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Factorizations of Matrices over Projective-free Rings*

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Abstract. An element of a ring R is called strongly $J^{\#}$-clean provided that it can be written as the sum of an idempotent and an element in $J^{\#}(R)$ that commute. In this paper, we characterize the strong $J^{\#}$-cleanness of matrices over projective-free rings. This extends many known results on strongly clean matrices over commutative local rings.

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1 Introduction

Let R be a ring with identity. We say that $x \in R$ is strongly clean provided that there exists an idempotent $e \in R$ such that $x - e \in U(R)$ and $ex = xe$. A ring R is strongly clean in case every element in R is strongly clean. We refer the reader to [7] and [8] for the general theory of such rings. In [2, Theorem 12], Borooah, Diesl and Dorsey proved that for a commutative local ring R and a monic polynomial $h \in R[t]$ of degree n, the following are equivalent: (1) h has an SRC-factorization in $R[t]$; (2) every $\varphi \in M_n(R)$ satisfying h is strongly clean. By [6, Example 3.1.7], the above statement (1) cannot be weakened from SRC-factorization to SR-factorization. The

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purpose of this paper is to investigate a subclass of strongly clean rings which behave like such ones but can be characterized by a kind of SR-factorizations, and so get more explicit factorizations for many class of matrices over projective-free rings.

Let $J(R)$ be the Jacobson radical of R. Set

$$J^{\#}(R) = \{x \in R \mid \exists\, n \in \mathbb{N} \text{ such that } x^n \in J(R)\}.$$

For instance, let $R = M_2(\mathbb{Z}_2)$. Then

$$J^{\#}(R) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\},$$

while $J(R) = 0$. Thus, $J^{\#}(R)$ and $J(R)$ are distinct in general. We say that an element $a \in R$ is strongly $J^{\#}$-clean provided that there exists an idempotent $e \in R$ such that $a - e \in J^{\#}(R)$ and $ea = ae$. If R is commutative, then $a \in R$ is strongly $J^{\#}$-clean if and only if a is strongly J-clean (cf. [3]). But they behave differently for matrices over commutative rings. A Jordan-Chevalley decomposition of an $n \times n$ matrix A over an algebraically closed field (e.g., the field of complex numbers) is an expression $A = E + W$, where E is semisimple, W is nilpotent, and E and W commute. The Jordan-Chevalley decomposition is extensively studied in Lie theory and operator algebra. As a corollary, we will completely determine when an $n \times n$ matrix over a filed is the sum of an idempotent matrix and a nilpotent matrix that commute. Thus, the strongly $J^{\#}$-clean factorization of matrices over rings is an analog of the Jordan-Chevalley decomposition for matrices over fields.

In this paper, we characterize the strong $J^{\#}$-cleanness of matrices over projective-free rings. Here, a commutative ring R is projective-free provided that every finitely generated projective R-module is free. For instance, every commutative local ring, every commutative semi-local ring, every principal ideal domain, every Bézout domain (e.g., the ring of all algebraic integers) and the polynomial ring $R[x]$ over a principal ideal domain R are all projective-free. We will show that strongly $J^{\#}$-clean matrices over projective-free rings are completely determined by a kind of "SC"-factorizations of the characteristic polynomials. These extend many known results on strongly clean matrices to such new factorizations of matrices over projective-free rings (cf. [1, 2, 5]).

Throughout this paper, all rings have an identity and all modules are unitary modules. We always use $U(R)$ to denote the set of all units in a ring R. If $\varphi \in M_n(R)$, we use $\chi(\varphi)$ to stand for the characteristic polynomial $\det(tI_n - \varphi)$.

2 Full Matrices over Projective-free Rings

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{Z}_2)$. It is directly verified that A is not strongly $J^{\#}$-clean, though A is strongly clean. It is hard to determine the strong cleanness even for matrices over the integers, but the strongly $J^{\#}$-clean case is a completely different situation. The aim of this section is to characterize a single strongly $J^{\#}$-clean $n \times n$ matrix over a projective-free ring. For a left R-module M, we denote the endomorphism ring of M by $\mathrm{End}(M)$.

