ :  $\Delta U_m / \Delta T_m$ ,  $m = 1, \dots, (M-1)$ .

#### Additional decision variables for concave welfare model

$u_{ik}$ : contribution to social welfare from the share of user  $k$  in benefit  $i$ , i.e.  $u_i(z_{ik})$

$\bar{Y}_{ikm}$ : amount of benefit type  $i$  obtained by entity  $k$  within interval  $m$

#### Concave welfare model (CW)

$$\text{Max} \quad \sum_{k \in K} u_{1k}, \dots, \sum_{k \in K} u_{nk} \quad (16)$$

$$\text{Subject to:} \quad x \in X \quad (17)$$

$$z_{ik} = g_{ik}(x) \quad \forall i \in I, \forall k \in K \quad (18)$$

$$z_{ik} = \sum_{m=1}^{M-1} \bar{Y}_{ikm} \quad \forall i \in I, \forall k \in K \quad (19)$$

$$u_{ik} = \sum_{m=1}^{M-1} W_m \bar{Y}_{ikm} \quad \forall i \in I, \forall k \in K \quad (20)$$

$$0 \leq \bar{Y}_{ikm} \leq \Delta T_m \quad \forall i \in I, \forall k \in K, m = 1, \dots, (M-1) \quad (21)$$

As in the aggregate efficiency-equality model,  $x$  and  $X$  denote the decision vector and feasible decision space, respectively. Constraint set (18) is used to calculate the amount of benefit  $i$  enjoyed by the entity  $k$  as a result of decision  $x$ . Constraint sets (19) and (20) are used to calculate the utility values of the entity  $k$ 's share from benefit type  $i$ . Finally, constraint set (21) ensures that  $\bar{Y}$  values are all in  $[0, \Delta T_m]$ .

The following section will discuss a real-life based problem that can be tackled using the structure discussed above.

## 4. Case study

We now introduce a real-life case study on a problem that public service planners face in Turkey. Public Education Centers (PEC) organize courses in almost all provinces of Turkey. These courses are offered year round, seven days a week and are free of charge. They can be offered as full day sessions as well as morning, lunch, evening and weekend sessions (groups).

Ankara, as the capital city of Turkey, hosts many of these courses. With a population of nearly 5.5 million [9], the city consists of many districts, each with residents of various demographic characteristics. Planning public education to ensure that the residents are offered courses based on their needs, in an equitable and efficient manner, is a substantial concern. Indeed, providing indiscriminate education support to all segments of the society is stated as a basic principle and a priority for these centers [54]. How to ensure this is a challenging question as it is not possible to offer all courses in all regions due to limited resources (such as budget, physical capacity and personnel) ([4], Directorate of Public Education Service). We, therefore, consider the problem of planning PECs in the districts of Ankara such that the benefits of the courses are distributed among different district groups in an equitable and efficient manner. In the current system, courses are offered mostly in the city centers, which conflicts with the "equitable education service for all segments of the society principle" ([11], Ministry of National Education); the low-income districts, which are typically located relatively far away from the city centers, are deprived of the service. Policymakers are planning to alleviate service deprivation in regions with low income levels and increase the welfare of their residents through these courses ([3], Non-formal Education Institutions of the Ministry of National Education). Our approach can be seen as a first step towards achieving this goal. For these reasons, we define the entities as district

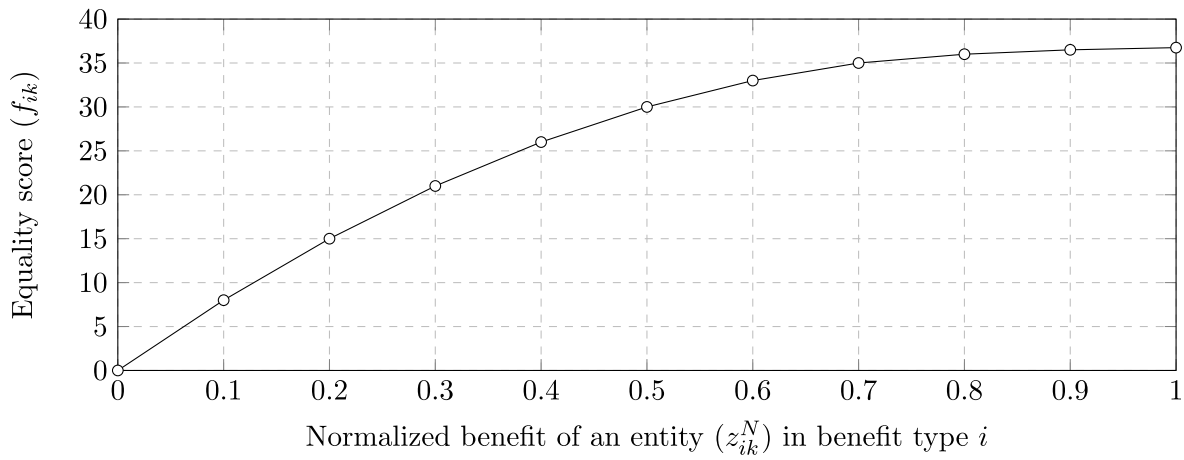


Fig. 2. An example equality score function.

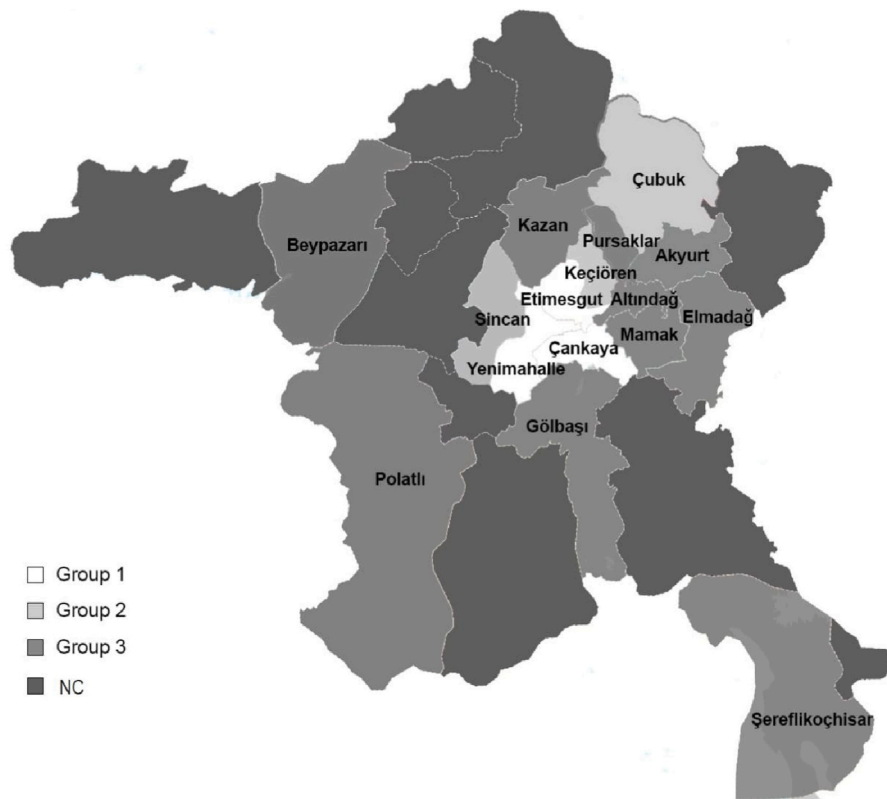


Fig. 3. Poverty rate level map of districts of Ankara. Group 1 (3) has the minimum (maximum) poverty rate. Ranges of the poverty rate levels: Group 1, [0.4–4]; Group 2, [5.9–6.7]; Group 3, [9.6–36.5]. NC: Not considered since data on poverty rates were not available for the area (since the population is below 20000) [1].

groups that are categorized based on poverty rates (see e.g., Fig. 3). It is, however, possible to use the proposed methodology on the same planning problem with different entity definitions (e.g., population groups can be constructed based on neighborhood, age or other attributes deemed relevant.) The problem is also challenging as there are multidimensional trade-offs, both between efficiency and equality levels of alternative plans for the same type of service, as well as across different types of services. Since the resources are limited, it is important for public planners to evaluate alternative solutions and make decisions based on a transparent mechanism that reveals *gains* and *losses*.

#### 4.1. Data for the case study

There are different types of possible courses that can be offered (see webpages [7,10] for the list of courses). These courses can be gathered into two main groups: hobby courses (H) and vocational assistance courses (VA).

The data for the model are based on factors deemed important for the illustrative case study and are estimated using publicly available information [12]. We consider 16 districts, which are: Etimesgut (ET), Çankaya (ÇA), Yenimahalle (YE), Sincan (Sİ), Çubuk (ÇU), Keçiören (KE),

**Table 1**

Categorization of 16 districts into poverty rate based groups.

District	Poverty Rate	Group	District	Poverty Rate	Group
ET	0.4	1	MA	10	
ÇA	2		BE	17	
YE	4		GÖ	17.1	
Sİ	5.9	2	AL	17.9	
ÇU	6.5		ŞE	25.9	3
KE	6.7		KA	30	
PO	9.6	3	EL	32.9	
PU	9.7		AK	36.5	

Polatlı (PO), Pursaklar (PU), Mamak (MA), Beypazarı (BE), Gölbaşı (GÖ), Altındağ (AL), Şereflikoçhisar (ŞE), Kahramankazan (KA), Elmadağ (EL) and Akyurt (AK). We divide these districts into three groups of similar population size based on relative poverty rate<sup>4</sup> [2,53] and consider these groups as entities of the allocation problem, as shown in Table 1 and Fig. 3. The sizes of district-based population groups are estimated using publicly available data [8].

