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Compound Poisson Disorder Problem with Uniformly Distributed Disorder Time

CAGIN URU[1,a], SAVAS DAYANIK[2,b] and SEMIH O. SEZER[3,c]

[1]*Duke University, Durham, NC, USA.* [a]*cagin.uru@duke.edu*
[2]*Bilkent University, Ankara, Turkey.* [b]*sdayanik@bilkent.edu.tr*
[3]*Sabancı University, Istanbul, Turkey.* [c]*sezer@sabanciuniv.edu*

Suppose that the arrival rate and the jump distribution of a compound Poisson process change suddenly at an unknown and unobservable time. We want to detect the change as quickly as possible to take counteractions, e.g., to assure top quality of products in a production system, or to stop credit card fraud in a banking system. If we have no prior information about future disorder time, then we typically assume that the disorder is equally likely to happen any time–or has uniform distribution–over a long but finite time horizon. We solve this so-called compound Poisson disorder problem for the practically important case of unknown, unobserved, but uniformly distributed disorder time. The solution hinges on the complete separation of information flow from the hard time horizon constraint, by describing the former with an autonomous time-homogeneous one-dimensional Markov process in terms of which the detection problem translates into a finite horizon optimal stopping problem. For any given finite horizon, the solution is two-dimensional. For cases where the horizon is large and one is unwilling to set a fixed value for it, we give a one-dimensional approximation. Also, we discuss an extension where the disorder may not happen on the given interval with a positive probability. In this extended model, if no detection decision is made by the end of the horizon, then a second level hypothesis testing problem is solved to determine the local parameters of the observed process.

Keywords: Compound Poisson process; optimal stopping; Poisson disorder problem; quickest detection

1. Introduction

A sudden change in the statistical behavior of a system can be catastrophic. A quickest detection rule may help decision makers alleviate some of the detrimental effects of the change by allowing them to react and take the necessary countermeasures in a timely manner. Monitoring the arrival process of seismic shocks and declaring an emergency when there is a change in their behavior may help with the efforts to foresee an emerging earthquake, and could potentially save lives and minimize the damage. Likewise, observing the arrival process of patients to medical centers can contribute to the disease outbreak investigations greatly. Therefore, detecting the change point as soon as it occurs can be a crucial task in certain applications.

Change detection problems have been widely studied in the literature. Basseville and Nikiforov (1993), Poor and Hadjiliadis (2008), and Tartakovsky, Nikiforov and Basseville (2014) show us a wide-range of applications and methods. Bayesian formulations of these problems date back to the works of Kolmogorov and Shiryaev in the early 1960s. Assuming some prior knowledge on the probability law governing the change point, the aim in these formulations is to minimize a cost function, known as the Bayes risk, consisting of the probability of falsely detecting a change and the average detection delay cost. The change point is assumed to have a geometric and exponential prior distribution in discrete time and continuous time models, respectively. The primary objects of study are the sequences of random variables and their continuous time analogs. The solution methods to those problems depend on the reformulation to an optimal stopping problem and reduction to a free-boundary problem. When solving the reformulated optimal stopping problem, both martingale and Markovian approaches are studied; see Shiryaev (2008), and Peskir and Shiryaev (2006) for a comprehensive review of the optimal stopping time problems. We invite the reader to Shiryaev (2019) for an extensive discussion on disorder problems, related results, and also for a comprehensive list of references from the literature.

For a Poisson process, the first formulation was given by Gal'chuk and Rozovskii (1971). Davis (1976) extended their results to a more general case. Three decades after its first formulation, the disorder problem for a simple Poisson process was completely solved by Peskir and Shiryaev (2002). Bayraktar, Dayanik and Karatzas (2005, 2006) considered the problem for different Bayes risks and with a random post-disorder rate, respectively. Taking jump observations into formulation, Gapeev (2005) gave a partial solution to the compound Poisson disorder problem. Dayanik and Sezer (2006a) solved the problem with general jump distributions by adapting a method of Gugerli (1986) and Davis (1993) for solving control problems of general piecewise-deterministic Markov processes. The results and the algorithm developed in Dayanik and Sezer (2006a) are employed in Buonaguidi et al. (2021) for a credit card fraud detection problem. Buonaguidi et al. (2021) can be consulted for practice related remarks and discussions. The formulations with observations at discrete points in time are studied by Brown (2008, 2016) and under different observation schemes by Herberts and Jensen (2004). In the above-mentioned literature, the disorder times have often a zero-modified exponential distribution. One exception, as a natural generalization of the exponential prior choice, is the phase-type distribution, which is studied by Bayraktar and Sezer (2009).

For related work on non-Bayesian minimax formulations with continuous-time jump processes, we refer the reader to El Karoui, Loisel and Salhi (2017) and Figueroa-López and Ólafsson (2019); El Karoui, Loisel and Salhi (2017) studies a problem where the stochastic intensity of a Poisson process undergoes a proportional change, and Figueroa-López and Ólafsson (2019) considers the detection problem for a Lévy process. These two papers can also be consulted for further references on other discrete and continuous time change detection problems in the non-Bayesian framework.

On the Bayesian side, the exponential distribution is useful to model machine part lifetimes in reliability theory. The end of life of a component in a sophisticated black-box system could be a potential change point in the behavior of the system. Thus, exponentially distributed change point makes sense when modeling a quickest detection problem due to the earliest failure of independent components in a large system with many components. More importantly, having an exponential prior proves useful when obtaining explicit results with several solution methods. For instance, the sufficient statistics are finite-dimensional (mostly one-dimensional) when the change point is exponentially distributed; see, for example, Peskir and Shiryaev (2002). Due to those analytical advantages, exponential distribution remained as the common assumption for the change point's prior in the literature.

Exponential prior is unfortunately inappropriate when the disorder is equally likely to happen within every infinitesimally small time epoch over a finite time interval. Although phase-type priors can be used to approximate a uniform or another prior distribution, it suffers from the high-dimensionality problem as the state space of the hidden Markov process gets larger. Also, in practice, if the system analyst does not have any prior information about time of the disorder other than the range of possible values, then she may choose to use a relatively uninformative prior for the disorder time. In that respect, one of the most straightforward candidates for a prior distribution is the uniform distribution. It is simple, puts equal weight to every small interval of the same length in a given larger interval, and it gives us on that larger interval the density with the maximum entropy. Thus, it can successfully express the notion of having little information and would be a good fit for the change point's prior. Unlike an exponential or phase-type distribution, however, the uniform distribution does not have forgetfulness property, and the likelihood of a disorder gradually increases as time without disorder progresses.

The uniform prior distribution has been recently introduced and used in Zhitlukhin and Shiryaev (2013, 2014) and Sokko (2015) in settings where the observations come from a Wiener process whose drift changes at the disorder time. Zhitlukhin and Shiryaev (2014) discusses an asset selling problem with a financial focus; see also Shiryaev, Zhitlukhin and Ziemba (2014, 2015) for some applications. Zhitlukhin and Shiryaev (2013) and Sokko (2015), on the other hand, study the statistical problem of detecting the change time. To our best knowledge, such a detection formulation with compound Poisson observations has not been studied yet. This is the contribution of the current paper.

In the paper, we denote by

$$X_t = X_0 + \sum_{k=1}^{N_t} Y_k, \quad t \geq 0, \tag{1}$$

a compound Poisson process with arrivals following a simple Poisson process $N = \{N_t;\, t \geq 0\}$ at a rate $\lambda_0 > 0$ and i.i.d. $\mathbb{R}^d$-valued random jumps $Y_1, Y_2, \ldots$ with common distribution $\nu_0(\cdot)$ independent of N. At some unknown and unobserved disorder time Θ, we assume that the arrival rate λ_0 and jump distribution ν_0 shift to λ_1 and ν_1, respectively. We assume that the disorder time Θ has a zero-modified uniform prior

$$\mathbb{P}\{\Theta = 0\} = \pi \quad \text{and} \quad \mathbb{P}\{\Theta > t\} = (1-\pi)\left(\frac{T-t}{T}\right), \quad t \in [0, T], \tag{2}$$

for known $T \in \mathbb{R}_+$ and $\pi \in [0, 1)$. We want to detect Θ with some $[0, T]$-valued stopping time τ adapted to the history $\mathcal{F}$ of the compound Poisson process X so as to minimize Bayes risk

$$B_\tau(T, \pi) := \mathbb{P}\{\tau < \Theta\} + c\mathbb{E}(\tau - \Theta)^+, \quad \pi \in [0, 1),\ \tau \in \mathcal{F}, \tag{3}$$

which is the sum of the false alarm probability $\mathbb{P}\{\tau < \Theta\}$ and the expected delay cost $c\mathbb{E}(\tau - \Theta)^+$. The parameter $c > 0$ is the unit delay cost relative to a false alarm. An $\mathcal{F}$-stopping time τ is called a Bayes-optimal alarm time if it attains the minimum Bayes risk

$$V(T, \pi) := \inf_{\tau \in \mathscr{S}_{[0,T]}} B_\tau(T, \pi), \tag{4}$$

where $\mathscr{S}_{[0,T]}$ is the collection of all $\mathcal{F}$-adapted $[0, T]$-valued stopping times.

Our analysis in the paper shows that the minimum Bayes risk problem in (4) is equivalent to a finite horizon optimal stopping of a one-dimensional piecewise-deterministic Markov sufficient statistic. We study properties of the value function of this stopping problem. We prove that the continuation and stopping regions are separated by a monotone boundary, and the optimal stopping time is the first crossing time of this boundary. We also show that the value function of the finite horizon problem can be approximated by that of the infinite horizon when T is large. This allows the decision maker to use the solution of the latter problem and thereby reduce the computations for large values of T to one dimension only. This infinite horizon approximation is also useful if T is known to be large but its exact value is difficult to specify.

The prior distribution in (2) above assumes that a disorder happens on the given range with probability one. When T is large, this can be a reasonable assumption. However, when T is moderate or small, it might be preferable from a practical point of view to consider the possibility that no disorder happens at all on $[0, T]$ with some non-zero probability. In Section 5, we discuss this extension, and we consider a model in which, if no detection decision is made by time T, a second level hypothesis testing problem is solved to determine the current (and future) arrival rate and jump distribution of the observed process. For this second level problem, we consider two formulations; in the first one, a decision must be made at time T, and in the second, this decision can be delayed and further observations can be collected after T at some additional cost per unit time of delay. We show that both formulations essentially give us similar auxiliary optimal stopping problems for the same Markov sufficient statistic, and in the second formulation, we apply the well-known *sequential probability ratio test* if we continue after T. One major difference of this extended setup compared to the case with the prior in (2) is that, in the auxiliary optimal stopping problems, the stopping boundaries may not be monotone anymore.

The remainder of the paper is organized as follows. In Section 2, we present the model containing the random elements of our problem, describe the quickest detection problem, and reformulate it to an optimal stopping problem for a suitable Markovian sufficient statistic. In Section 3, we investigate the properties of the value function of this stopping problem and show that the continuation and stopping regions are separated by a non-decreasing boundary. We also show that the infinite horizon problem can approximate the finite horizon one when T is large. In Section 4, we introduce the successive approximations of the value function and derive some useful results. In Section 5, we discuss the extension where the change may not happen at all on the given interval with a positive probability. Section 6 concludes the paper with final remarks. Lengthy derivations and auxiliary proofs are given in the appendices at the end.