Lemma 2.1. *Let M be a left R-module, $E = \mathrm{End}(M)$ and let $\alpha \in E$. Then the following are equivalent:*

(1) *$\alpha \in E$ is strongly $J^{\#}$-clean.*

(2) *There exists a left R-module decomposition $M = P \oplus Q$, where P, Q are α-invariant, $\alpha|_P \in J^{\#}(\mathrm{End}(P))$ and $(1_M - \alpha)|_Q \in J^{\#}(\mathrm{End}(Q))$.*

Proof. (1)⇒(2) Since α is strongly $J^{\#}$-clean in E, there exists an idempotent $\pi \in E$ and $u \in J^{\#}(E)$ such that $\alpha = (1-\pi)+u$ and $\pi u = u\pi$. Thus, $\pi\alpha = \pi u \in J^{\#}(\pi E\pi)$. Further, $1-\alpha = \pi - u$, and so $(1-\pi)(1-\alpha) = (1-\pi)(-u) \in J^{\#}((1-\pi)E(1-\pi))$. Set $P = M\pi$ and $Q = M(1-\pi)$. Then $M = P \oplus Q$. As $\alpha\pi = \pi\alpha$, we see that P and Q are α-invariant. As $\alpha\pi \in J^{\#}(\pi E\pi)$, we can find $t \in \mathbb{N}$ such that $(\alpha\pi)^t \in J(\pi E\pi)$. Let $\gamma \in \mathrm{End}(P)$. For any $x \in M$, it is easy to see that $(x)\pi\big(1_P - \gamma(\alpha|_P)^t\big) = (x)\pi\big(\pi - (\pi\overline{\gamma}\pi)(\pi\alpha\pi)^t\big)$, where $\overline{\gamma} : M \to M$ is given by $(m)\overline{\gamma} = (m)\pi\gamma$ for any $m \in M$. Hence, $1_P - \gamma(\alpha|_P)^t \in \mathrm{Aut}(P)$ and so $(\alpha|_P)^t \in J(\mathrm{End}(P))$. This implies that $\alpha|_P \in J^{\#}(\mathrm{End}(P))$. Likewise, we verify that $(1-\alpha)|_Q \in J^{\#}(\mathrm{End}(Q))$.

(2)⇒(1) For any $\lambda \in \mathrm{End}(Q)$, we construct an R-homomorphism $\overline{\lambda} \in \mathrm{End}(M)$ given by $(p+q)\overline{\lambda} = (q)\lambda$. By hypothesis, $\alpha|_P \in J^{\#}(\mathrm{End}(P))$ and $(1_M - \alpha)|_Q \in J^{\#}(\mathrm{End}(Q))$. Thus, $\alpha = \overline{1_Q} + \overline{\alpha|_P} - \overline{(1_M - \alpha)|_Q}$. As P and Q are α-invariant, we see that $\alpha\overline{1_Q} = \overline{1_Q}\,\alpha$. In addition, $\overline{1_Q} \in \mathrm{End}(M)$ is an idempotent. Since $\overline{\alpha|_P}\,\overline{(1_M - \alpha)|_Q} = 0 = \overline{(1_M - \alpha)|_Q}\,\overline{\alpha|_P}$, we have $\overline{\alpha|_P} - \overline{(1_M - \alpha)|_Q} \in J^{\#}(\mathrm{End}(M))$, as required. □

Lemma 2.2. [6, Lemma 3.2.6] *Let R be a ring and M a left R-module. Suppose that $x, y, a, b \in \mathrm{End}(M)$ such that $xa + yb = 1_M$, $xy = yx = 0$, $ay = ya$ and $xb = bx$. Then $M = \ker(x) \oplus \ker(y)$ as left R-modules.*

Lemma 2.3. *Let R be a commutative ring and $\varphi \in M_n(R)$. Then the following are equivalent:*

(1) *$\varphi \in J^{\#}(M_n(R))$.*

(2) *$\chi(\varphi) \equiv t^n \pmod{J(R)}$, i.e., $\chi(\varphi) - t^n \in J(R)[t]$.*

(3) *There exists a monic polynomial $h \in R[t]$ such that $h \equiv t^{\deg h} \pmod{J(R)}$ and $h(\varphi) = 0$.*

Proof. (1)⇒(2) Since $\varphi \in J^{\#}(M_n(R))$, there exists some $m \in \mathbb{N}$ such that $\varphi^m \in J(M_n(R))$. As $J(M_n(R)) = M_n(J(R))$, we get $\overline{\varphi} \in N\big(M_n(R/J(R))\big)$. In view of [6, Proposition 3.5.4], $\chi(\overline{\varphi}) \equiv t^n \pmod{N(R/J(R))}$. Write $\chi(\varphi) = t^n + a_1t^{n-1} + \cdots + a_n$. Then $\chi(\overline{\varphi}) = t^n + \overline{a_1}\,t^{n-1} + \cdots + \overline{a_n}$. We infer that each $a_i^{m_i} + J(R) = 0 + J(R)$ where $m_i \in \mathbb{N}$. This implies that $a_i \in J^{\#}(R)$. That is, $\chi(\varphi) \equiv t^n \pmod{J^{\#}(R)}$. Obviously, $J(R) \subseteq J^{\#}(R)$. For any $x \in J^{\#}(R)$, there exists some $m \in \mathbb{N}$ such that $x^n \in J(R)$. For any maximal ideal M of R, M is prime, and so $x \in M$. This implies that $x \in J(R)$, hence $J^{\#}(R) \subseteq J(R)$. Therefore, $J^{\#}(R) = J(R)$, as required.