Each public educational unit serves only the region it is located in. Course participant numbers for each district between 2009 and 2018 are publicly available in Ref. [5], which corresponds to approximately 1% of the total population for each year. In line with this, we assume that the demand for the courses will be 1% of the district's population and that the courses have the capacity to satisfy this demand. The cost of opening a new course mainly consists of venue and instructor costs, and varies from district to district. We estimate the venue cost by calculating an approximate rental cost of the area required for the course, which is directly proportional to the district's population. We also use district-based rental rates for this estimation. The teaching cost is calculated by multiplying permanent instructors' salaries by the number of instructors that would be needed in the district.<sup>5</sup> In districts with a higher population, the course classrooms are assumed to accommodate 25 people; otherwise they are assumed to serve 20 people, based on the current practice (see Ref. [7], Ankara Directorate-General Public Education Service). We assume that the available budget is approximately 50% of the total cost of the courses.

We expect that whether people would prefer hobby or vocational courses would depend on the average poverty level of the district residents. Therefore, we estimate attendance rates to the hobby and vocational assistance courses accordingly, as presented in Table 2. To be more specific, we assume that around 1% of the district's population (see Table 3) would be attending the courses, and the participants' preference for hobby and vocational courses would be determined by the rates given in Table 2. For example, in the second group, a total of 15,177 people would be participating, 8965 of whom are expected to choose hobby courses (see Table 3). The cost of opening a new course is district-specific since it is evaluated by examining both the rental prices and the salaries of the course instructors in the districts. We measure the benefit of a course in terms of the number of participants.

<sup>4</sup> The individual or household that has income and spending below a certain limit (a specific rate of the average welfare level of the society) is defined as relatively poor. The relative poverty rate, in our data, is the share of individuals or households living with less than half of the median disposable personal income in Ankara.

<sup>5</sup> We assume that the instructors are already recruited. Hence costs associated with recruiting and training teachers are not taken into account. In accordance with the Regulations on Non-formal Education Institutions of the Ministry of National Education published in the Official Gazette [6], the salaries of public education center (non-formal) teachers are the same as the formal education teachers. Every public personnel in the teacher status should have the same base salary. This situation reveals salary equality regardless of the course given.

## 4.2. Results

We formulate the public education provision problem described above as a binary knapsack problem. Note that the aggregation based framework is a biobjective framework, irrespective of the number of benefit types. In the second framework, though, the number of objective functions is equal to the number of benefit types. Since we consider two benefit types (hobby and vocational-related benefit), all formulations result in biobjective programming problems, the details of which are provided in Appendix A. We solve these models using the epsilon-constraint approach [29].<sup>6</sup> All models are coded in Eclipse JAVA Oxygen and solved by CPLEX 12.6 on a dual core (Intel Core i7 2.81 GHz) computer with 16 GB RAM. All solution times are reported in central processing unit (CPU) seconds.

A common drawback to all approaches relying on inequality indices is the fact that these indices only focus on how the total benefit is distributed regardless of the level of total benefit: even an allocation not distributing any benefit to any of the entities is a perfectly equal one. The aggregated efficiency-equality approaches suffer from the same drawback as they quantify (in)equality based on proportions of benefit distributed to the entities. To give an example, consider the following two solutions that allocate two benefits and three entities (the rows correspond to the allocation vectors of the benefits):

$\begin{bmatrix} 100 & 300 & 200 \\ 100 & 300 & 200 \end{bmatrix}$ , when AEE-C approach is used, the aggregated equality and efficiency scores of these matrices are (73.50, 130.64) and (65.00, 134.86), respectively; hence none dominates the other in the bicriteria sense. However, when the allocation matrices are investigated in detail, it is observed that the first allocation is better. This is because anonymity is assumed across entities, implying that an allocation matrix is equally good as its column permutations. When we permute the second matrix swapping the first and the second columns, it is seen that in the first alternative each element is at least as much as its counterpart in the second one. Since the second matrix distributes the benefits in a less efficient but relatively more equal way, the equality score of this option is higher than that of the first, making it a nondominated alternative. This can also be observed looking at the fractional distributions as follows:  $\begin{bmatrix} 0.17 & 0.5 & 0.33 \\ 0.17 & 0.5 & 0.33 \end{bmatrix}$ ,  $\begin{bmatrix} 0.42 & 0.29 & 0.29 \\ 0.42 & 0.29 & 0.29 \end{bmatrix}$ ; this leads to the second matrix having a higher aggregated equality score. This is the reason why any approach incorporating equality concerns using inequality indices should also account for efficiency.

The relation between the above allocation matrices is called equitable matrix dominance [36] and is defined below.

**Definition 1.** Given two alternatives  $f^j, f^{j'} \in \mathbb{R}^{(n \times l)}$  where  $n$  is the number of benefits and  $l$  is the number of entities,  $I = \{1, 2, \dots, n\}$  and  $K = \{1, 2, \dots, l\}$ :

$f^j \preceq_d f^{j'}$  ( $f^j$  weakly matrix dominates  $f^{j'}$ )  $\iff f^j_{ik} \leq f^{j'}_{ik}$  for all  $i \in I, k \in K$ .

Let  $\pi_r(f^{j'})$  be a column permutation of  $f^{j'}$  and  $R = \{1, 2, \dots, l\}$ :  $f^j \preceq_{em} f^{j'}$  ( $f^j$  equitably matrix weak dominates (em-dominates)  $f^{j'}$ )  $\iff \pi_r(f^{j'}) \preceq_d f^j$  for at least one  $r \in R$ .

To remedy the issue of obtaining em-dominated alternatives, we perform post-processing and eliminate such solutions. All the results

<sup>6</sup> Let the biobjective programming problem be:  $\max[z_1(x), z_2(x)]$  s.t.  $x \in X$ . We use augmented epsilon constraint models with an augmentation term set as:  $10^{-7}(\frac{z_1^i - z_1^N}{z_2^i - z_2^N})$  where  $x^i \in \arg\max_{x \in X} z_1(x)$ ,  $z_1^i = z_1(x^i)$ ,  $z_1^N = \arg\min_{x \in X} z_1(x)$  for  $i = 1, 2$ . The stepsize used in the constraints controlling the level of the second objective function is set as:  $\frac{z_1^i - z_1^N}{100}$ .

**Table 2**

Rates used to reflect course participation preferences of districts. (We assume that as poverty rate increases in a district, the residents' participation to vocational assistance courses will increase since there will be a need to gain skills for employability.)

District Group 1 (G1)			District Group 2 (G2)			District Group 3 (G3)								
District	VA	H	District	VA	H	District	VA	H	District	VA	H	District	VA	H
ET	0.35	0.65	Si	0.4	0.6	PO	0.45	0.55	GÖ	0.45	0.55	EL	0.45	0.55
ÇA	0.35	0.65	ÇU	0.4	0.6	PU	0.45	0.55	AL	0.45	0.55	AK	0.45	0.55
YE	0.35	0.65	KE	0.4	0.6	MA	0.45	0.55	ŞE	0.45	0.55			
						BE	0.45	0.55	KA	0.45	0.55			

**Table 3**

Cost and benefit (outcome) values of courses. Cost consists of salary and facility rental cost and is given in  $\times 1000$  TL; outcome is measured in terms of the number of participants.

Group	District	Cost	Outcome		Group	District	Cost	Outcome	
			H	VA				H	VA
G1	ET	1947	3339	1998	G3	PO	379	673	550
	ÇA	3740	5387	3223		PU	514	787	644
	YE	2429	4313	2323		MA	1877	3560	2913
G2	Si	1480	3061	2127		BE	153	266	217
	ÇU	275	525	365		GÖ	584	739	605
	KE	2803	5368	3730		AL	1128	2035	1665
						ŞE	106	188	154
						KA	173	294	241
						EL	140	249	204
						AK	100	190	156

reported in the following analysis correspond to the solutions obtained after post-processing and hence to nondominated solutions in the em-dominance sense. For each method, we perform a dominance check among the solution set returned by that specific method (i.e., we do not cross-check with the solutions returned by the other methods). That is, if a solution is em-dominated by solutions returned by an alternative approach we do not eliminate it since it would not be traceable in a real-life application.

In our case study, we assume that the concave equality score functions are of the following form [16]:

$$f(x) = \frac{x^{(1-\alpha)} - 1}{1 - \alpha}$$

, where  $x = z_{ik}^N$ .

This is a well-known function in the income inequality literature and is discussed in detail in Ref. [16]. The central DM's preferences for equality are incorporated through the degree of concavity in this function, hence through parameter  $\alpha = [0, 1)$ . The function represents preference models for decision makers with less equality concerns when  $\alpha$  approaches zero (less concavity). For our demonstration, we assume that the DM's preference for equality is at an upper-middle level and set  $\alpha = 0.7$ .

In AEE-C the concave efficiency score function and in CW the utility functions  $u(\cdot)$  are also taken in the same form, again with  $\alpha = 0.7$ . Note that  $x$  corresponds to benefit proportions enjoyed by entities ( $z_{ik}^N$ ) in the equality score function, while it is the (scalarized) total benefit enjoyed ( $t_i$ ) in the efficiency score function. In  $u(\cdot)$ ,  $x$  corresponds to the amount enjoyed by each entity ( $z_{ik}$ ). As discussed before, we use piecewise linear approximation to approximate these concave functions within the model.

We first investigate the Pareto solutions returned by the aggregate efficiency-equality approach. Recall that we have two variants of this approach based on the form of aggregate efficiency function: concave and linear. Figs. 4 and 5 show the Pareto solutions of concave (AEE-C) and linear (AEE-L) efficiency function variants, respectively; Fig. 6 shows the solutions of the concave welfare approach (CW). In each figure, in addition to the Pareto solutions of the corresponding approach, we provide the images of the solutions returned by the other

two approaches so as to enable a detailed comparative analysis.