2. Model and problem formulation

Let $(\Omega, \mathcal{F}, \mathbb{P}_0)$ be a probability space hosting the following independent stochastic elements:

(i) a standard Poisson process $N = \{N_t;\, t \geq 0\}$ with the arrival rate λ_0,
(ii) i.i.d. $\mathbb{R}^d$-valued random variables $Y_1, Y_2, \ldots$ with a common distribution $\nu_0(A) := \mathbb{P}_0\{Y_1 \in A\}$ for every Borel set A in the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ and $\nu_0(\{0\}) = 0$,
(iii) a random variable Θ with the distribution

$$\mathbb{P}_0\{\Theta = 0\} = \pi \quad \text{and} \quad \mathbb{P}_0\{\Theta > t | \Theta > 0\} = 1 - \frac{t}{T} \quad \text{for } 0 \leq t \leq T,\ \pi \in [0,1),\ T \in \mathbb{R}_+. \tag{5}$$

Let $X = \{X_t;\, t \geq 0\}$ defined by (1) be a compound Poisson process with the arrival rate λ_0, jump distribution $\nu_0(\cdot)$, and the jump times

$$\sigma_n := \inf\{t > \sigma_{n-1} : X_t \neq X_{t-}\}, \quad n \geq 1 \quad (\sigma_0 \equiv 0). \tag{6}$$

Let us denote the augmentation of its natural filtration $\sigma(X_s,\, s \leq t)$, $t \geq 0$, with $\mathbb{P}_0$-null sets by $\mathcal{F} = \{\mathcal{F}_t; t \geq 0\}$. We will describe the enlargement of this filtration $\mathcal{F}$ with the sigma-algebra $\sigma(\Theta)$ generated by Θ with $\mathcal{G} = \{\mathcal{G}_t; t \geq 0\}$. That is, $\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\Theta)$ for every $t \geq 0$.

Let $\lambda_1 > 0$ be a constant, and $\nu_1(\cdot)$ be a probability measure on the measurable space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ absolutely continuous with respect to the distribution $\nu_0(\cdot)$. Radon-Nikodym derivative

$$f(y) := \frac{d\nu_1}{d\nu_0}\Big|_{\mathcal{B}(\mathbb{R}^d)}(y), \quad y \in \mathbb{R}^d, \tag{7}$$

of $\nu_1(\cdot)$ with respect to $\nu_0(\cdot)$ exists and is a $\nu_0 - a.e.$ non-negative Borel function.

We describe a new probability measure $\mathbb{P}$ on the measurable space $(\Omega, \vee_{s \geq 0}\mathcal{G}_s)$ by a change of measure with Radon-Nikodym derivative Z_t of $\mathbb{P}$ with respect to $\mathbb{P}_0$

$$\frac{d\mathbb{P}}{d\mathbb{P}_0}\Big|_{\mathcal{G}_t} = Z_t := 1_{\{t < \Theta\}} + 1_{\{t \geq \Theta\}} \frac{R_t}{R_\Theta}, \quad t \geq 0, \tag{8}$$

in terms of the *likelihood ratio process* $R = \{R_t; t \geq 0\}$ given by

$$R_t := e^{-(\lambda_1-\lambda_0)t} \prod_{k=1}^{N_t} \left[\frac{\lambda_1}{\lambda_0} f(Y_k) \right], \quad t \geq 0. \tag{9}$$

Probability measures $\mathbb{P}_0$ and $\mathbb{P}$ are called the reference and physical probability measures, respectively. Observe that since $Z_0 = 1$ $\mathbb{P}_0$-a.s. and the probability measures $\mathbb{P}$ and $\mathbb{P}_0$ agree on $\mathcal{G}_0 = \sigma(\Theta)$, the disorder time Θ has the same distribution under both probability measures. Calculations which are difficult under $\mathbb{P}$ become easier under $\mathbb{P}_0$ due to the independence of the previously defined stochastic elements.

Our problem is to find an $\mathcal{F}$-stopping time $\tau \leq T$ that minimizes the Bayes risk in (3) and detect the disorder time Θ as soon as possible while observing the process X. As shown in Appendix A, the Bayes risk can be rewritten as

$$B_\tau(T,\pi) = 1 - \pi + \frac{c(1-\pi)}{T} \mathbb{E}_0 \left[\int_0^\tau \left(\frac{\Phi_t}{D_t} - \frac{1}{c} \right) dt \right], \tag{10}$$

for $\pi \in [0,1)$ and $\tau \leq T$, in terms of the *odds-ratio process* Φ defined by

$$\Phi_t := \frac{\Pi_t}{1-\Pi_t} = \frac{\mathbb{P}\{\Theta \leq t | \mathcal{F}_t\}}{\mathbb{P}\{\Theta > t | \mathcal{F}_t\}}, \qquad t \geq 0, \tag{11}$$

where $\Pi_t := \mathbb{P}\{\Theta \leq t | \mathcal{F}_t\}$, $t \geq 0$, is the *posterior probability process*. The deterministic process $D = \{D_t; t \in [0,T)\}$ is given by

$$D_t := \frac{1}{T-t}. \tag{12}$$

Let us define

$$Q_t := R_t \left[q + \int_0^t \frac{1}{R_s} ds \right], \qquad t \geq 0. \tag{13}$$

In Appendix A.2, starting with $Q_0 = q = T\frac{\pi}{1-\pi}$, we show that $Q_t = \frac{\Phi_t}{D_t}$, for $t < T$. In the appendices (see Appendix B), we also present the dynamics of the process Q. As it turns out, the process Q belongs to the family of piecewise-deterministic Markov processes. If we define

$$x(q,t) := \begin{cases} \dfrac{1}{\lambda_1-\lambda_0} + \left(q - \dfrac{1}{\lambda_1-\lambda_0} \right) e^{-(\lambda_1-\lambda_0)t}, & \lambda_1 \neq \lambda_0, \\ q + t, & \lambda_1 = \lambda_0, \end{cases} \tag{14}$$

for $t \in \mathbb{R}_+$ and $q \in \mathbb{R}_+$, then for $n \geq 0$

$$Q_t = \begin{cases} x(Q_{\sigma_n}, t - \sigma_n), & \sigma_n \leq t < \sigma_{n+1}, \\ x(Q_{\sigma_n}, \sigma_{n+1} - \sigma_n) \dfrac{\lambda_1}{\lambda_0} f(Y_{n+1}), & t = \sigma_{n+1}. \end{cases} \tag{15}$$

This means that the process Q follows the deterministic curves $t \mapsto x(q,t)$ in (14) between two consecutive jumps of the process X and jumps instantly at the jump times of the process X as described in (15). Using (13) and the martingale property under $\mathbb{P}_0$ of the non-negative process R, it is easy to verify that Q is a non-negative supermartingale, again under $\mathbb{P}_0$. We also have $\mathbb{E}_{0,q}[Q_t] = q + t$, for $t \geq 0$. Throughout, $\mathbb{E}_{0,q}$ denotes the expectation operator under the probability measure $\mathbb{P}_0$ with $Q_0 = q \in \mathbb{R}_+$ with probability one.

The result below shows that the exit time of the process Q from an interval of the form $[0, r)$, for $r < \infty$, has a uniformly bounded first moment. The proof is deferred to Appendix C. As the proof illustrates, it is possible to find explicit upper bounds on the moment depending on the boundedness of the random variables $f(Y_k)$s and/or the ordering of λ_1 and λ_0. These expressions are omitted here for conciseness and clarity of the presentation.

Lemma 2.1. *Define* $\tau_r := \inf\{t \geq 0 : Q_t \geq r\}$ *for* $r < +\infty$. *Then, we have* $\mathbb{E}_{0,q}[\tau_r] \leq \mathbb{E}_{0,0}[\tau_r] < \infty$ *for every* $q \in \mathbb{R}_+$.

Substituting the process Q in the Bayes risk yields

$$B_\tau(T,\pi) = 1 - \pi + \frac{c(1-\pi)}{T}\,\mathbb{E}_0\left[\int_0^\tau \left(Q_t - \frac{1}{c}\right) dt\right], \qquad \text{with } \; Q_0 = T\frac{\pi}{1-\pi}. \tag{16}$$

Hence, the minimum Bayes risk in (4) is given by

$$V(T,\pi) = 1 - \pi + \frac{c(1-\pi)}{T}\,U\left(T, T\frac{\pi}{1-\pi}\right) \tag{17}$$

in terms of the value function

$$U(T,q) := \inf_{\tau \in \mathscr{S}_{[0,T]}} \mathbb{E}_{0,q}\left[\int_0^\tau g(Q_t)dt\right], \qquad q = Q_0 \in \mathbb{R}_+, \tag{18}$$

with the running cost $g(q) := q - \frac{1}{c}$, $q \in \mathbb{R}_+$. The quickest detection problem for a compound Poisson process can thus be reformulated equivalently as a finite horizon optimal stopping problem for the process Q.

3. Properties of the value function

The following remark is useful in establishing the concavity and Lipschitz continuity of the function U in (18) in the q variable. For a discussion on the concavity of the value functions of detection and testing problems for Wiener processes, we refer the reader to the recent paper Ekström and Wang (2022).

Remark 3.1. For every bounded stopping time τ, we have

$$\mathbb{E}_{0,q}\int_0^\tau \left(Q_t - \frac{1}{c}\right)dt = \mathbb{E}_{0,q}\left[\tau\left(Q_\tau - \frac{1}{c}\right) - \frac{\tau^2}{2}\right] = q\,\mathbb{E}_0[\tau R_\tau] + \mathbb{E}_0\left[\tau R_\tau \int_0^\tau \frac{1}{R_s}ds - \frac{\tau}{c} - \frac{\tau^2}{2}\right]. \tag{19}$$

Proof. Applying the chain rule for the process $t \mapsto t(Q_t - 1/c)$ and using the dynamics of the process Q in (62), we obtain

$$t\left(Q_t - \frac{1}{c}\right) - 0 = \int_0^t \left(Q_u - \frac{1}{c}\right)du + \int_0^t u dQ_u = \int_0^t \left(Q_u - \frac{1}{c}\right)du + \int_0^t u du + \int_0^t u dM_u,$$

where the last integral term is a $(\mathbb{P}_0, \mathcal{F})$-martingale; see (59) and (62) in Appendix B for the process M. Evaluating all expressions for a bounded stopping time τ and taking the expectations, we obtain the first identity in (19) thanks to optional sampling theorem. The second now follows from (13). $\square$

Lemma 3.2. *The mapping $q \mapsto U(T,q)$ is non-decreasing and concave for every fixed T, and $T \mapsto U(T,q)$ is non-increasing for every fixed q. Moreover,*

$$-\frac{T}{c} \leq U(T,q) \leq \bar{U}(T,q) \leq 0, \tag{20}$$

where

$$\bar{U}(T,q) := \inf_{t \leq T} \mathbb{E}_{0,q} \int_0^t \Big(Q_u - \frac{1}{c}\Big) du = -\int_q^\infty \min\Big\{\Big(\frac{1}{c} - y\Big)^+, T\Big\} dy. \tag{21}$$

Proof. Concavity and monotonicity in q are direct consequences of the second equality in (19) (recall that R is a non-negative process). It is also clear from the definition in (18) that $T \mapsto U(T,q)$ is non-increasing for every fixed q, and $U(T,q) \geq \inf_{\tau \leq T} \mathbb{E}_{0,q} \int_0^\tau \Big(-\frac{1}{c}\Big) du = -\frac{T}{c}$. Since every deterministic time is also a stopping time, we have

$$U(T,q) \leq \bar{U}(T,q) = \inf_{t \leq T} \mathbb{E}_{0,q} \int_0^t \Big(Q_u - \frac{1}{c}\Big) du = \inf_{t \leq T} \mathbb{E}_{0,q} \Big[t\Big(Q_t - \frac{1}{c}\Big) - \frac{t^2}{2}\Big],$$

where the last equality is due to the first equality in (19). Using $\mathbb{E}_0 Q_t = q + t$, the deterministic minimization problem can be solved easily, and this gives the explicit expressions in (21). The bound $\bar{U}(T,q) \leq 0$ follows by taking $t = 0$. □

