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DUALITY FOR LINEAR MULTIPLICATIVE PROGRAMS

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Abstract

Linear multiplicative programs are an important class of nonconvex optimisation problems that are currently the subject of considerable research as regards the development of computational algorithms. In this paper, we show that mathematical programs of this nature are, in fact, a special case of more general signomial programming, which in turn implies that research on this latter problem may be valuable in analysing and solving linear multiplicative programs. In particular, we use signomial programming duality theory to establish a dual program for a nonconvex linear multiplicative program. An interpretation of the dual variables is given.

1. Introduction

We consider mathematical programs of the form

$$\text{Minimise} \quad \prod_{i=1}^{n}(a_i^T x + b_i) \quad \text{subject to} \;\; Dx \geq c, \tag{P_1}$$

where $x \in \mathbb{R}^m$ is a vector of variables and $a_i \in \mathbb{R}^m$, $b_i \in \mathbb{R}$, $i = 1, \ldots, n$, $c \in \mathbb{R}^K$ and $D \in \mathbb{R}^{K \times m}$ are constants. We assume that the feasible region $\{x \mid Dx \geq c\}$ is nonempty and bounded so that program (P_1) has a finite optimal solution.

We call program (P_1) a linear multiplicative program. It is a nonconvex program with multiple local optima. Applications include economic analysis [6], bond portfolio optimisation [7] and VLSI chip design [12]. Matsui [13] shows that this program is NP-hard. Extensive analysis of this problem was first carried out for $n = 2$ by Forgo [5], Swarup [16] and Konno *et al*. [8, 9], where several earlier references may

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be found. Subsequently further development particularly with regard to computational methods for $n > 2$ occurred (see, for example, [1, 2, 10, 11, 15, 17]).

In this paper, we show that a linear multiplicative program is a particular case of a signomial program and hence theory developed for signomial programs is transferable to linear multiplicative programs. In particular, by making this correspondence, we develop a dual program for a linear multiplicative program. An interesting interpretation is given for the dual variables which is similar to that in prototype geometric programming.

2. Signomial programming and duality

A general signomial problem is of the form

$$\text{Minimise}\ \ g_0(t)\ \ \text{subject to}\ \ g_k(t) \le \theta_k (\equiv \pm 1), k = 1, \dots, p, \quad \text{and}\ \ t > 0,$$

where $g_k(t) = \sum_{i \in [k]} \sigma_i c_i \sum_{j=1}^{m} t_j^{a_{ij}}$, $k = 0, \dots, p$, are signomial functions, which are in general nonconvex. The index sets $[k]$, $k = 0, \dots, p$, form a sequential partition of the integers 1 to n, that is, $[0] = \{1, \dots, n_1\}$, $[1] = \{n_1 + 1, \dots, n_2\}, \dots,$ $[p] = \{n_p+1, \dots, n\}$. Here c_i, $i = 1, \dots, n$, are strictly positive and a_{ij}, $i = 1, \dots, n$, $j = 1, \dots, m$, are arbitrary coefficients. Further, $\sigma_i = \pm 1$, $i = 1, \dots, n$, and consequently, signomial programs are nonconvex programs with multiple local optima. Note that signomial programs are an extension of prototype geometric programs [4] from posynomial functions to signomials [14].

The corresponding dual program [14] is

$$\text{Maximise} \quad \prod_{k=0}^{p} \prod_{i \in [k]} \left(\frac{c_i}{\delta_i} \right)^{\sigma_i \delta_i} \prod_{k=1}^{b} \lambda_k^{\lambda_k}$$

subject to a generalised normality condition $\sum_{i \in [0]} \sigma_i \delta_i = 1$, orthogonality conditions $\sum_{i=1}^{n} \sigma_i a_{ij} \delta_i = 0$, $j = 1, \dots, m$, linear inequality constraints $\theta_k \sum_{i \in [k]} \sigma_i \delta_i = \lambda_k \ge 0$, $k = 1, \dots, p$, and nonnegativity constraints

$$\delta_i \ge 0, \quad i = 1, \dots, n, \qquad \lambda_k \ge 0, \quad k = 1, \dots, p.$$

For every point t^0 where $g_0(t)$ is a local minimum there exists a set of dual variables δ^0, λ^0 such that $v(\delta^0, \lambda^0) = g_0(t^0)$.