AEE-C, AEE-L and CW return 29, 19 and 35 solutions, with total solution times of 5.02, 5.53 and 1.38 s, respectively. For clarity purposes we excluded two of the most equal solutions of AEE-C and AEE-L from all of the graphs. These two solutions were also problematic as they failed to provide any vocational benefit to the groups, hence were removed from the analysis.<sup>7</sup> Three of these problematic solutions were also obtained in CW (at the hobby and vocational welfare maximizing extremes), hence they were also removed from the CW set. We provide the results of the three approaches in detail in Figs. 21–23 in Appendix B.

When aggregate efficiency-equality approach variants are compared, the variant using a concave efficiency aggregation function outperforms the other variant in terms of the number and diversity of the solutions obtained. Specifically, AEE-C helps us to find more solutions on both edges of the Pareto frontier. Moreover, while CW provides solutions covering the range between the vocational benefit maximizing solution and the hobby benefit maximizing one, hence illustrating the trade-off between the two benefit types, AEE-C and AEE-L reveal the trade-off between (aggregated) equality and efficiency.

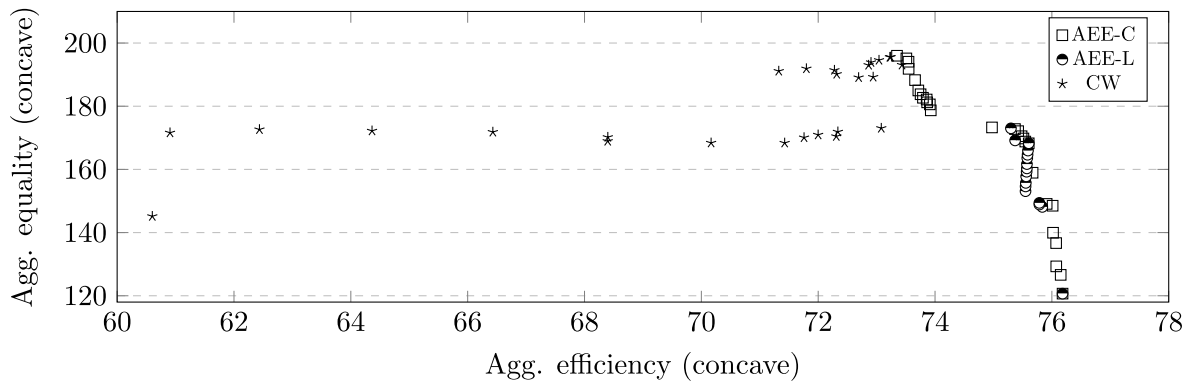
It is observed that all approaches return a core set of solutions, which are common, as well as additional solutions which perform worse with respect to the objective functions of the other methods (as seen in Figs. 4–6.)

One sees in Fig. 6 that AEE-C avoids extreme solutions that put too much emphasis on one benefit while sacrificing from other and provides solutions which have similar welfare scores for the two benefit types. As a result, the range of AEE-C solutions are narrower than that of CW. The proposed solutions are around the center of the CW Pareto frontier. Similarly, since CW does not put emphasis on balance, the resulting aggregated efficiency and equality scores of CW solutions are lower compared to AEE variants, making most of these solutions dominated in Figs. 4 and 5.

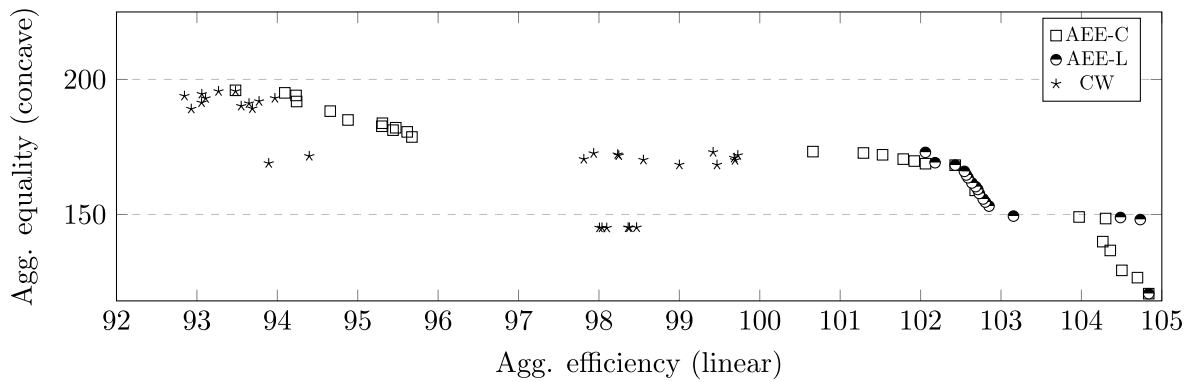
To investigate the trade-off between efficiency and equality, we present the two extreme solutions (1 and 27) for AEE-C, which

<sup>7</sup> Recall the discussion we had on the drawback of using an equality indicator that only focuses on the proportions. Even an allocation in which no one receives anything is considered perfectly equal.

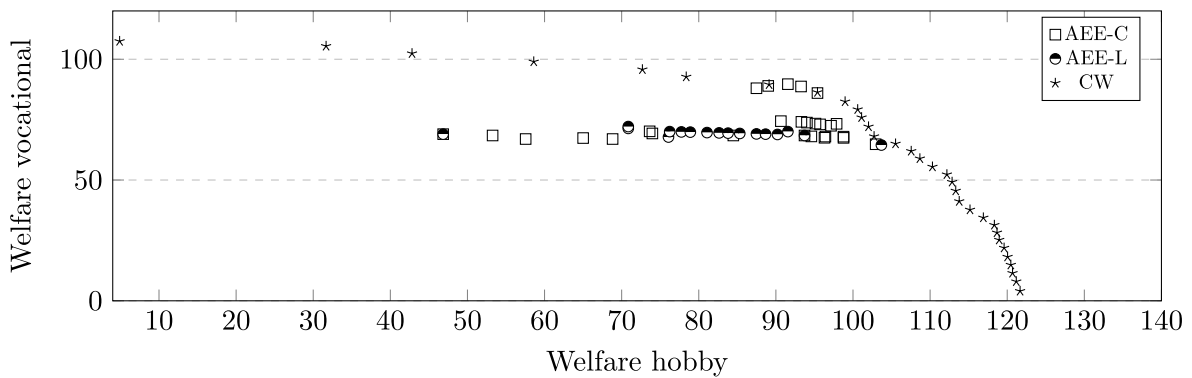




**Fig. 4.** Pareto optimal solutions of the aggregate efficiency-equality model: AEE-C. Aggregate efficiency and equality values corresponding to the solutions found by the other two methods are also given for comparability purposes.



**Fig. 5.** Pareto optimal solutions of the aggregate efficiency-equality Model: AEE-L. Aggregate efficiency and equality values corresponding to the solutions found by the other two methods are also given for comparability purposes.



**Fig. 6.** Pareto optimal solutions of the concave welfare model: CW. Hobby and vocational welfare values corresponding to the solutions found by the other two methods are also given for comparability purposes.

<i>Solution 1</i>				<i>Solution 15</i>				<i>Solution 27</i>			
	<i>G1</i>	<i>G2</i>	<i>G3</i>		<i>G1</i>	<i>G2</i>	<i>G3</i>		<i>G1</i>	<i>G2</i>	<i>G3</i>
<i>H</i>	0	8429	0	<i>H</i>	3339	3061	3335	<i>H</i>	3339	3061	1611
<i>VA</i>	0	2127	4782	<i>VA</i>	0	2492	2913	<i>VA</i>	1998	2127	1665

**Fig. 7.** Extreme and moderate solutions of AEE-C. solution 1 has the highest efficiency value, solution 27 has the highest equality value and solution 15 is a compromise solution in between.

correspond to the solutions with the best levels of efficiency and equality in Fig. 4, respectively, as well as a moderate solution (15) in Fig. 7.

There are striking differences between the solutions. Solution 27 uses the budget to ensure a more equal distribution of the benefits across groups. As one approaches the maximum efficiency extreme, solution 1, the total number of people served increases (15,338 vs. 13,801), but this occurs at the expense of equality as we see that no benefit is provided to group 1 and no hobby courses are offered to group 3. Solution 15 is in between the two extremes: it offers a more balanced distribution of hobby course service to the population groups compared to solution 1, while still suffering from imbalance in vocational assistance courses. This solution uses a higher portion of the budget on hobby courses compared to solution 27, resulting in an increase in the number of people offered hobby courses while sacrificing from the benefit of vocational assistance courses.

The extreme solutions of AEE-L are shown in Fig. 8. Similar observations can be made regarding this variant. However, unlike AEE-C solutions, even the most equal solution (Solution 17) cannot ensure that all groups have at least benefited from both benefit types.

Solutions 2 and 35, which correspond to the solutions with the best hobby welfare and vocational welfare in Fig. 6, are given below in Fig. 9. Solution 18 is an example solution lying in the middle of the Pareto frontier. This approach moves from a hobby welfare maximizing solution towards a vocational welfare maximizing one, hence the interpretation of the extreme solutions is different. The Pareto frontier shows the trade-off between a hobby course service prioritizing approach and a vocational course service prioritizing one, rather than the trade-off between overall equality and efficiency. As expected, in solution 2 almost all budget is devoted to hobby courses while in solution 35, all of it is used for vocational courses. Solution 18 provides a compromise between these two extremes, offering a more balanced distribution of benefits.