Lemma 3.3. *The function U is locally (jointly) Lipschitz continuous. We have*

$$0 \leq U(T_1,q) - U(T_2,q) \leq \frac{1}{c}(T_2 - T_1), \qquad \text{for } T_1 < T_2, \tag{22}$$

$$0 \leq U(T,q_2) - U(T,q_1) \leq (q_2 - q_1)T, \qquad \text{for } q_1 < q_2. \tag{23}$$

Proof. Non-negativity of the differences are immediate consequences of the monotonicity properties discussed in Lemma 3.2. To show the upper bound in (22), fix $T_1 < T_2$ and q, and for $\epsilon > 0$ let $\tau_\epsilon \leq T_2$ be an ϵ-optimal stopping time starting from the point (T_2, q). That is, $\mathbb{E}_{0,q} \int_0^{\tau_\epsilon} \Big(Q_u - \frac{1}{c}\Big) du \leq U(T_2,q) + \epsilon$. Since $\tau_\epsilon \wedge T_1$ is a feasible solution (stopping time) starting from (T_1, q), we obtain

$$U(T_1,q) - U(T_2,q) \leq -\mathbb{E}_{0,q} 1_{\{\tau_\epsilon > T_1\}} \int_{T_1}^{\tau_\epsilon} \Big(Q_u - \frac{1}{c}\Big) du + \epsilon \leq \mathbb{E}_{0,q} \int_{T_1}^{T_2} \frac{1}{c} du + \epsilon = \frac{T_2 - T_1}{c} + \epsilon$$

from which (22) follows since $\epsilon > 0$ is arbitrary.

To establish the upper bound in (23), for $q_1 < q_2$ and T fixed, let $\tilde{\tau}_\epsilon$ denote an ϵ-optimal rule starting from the point (T, q_1), which is also a feasible rule starting from (T, q_2). Using Remark 3.1, we have

$$U(T,q_2) - U(T,q_1) \leq (q_2 - q_1)\mathbb{E}_{0,\cdot}[\tilde{\tau}_\epsilon R_{\tilde{\tau}_\epsilon}] + \epsilon. \tag{24}$$

Since $\tilde{\tau}_\epsilon \leq T$ and R is a non-negative martingale, it follows by the optional sampling theorem that $\mathbb{E}_0[\tilde{\tau}_\epsilon R_{\tilde{\tau}_\epsilon}] \leq T\mathbb{E}_0[R_{\tilde{\tau}_\epsilon}] = TR_0 = T$. This shows (23) by the arbitrariness of ϵ again. □

We can extend the definition of $U(T,q)$ for $T = \infty$ to obtain the infinite horizon version of the problem in (18) without the constraint $\tau \leq T$. This gives us a one-dimensional problem, and it can be solved as in Dayanik and Sezer (2006a). Alternatively, the arguments of Section 4 can be modified (with the time dimension removed) to account for this infinite horizon problem. Here, we omit the details for conciseness. Instead, we summarize the useful results in the remark below.

Remark 3.4. The function $q \mapsto U(\infty, q)$ is a continuous, non-decreasing, and concave function with $U(\infty, 0) > -\infty$. There exists a point $q_\infty \geq \frac{1}{c}$ such that $U(\infty, q) < 0$ for $q < q_\infty$ and $U(\infty, q) = 0$ for $q \geq q_\infty$. The first entrance time τ_{q_∞} of the process Q into the region $[q_\infty, \infty)$ is an optimal stopping time for the infinite horizon problem.

The following lemma is non-trivial. Its proof is left to Appendix C. The result implies that when T is large, one could use the solution of the infinite horizon problem. Its optimal stopping time τ_{q_∞} has finite expectation by Lemma 2.1.

Lemma 3.5. *As $T \nearrow \infty$, $U(T, q)$ converges to $U(\infty, q)$ uniformly (over q) with convergence bounds*

$$0 \leq U(T, q) - U(\infty, q) \leq -U(\infty, 0)\, \frac{\mathbb{E}_{0,0}[\tau_{q_\infty}]}{T}. \tag{25}$$

We conclude this section with the following corollary, which shows the stopping region is separated from the continuation region by a monotone boundary. The result follows directly from Lemmas 3.2 and 3.3 and Remark 3.4. No proof is needed.

Corollary 3.6. *Since*

(i) *U is (jointly) continuous*
(ii) *$q \mapsto U(T, q)$ is non-decreasing and concave for all $T \geq 0$*
(iii) *$T \mapsto U(T, q)$ is non-increasing for all $q \geq 0$*
(iv) *$U(\infty, q) \leq U(T, q)$ for all $T \geq 0$ and $q \geq 0$*

it follows that

$$\Gamma := \{(T, q) : U(T, q) = 0\} \tag{26}$$

is a closed region with a non-decreasing lower boundary

$$b(T) := \min\{q : U(T, q) = 0\} \tag{27}$$

uniformly bounded from above by q_∞. The upper bound function $\bar{U}(T, q)$ with its explicit form in (21) implies that $b(0+) \geq \frac{1}{c}$.

4. Successive approximations

In this section, we show how the function U in (18) can be constructed sequentially. Such a construction follows naturally from the observation that the problem regenerates at every jump time. This is indeed a well-known approach for problems with jump processes; see Dayanik, Poor and Sezer (2008), Ludkovski and Sezer (2012), Dayanik and Parlar (2013), Arslan, Frenk and Sezer (2015), Kilic, Saygi and Sezer (2017) for some other examples. We repeat some of the related arguments here for completeness, and we omit others for conciseness. In particular, we omit the proofs Lemmas 4.4 and 4.10, and that of Proposition 4.9. The reader may refer to Ludkovski and Sezer (2012) for the proofs of similar results.

Let us define the family of optimal stopping problems

$$U_n(T, q) := \inf_{\tau \in \mathscr{S}_{[0,T]}} \mathbb{E}_{0,q}\left[\int_0^{\tau \wedge \sigma_n} \left(Q_t - \frac{1}{c}\right) dt\right], \quad T \in \mathbb{R}_+,\ q \in \mathbb{R}_+,\ n \in \mathbb{N}, \tag{28}$$

obtained from stopping the process Q at the nth jump time σ_n of the observation process X. With $n = 0$, we obviously have $\sigma_0 = 0$ and $U_0(T, q) = 0$. Since the integrand in (28) is bounded from below by $-\frac{1}{c}$, the expectation in (28) is well defined for every $\tau \in \mathscr{S}_{[0,T]}$.

For any n, stopping the process Q at time 0, i.e., taking $\tau = 0$, results in $U_n(T, q) = 0$. Hence, 0 is an upper bound for the $(U_n(T, q))_{n\in\mathbb{N}}$ sequence. Since $Q_t \geq 0$ for all $t \in \mathbb{R}_+$ and $\tau \leq T$ almost surely,

$$\int_0^{\tau\wedge\sigma_n} \left(Q_t - \frac{1}{c}\right) dt \geq -\int_0^{\tau\wedge\sigma_n} \frac{1}{c} dt \geq -\int_0^{\tau} \frac{1}{c} dt \geq -\int_0^{T} \frac{1}{c} dt = -\frac{T}{c}.$$

Taking the expectation and infimum on both sides over $\tau \in \mathscr{S}_{[0,T]}$ shows $-\frac{T}{c}$ is a lower bound for the $(U_n(T, q))_{n\in\mathbb{N}}$ sequence. Hence, $-\frac{T}{c} \leq U_n(T, q) \leq 0$ for all $n \in \mathbb{N}$.

The sequence $(U_n(T, q))_{n\in\mathbb{N}}$ is decreasing because the sequence $(\sigma_n)_{n\in\mathbb{N}}$ of jump times of the process X is increasing almost surely. Therefore, $\lim_{n\to\infty} U_n(T, q)$ exists everywhere. Clearly, we also have $U_n(T, q) \geq U(T, q)$ for all $n \in \mathbb{N}$.

Proposition 4.1. *For every $T_{\max} \in \mathbb{R}_+$, as $n \to \infty$, the sequence $(U_n(T, q))_{n\in\mathbb{N}}$ converges to $U(T, q)$ uniformly in $T \in [0, T_{\max}]$, $q \in \mathbb{R}_+$. In fact, for every $n \in \mathbb{N}$, $T \in [0, T_{\max}]$, and $q \in \mathbb{R}_+$, we have*

$$-\frac{T}{c}\left(1 - e^{-\lambda_0 T}\right)^n \leq U(T, q) - U_n(T, q) \leq 0. \tag{29}$$

Before proceeding with the proof let us define $\|f\|_{T_{\max},\infty} := \sup_{(T,q)\in[0,T_{\max}]\times\mathbb{R}_+} |f(T, q)|$, $\forall T_{\max} \in \mathbb{R}_+$, for a function $f : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_-$.

Proof. The second inequality in (29) is already established, and the first inequality is immediate for $n = 0$. To show the first inequality for $n \geq 1$, we note that, because $Q_t \geq 0$ for all $t \in \mathbb{R}_+$ and $\tau \leq T$ almost surely, we have

$$\begin{aligned}
\mathbb{E}_{0,q}\left[\int_0^{\tau}\left(Q_t - \frac{1}{c}\right)dt\right] &= \mathbb{E}_{0,q}\left[\int_0^{\tau\wedge\sigma_n}\left(Q_t - \frac{1}{c}\right)dt + 1_{\{\tau>\sigma_n\}}\int_{\sigma_n}^{\tau}\left(Q_t - \frac{1}{c}\right)dt\right] \\
&\geq U_n(T, q) + \mathbb{E}_{0,q}\left[1_{\{\tau>\sigma_n\}}\int_{\sigma_n}^{\tau}\left(Q_t - \frac{1}{c}\right)dt\right] \geq U_n(T, q) - \frac{1}{c}\mathbb{E}_0\left[1_{\{\tau>\sigma_n\}}(\tau - \sigma_n)\right] \\
&\geq U_n(T, q) - \frac{1}{c}\mathbb{E}_0\left[1_{\{T>\sigma_n\}}(T - \sigma_n)\right] \geq U_n(T, q) - \frac{T}{c}\mathbb{P}_0\{\sigma_n \leq T\}.
\end{aligned}$$

Since the nth jump time σ_n of the process X is the sum of n i.i.d. interarrival times, denoted by the sequence $(\widetilde{\sigma_n})_{n\geq1}$, having exponential distribution with common parameter λ_0 under $\mathbb{P}_0$, we obtain $\mathbb{P}_0\{\sigma_n \leq T\} = \mathbb{P}_0\{\widetilde{\sigma_1} + \ldots + \widetilde{\sigma_n} \leq T\} \leq \mathbb{P}_0\{\widetilde{\sigma_1} \leq T, \ldots, \widetilde{\sigma_n} \leq T\} = (\mathbb{P}_0\{\widetilde{\sigma_1} \leq T\})^n = \left(1 - e^{-\lambda_0 T}\right)^n$. Hence,

$$\mathbb{E}_{0,q}\left[\int_0^{\tau}\left(Q_t - \frac{1}{c}\right)dt\right] \geq U_n(T, q) - \frac{T}{c}\left(1 - e^{-\lambda_0 T}\right)^n.$$

Taking the infimum of both sides over $\tau \in \mathscr{S}_{[0,T]}$ gives the first inequality in (29) for $n \geq 1$.