Since a weak duality theorem does not hold, this dual is termed a pseudo-dual. The global minimum is obtained through a process called pseudominimisation [14] whereby all local maxima of the dual are obtained with the global minimum being the minimum of these local maxima. This concept of "pseudo-duality" is similar to

Craven's concept of "quasi-duality" [3] which shows the existence of points termed "quasimin" and "quasimax" where the duality gap is zero. In both of the above cases which deal with nonconvex problems, a strong duality result holds without a weak duality result.

A locally optimal primal solution can be constructed from a locally optimal dual solution from the following relations between the primal and dual variables:

$$\frac{c_i \prod_{j=1}^m t_j^{a_{ij}}}{g_0(t)} = \delta_i, \qquad i \in [0]$$

and

$$c_i \prod_{j=1}^m t_j^{a_{ij}} = \frac{\delta_i}{\lambda_k}, \quad i \in [k],\ k = 1, \dots, p.$$

3. Dual linear multiplicative program

For notational convenience, we assume that D, c, b_i, $i = 1, \dots, n$, are nonnegative and $a_{ij} = \sigma_{ij} a_{ij}^+$, $i = 1, \dots, n$, $j = 1, \dots, m$, where σ_{ij} is a sign function defined by

$$\sigma_{ij} = \begin{cases} +1, & \text{if } a_{ij} > 0, \\ -1, & \text{otherwise.} \end{cases}$$

Note that $a_{ij}^+ > 0$. Further, without loss of generality, we require that $a_i^T x + b_i > 0$ and $x_i > 0$, $i = 1, \dots, n$.

Program (P_1) may be written in the following form:

$$\text{Minimise} \quad \prod_{i=1}^m s_i \quad \text{subject to} \quad \begin{cases} \sum_{j=1}^m \sigma_{ij} a_{ij}^+ x_j + b_i \le s_i, & i = 1, \dots, n, \\ \sum_{j=1}^m d_{kj} x_j \ge c_k, & k = 1, \dots, K \end{cases} \tag{P_2}$$

and finally as a signomial program:

$$\text{Minimise} \quad \prod_{i=1}^m s_i \quad \text{subject to} \quad \begin{cases} s_i^{-1} \sum_{j=1}^m \sigma_{ij} a_{ij}^+ x_j + b_i s_i^{-1} \le 1, & i = 1, \dots, n, \\ c_k^{-1} \sum_{j=1}^m d_{kj} x_j \ge 1, & k = 1, \dots, K, \end{cases} \tag{P_3}$$

with $s_i > 0$, $i = 1, \dots, n$ and $x_j > 0$, $j = 1, \dots, m$.

Using the prescription in Section 2, we may construct the following dual to program (P_3). This is

$$\text{Maximise}\quad \left(\frac{1}{\delta_0}\right)^{\delta_0} \prod_{i=1}^{n}\prod_{j=1}^{m}\left(\frac{a_{ij}^{+}}{\delta_{ij}}\right)^{\sigma_{ij}\delta_{ij}} \prod_{i=1}^{n}\left(\frac{b_i}{\beta_i}\right)^{\beta_i} \prod_{k=1}^{K}\prod_{j=1}^{m}\left(\frac{c_k^{-1}d_{kj}}{\gamma_{kj}}\right)^{-\gamma_{kj}} \times \prod_{i=1}^{n}\delta_i^{\delta_i}\prod_{k=1}^{K}\gamma_k^{\gamma_k} \tag{D_3}$$