To observe how the two benefits (H and VA) are allocated across the three groups, we summarize all allocations in Figs. 10 and 11, which show allocations of the hobby course and vocational course benefits in AEE-C solutions, respectively. As seen in these figures, the shares of the groups get closer as one moves from most efficient solution (1) to the most equal one (27) in both benefit types.

Figs. 12 and 13 show the allocations of two benefits in CW solutions. To ensure a better display, we only show solutions with odd indices, (i. e., half of the solutions). This provides sufficient information to observe the trend. As expected, starting from the first solution, which has the highest hobby course welfare (resp. the lowest level of vocational course welfare), the level of total hobby benefit decreases (resp. total vocational course benefit increases) as one moves toward the other edge of the frontier. There are a few exceptions to this observation, (e.g., solution 19). In this solution, vocational welfare decreases as a result of obtaining a more equal allocation of a benefit rather than a higher total. This is desired and is a result of using a concave welfare function: more equal allocations with little sacrifice from the total amount result in higher welfare.

We also investigate the solutions in more detail to observe the district-specific recommendations of the alternative approaches. For each district-course pair, we calculate in how many of the Pareto optimal solutions the corresponding course is offered in that district. We then find the frequency of that decision by dividing it by the total number of Pareto solutions found. For example, in AEE-C, 29 Pareto

solutions are found; in 22 of these, a vocational course is opened in Etimesgut (ET), resulting in a percentage value of 75.86. We report these percentage values in Table 4. We highlight the cases for which a district-course pair's percentage is considerably high (higher than 40%) and considerably low (lower than 20%) across all solutions in boldface. Such an analysis may help decision makers determine which district-course pairs to prioritize when making decisions. For example, there is a tendency to open hobby and vocational courses to Elmadağ and Beypazarı in all models while no courses are offered in Sincan, Mamak and Altındağ in any of the solutions. Since the problem is a knapsack problem, benefit/cost ratios, which are provided in Table 5, may be an indicator of how frequent a district-course pair will be observed in the Pareto set. To see this, we doubled Sincan's hobby course demand, hence doubled its benefit/cost ratio. We see that all three models encourage opening hobby courses in the district; the percentages were around 80%. However, the frequencies are not directly proportional to the ratios since the problem is combinatorial: i) it is an integer knapsack problem and ii) the evaluation is group-based (i.e., we consider allocations over district groups). Even when a district has the highest benefit/cost ratio within its group, it is not necessarily included in the solutions: this is the case for Sincan, Mamak and Akyurt. Combinations of other districts in the group are chosen over these districts.

Overall, we observe that one can obtain solutions, with various degrees of efficiency and equality using the proposed methodology. Among the aggregate efficiency-equality variants, AEE-C (which relies on concave functions for aggregating the normalized efficiency and equality scores) and AEE-L (which relies on linear and concave functions for aggregating the normalized efficiency and equality scores, respectively) find similar solutions and both effectively distribute the total benefit performances across different benefit types. However, AEE-C finds more solutions on both the efficiency and equality edges of the Pareto-frontier. When aggregation based methods and the concave welfare method (which relies on defining a welfare function for each benefit are compared) we observe that the solution sets may differ in line with how the method is structured. Indeed, they all suggest a set of core solutions, which are (almost) the same.

Recall that this problem is motivated by the observation that in the current system, most of the course service is offered at the city center (i. e., mostly to group 1 districts in our categorization). Decision makers would like to make improvements, especially in terms of equality in the service provision. We believe that our work provides a way to improve the current system in a structured manner. To illustrate, we provide a piechart showing the number of people benefiting from the services in each group in the current system, which is generated based on data from 2009 in Fig. 14a. We also give the piechart of a solution that is common to AEE-C and CW Pareto sets in Fig. 14b. We chose 2009 as an example because the total number of participants was similar to our solution. In Fig. 14c, we provide the distribution over groups for the period 2014–2018. As seen in Fig. 14a and c, the past levels of service are highly imbalanced across groups compared to what can be achieved through our methods.

#### 4.3. Sensitivity analysis

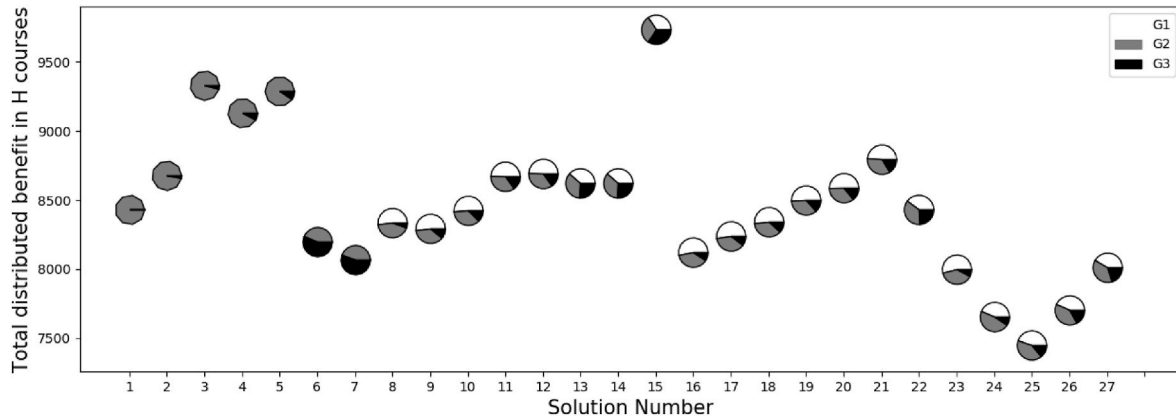
We now discuss some aspects that may affect the performance and behavior of the solution approaches. We will first investigate the effect

Solution 1				Solution 9				Solution 17			
	G1	G2	G3		G1	G2	G3		G1	G2	G3
H	0	8429	0	H	4313	3061	439	H	3339	3061	3560
VA	0	2127	4782	VA	0	2127	4975	VA	0	2127	3286

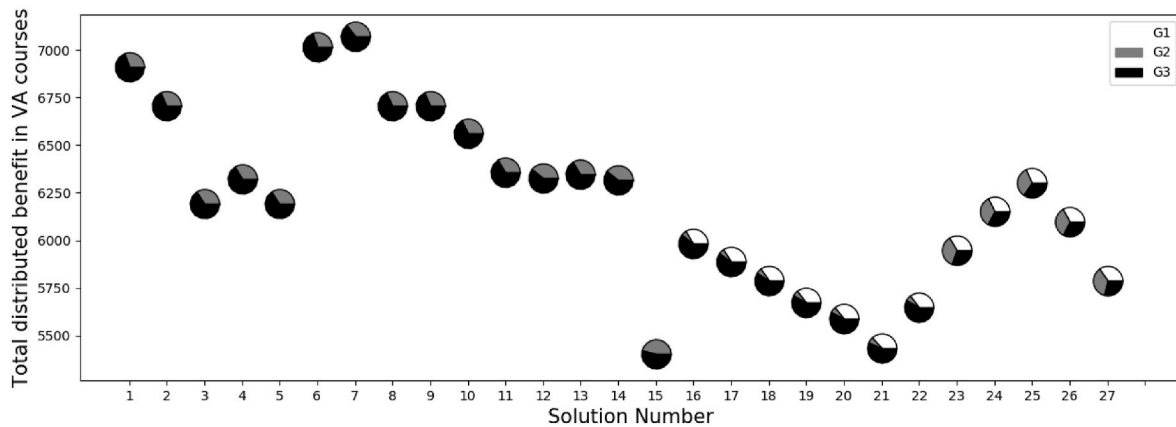
Fig. 8. Extreme and moderate solutions of AEE-L. Solution 1 has the highest efficiency value, solution 17 has the highest equality value and solution 9 is a compromise solution in between.

Solution 2				Solution 18				Solution 35			
	G1	G2	G3		G1	G2	G3		G1	G2	G3
$H$	4313	5893	6110	$H$	4313	3061	4187	$H$	0	0	0
$VA$	0	0	156	$VA$	1998	365	791	$VA$	1998	4095	5859

**Fig. 9.** Extreme and moderate solutions of CW. Solution 2 has the highest hobby welfare, solution 35 has the highest vocational welfare and solution 18 is a compromise solution in between.



**Fig. 10.** Hobby course benefit distribution across groups in AEE-C solutions. Towards the most equal edge of the Pareto (to the right), the benefit levels enjoyed by the three groups become gradually closer.



**Fig. 11.** Vocational assistance course benefit distribution across groups in AEE-C solutions. Towards the most equal edge of the Pareto (to the right), the benefit levels enjoyed by the three groups become gradually closer.

of the degree of concavity of the functions and the number of thresholds used in the linearization. We then provide the results of our numerical study to show how the computation times are affected by the size of the problem. Lastly, we solve a scenario with three benefit types so as to observe the performance of the two approaches in more detail.

We first discuss the effect of the degree of concavity used in the equality and efficiency score functions in AEE methods, and the utility functions in CW. To demonstrate how the degree of inequity aversion affects the proposed solution, we performed a new set of experiments for two more settings. We used concave functions with  $\alpha$  values of 0.3 and 0.9 to represent cases in which the DM is mildly and highly averse to inequality, respectively.

Recall that we use piecewise linearization so as to approximate these concave functions. To investigate the effect of the number of thresholds used on the quality of the approximation and the resulting Pareto frontier, we repeated the experiments for two more levels (5 and 20) in addition to the level we use in the main runs (10).

Table 6 summarizes these results. As expected, for a given concave function with parameter  $\alpha$ , increasing the number of thresholds improves the approximation of the underlying concave functions, resulting in the model better distinguishing different solutions from one another. Thus, more Pareto solutions are found.