It now follows from (29) that we have $\lim\limits_{n\to\infty} \|U_n(T, q) - U(T, q)\|_{T_{\max},\infty} = 0$, and the sequence $(U_n(T, q))_{n\in\mathbb{N}}$ converges to $U(T, q)$ uniformly in $T \in [0, T_{\max}]$, $q \in \mathbb{R}_+$. □

Acting on bounded Borel functions $w : [0, T_{\max}] \times \mathbb{R}_+ \mapsto \mathbb{R}_-$, let us define the operators

$$(J_t w)(T, q) := \mathbb{E}_{0,q}\left[\int_0^{t\wedge\sigma_1}\left(Q_s - \frac{1}{c}\right)ds + 1_{\{t\geq\sigma_1\}} w(T-\sigma_1, Q_{\sigma_1})\right], \quad t \in [0, T], \tag{30}$$

$$(Jw)(T, q) := \inf_{t\in[0,T]} (J_t w)(T, q). \tag{31}$$

Note that σ_1 is exponentially distributed with parameter λ_0 under $\mathbb{P}_0$. Hence, using the Fubini theorem, for $t \in [0, T]$ we can write

$$\begin{aligned}
(J_t w)(T, q) &= \int_0^t \mathbb{P}_0\{s < \sigma_1\}\Big(x(q, s) - \frac{1}{c}\Big)ds + \mathbb{E}_{0,q}\left[1_{\{t\geq\sigma_1\}} w\left(T - \sigma_1, x(q, \sigma_1)\frac{\lambda_1}{\lambda_0} f(Y_1)\right)\right] \\
&= \int_0^t e^{-\lambda_0 s}\left(x(q, s) - \frac{1}{c}\right)ds + \int_0^t \lambda_0 e^{-\lambda_0 s}\left[\int_{y\in\mathbb{R}^d} w\left(T - s, x(q, s)\frac{\lambda_1}{\lambda_0} f(y)\right)\nu_0(dy)\right]ds \\
&= \int_0^t e^{-\lambda_0 s}\Big(g(x(q, s)) + \lambda_0 \cdot (Sw)(T - s, x(q, s))\Big)ds,
\end{aligned} \tag{32}$$

where S is the linear operator

$$(Sw)(t, x) := \int_{y\in\mathbb{R}^d} w\left(t, x\frac{\lambda_1}{\lambda_0} f(y)\right)\nu_0(dy), \quad t \in [0, T],\ x \in \mathbb{R}_+, \tag{33}$$

defined on the collection of bounded Borel functions $w : [0, T_{\max}] \times \mathbb{R}_+ \mapsto \mathbb{R}_-$.

Remark 4.2. For every $T \leq T_{\max} < \infty$ and bounded Borel function $w : [0, T_{\max}] \times \mathbb{R}_+ \mapsto \mathbb{R}_-$, the last integrand of (32) is also bounded. Therefore, the mapping $t \mapsto (J_t w)(T, q) : [0, T] \mapsto \mathbb{R}_-$ is continuous and bounded. Hence, the infimum $(Jw)(T, q)$ in (31) is attained.

Next two lemmas show that the operator J preserves useful properties of the function w. Lemma 4.3 is for the joint behavior of Jw on $[0, T_{\max}] \times \mathbb{R}_+$ and Lemma 4.4 is for the variables T and q isolated.

Lemma 4.3. *For every bounded continuous Borel function w on $[0, T_{\max}] \times \mathbb{R}_+$ satisfying $-\frac{T}{c} \leq w(T, \cdot) \leq 0$ for $T \leq T_{\max}$, it follows that Jw is also a bounded continuous function on $[0, T_{\max}] \times \mathbb{R}_+$ with the same bounds*

$$-\frac{T}{c} \leq (Jw)(T, \cdot) \leq 0, \qquad T \leq T_{\max}, \tag{34}$$

and we have $(Jw)(\cdot, q) = 0$ for all $q \geq \bar{q}$ for

$$\bar{q} := \frac{(1 + \lambda_0 T_{\max})\ e^{|\lambda_1-\lambda_0|T_{\max}}}{c} + 1_{\{\lambda_1\neq\lambda_0\}}\frac{(e^{|\lambda_1-\lambda_0|T_{\max}} + 1)}{|\lambda_1 - \lambda_0|}. \tag{35}$$

Also, if $w_1(\cdot, \cdot) \leq w_2(\cdot, \cdot)$ are two bounded Borel functions defined on $[0, T_{\max}] \times \mathbb{R}_+$, then $(Jw_1)(\cdot, \cdot) \leq (Jw_2)(\cdot, \cdot)$.

Proof. The lower bound $w(T, \cdot) \geq -\frac{T}{c}$ implies that $(Sw)(T - s, x(q, s)) \geq -\frac{T-s}{c}$ holds for $s \in [0, T] \subseteq [0, T_{\max}]$, and $q \in \mathbb{R}_+$. Then, as $x(\cdot, \cdot)$ is positive and $t < T$,

$$(J_t w)(T,q) \geq \int_0^t e^{-\lambda_0 s}\left(x(q,s)-\frac{1}{c}\right)ds - \int_0^t \lambda_0 e^{-\lambda_0 s}\left(\frac{T-s}{c}\right)ds$$
$$\geq -\frac{1}{c\lambda_0}\int_0^t \lambda_0 e^{-\lambda_0 s}ds - \frac{T}{c}\int_0^t \lambda_0 e^{-\lambda_0 s}ds + \frac{1}{c}\int_0^t s\lambda_0 e^{-\lambda_0 s}ds \geq -\frac{T}{c}.$$

Taking the infimum over $t \in [0,T]$ gives the lower bound in (34). The upper bound follows since $t=0$ gives $(J_0 w)(T,q)=0$. Also, it is easy to verify that $g(x(q,s))+\lambda_0 \cdot (Sw)(T-s,x(q,s)) \geq 0$ for $q \geq \bar{q}$, $s \leq t \leq T$, and therefore $(J_t w)(T,q) \geq 0$ for all $t \leq T$. This gives $(Jw)(T,q) = (J_0 w)(T,q) = 0$ for all $q \geq \bar{q}$.

On $[0,T_{\max}] \times [\bar{q},+\infty)$, $Jw = 0$, and the continuity is immediate. For $(T,q) \in [0,T_{\max}] \times [0,\bar{q}]$, the continuity follows from the continuity of Sw (thanks to bounded convergence theorem) and the regularity of the paths $t \mapsto x(q,t)$.

Finally, to show the monotonicity of Jw in w, for two functions $w_1 \leq w_2$, we have $(Sw_1)(t,q) \leq (Sw_2)(t,q)$ for $t \in [0,T]$ and $q \in \mathbb{R}_+$ due to the monotonicity of the S operator. As the constant λ_0 is positive, $(J_t w_1)(T,q) \leq (J_t w_2)(T,q)$ for all $t \in [0,T]$. Taking the infimum over $t \in [0,T]$, we have $(Jw_1)(T,q) \leq (Jw_2)(T,q)$. □

Lemma 4.4. *For every fixed $T \in [0,T_{\max}]$, if the mapping $q \mapsto w(T,q)$ is non-decreasing and concave, then so is $q \mapsto (Jw)(T,q)$. For every fixed $q \in \mathbb{R}_+$, if $T \mapsto w(T,q)$ is non-increasing, then so is $T \mapsto (Jw)(T,q)$.*

The following result is a direct consequence of Lemmas 4.3 and 4.4. No proof is needed. It is analogous to Corollary 3.6.

Corollary 4.5. *Let w be a bounded continuous function on $[0,T_{\max}]\times\mathbb{R}_+$ with the following properties: i) $-\frac{T}{c} \leq w(T,\cdot) \leq 0$ for $T \leq T_{\max}$, ii) $T \mapsto w(T,q)$ is non-increasing for every $q \in \mathbb{R}_+$, and iii) $q \mapsto w(T,q)$ is non-decreasing and concave for every $T \leq T_{\max}$. Then, $T \mapsto b_{[w]}(T) := \min\{q \geq 0 : (Jw)(T,q) = 0\}$ is a non-decreasing function on $[0,T_{\max}]$.*

Let us next define the successive approximations $\bar{u}_n : [0,T]\times\mathbb{R}_+ \mapsto \mathbb{R}_-$ and $\underline{u}_n : [0,T]\times\mathbb{R}_+ \mapsto \mathbb{R}_-$, $n \in \mathbb{N}$, by

$$\bar{u}_0 \equiv 0, \quad \bar{u}_n := J\bar{u}_{n-1}, \quad n \geq 1, \quad \text{and} \quad \underline{u}_0 \equiv -\frac{T}{c}, \quad \underline{u}_n := J\underline{u}_{n-1}, \quad n \geq 1. \tag{36}$$

Proposition 4.6. *The functions defined in (36) are ordered as*

$$-\frac{T}{c} \equiv \underline{u}_0 \leq \underline{u}_1 \leq \underline{u}_2 \leq \ldots \leq \ldots \leq \bar{u}_2 \leq \bar{u}_1 \leq \bar{u}_0 \equiv 0.$$

Proof. Observe that as $\bar{u}_0 \equiv 0 \in [-\frac{T}{c},0]$, we have $-\frac{T}{c} \leq \bar{u}_1 \equiv J\bar{u}_0 \leq \bar{u}_0 = 0$ as the operator J preserves boundedness by Lemma 4.3. Hence, $\bar{u}_1 \leq \bar{u}_0$ holds. Applying the operator J to $\bar{u}_1$ and $\bar{u}_0$, we have $-\frac{T}{c} \leq \bar{u}_2 \equiv J\bar{u}_1 \leq J\bar{u}_0 \equiv \bar{u}_1 \leq 0$ as the operator J preserves monotonicity and boundedness by Lemma 4.3. Successively applying the operator J to $\bar{u}_n$'s leads us to the decreasing sequence $-\frac{T}{c} \leq \ldots \leq \bar{u}_2 \leq \bar{u}_1 \leq \bar{u}_0 \equiv 0$.

Similarly, as the operator J preserves monotonicity and boundedness by Lemma 4.3, continuously applying the operator J to $\underline{u}_n$'s gives the increasing sequence $-\frac{T}{c} \equiv \underline{u}_0 \leq \underline{u}_1 \leq \underline{u}_2 \leq \ldots \leq 0$.

Let us show $\underline{u}_n \leq \bar{u}_n$ for $n \in \mathbb{N}$ by an induction argument. Note that $-\frac{T}{c} \equiv \underline{u}_0 \leq \bar{u}_0 \equiv 0$ by construction. Next, let us assume $\underline{u}_k \leq \bar{u}_k$ holds for some $k \in \mathbb{N}$. Since the operator J preserves monotonicity by Lemma 4.3, when we apply the operator J to $\underline{u}_k$ and $\overline{u}_k$ we have $\underline{u}_{k+1} \equiv J\underline{u}_k \leq J\overline{u}_k \equiv \overline{u}_{k+1}$ which implies $\underline{u}_{k+1} \leq \overline{u}_{k+1}$. This shows $\underline{u}_n \leq \overline{u}_n$ for $n \in \mathbb{N}$ and completes the proof. □

With the next proposition we will show that the limit of the bounded increasing sequence $(\underline{u}_n)_{n\geq 0}$ and that of the bounded decreasing sequence $(\overline{u}_n)_{n\geq 0}$ coincide.