(2)⇒(3) Choose $h = \chi(\varphi)$. Then $h \equiv t^{\deg h} \pmod{J(R)}$. In light of the Cayley-Hamilton theorem, $h(\varphi) = 0$, as required.

(3)⇒(1) By hypothesis, there exists a monic polynomial $h \in R[t]$ such that $h \equiv t^{\deg h} \pmod{J(R)}$ and $h(\varphi) = 0$. Write $h = t^n + a_1t^{n-1} + \cdots + a_n$. Choose $\overline{h} = t^n + \overline{a_1}\,t^{n-1} + \cdots + \overline{a_n} \in (R/J(R))[t]$. Then $\overline{h} \equiv t^n \pmod{N(R/J(R))}$ and $\overline{h}(\overline{\varphi}) = 0$. According to [6, Proposition 3.5.4], there exists some $m \in \mathbb{N}$ such that $(\overline{\varphi})^m = \overline{0}$ in $R/J(R)$. Therefore, $\varphi^m \in M_n(J(R))$, and so $\varphi \in J^{\#}(M_n(R))$. □

Definition 2.4. For $r \in R$, define

$$\mathbb{J}_r = \{f \in R[t] \mid f \text{ monic and } f \equiv (t-r)^{\deg f} \ (\mathrm{mod}\ J^{\#}(R))\}.$$

Lemma 2.5. *Let R be a projective-free ring, $\varphi \in M_n(R)$, and let $h \in R[t]$ be a monic polynomial of degree n. If $h(\varphi) = 0$ and there exists a factorization $h = h_0h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$, then φ is strongly $J^{\#}$-clean.*

Proof. Write $h_0 = t^p + a_1t^{p-1} + \cdots + a_p$ and $h_1 = (t-1)^q + b_1t^{q-1} + \cdots + b_q$. Then $a_i, b_j \in J^{\#}(R)$ for all i, j. Since R is commutative, we get $a_i, b_j \in J(R)$. Thus, $\overline{h_0} = t^p$ and $\overline{h_1} = (t-\overline{1})^q$ in $(R/J(R))[t]$. Hence, $(\overline{h_0}, \overline{h_1}) = \overline{1}$. In virtue of [6, Lemma 3.5.10], we have some $u_0, u_1 \in R[t]$ such that $u_0h_0 + u_1h_1 = 1$. Then we obtain $u_0(\varphi)h_0(\varphi) + u_1(\varphi)h_1(\varphi) = 1_{nR}$. By hypothesis, $h(\varphi) = h_0(\varphi)h_1(\varphi) = h_1(\varphi)h_0(\varphi) = 0$. Clearly, $u_0(\varphi)h_1(\varphi) = h_1(\varphi)u_0(\varphi)$ and $h_0(\varphi)u_1(\varphi) = u_1(\varphi)h_0(\varphi)$. In light of Lemma 2.2, $nR = \ker(h_0(\varphi)) \oplus \ker(h_1(\varphi))$. As $h_0t = th_0$ and $h_1t = th_1$, we have $h_0(\varphi)\varphi = \varphi h_0(\varphi)$ and $h_1(\varphi)\varphi = \varphi h_1(\varphi)$, and so $\ker(h_0(\varphi))$ and $\ker(h_1(\varphi))$ are both φ-invariant. It is easy to verify that $h_0(\varphi|_{\ker(h_0(\varphi))}) = 0$. Since $h_0 \in \mathbb{J}_0$, we see that $h_0 \equiv t^{\deg h_0}$ (mod $J^{\#}(R)$), hence $\varphi|_{\ker(h_0(\varphi))} \in J^{\#}\big(\mathrm{End}(\ker(h_0(\varphi)))\big)$.