Moreover, as the concavity of the functions increases, the models tend to prioritize solutions in which the allocations are more equal. For example, in CW, the solutions that maximize hobby welfare have the following hobby benefit distributions (4313, 8429, 4103) and (4313, 5893, 6300) for  $\alpha$  values of 0.3 and 0.9 (both with threshold 20), respectively. In these solutions, vocational benefit is zero as the focus is on hobby. Similarly, the solutions that maximize vocational welfare have the following vocational benefit distributions (1998, 4095, 5859) and (2323, 4095, 5128) for  $\alpha$  values of 0.3 and 0.9, respectively. One can see that using higher  $\alpha$  values treats solutions that distribute the benefits more evenly as ones with higher welfare.

In our case study example, the solution times required to find the set

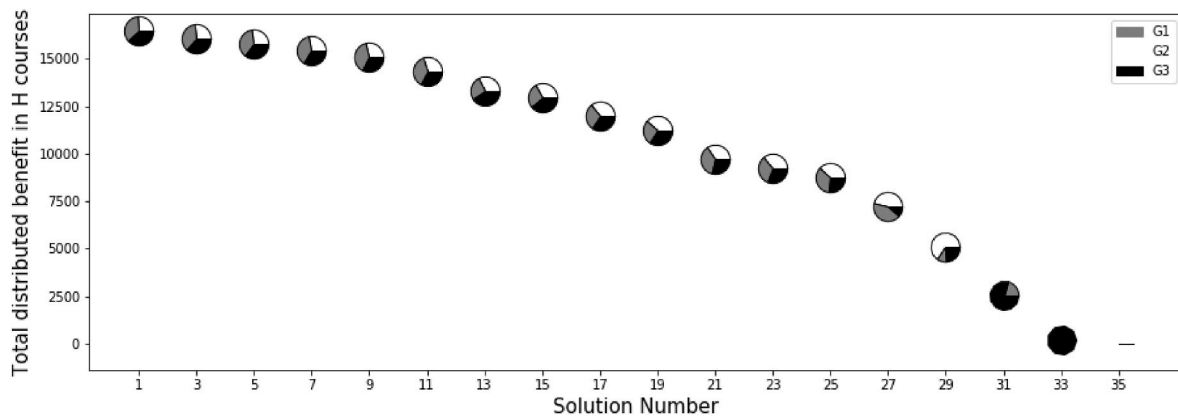


Fig. 12. Hobby course benefit distribution across groups in CW solutions. Total hobby benefit gradually decreases. Towards the right, the benefit levels enjoyed by the three groups get imbalanced.

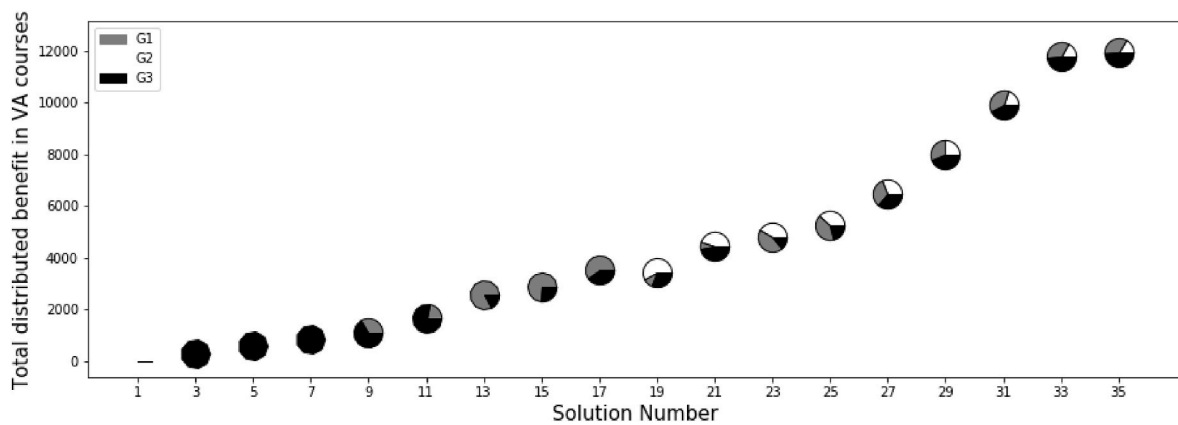


Fig. 13. Vocational assistance course benefit distribution across groups in CW solutions. Total vocational benefit gradually increases. Towards the left, the benefit levels enjoyed by the three groups get imbalanced.

Table 4

District-specific course recommendations in each model. Percentages show the frequency of the district-course pair in the Pareto solutions. High and low levels are indicated in boldface.

District	H (%)			VA (%)		
	AEE-C	AEE-L	CW	AEE-C	AEE-L	CW
YE	<b>55.17</b>	21.05	<b>42.86</b>	31.03	<b>10.53</b>	<b>40.00</b>
ÇA	27.59	<b>10.53</b>	28.57	<b>41.38</b>	<b>0.00</b>	<b>51.43</b>
ET	<b>13.79</b>	31.58	<b>57.14</b>	<b>75.86</b>	<b>89.47</b>	22.86
ÇU	24.14	31.58	<b>45.71</b>	24.14	<b>42.11</b>	<b>42.86</b>
KE	<b>17.24</b>	<b>5.26</b>	34.29	37.93	<b>78.95</b>	25.71
Sİ	<b>0.00</b>	<b>0.00</b>	<b>0.00</b>	<b>0.00</b>	<b>0.00</b>	<b>0.00</b>
EL	<b>96.55</b>	<b>94.74</b>	<b>45.71</b>	<b>68.97</b>	<b>89.47</b>	<b>40.00</b>
BE	<b>72.41</b>	<b>47.37</b>	<b>51.43</b>	<b>68.97</b>	<b>63.16</b>	<b>65.71</b>
KA	<b>48.28</b>	<b>68.42</b>	<b>54.29</b>	<b>0.00</b>	<b>0.00</b>	<b>0.00</b>
PO	<b>44.83</b>	<b>47.37</b>	<b>40.00</b>	34.48	<b>47.37</b>	<b>45.71</b>
PU	34.48	21.05	22.86	31.03	26.32	<b>40.00</b>
GÖ	24.14	<b>10.53</b>	<b>62.86</b>	20.69	<b>5.26</b>	<b>45.71</b>
AK	24.14	<b>15.79</b>	34.29	<b>0.00</b>	<b>0.00</b>	<b>14.29</b>
ŞE	20.69	21.05	<b>54.29</b>	37.93	<b>10.53</b>	42.86
MA	<b>0.00</b>	<b>0.00</b>	<b>0.00</b>	<b>0.00</b>	<b>0.00</b>	<b>0.00</b>
AL	<b>0.00</b>	<b>0.00</b>	<b>0.00</b>	<b>0.00</b>	<b>0.00</b>	<b>0.00</b>

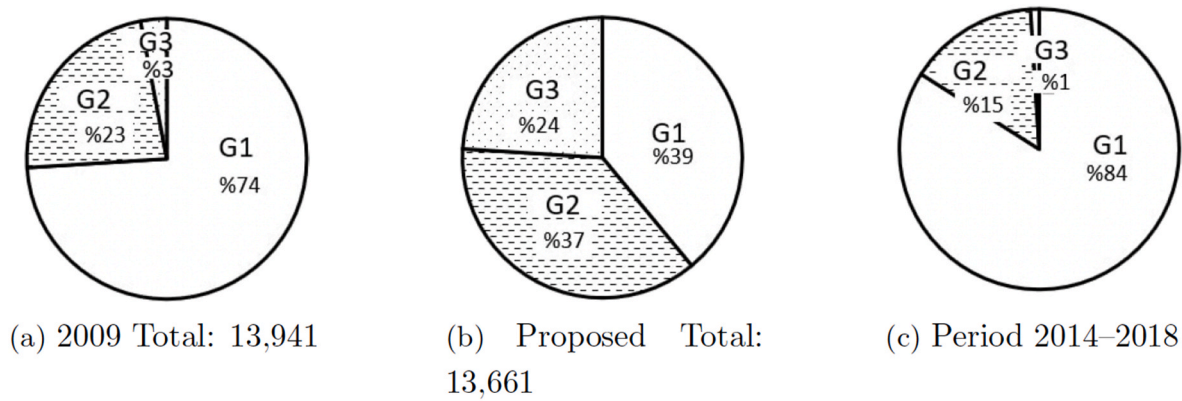
of Pareto solutions are negligible. To observe the scalability of the approach, we created a set of larger instances using randomly generated data and observed the solution times. Table 7 reports solution times with 20, 35, 50 districts and 3, 5 groups. We first implemented the epsilon-constraint approach as in the previous section. We observed that the

Table 5

Benefit/Cost ratios of districts.

Group	District	Benefit/Cost		Group	District	Benefit/Cost	
		H	VA			H	VA
G1	ET	1.71	1.03	G3	PO	1.78	1.45
	ÇA	1.44	0.86		PU	1.53	1.25
	YE	1.78	0.96		MA	1.90	1.55
G2	Sİ	2.07	1.44	G3	BE	1.74	1.00
	ÇU	1.91	1.33		GÖ	1.27	1.04
	KE	1.92	1.33		AL	1.80	1.48
					ŞE	1.78	1.45
					KA	1.70	1.39
					EL	1.78	1.46
					AK	1.90	1.56

CW approach returns solutions in reasonable time while AEE approaches may run into computational difficulties. When the number of districts is 50, AEE-C and AEE-L can not terminate in 1 h, which is our time limit. The epsilon constraint algorithm (augmented variant) is widely preferred for biobjective programming problems due to its ability to return the whole Pareto set (both supported and unsupported non-dominated points) in an efficient manner (i.e., by solving  $N+1$  models when the number of Pareto solutions is  $N$ ). However, the augmentation term used in the epsilon-constraint approach may cause numerical



**Fig. 14.** Distribution of the number of people benefiting from the courses across groups (a) in an example year, 2009 (b) in one of the Pareto solutions common to both AEE-C and CW (c) in period 2014–2018.