Proposition 4.7. *For every $T_{\max} \in \mathbb{R}_+$, as $n \to \infty$, the sequences of successive approximations $(\underline{u}_n)_{n\geq 0}$ and $(\overline{u}_n)_{n\geq 0}$ defined in (36) converge to the same limit*

$$u := \lim_{n\to\infty} \underline{u}_n = \lim_{n\to\infty} \overline{u}_n \tag{37}$$

uniformly in $T \in [0, T_{\max}]$, $q \in \mathbb{R}_+$. In fact, for every $n \in \mathbb{N}$, $T \in [0, T_{\max}]$ and $q \in \mathbb{R}_+$, we have

$$0 \leq \overline{u}_n(T,q) - \underline{u}_n(T,q) \leq \frac{T}{c}(1 - e^{-\lambda_0 T})^n. \tag{38}$$

Limit function u is a fixed point of the operator J, and it satisfies the properties listed in Corollary 4.5.

Proof. In Proposition 4.6, we have shown that the first inequality in (38) holds for all $n \in \mathbb{N}$. Now, let us prove the second inequality. By Remark 4.2, we know that

$$\overline{u}_n(T,q) = J\overline{u}_{n-1}(T,q) = J_{\overline{t}_n}\overline{u}_{n-1}(T,q), \quad \text{and} \quad \underline{u}_n(T,q) = J\underline{u}_{n-1}(T,q) = J_{\underline{t}_n}\underline{u}_{n-1}(T,q)$$

for some $\overline{t}_n$ and $\underline{t}_n$ in the interval $[0, T]$. Then by the linearity of the operator S

$$\begin{aligned}\overline{u}_n - \underline{u}_n &\leq \int_0^{\underline{t}_n} e^{-\lambda_0 s}\lambda_0 (S(\overline{u}_{n-1} - \underline{u}_{n-1}))(T-s, x(q,s))ds \\ &\leq \|\overline{u}_{n-1} - \underline{u}_{n-1}\|_{T_{\max},\infty} \int_0^{\underline{t}_n} e^{-\lambda_0 s}\lambda_0 ds \leq \|\overline{u}_{n-1} - \underline{u}_{n-1}\|_{T_{\max},\infty}(1 - e^{-\lambda_0 T_{\max}})\end{aligned}$$

holds for all $n \in N_0$. Taking the supremum of $\overline{u}_n - \underline{u}_n$ over $(T,q) \in [0, T_{\max}] \times \mathbb{R}_+$ gives

$$\|\overline{u}_n - \underline{u}_n\|_{T_{\max},\infty} \leq \|\overline{u}_{n-1} - \underline{u}_{n-1}\|_{T_{\max},\infty}(1 - e^{-\lambda_0 T_{\max}}).$$

This shows that the gap between the terms of the sequences $(\underline{u}_n)_{n\geq 0}$ and $(\overline{u}_n)_{n\geq 0}$ contracts by the factor $(1 - e^{-\lambda_0 T_{\max}})$ with each new term. Hence,

$$\|\overline{u}_n - \underline{u}_n\|_{T_{\max},\infty} \leq \|\overline{u}_{n-1} - \underline{u}_{n-1}\|_{T_{\max},\infty}(1 - e^{-\lambda_0 T_{\max}}) \leq \frac{T}{c}(1 - e^{-\lambda_0 T_{\max}})^n.$$

Thus, (38) holds for $\forall n \in \mathbb{N}$, and as $n \to \infty$, $(\underline{u}_n)_{n\geq 0}$ and $(\overline{u}_n)_{n\geq 0}$ converge to common limit u in (37).

The proof of u being a fixed point of J is similar to the proof of Proposition 3.6 in Dayanik and Sezer (2006a), hence omitted. Since $\overline{u}_0 = 0$, properties listed in Corollary 4.5 hold inductively for every $\overline{u}_n$ and therefore for $\overline{u}$ as well (for continuity we use the uniform convergence). □

This means that whether we start with $\underline{u}_0(T,\cdot) \equiv -\frac{T}{c}$ or $\overline{u}_0 \equiv 0$, we will end up eventually at the fixed limit u as $n \to \infty$. In fact, we can reach to this limit u starting with any bounded function f on $[0, T_{\max}] \times \mathbb{R}_+$ with bounds $-\frac{T}{c} \leq f(T,\cdot) \leq 0$ for every $T \in [0, T_{\max}]$. We state this claim in the next proposition. The proof follows immediately from repeated applications of operator J, Lemma 4.3, and Proposition 4.7. For such a function f, let us introduce the f-successive approximation sequence $u_n(f)$, $n \in \mathbb{N}$, as

$$u_0(f)(T,q) := f(T,q), \quad \text{and} \quad u_n(f)(T,q) := J(u_{n-1}(f))(T,q), \quad T \in [0, T_{\max}],\ q \in \mathbb{R}_+.$$

Proposition 4.8. *For any $f : [0, T_{\max}] \times \mathbb{R}_+ \mapsto \mathbb{R}_-$ bounded as $f(T,\cdot) \in [-\frac{T}{c}, 0]$ and bounded functions $\overline{u}_n$ and $\underline{u}_n$, $n \in \mathbb{N}$, we have*

$$\underline{u}_n \equiv u_n\Big(-\frac{T}{c}\Big) \le u_n(f) \le u_n(0) \equiv \overline{u}_n, \quad n \in \mathbb{N}.$$

Moreover, $0 \le u_n(f) - \underline{u}_n \le \overline{u}_n - \underline{u}_n \le \frac{T}{c}(1 - e^{-\lambda_0 T})^n$, $n \in \mathbb{N}$, *and* $\lim_{n\to\infty} u_n(f) = u$ *of* (37).

Proposition 4.9. *We have $\overline{u}_n = U_n$ for every $n \in \mathbb{N}$, and $u = U$.*

The following lemma gives us ϵ-optimal stopping times for $\epsilon \ge 0$. For $\epsilon = 0$, we obtain an optimal rule.

Lemma 4.10. *For every $\epsilon \ge 0$, define $\tau_\epsilon(T,q) := \inf\{t \in [0,T] : U(T-t, Q_t) \ge -\epsilon\}$. Then, we have*

$$\mathbb{E}_{0,q}\Big[\int_0^{\tau_\epsilon(T,q)} \Big(Q_t - \frac{1}{c}\Big) dt\Big] \le U(T,q) + \epsilon. \tag{39}$$

Since $U(0,q) = 0$ for all $q \ge 0$, the stopping time $\tau_\epsilon(T,q)$ is less than or equal to T. Observe that for $\epsilon = 0$, we obtain the optimal stopping rule as the first entrance time of the sample paths $t \mapsto (T-t, Q_t)$ into the closed set in (26).

The results in Propositions 4.1 and 4.9 yield a natural numerical algorithm to compute/approximate the function U. We simply start with $u_0 = 0$ and successively iterate the operator J numerically. We set the number of iterations so that the error bound in (29) is negligible. Detailed discussions on how to select a grid on $[0, q_{\max}] \times [0, T_{\max}]$, conveniently evaluate function values at the grid points, and make interpolations at other points are available in Çağın Ürü (2019). The same reference also gives numerical examples and additional observations on the effects of different problem parameters (on the optimal stopping time). In Figure 1, we present one example in which jumps before and after the change are normally distributed. Prior to the change-point, they have the standard normally distribution, and after the change, the mean shifts to $\mu = 1$ while the standard deviation remains the same. Other parameters are $T = 1.5351$, $\lambda_0 = 3$, $\lambda_1 = 1.5$, and $c = 1$. The left panel in the figure shows the value function, and the right panel illustrates the stopping boundary and a sample path of the process Q starting from $q = 0$.

An alternative numerical method would be obtained by discretizing the time horizon $[0, T_{max}]$. That is, for a small step size $\delta = T_{\max}/N_\delta$ for some large integer N_δ, we can introduce the finite set $\mathcal{D} = \{0, \delta, 2\delta, \ldots, T_{\max}\}$, and for every fixed $T \in \mathcal{D}$, we can search for the infimum in (18) over all stopping times taking values in $\mathcal{D} \in [0,T]$. Naturally, for $T \in \mathcal{D}$, the corresponding discrete-step dynamic programming operator can be used to carry out function evaluations, and for $T \notin \mathcal{D}$ an interpolation method can be considered (if δ is very small, we may even take the function values on the left or right neighbor point in $\mathcal{D}$). In the numerical examples in Figure 2 in the next section, we use this discretization idea.

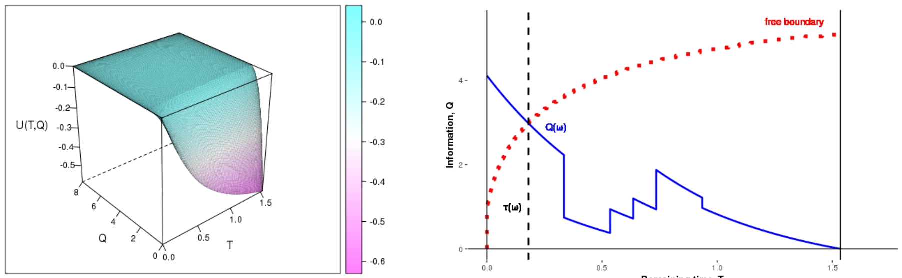

Figure 1: A numerical illustration of the value function U (on the left) and the stopping rule (on the right).

5. An extended model where the change fails to happen on the given interval with a positive probability

Suppose there is a positive probability that the change does not happen at all on the interval $[0, T]$, in which case we say that the random variable Θ takes the value $+\infty$. We let $\psi \in (0, 1)$ denote the probability of the event $\{\Theta = +\infty\}$, and we rewrite our prior distribution as

$$\mathbb{P}(\Theta = 0) = \pi, \quad \mathbb{P}(\Theta = +\infty) = \psi, \quad \text{and} \quad \mathbb{P}(\Theta \in (t, T]) = (1 - \pi - \psi)\frac{T - t}{T}, \quad t \in (0, T]. \tag{40}$$

In this extended setup, the interval $[0, T]$ can be monitored for a potential disorder as before; that is, one can stop at any time before T and announce the onset of the new regime. If, however, such a detection decision is not made by time T, further actions/decisions can be considered as a second level hypothesis testing problem for which two formulations are given in the subsections below.

5.1. Making a one-time decision at T

In this first formulation, a terminal decision is made at T regarding whether the change has never happened or it has already happened but missed. In other words, we test whether the local parameters of the observed compound Poisson process after T are given by the pair (λ_0, ν_0) or (λ_1, ν_1). We represent this terminal decision with a $\{0, 1\}$-valued $\mathcal{F}_T$-measurable random variable

$$d = \begin{cases} 1, & \text{when it is decided that the change has happened on } [0, T], \\ 0, & \text{when } \Theta \text{ is estimated to be } +\infty. \end{cases}$$

On the event $\{\Theta \leq T\}$, if we stop before T (with a detection decision), we incur the false alarm cost $1_{\{\tau<\Theta\}}$ and the relative delay cost $c(\tau - \Theta)^+$ as in (3). If we do not stop before T but select $d = 1$ at T, then our choice of d is correct and we incur only the delay cost $c\,(T - \Theta)^+$. Otherwise (with $d = 0$ at T), the cost is given by $a_0 + c\,(T - \Theta)^+$ in which a_0 represents the relative cost of misdiagnosis associated with $d = 0$ when this is in fact a wrong decision. On the event $\{\Theta = +\infty\}$, on the other hand, the decision to stop before T or selecting $d = 1$ at T has a relative misdiagnosis cost of a_1. Choosing $d = 0$ at T is a correct decision, and therefore there is no penalty associated with it.