subject to the normality condition

$$\delta_0 = 1, \tag{3.1}$$

the orthogonality conditions

$$\delta_0 - \sum_{j=1}^{m}\sigma_{ij}\delta_{ij} - \beta_i = 0, \quad i = 1, \ldots, n, \tag{3.2}$$

$$\sum_{i=1}^{n}\sigma_{ij}\delta_{ij} - \sum_{k=1}^{K}\gamma_{kj} = 0, \quad j = 1, \ldots, m$$

and

$$\delta_i = \sum_{j=1}^{m}\sigma_{ij}\delta_{ij} + \beta_i, \quad i = 1, \ldots, n, \tag{3.3}$$

$$\gamma_k = \sum_{j=1}^{m}\gamma_{kj}, \quad \delta_{ij} \ge 0, \quad \gamma_{kj} \ge 0, \quad \delta_i \ge 0, \quad \beta_i \ge 0,$$

where $i = 1, \ldots, n$, $j = 1, \ldots, m$, $k = 1, \ldots, K$.

Combining results (3.1)–(3.3) shows that $\sum_{j=1}^{m}\sigma_{ij}\delta_{ij} = 1 - \beta_i$, $i = 1, \ldots, n$ and $\delta_i = 1$. Hence the dual program (D_3) may be simplified somewhat to yield:

$$\text{Minimise}\quad \prod_{i=1}^{n}\prod_{j=1}^{m}\left(\frac{a_{ij}^{+}}{\delta_{ij}}\right)^{\sigma_{ij}\delta_{ij}} \prod_{i=1}^{n}\left(\frac{b_i}{\beta_i}\right)^{\beta_i} \prod_{k=1}^{K}\prod_{j=1}^{m}\left(\frac{c_k^{-1}d_{kj}}{\gamma_{kj}}\right)^{-\gamma_{kj}} \prod_{k=1}^{K}\gamma_k^{\gamma_k} \tag{D_4}$$

subject to the linear constraints $\sum_{j=1}^{m}\sigma_{ij}\delta_{ij} + \beta_i = 1$, $\sum_{i=1}^{n}\sigma_{ij}\delta_{ij} - \sum_{k=1}^{K}\gamma_{kj} = 0$, $\sum_{j=1}^{m}\gamma_{kj} - \gamma_k = 0$, $\delta_{ij} \ge 0$, $\gamma_{kj} \ge 0$, $\beta_i \ge 0$, where $i = 1, \ldots, n$, $j = 1, \ldots, m$, $k = 1, \ldots, K$.

Further, at optimality, the primal and dual variables are related by

$$s_i^{-1}a_{ij}^{+}x_j = \delta_{ij}/\delta_i, \quad s_i^{-1}b_i = \beta_i/\delta_i, \quad c_k^{-1}d_{kj}x_j = \gamma_{kj}/\gamma_k,$$

where $i = 1, \ldots, n$, $j = 1, \ldots, m$, $k = 1, \ldots, K$.

Since $\delta_i = 1$, $i = 1, \ldots, n$, it follows that

$$\delta_{ij} = \frac{a_{ij}^{+} x_j}{\sum_{j=1}^{m} \sigma_{ij} a_{ij}^{+} x_j + b_i}, \quad \beta_i = \frac{b_i}{\sum_{j=1}^{m} \sigma_{ij} a_{ij}^{+} x_j + b_i}$$

and

$$x_j = c_k d_{kj}^{-1} \frac{\gamma_{kj}}{\gamma_k}. \tag{3.4}$$

Note that the dual variables δ_{ij} and β_i may be interpreted as the relative contribution of each variable x_j and parameter b_i respectively to term i in the multiplicative objective at optimality. In polynomial geometric programming, the dual variables δ_i have an interpretation as the relative contribution of each term i to the optimal objective value. Hence in both cases they have an interpretation in terms of a relative contribution. Note also that the optimal primal variables are readily calculated from (3.4).

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