**Table 6**

Effect of  $\alpha$  and the number of thresholds on the number of solutions and solution time.

Method	$\alpha$	# Thresholds	# of Solutions	Solution Time
AEE-C	0.9	5	8	2.24
		10	29	5.49
		20	71	8.5
	0.7	5	15	3.45
		10	29	5.02
		20	82	25.02
	0.3	5	52	35.01
		10	66	50.32
		20	78	58.34
AEE-L	0.9	5	6	0.8
		10	14	2.51
		20	55	12.74
	0.7	5	10	1.37
		10	19	5.53
		20	50	15.22
	0.3	5	35	21.18
		10	43	23.88
		20	72	38.88
CW	0.9	5	19	0.97
		10	37	2.07
		20	82	6.71
	0.7	5	25	1.2
		10	35	1.38
		20	74	5.77
	0.3	5	33	1.39
		10	66	3.78
		20	87	5.08

issues. Therefore, we also solved the same problems with an algorithm using weighted sum scalarization.<sup>8</sup> We determined weights by discretizing the feasible weight space. We used a stepsize of 0.01 and solved the corresponding 100 scalarization problems so as to have a similar number of iterations to the epsilon constraint approach. Note that multiple weight vectors may provide the same solution, hence the number of solutions is less than 100. Moreover, the weighted sum approach only returns the supported nondominated points; hence, the number of solutions returned are less compared to epsilon constraint. Nevertheless, one can obtain a representative set of solutions from the Pareto frontier in reasonable time with this approach, as seen in Table 7.

One technical difference between the aggregated approach and the

CW approach is that the former always solves biobjective models while in the latter the number of objectives is equal to the number of benefit types. To see how these approaches would work in a setting with more benefits, we created problem instances using randomly generated data for a setting with three benefits. We solved the resulting bi/triobjective programming problems with an algorithm using weighted sum scalarization, where weights are set by discretizing the feasible weight space.<sup>9</sup> We obtained 10 Pareto solutions with AEE-C and AEE-L in 2.34 and 1.73 s, respectively. We obtained 15 Pareto solutions for the triobjective model of CW in 51.32 s. We provide these solutions in Figs. 15–20.

Figs. 15–17 show the 10 solutions obtained with AEE-C (the ones obtained with AEE-L were similar, hence we omit them for brevity.) This approach obtains Pareto solutions in a spectrum lying between the most equal and the most efficient solutions, and this is mirrored in the results: solution 1 has the highest total benefit while solution 10 allocates the three benefits to the groups in a more equal manner (see the piecharts of Solution 10 in Figs. 15–17).

Figs. 18–20 show the 15 solutions obtained with CW. In the weighted sum approach, we first solve the scalarization problem with weights (0,0,1); hence, the first solution maximizes the welfare in the third benefit, as expected (see solution 1 in Figs. 18–20). Then, keeping  $w_1$  the same, the weight of the second benefit,  $w_2$ , is gradually increased, leading to solutions 2–5, providing more (resp. less) welfare in benefit 2 (resp. 3). Afterwards, the welfare of benefit 1 has an increasing trend as a result of the increase in  $w_1$ . By structure, this approach demonstrates the trade-off between multiple benefits by showing the three extremes focusing on only one benefit (solutions 1, 5 and 15) and other Pareto solutions in between.

## 5. Conclusion

We consider optimization problems in which the decisions made result in allocations of multiple benefits to multiple entities in various degrees. Such problems are highly relevant in many public sector decision making problems, and they are generalizations of equitable optimization problems in which only a single good is allocated. Ensuring equity is important for obtaining implementable solutions that will be accepted for all stakeholders (these may be population groups) in many real life resource allocation settings. We solve these problems under the assumption that entities are indistinguishable, hence no group is prioritized over another.

We suggest two modeling approaches to be used in any such

<sup>8</sup> Let the multiobjective programming problem be:  $\max[z_1(x), \dots, z_p(x)]$  s.t.  $x \in X$ . In the weighted sum algorithm we repetitively solve problems with the following aggregated objective function:  $\max \sum_{i=1}^p w_i z_i(x)$ , where  $w: \sum_{i=1}^p w_i = 1$  is a weight vector.

<sup>9</sup> We used a stepsize of 0.1 for the biobjective models of AEE-C and AEE-L and solved the corresponding 10 scalarization problems. For the triobjective problems we discretized each weight value with a stepsize of 0.25, which led to solving 15 scalarization problems.



**Table 7**  
Numerical results for larger instances.

Method	# of Districts	# of Groups	Epsilon Constraint		Weighted Sum	
			# of Solutions	Solution Time	# of Solutions	Solution Time
AEE-C	20	3	7	2981.00	5	30.55
		5	9	1200.00	8	6.31
	35	3	16	2700.00	11	22.78
		5	49	1980.00	45	382.55
	50	3	7	3600.00	34	834.42
		5	9	3600.00	63	389.67
AEE-L	20	3	11	168.90	4	10.93
		5	7	27.90	5	7.26
	35	3	14	759.41	9	26.44
		5	43	418.33	41	27.67
	50	3	8	3600.00	24	21.29
		5	35	3600.00	51	139.86
CW	20	3	7	0.49	7	4.1
		5	15	0.81	13	4.09
	35	3	17	2.03	6	4.57
		5	46	15.91	44	6.66
	50	3	29	3.00	26	11.74
		5	67	360.62	59	170.45

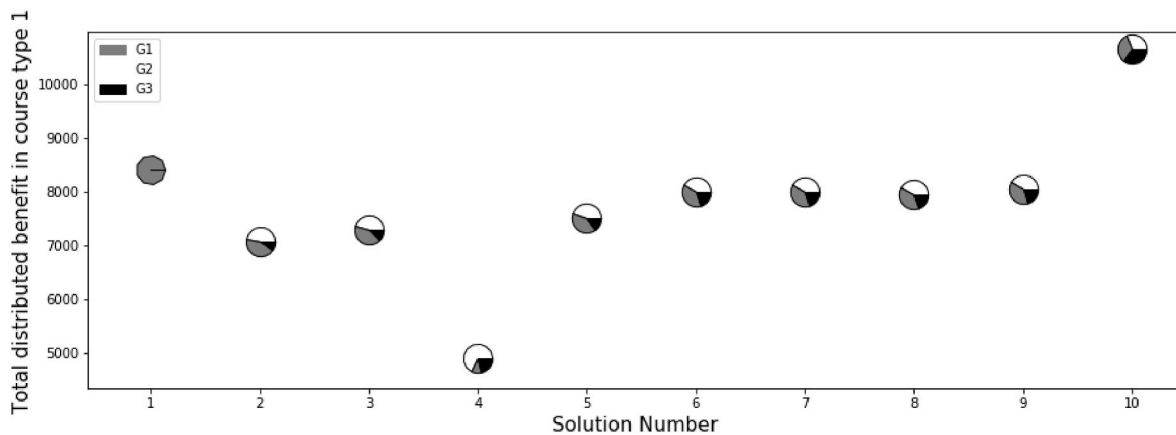


Fig. 15. Benefit 1 distribution across groups in AEE-C solutions.

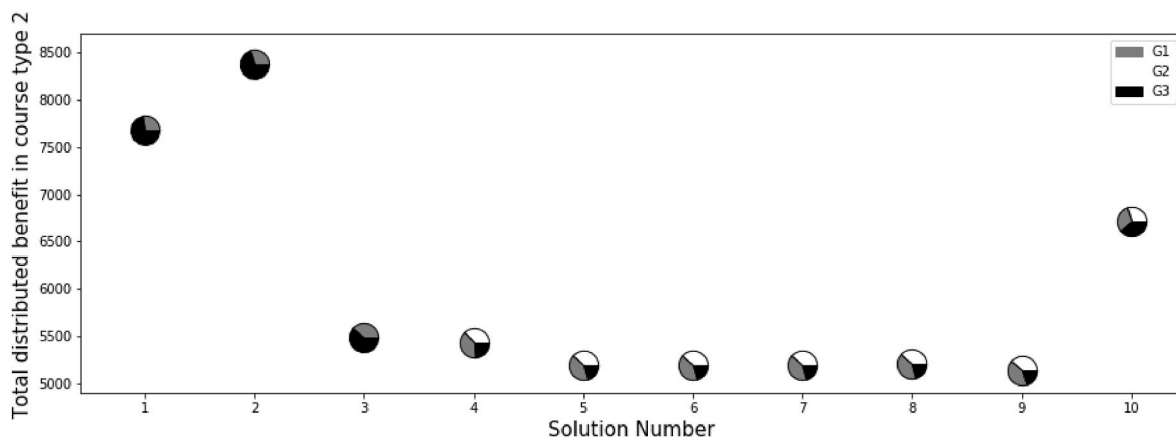


Fig. 16. Benefit 2 distribution across groups in AEE-C solutions.

optimization problem. The first of these approaches tackles the efficiency and equality concerns separately, hence provides a tool to observe the trade-off between these. The second approach aggregates efficiency and equality concerns for each benefit using a concave (hence Schur-concave) function; therefore, it defines a welfare function that demonstrates how good a decision is with respect to the allocation of the

corresponding benefit. The trade-off is observed between welfares of different benefit types.