Collecting all these cost terms together, for a stopping time $\tau \le T$ and decision $d \in \{0,1\}$, our new objective function is given by

$$\mathbb{E}1_{\{\Theta\le T\}}\Big\{1_{\{\tau<T\}}\big[1_{\{\tau<\Theta\}}+c\,(\tau-\Theta)^+\big]+1_{\{\tau=T\}}1_{\{d=1\}}c\,(T-\Theta)^+ \\ +1_{\{\tau=T\}}1_{\{d=0\}}\big[a_0+c\,(T-\Theta)^+\big]\Big\}+a_1\mathbb{E}1_{\{\Theta=+\infty\}}\Big\{1_{\{\tau<T\}}+1_{\{\tau=T\}}1_{\{d=1\}}\Big\}.$$

In this new setup, we have the Radon-Nikodym derivative in (8) between the auxiliary and physical measures as

$$Z_t = 1_{\{\Theta\le T\}}\Big(1_{\{t<\Theta\}}+1_{\{\Theta\le t\}}\frac{R_t}{R_\Theta}\Big)+1_{\{\Theta=+\infty\}} = 1_{\{t<\Theta\}}+1_{\{\Theta\le t\}}\frac{R_t}{R_\Theta}, \qquad t\ge 0,$$

and using similar arguments as in Appendix A.1, we can rewrite the objective function as

$$a_1\psi+\frac{(1-\pi-\psi)}{T}c\,\mathbb{E}_0\bigg[\int_0^\tau \Big(Q_t-\frac{1}{c}\Big)dt+1_{\{\tau=T\}}1_{\{d=0\}}\frac{1}{c}\Big(a_0Q_T-\frac{Ta_1\psi}{(1-\pi-\psi)}\Big)\bigg] \tag{41}$$

for the process Q in (13) with $Q_0 = T\frac{\pi}{1-\pi-\psi}$. From the expectation in (41), it is easy to see that, for any given stopping time $\tau\le T$, the best decision on the event $\{\tau=T\}$ is simply $d^*(Q_T)\in\mathcal{F}_T$, where

$$d^*(q) := \begin{cases} 1 & \text{if } q \ge \dfrac{Ta_1\psi}{a_0(1-\pi-\psi)}, \\ 0 & \text{otherwise.}\end{cases} \tag{42}$$

Then, to obtain the best stopping time, we need to study and solve the optimal stopping problem

$$\inf_{\tau\le T}\mathbb{E}_0\Big[\int_0^\tau \Big(Q_t-\frac{1}{c}\Big)dt-1_{\{\tau=T\}}\big(\alpha_1 Q_T-\alpha_2\big)^-\Big] \tag{43}$$

defined for positive constants α_1,α_2, and we need to apply its solution with the particular choices

$$\alpha_1=\frac{a_0}{c} \quad \text{and} \quad \alpha_2=\frac{Ta_1\psi}{c(1-\pi-\psi)} \quad \text{starting from the point } Q_0 = T\frac{\pi}{1-\pi-\psi}.$$

5.2. Collecting further observations after T

Next, we consider a second formulation in which the terminal diagnosis d can be delayed and additional observations can be collected after T prior to such a decision. It is natural to assume that such an option comes with a relative delay cost of a_d per unit time. Therefore, in addition to the penalties considered above in Section 5.1, we have the cost $a_d(\tau-T)$ of collecting more observations on the event $\{\tau\ge T\}$. As a result, the new Bayes risk for an almost surely stopping time τ (not necessarily less than or equal to T) is

$$\mathbb{E}1_{\{\Theta\le T\}}\Big\{1_{\{\tau<T\}}\big[1_{\{\tau<\Theta\}}+c\,(\tau-\Theta)^+\big]+1_{\{\tau\ge T\}}\big[c\,(T-\Theta)+a_d(\tau-T)+1_{\{d=0\}}a_0\big]\Big\} \\ +\mathbb{E}1_{\{\Theta=+\infty\}}\Big\{1_{\{\tau<T\}}a_1+1_{\{\tau\ge T\}}\big[a_d(\tau-T)+a_1 1_{\{d=1\}}\big]\Big\},$$

where the diagnosis decision d is now an $\mathcal{F}_\tau$-measurable variable. Once again, using similar arguments as in Appendix A.1, we can rewrite this Bayes risk in terms of the process Q with $Q_0 = T\frac{\pi}{1-\pi-\psi}$ as

$$a_1\psi + (1-\pi-\psi) + \frac{(1-\pi-\psi)}{T} c\,\mathbb{E}_0\Big[\int_0^{\tau\wedge T} (Q_t - \frac{1}{c})dt \\ + 1_{\{\tau\geq T\}}\frac{a_d}{c}\Big\{\int_T^{\tau} (Q_T\frac{R_t}{R_T} + \frac{T}{1-\pi-\psi}\psi)dt + 1_{\{d=0\}}\frac{a_0}{a_d}\big[Q_T\frac{R_\tau}{R_T} - \frac{a_1}{a_0}\frac{T\psi}{1-\pi-\psi}\big]\Big\}\Big]. \tag{44}$$

It is easy to verify that the expectations in (41) and (44) agree for stopping times less than or equal to T almost surely. Also, from the last term in (44), the optimal diagnosis d on the event $\{\tau \geq T\}$ can easily be identified as $d^*\big(Q_T\frac{R_\tau}{R_T}\big) \in \mathcal{F}_\tau$ with d^* defined in (42). Therefore, our problem reduces to finding the stopping rule attaining the infimum

$$\inf_\tau \mathbb{E}_0\Big[\int_0^{\tau\wedge T} (Q_t - \frac{1}{c})dt \\ + 1_{\{\tau\geq T\}}\frac{a_d}{c}\Big\{\int_T^{\tau} (Q_T\frac{R_t}{R_T} + \frac{T}{1-\pi-\psi}\psi)dt - \frac{a_0}{a_d}\Big[Q_T\frac{R_\tau}{R_T} - \frac{a_1}{a_0}\frac{T\psi}{1-\pi-\psi}\Big]^-\Big\}\Big]. \tag{45}$$

To address the problem in (45), we introduce a new stochastic process $P_t = qR_t$, $t \geq 0$, which is a $(\mathbb{P}_0, \mathcal{F})$-Markov process with piecewise-deterministic sample paths; see (57-58) in Appendix B. For this process, we define the optimal stopping problem

$$W_{(\beta_1,\beta_2,\beta_3)}(q) := \inf_\tau \mathbb{E}_0\Big[\int_0^{\tau} (P_t + \beta_1)dt - (\beta_2 P_\tau - \beta_3)^-\Big], \qquad q \geq 0, \tag{46}$$

for given positive constants $\beta_1, \beta_2, \beta_3$. The problem in (46) is very similar to the one studied Dayanik and Sezer (2006b); compare with the problem defined in (2.9) in the cited paper. The paper analyzes the case with $\beta_1 = 1$ and arbitrary positive constants β_2 and β_3. It is relatively straightforward to verify that the same analysis also holds when $\beta \neq 1$ and positive. Below, we summarize some useful results.

Remark 5.1. The function $q \mapsto W_{(\beta_1,\beta_2,\beta_3)}(q)$ is non-positive, non-decreasing, and concave. Its continuation region $C_W := \{q \geq 0 : W_{(\beta_1,\beta_2,\beta_3)}(q) < -(\beta_2 q - \beta_3)^-\}$, if non-empty, is a bounded interval away from the origin, which implies that i) $W_{(\beta_1,\beta_2,\beta_3)}(0) > -\infty$, ii) the right derivative (which exists thanks to concavity) at $q = 0$ is finite, and iii) irrespective of whether the continuation region is empty, there exists a finite point q beyond which the function is equal to zero. Furthermore, the first exit time τ_{C_W} of the process P from the region C_W is an optimal stopping time for (46).

When $\lambda_0 \neq \lambda_1$, provided that the continuation region is non-empty, the function is continuously differentiable except at one of the boundaries (which one being dependent on the ordering of λ_0 and λ_1). When $\lambda_0 = \lambda_1$, on the other hand, the process P remains constant between two jumps and the differentiability may fail on the interior of the continuation region. The function $W_{(\beta_1,\beta_2,\beta_3)}$ is the unique solution f in some suitable sense (as described in Proposition 6.2 in Dayanik and Sezer (2006b)) of the equation

$$\min\{\mathcal{A}f(q) + (q + \beta_1)\,,\, -(\beta_2 q - \beta_3)^- - f(q)\} = 0, \tag{47}$$

where $\mathcal{A}$ denotes the infinitesimal generator of the process P (wherever $\mathcal{A}f$ is well-defined).

Using the inequalities $\mathcal{A}W_{(\beta_1,\beta_2,\beta_3)}(q)+(q+\beta_1)\geq 0$ and $W_{(\beta_1,\beta_2,\beta_3)}(q)\leq -(\beta_2 q-\beta_3)^-$ following from (47), one can prove by chain rule (with a time truncation idea followed by an application of monotone and bounded convergence theorems) that the inequality

$$1_{\{\tau\geq T\}}\,\mathbb{E}_0\Big[\int_T^\tau (P_t+\beta_1)dt-\big[\beta_2 P_\tau-\beta_3\big]^-\Big|\mathcal{F}_T\Big]\geq 1_{\{\tau\geq T\}}\,W_{(\beta_1,\beta_2,\beta_3)}(P_T) \tag{48}$$

holds for a given a.s. finite stopping time τ. Moreover, similar arguments as in the proof of Proposition 3.13 in Dayanik and Sezer (2006b) and the Markov property of the process P imply that we have

$$W_{(\beta_1,\beta_2,\beta_3)}(P_T)=\mathbb{E}_0\Big[\int_T^{\tau_{C_W}\circ\Theta_T}(P_t+\beta_1)dt-\big(\beta_2 P_{\tau_{C_W}\circ\Theta_T}-\beta_3\big)^-\Big|\mathcal{F}_T\Big]\qquad \mathbb{P}_0\text{-a.s.}, \tag{49}$$

where $\tau_{C_W}\circ\theta_T$ denotes the first exit time of P from the region C_W after time T. Note that $P_t=P_T\frac{R_t}{R_T}$ for $t\geq T$. Hence, with P_T replaced by Q_T, the results in (48-49) imply that, in the problem

$$\inf_\tau \mathbb{E}_0\Big[\int_0^{\tau\wedge T}(Q_t-\frac{1}{c})dt+1_{\{\tau\geq T\}}\frac{a_d}{c}\Big\{\int_T^\tau\big(Q_T\frac{R_t}{R_T}+\beta_1\big)dt-\big[\beta_2 P_\tau-\beta_3\big]^-\Big\}\Big],$$

the optimal choice on the event $\{\tau\geq T\}$ is to stop at the first exit time after T of the process $t\mapsto Q_t 1_{\{t\leq T\}}+Q_T\frac{R_t}{R_T}1_{\{t>T\}}$ from the region C_W. This gives us the well-known *sequential probability ratio test* (SPRT) applied after time T with the starting odds-ratio given by Q_T. With this observation, the problem in (45) then simplifies to finding the stopping time τ attaining

$$\inf_\tau \mathbb{E}_0\Big[\int_0^{\tau\wedge T}(Q_t-\frac{1}{c})dt+1_{\{\tau\geq T\}}\frac{a_d}{c}W_{(\beta_1,\beta_2,\beta_3)}(Q_T)\Big] \tag{50}$$

with the choices

$$\beta_1=\frac{T\psi}{1-\pi-\psi},\quad \beta_2=\frac{a_0}{a_d},\quad\text{and}\quad \beta_3=\frac{a_1 T\psi}{a_d(1-\pi-\psi)},$$

and the solution is implemented starting from $Q_0=T\frac{\pi}{1-\pi-\psi}$.