We demonstrate the usability of the approaches on a real-life based application example, in which the decision makers seek to allocate various public education services as equally and efficiently as possible. The results exhibit the computational feasibility of the suggested

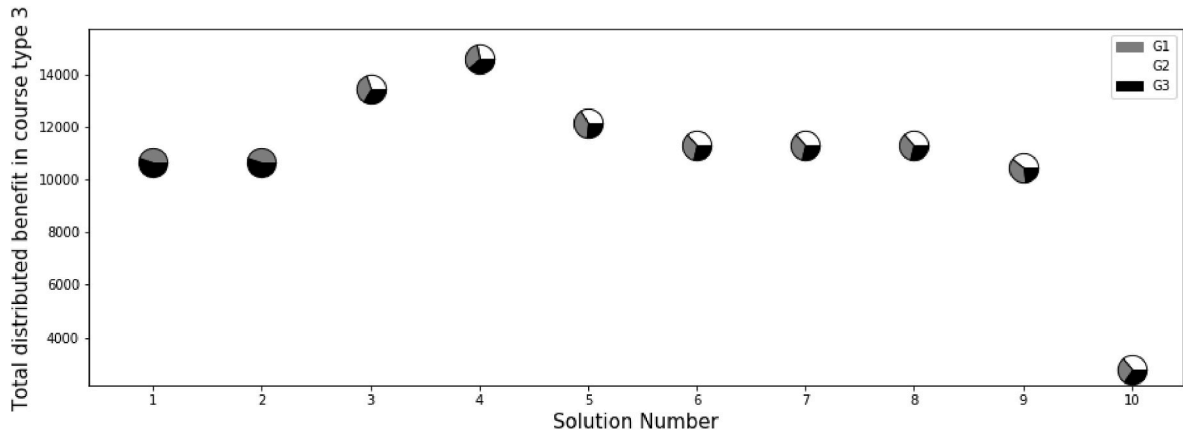


Fig. 17. Benefit 3 distribution across groups in AEE-C solutions.

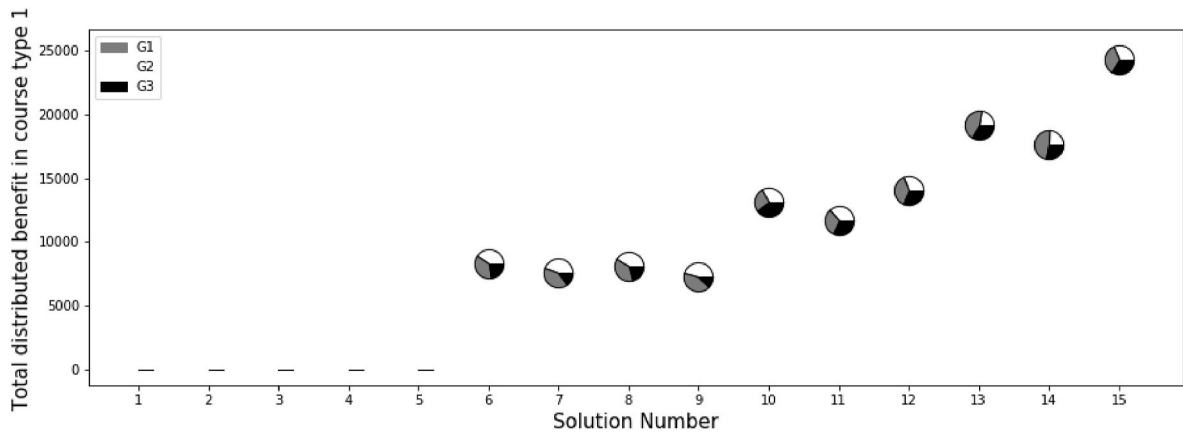


Fig. 18. Benefit 1 distribution across groups in CW solutions.

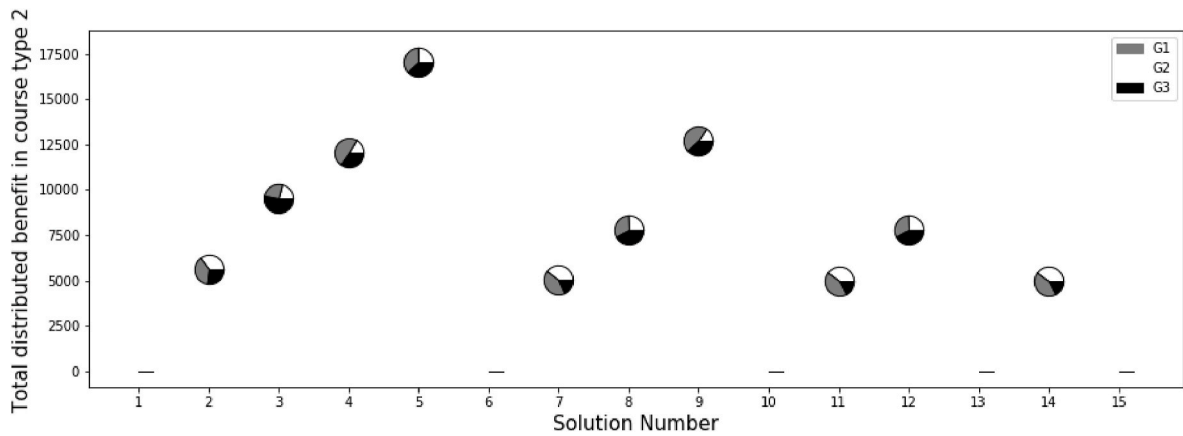


Fig. 19. Benefit 2 distribution across groups in CW solutions.

methods on the case study problem considered. Note that our generic frameworks to capture equality concerns in multiple dimensions could be adapted to various application problems. Hence, if the application-specific optimization problems are difficult to solve, the required computation effort can be compounded. However, this could be tackled by, as an example finding a set of representative points in the Pareto set rather than the whole set and/or using heuristic or metaheuristic approaches to find approximate Pareto solutions.

The case study also demonstrates that considering both equality and efficiency in the resulting benefit distributions can have a significant

impact on how the resources are allocated. Ignoring equality can lead to some district groups suffering from a lack of educational services. Since the problem setting is relevant for most real-life public sector decision making settings, the suggested models would provide useful insights and hence contribute to the relevant literature. Future research could focus on developing exact and/or heuristic solution algorithms to be able to solve the resulting multiobjective optimization models of larger-scale problems. Moreover, how to ensure balance in the asymmetric case in which the groups are not anonymous is an interesting yet challenging question from both research and application aspects. Specifically,

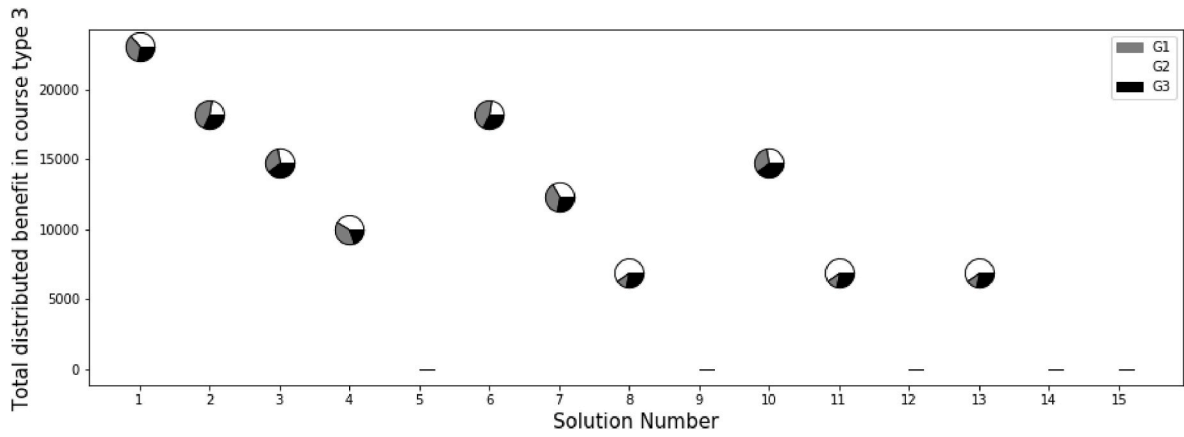


Fig. 20. Benefit 3 distribution across groups in CW solutions.

instead of aiming for equality (equitability) when distributing benefits in the case study, one can aim for ensuring a balance in which more impoverished communities receive more support than wealthy districts. There are methods in the literature designed to address such balance concerns in which the most fair allocation is not necessarily on the equality line [32,33]. These methods are designed for the case where a single benefit is allocated. Extending these to the multibenefit case and designing solution methods that are not just computationally feasible but also easy-to-grasp for decision makers is an open direction for research. Future research can also be conducted on distinguishing

benefits with respect to their relevance in the aggregated efficiency-equality approach, for example by incorporating weights in the scalarization.

#### CRediT authorship contribution statement

**Damla Akoluk:** Conceptualization; Methodology; Software; Writing.

**Özlem Karsu:** Conceptualization; Methodology; Supervision; Writing.

#### A. Models used in the case study

##### Problem Parameters

$K$ : the set of district groups, index  $k$

$I$ : the set of course types, index  $i$

$J$ : the set of districts, index  $j$

$M$ : number of thresholds used for piecewise linearization

$L_i$ : a lower bound on the total amount of enrollment in course  $i$

$H_i$ : an upper bound on the total amount of enrollment in course  $i$

$\Delta T_m^f$ : difference between two consecutive normalized benefit thresholds defining interval  $m$  in equality aggregation function,  $m = 1, \dots, (M - 1)$

$\Delta T_m^e$ : difference between two consecutive normalized benefit thresholds defining interval  $m$  in efficiency aggregation function,  $m = 1, \dots, (M - 1)$

$\Delta U_m^f$ : difference between equality scores of benefit thresholds defining interval  $m$ , in equality aggregation function,  $m = 1, \dots, (M - 1)$

$\Delta U_m^e$ : difference between efficiency scores of benefit thresholds defining interval  $m$  in efficiency aggregation function,  $m = 1, \dots, (M - 1)$

$W_m^f$ :  $\Delta U_m^f / \Delta T_m^f$ ,  $m = 1, \dots, (M - 1)$ .