5.3. A common framework

Optimal stopping problems in (43) and (50) are finite horizon problems for the process Q. Compared to the problem in (18), we have the same running cost but now there is an additional (non-positive) terminal term given by a non-decreasing and concave function evaluated at Q_T. We collect this terminal term only if we wait until T.

We denote the common form of (43) and (50) as

$$U(h)(T,q):=\inf_{\tau\leq T}\mathbb{E}_0\Big[\int_0^\tau(Q_t-\frac{1}{c})dt+1_{\{\tau=T\}}h(Q_T)\Big],\qquad q\in\mathbb{R}_+,\ T\in[0,T_{\max}], \tag{51}$$

for a non-positive, non-decreasing, and concave function h which is equal to zero after some strictly positive point q. We also have $h(0)>-\infty$ and $D^+h(0)<\infty$, where $D^+h(0)$ denotes the right derivative of the concave function h at the origin. Clearly, with $h(\cdot)=0$, we recover the function U in (18).

At $T=0$, we have the boundary condition $U(h)(0,q)=h(q)$, and for $T>0$, we have the easy bounds

$$-\frac{T}{c}+h(0)\leq U(T,q)+h(0)\leq U(h)(T,q)\leq U(T,q),\qquad q\geq 0, \tag{52}$$

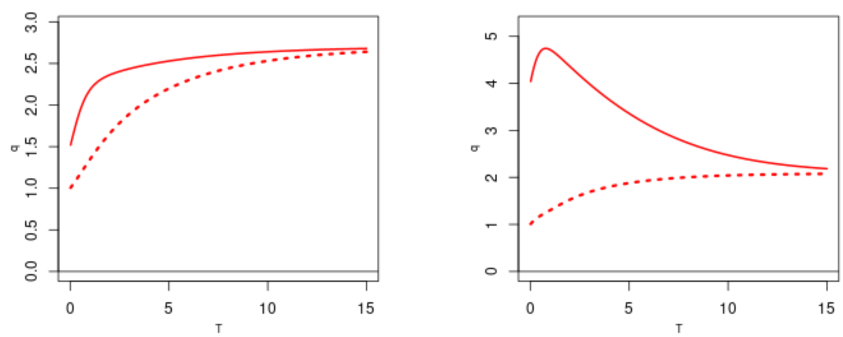

Figure 2: Examples of stopping boundaries for the problem in (51) in different problem instances.

Using arguments similar to those in Remark 3.1 and Lemma 3.2, it is relatively straightforward to verify that $q \mapsto U(h)(T,q)$ is again non-decreasing and concave for every T.

A similar analysis as in Sections 3 and 4 can be given to study the properties of the function $U(h)$. Since the arguments and steps are similar, we omit them here. Notably, it can be shown that $U(h)$ is a (jointly) continuous function of (T,q), and the first entrance time of the process Q into the region $\{(T,q) : T=0\} \cup \{(T,q) : T>0, U(h)(T,q)=0\}$ is an optimal stopping time. Note that since $q \mapsto U(h)(T,q)$ is non-decreasing and concave, it follows that the stopping region and continuation regions are separated by a boundary for $T>0$ below which it is optimal to continue. However, a major difference compared to Section 3 is that the boundary may be non-monotone with non-zero terminal functions. In Figure 2, we give two numerical examples with different boundary behaviors. In both problems, as in the example in Figure 1, marks are normally distributed before and after the change. Before the change, they have the standard normal distribution, and at the change time, the mean becomes $\mu = 1$ whereas the standard deviation remains the same. We also have $c = 1$ in both panels of Figure 2. The panels differ by the choices of the rates λ_0, λ_1, and the function h. On the left panel, we have $\lambda_0 = 1$, $\lambda_1 = 2.5$, and $h(q) = -(1.5-q)^-$; and on the right, we set $\lambda_0 = 2$, $\lambda_1 = 3$, and $h(q) = -(4-q)^-$. The solid curves show the stopping boundaries with the given h functions, and the dotted curves are the boundaries for $h(\cdot) = 0$. We observe that solid and dotted curves converge to each other as T grows, in which case the values of the functions $U(h)$ and U are also comparable as expected. Any negative reward given by the terminal function is simply offset by the waiting costs when T is large.

6. Concluding remarks

In this paper, we revisit the compound Poisson disorder problem in which the aim is to detect a change in the probability law of a compound Poisson process. In practice, the identification of a quickest detection rule may help with mitigating the detrimental effects of an unfavorable regime change. Hence, detecting the change point as soon as it occurs is an important task. The novelty of our work is that we solve the problem when the change is equally likely to be anywhere on an interval. This prior is a natural choice for a decision maker who is relatively uninformed about the time of disorder.

We show that the quickest detection problem can be reformulated as a finite horizon optimal stopping problem for a piecewise-deterministic Markov process. Since the sufficient statistic has piecewise-deterministic sample paths, the optimal stopping problem can be solved with successive approximations. It is interesting to see that the problem admits such a two-dimensional representation. When the change point is not exponentially distributed, this is generally not the case. Our one-dimensional infinite horizon approximation provides further computational advantages when the horizon is large.

We also investigate the case where a change may not happen on the given interval. In this case, if no detection decision is made by the end of the horizon, a second decision is needed to identify the current probability law of the observations. This is a hypothesis testing problem with two simple hypotheses on the local parameters. When an immediate decision is needed, a decision is made based on the value of the sufficient statistic. If further observations are allowed, then we apply the sequential probability ratio test with the value of the sufficient statistic as our initial odds-ratio.

As future work, one may study the compound Poisson disorder problem under other relatively uninformative priors, such as the Jeffrey's prior. Note that the uniform prior is not the only way to describe the absence of a prior knowledge on the change point. In fact, there are other priors that can be used which are invariant to reparametrization unlike the uniform prior.

Appendix A: Calculations

A.1. Reformulation of the Bayes risk

Recall that Bayes risk consists of the probability of a false alarm and the expected detection delay cost. Below $\mathbb{E}_0$ is with respect to the reference probability measure $\mathbb{P}_0$.

False alarm frequency is given by

$$\mathbb{P}\{\tau < \Theta\} = \mathbb{E}[1_{\{\tau<\Theta\}}] = \mathbb{E}_0[Z_\tau 1_{\{\tau<\Theta\}}] = \mathbb{P}_0\{\tau < \Theta\} = 1 - \mathbb{P}_0\{\tau \geq \Theta\} = 1 - \pi - \frac{(1-\pi)}{T}\mathbb{E}_0[\tau]. \tag{53}$$

Generalized Bayes theorem (Shiryaev (1996), pp. 230-231) implies that

$$\Phi_t = \frac{\mathbb{E}[1_{\{\Theta\leq t\}}|\mathcal{F}_t]}{\mathbb{E}[1_{\{\Theta>t\}}|\mathcal{F}_t]} = \frac{\mathbb{E}_0[Z_t 1_{\{\Theta\leq t\}}|\mathcal{F}_t]}{\mathbb{E}_0[Z_t|\mathcal{F}_t]}\left(\frac{\mathbb{E}_0[Z_t 1_{\{\Theta>t\}}|\mathcal{F}_t]}{\mathbb{E}_0[Z_t|\mathcal{F}_t]}\right)^{-1}. \tag{54}$$

Since τ is an $\mathcal{F}$-stopping time, and Θ is $\mathcal{G}_t$-measurable, we have

$$\mathbb{E}[(\tau-\Theta)^+] = \mathbb{E}\left[1_{\{\tau\geq\Theta\}}\int_\Theta^\tau dt\right] = \mathbb{E}_0\int_0^\tau \Phi_t \mathbb{E}_0[Z_t 1_{\{\Theta>t\}}]dt = (1-\pi)\mathbb{E}_0\left[\int_0^\tau \left(\frac{T-t}{T}\right)\Phi_t\, dt\right], \tag{55}$$

where the last expression follows since Θ is independent of the process X and has the uniform distribution in (2) under $\mathbb{P}_0$. When we combine the two components of the Bayes risk in (53) and (55), we obtain (3).

A.2. Explicit form of the process Φ and its relation to Q

Under $\mathbb{P}_0$, we have $\mathbb{E}_0[Z_t 1_{\{\Theta\le t\}}|\mathcal{F}_t] = \mathbb{E}_0\left[\frac{R_t}{R_\tau} 1_{\{\Theta\le t\}}|\mathcal{F}_t\right] = R_t\left(\pi + (1-\pi)\int_0^t \frac{1}{R_s}\frac{ds}{T}\right)$, giving us the ratio in (54) as

$$\Phi_t = \frac{R_t}{T-t}\left(T\frac{\pi}{1-\pi} + \int_0^t \frac{1}{R_s}ds\right) \tag{56}$$

which is equal to $D_t Q_t = Q_t/(T-t)$ for $t<T$ with $Q_0 = T\pi/(1-\pi)$.

Appendix B: Dynamics of the sufficient statistic

Let us show the dynamics of the likelihood ratio process R given in (9). Observe that, at every time t between any two consecutive jumps σ_n and σ_{n+1} of the process X, we have

$$\frac{R_t}{R_{\sigma_n}} = \frac{e^{-(\lambda_1-\lambda_0)t}\prod_{k=1}^{N_t}\left[\frac{\lambda_1}{\lambda_0}f(Y_k)\right]}{e^{-(\lambda_1-\lambda_0)\sigma_n}\prod_{k=1}^{N_{\sigma_n}}\left[\frac{\lambda_1}{\lambda_0}f(Y_k)\right]} = \frac{e^{-(\lambda_1-\lambda_0)t}}{e^{-(\lambda_1-\lambda_0)\sigma_n}}, \quad \sigma_n \le t < \sigma_{n+1},$$

which implies

$$R_t = R_{\sigma_n}e^{-(\lambda_1-\lambda_0)(t-\sigma_n)}, \quad \text{and thus} \quad dR_t = -(\lambda_1-\lambda_0)R_t\,dt, \quad \sigma_n \le t < \sigma_{n+1}.$$

At the jump time σ_{n+1}, we have $dR_{\sigma_{n+1}} = R_{\sigma_{n+1}} - R_{\sigma_{(n+1)-}} = R_{\sigma_{(n+1)-}}\left(\frac{\lambda_1}{\lambda_0}f(Y_{n+1})-1\right)$. Therefore, the dynamics of the process R between the jumps and at the jump times are given by

$$dR_t = \begin{cases} -(\lambda_1-\lambda_0)R_t\,dt, & t\in[\sigma_n,\sigma_{n+1}), \\ R_{t-}\left(\dfrac{\lambda_1}{\lambda_0}f(Y_{n+1})-1\right), & t=\sigma_{n+1}. \end{cases} \tag{57}$$

In terms of the point process $p(dt,dy)$ of our observations (with $(\mathbb{P}_0,\mathcal{F})$-mean measure $\lambda_0 dt\nu_0(dy)$), the dynamics of the process R above can also be written as

$$dR_t = -(\lambda_1-\lambda_0)R_{t-}dt + R_{t-}\int_{\mathbb{R}^d}\big(\frac{\lambda_1}{\lambda_0}f(y)-1\big)p(dt,dy) = R_{t-}\int_{\mathbb{R}^d}\big(\frac{\lambda_1}{\lambda_0}f(y)-1\big)m(dt,dy) \tag{58}$$

with the compansated point process

$$m(dt,dy) := p(dt,dy) - \lambda_0 dt\nu_0(dy). \tag{59}$$

It follows from the explicit form in (13) that

$$dQ_t = \left[q + \int_0^t \frac{1}{R_s}ds\right]dR_t + dt. \tag{60}$$

The dynamics of R in (58) now yields

$$dQ_t = [1-(\lambda_1-\lambda_0)Q_t]dt + Q_{t-}\int_{\mathbb{R}^d}\Big(\frac{\lambda_1}{\lambda_0}f(y)-1\Big)p(dt,dy). \tag{61}$$

After rearranging the terms, these dynamics can also be written more compactly as

$$dQ_t = dt + dM_t \quad \text{with the } (\mathbb{P}_0, \mathcal{F})\text{-martingale } M_s := \int_{(0,s]\times\mathbb{R}^d} Q_{u-}\big(\frac{\lambda_1}{\lambda_0} f(y) - 1\big) m(du, dy), \tag{62}$$

for $s \geq 0$. The dynamics in (61) suggest that the process Q is an autonomous process: it is a piecewise-deterministic Markov process driven by the point process $p(dt, dy)$. Between the jumps of the process X, the process Q follows the integral curves of the continuous part of (61). At every jump of X, the process Q is updated instantaneously.

By (60), for $\sigma_n \leq t < \sigma_{n+1}$ we have $\frac{dQ_t}{dt} = 1 - (\lambda_1 - \lambda_0)Q_t$, which implies $Q_t' + (\lambda_1 - \lambda_0)Q_t = 1$. Solving this equation, we have $d\left(Q_t e^{(\lambda_1-\lambda_0)t}\right) = e^{(\lambda_1-\lambda_0)t}dt$. Integrating both sides on $[\sigma_n, t]$ gives

$$Q_t e^{(\lambda_1-\lambda_0)t} - Q_{\sigma_n} e^{(\lambda_1-\lambda_0)\sigma_n} = \begin{cases} \dfrac{1}{\lambda_1 - \lambda_0}\left(e^{(\lambda_1-\lambda_0)t} - e^{(\lambda_1-\lambda_0)\sigma_n}\right), & \lambda_1 \neq \lambda_0, \\ t - \sigma_n, & \lambda_1 = \lambda_0. \end{cases}$$

Then, Q_t is given by

$$Q_t = \begin{cases} \dfrac{1}{\lambda_1 - \lambda_0} + \left(Q_{\sigma_n} - \dfrac{1}{\lambda_1 - \lambda_0}\right) e^{-(\lambda_1-\lambda_0)(t-\sigma_n)}, & \lambda_1 \neq \lambda_0, \\ Q_{\sigma_n} + t - \sigma_n, & \lambda_1 = \lambda_0. \end{cases}$$

Likewise, for $t = \sigma_{n+1}$ we have $Q_t - Q_{t-} \equiv dQ_t = Q_{t-}\left(\frac{\lambda_1}{\lambda_0} f(Y_{n+1}) - 1\right)$, which implies $Q_t = Q_{t-}\frac{\lambda_1}{\lambda_0} f(Y_{n+1})$. Hence, $Q_{\sigma_{n+1}}$ is given by

$$Q_{\sigma_{n+1}} = Q_{\sigma_{(n+1)-}} \frac{\lambda_1}{\lambda_0} f(Y_{n+1}) = \begin{cases} \dfrac{\lambda_1}{\lambda_0} f(Y_{n+1}) \left[\dfrac{1}{\lambda_1 - \lambda_0} + \left(Q_{\sigma_n} - \dfrac{1}{\lambda_1 - \lambda_0}\right) e^{-(\lambda_1-\lambda_0)(\sigma_{n+1}-\sigma_n)}\right], \\ \dfrac{\lambda_1}{\lambda_0} f(Y_{n+1}) \left[Q_{\sigma_n} + \sigma_{n+1} - \sigma_n\right], \end{cases}$$

separately for the cases $\lambda_1 \neq \lambda_0$ and $\lambda_1 = \lambda_0$ respectively.

Appendix C: Auxiliary proofs

Proof of Lemma 2.1. The ordering $\mathbb{E}_{0,q}[\tau_r] \leq \mathbb{E}_{0,0}[\tau_r]$ is an immediate consequence of the monotonicity of Q in the initial point $Q_0 = q$; see (13).

To show the finiteness of $\mathbb{E}_{0,0}[\tau_r]$, we first consider the easy case where the non-negative i.i.d. random variables $f(Y_k)$s are bounded. That is, there exists some $L < \infty$ such that $f(Y_1) \leq L$ with probability one under $\mathbb{P}_0$. Then, using the dynamics of the process Q in (62), we have

$$\mathbb{E}_{0,0}[Q_{\tau_r\wedge t}] = \mathbb{E}_{0,0}[\tau_r \wedge t + M_{\tau_r\wedge t}] = \mathbb{E}_{0,0}[\tau_r \wedge t], \tag{63}$$

where the last equality is thanks to Doob's stopping theorem. Since $Q_{\tau_r\wedge t} \leq r\max\{\frac{\lambda_1}{\lambda_0}, 1\}L$, we have $\mathbb{E}_{0,0}[\tau_r \wedge t] \leq r\max\{\frac{\lambda_1}{\lambda_0}, 1\}L$. The result follows now by letting $t \to \infty$ and using the monotone convergence theorem.

Other easy cases are i) $\lambda_1 \le \lambda_0$, and ii) $\lambda_1 > \lambda_0$ and $r < \frac{1}{\lambda_1 - \lambda_0}$ (see (14)). For notational convenience, let us define

$$\zeta(q,r) := \inf\{t \ge 0 : x(q,t) \ge r\} \qquad \text{and} \qquad m(q,r) := \mathbb{E}_{0,q}[\tau_r], \qquad q, r \in \mathbb{R}_+. \tag{64}$$

It is easy to verify that $\zeta(0,r) < \infty$ for these cases. Then, by strong Markov property, we write

$$m(0,r) = \mathbb{E}_{0,0}\big[(\zeta(0,r) \wedge \sigma_1) + 1_{\{\sigma_1 \le \zeta(0,r)\}}\, m(Q_{\sigma_1}, r)\big] \le \zeta(0,r) + m(0,r)\,\mathbb{P}_0\{\sigma_1 \le \zeta(0,r)\}.$$

Re-arranging the terms yields $m(0,r) \le \zeta(0,r)/\mathbb{P}_0\{\sigma_1 > \zeta(0,r)\} = \zeta(0,r)e^{\lambda_0 \zeta(0,r)} < \infty$ showing the claim.

Finally, the remaining case is the one where $f(Y_1)$ is not almost surely bounded, $\lambda_1 > \lambda_0$, and $r \ge \frac{1}{\lambda_1 - \lambda_0}$. For this case, fix two points $0 < q_1 < \frac{1}{\lambda_1 - \lambda_0} \le q_2 < r$. Since $\zeta(0,q_1) < \infty$, it follows from our arguments above that $m(0,q_1) = \zeta(0,q_1)e^{\lambda_0\zeta(0,q_1)} < \infty$. Also note that

$$\begin{aligned}\mathbb{P}_{0,q_1}(\tau_{q_2} > t) &= \sum_{k\in\mathbb{N}} \mathbb{P}_{0,q_1}(\tau_{q_2} > t, N_t = k) \le \sum_{k\in\mathbb{N}} \mathbb{P}_{0,q_1}\big(\cap_{i=1}^k \{f(Y_i) \le \frac{\lambda_0 q_2}{\lambda_1 q_1}\} \cap \{N_t = k\}\big) \\ &\le \sum_{k\in\mathbb{N}} \big[\mathbb{P}_0\big(f(Y_1) \le \frac{\lambda_0 q_2}{\lambda_1 q_1}\big)\big]^k e^{-\lambda_0 t}(\lambda_0 t)^k/k! = e^{-\lambda_0 t\left[1 - \mathbb{P}_0\left(f(Y_1) \le \frac{\lambda_0 q_2}{\lambda_1 q_1}\right)\right]}.\end{aligned} \tag{65}$$

The probability $\mathbb{P}_0(f(Y_1) \le \frac{\lambda_0 q_2}{\lambda_1 q_1})$ is strictly less than one by the lack of an upper bound on $f(Y_1)$. Using (65), we then obtain $m(q_1,q_2) =$

$$\int_0^\infty \mathbb{P}_{0,q_1}(\tau_{q_2} > t)dt \le \int_0^\infty e^{-\lambda_0 t\left[1 - \mathbb{P}_0\left(f(Y_1) \le \frac{\lambda_0 q_2}{\lambda_1 q_1}\right)\right]} dt = \Big(\lambda_0\big[1 - \mathbb{P}_0\big(f(Y_1) \le \frac{\lambda_0 q_2}{\lambda_1 q_1}\big)\big]\Big)^{-1} < \infty.$$

Similar arguments also yields $\mathbb{P}_{0,q_2}(\tau_r > t) \le \exp\big(-\lambda_0 t\big[1 - \mathbb{P}_0\big(f(Y_1) \le \frac{\lambda_0 r}{\lambda_1 q_2}\big)\big]\big)$, with $\mathbb{P}_0\big(f(Y_1) \le \frac{\lambda_0 r}{\lambda_1 q_2}\big) < 1$, and $m(q_2,r) \le \Big(\lambda_0\big[1 - \mathbb{P}_0\big(f(Y_1) \le \frac{\lambda_0 r}{\lambda_1 q_2}\big)\big]\Big)^{-1} < \infty$. Finally, it follows by the strong Markov property and the monotonicity of $q \mapsto m(q,\cdot)$ that $m(0,r) \le m(0,q_1) + m(q_1,q_2) + m(q_2,r) < \infty$, establishing the result for the last case and completing the proof. $\square$

Proof of Lemma 3.5. No proof is needed for the first inequality in (25). To establish the second inequality, we note that

$$\begin{aligned}U(\infty,q) = \mathbb{E}_{0,q}\int_0^{\tau_{q_\infty}} (Q_t - \frac{1}{c})dt &= \mathbb{E}_{0,q}\Big[\int_0^{\tau_{q_\infty}\wedge T} (Q_t - \frac{1}{c})dt + 1_{\{\tau_{q_\infty} > T\}} \int_T^{\tau_{q_\infty}} (Q_t - \frac{1}{c})dt\Big] \\ &= \mathbb{E}_{0,q}\Big[\int_0^{\tau_{q_\infty}\wedge T} (Q_t - \frac{1}{c})dt + 1_{\{\tau_{q_\infty} > T\}} U(\infty, Q_T)\Big],\end{aligned}$$

where the last line is by the Markov property. Since $q \mapsto U(\infty,q)$ is non-decreasing and $U(T,q) \le \mathbb{E}_{0,q}\int_0^{\tau_{q_\infty}\wedge T}(Q_t - \frac{1}{c})dt$, we obtain

$$U(\infty,q) \ge U(T,q) + U(\infty,0)\,\mathbb{P}_{0,q}(\tau_{q_\infty} > T) \ge U(T,q) + U(\infty,0)\,\mathbb{E}_{0,q}[\tau_{q_\infty}]/T$$

thanks to Markov inequality. The second inequality in (25) now follows since $\mathbb{E}_{0,0}[\tau_{q_\infty}] \ge \mathbb{E}_{0,q}[\tau_{q_\infty}]$ for any $q \in \mathbb{R}_+$. $\square$

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