$W_m^e$ :  $\Delta U_m^e / \Delta T_m^e$ ,  $m = 1, \dots, (M - 1)$ .

$p_{ijk}$ : Number of participants from district group  $k$  to course type  $i$  in district  $j$

$c_{ij}$ : cost of opening course type  $i$  in district  $j$

$C$ : total available budget

##### Decision Variables

$z_{ik}$ : total number of people in district group  $k$  that are enrolled in course type  $i$

$z_{ik}^N$ : normalized  $z_{ik}$  ( $\frac{z_{ik}}{\sum_{k \in K} z_{ik}}$ )

$f_{ik}$ : equality score contribution of  $z_{ik}^N$

$t_i$ : scalarized value of total amount of benefit in course type  $i$

$e_i$ : efficiency score contribution of  $t_i$

$\bar{X}_{ikm}^f$ : amount of normalized benefit obtained within interval  $m = 1, \dots, (M - 1)$  for district group  $k$  from course type  $i$  in equality aggregation function

$\bar{X}_{im}^e$ : amount of normalized total benefit obtained within interval  $m = 1, \dots, (M - 1)$  from course type  $i$  in efficiency aggregation function

$$y_{ij} = \begin{cases} 1, & \text{if course type } i \text{ is offered at district } j \\ 0, & \text{otherwise.} \end{cases}$$

$a_{ikj}$  : auxiliary variable( $y_{ij} \times z_{ik}^N$ )

**Aggregate efficiency-equality model (AEE-C);**

$$\begin{aligned} & \max \sum_{i \in I} e_i, \quad \max \sum_{k \in K} \sum_{i \in I} f_{ik} \\ & \text{Subject to :} \\ & (8) - (15) \end{aligned} \quad (22)$$

$$\sum_{j \in J} \sum_{i \in I} c_{ij} y_{ij} \leq C$$

$$z_{ik} = \sum_{j \in J} p_{ijk} y_{ij} \quad \forall i \in I, \forall k \in K \quad (23)$$

$$\sum_{j \in J} \sum_{k' \in K} p_{ijk'} a_{ikj} = z_{ik} \quad \forall i \in I, \forall k \in K \quad (24)$$

$$a_{ikj} \leq y_{ij} \quad \forall i \in I, \forall k \in K, \forall j \in J \quad (25)$$

$$a_{ikj} \leq z_{ik}^N \quad \forall i \in I, \forall k \in K, \forall j \in J \quad (26)$$

$$a_{ikj} \geq z_{ik}^N - (1 - y_{ij}) \quad \forall i \in I, \forall k \in K, \forall j \in J \quad (27)$$

$$z_{ik}, f_{ik}, e_i \geq 0 \quad \forall i \in I, \forall k \in K, \forall j \in J \quad (28)$$

$$y_{ij} \in \{0, 1\} \quad \forall i \in I, \forall j \in J \quad (29)$$

Constraint (22) ensures that the budget is not exceeded. The set of constraints (23) is used to calculate the total number of district group members benefiting from a specific course type, for each course type-district group pair. Constraint sets (24)–(27) are for linearization. We set  $\Delta T_m^f = \Delta T_m^e = 0.1 \quad \forall m = 1, \dots, (M-1)$  in constraint sets (14) and (15). Finally, constraints (29) define the binary variables.

Parameters and decision variables are the same as in the concave efficiency model for variant (AEE-L). We delete constraints (11), (12), (13), (15) and replace the first objective function with  $\sum_{i \in I} t_i$ .

**Additional parameters for concave welfare model**

$\Delta T_m$ : difference between two consecutive benefit thresholds defining interval  $m$ ,  $m = 1, \dots, (M-1)$ .

$\Delta U_m$ : difference between welfare scores of benefit thresholds defining interval  $m$ ,  $m = 1, \dots, (M-1)$ .

$W_m$ :  $\Delta U_m / \Delta T_m$ ,  $m = 1, \dots, (M-1)$ .

**Additional decision variables for concave welfare model**

$u_{ik}$ : contribution to social welfare from the share of district group  $k$  in service related to course type  $i$ , i.e.,  $u_i(z_{ik})$

$\bar{Y}_{ikm}$ : amount of benefit from course type  $i$  obtained by district group  $k$  within interval  $m$

**Concave welfare model (CW);**

$$\begin{aligned} & \max \quad \left( \sum_{k \in K} u_{1k}, \quad \sum_{k \in K} u_{2k} \right) \\ & \text{Subject to :} \\ & (19), (20), (21), (23) \end{aligned} \quad (30)$$

$$\sum_{j \in J} \sum_{i \in I} c_{ij} y_{ij} \leq C$$

$$u_{ik} \geq 0 \quad \forall i \in I, \forall k \in K \quad (31)$$

$$y_{ij} \in \{0, 1\} \quad \forall i \in I, \forall j \in J \quad (32)$$

Constraint (30) ensures that the budget is not exceeded. Constraint set (32) defines the binary variables.

## B. Results of the case study

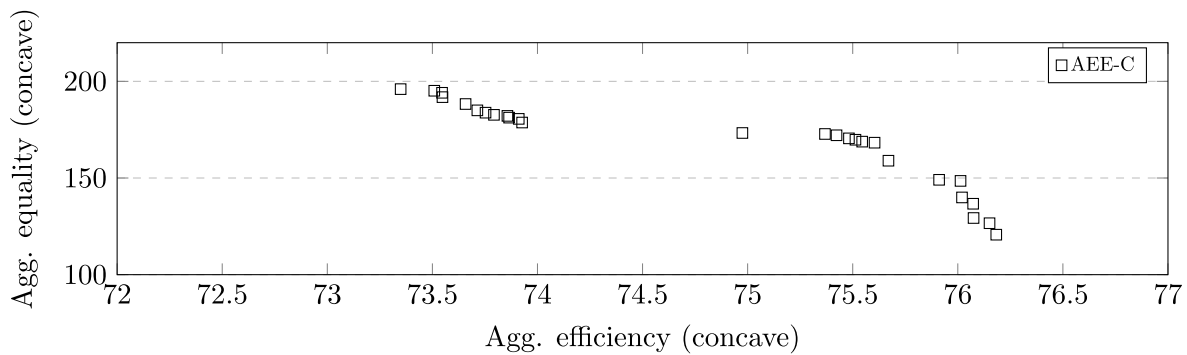


Fig. 21. Pareto optimal solutions of the aggregate efficiency-equality model: AEE-C.

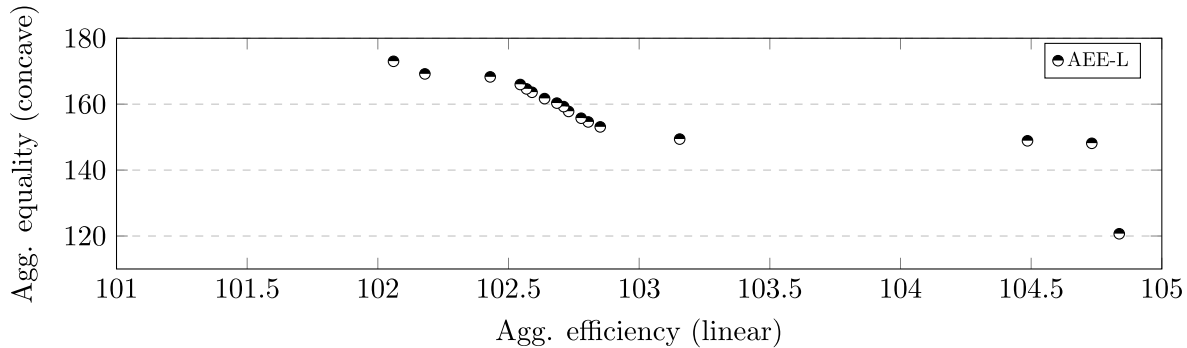


Fig. 22. Pareto optimal solutions of the aggregate efficiency-equality Model: AEE-L.

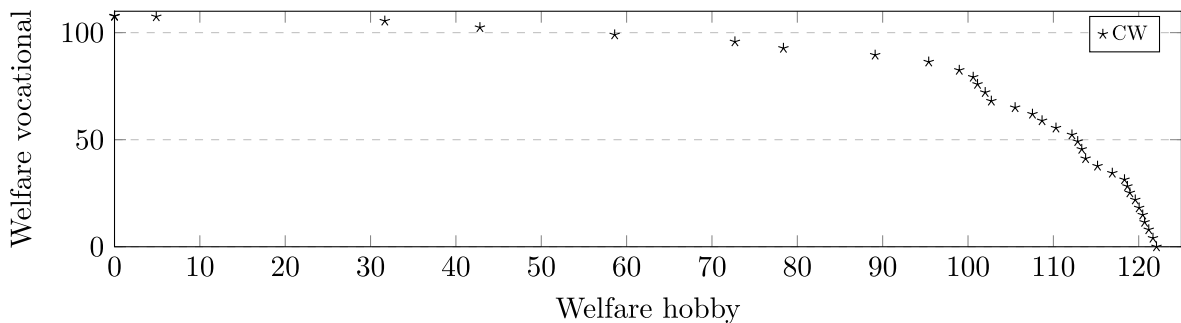


Fig. 23. Pareto optimal solutions of the concave welfare model: CW.

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