It is easy to verify that $h_1(\varphi|_{\ker(h_1(\varphi))}) = 0$. Set $g(u) = (-1)^{\deg h_1}h_1(1-u)$. Then $g((1-\varphi)|_{\ker(h_1(\varphi))}) = 0$. Since $h_1 \in \mathbb{J}_1$, we see that $h_1 \equiv (t-1)^{\deg h_1}$ (mod $J^{\#}(R)$). Hence, $g(u) \equiv (-1)^{\deg h_1}(-u)^{\deg g}$ (mod $J(R)$). This implies that $g \in \mathbb{J}_0$. By virtue of Lemma 2.3, $(1-\varphi)|_{\ker(h_1(\varphi))} \in J^{\#}\big(\mathrm{End}(\ker(h_1(\varphi)))\big)$. According to Lemma 2.1, $\varphi \in M_n(R)$ is strongly $J^{\#}$-clean. □

For $h = t^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0 \in R[t]$, the matrix

$$C_h = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix} \in M_n(R)$$

is called the companion matrix of h.

Theorem 2.6. *Let R be a projective-free ring and $h \in R[t]$ a monic polynomial of degree n. Then the following are equivalent:*

1. *Every $\varphi \in M_n(R)$ with $\chi(\varphi) = h$ is strongly $J^{\#}$-clean.*
2. *The companion matrix C_h of h is strongly $J^{\#}$-clean.*
3. *There exists a factorization $h = h_0h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$.*

Proof. (1)⇒(2) Write $h = t^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0 \in R[t]$. Choose C_h as above. Then $\chi(C_h) = h$. By hypothesis, $C_h \in M_n(R)$ is strongly $J^{\#}$-clean.

(2)⇒(3) In view of Lemma 2.1, there exists a decomposition $nR = A \oplus B$ such that A and B are φ-invariant, $\varphi|_A \in J^{\#}(\mathrm{End}_R(A))$ and $(1-\varphi)|_B \in J^{\#}(\mathrm{End}_R(B))$. Since R is a projective-free ring, there exist $p, q \in \mathbb{N}$ such that $A \cong pR$ and $B \cong qR$. Regarding $\mathrm{End}_R(A)$ as $M_p(R)$, we see that $\varphi|_A \in J^{\#}(M_p(R))$. By virtue of Lemma 2.3, $\chi(\varphi|_A) \equiv t^p$ (mod $J^{\#}(R)$). Thus $\chi(\varphi|_A) \in \mathbb{J}_0$. Analogously, $(1-\varphi)|_B \in J^{\#}(M_q(R))$. It follows from Lemma 2.3 that $\chi((1-\varphi)|_B) \equiv t^q$ (mod $J^{\#}(R)$). This implies that $\det(\lambda I_q - (1-\varphi)|_B) \equiv \lambda^q \pmod{J^\#(R)}$. Hence, $\det((1-\lambda)I_q - \varphi|_B) \equiv (-\lambda)^q \pmod{J^\#(R)}$. Set $t = 1-\lambda$. Then $\det(tI_q - \varphi|_B) \equiv (t-1)^q \pmod{J^\#(R)}$. Therefore, we get $\chi(\varphi|_B) \equiv (t-1)^q \pmod{J^\#(R)}$. We infer that $\chi(\varphi|_B) \in \mathbb{J}_1$. Clearly, $\chi(\varphi) = \chi(\varphi|_A)\chi(\varphi|_B)$. Choose $h_0 = \chi(\varphi|_A)$ and $h_1 = \chi(\varphi|_B)$. Then there exists a factorization $h = h_0 h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$, as desired.

(3)⇒(1) For every $\varphi \in M_n(R)$ with $\chi(\varphi) = h$, it follows by the Cayley-Hamilton theorem that $h(\varphi) = 0$. Therefore, φ is strongly $J^\#$-clean by Lemma 2.5. □

Corollary 2.7. *Let F be a field and $A \in M_n(F)$. Then the following are equivalent:*

(1) *A is the sum of an idempotent matrix and a nilpotent matrix that commute.*

(2) *$\chi(A) = t^k(t-1)^l$ for some $k, l \geq 0$.*

Proof. As $J(M_n(F)) = 0$, we see that an $n \times n$ matrix is contained in $J^\#(M_n(F))$ if and only if it is a nilpotent matrix. So $A \in M_n(F)$ is strongly $J^\#$-clean if and only if A is the sum of an idempotent matrix and a nilpotent matrix that commute. By virtue of Theorem 2.6, this is the case if and only if $\chi(A) = h_0 h_1$, where $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$. Clearly, $h_0 \in \mathbb{J}_0$ if and only if $h_0 \equiv t^{\deg h_0} \pmod{J^\#(F)}$. But $J^\#(F) = 0$, and so $h_0 = t^k$, where $k = \deg h_0$. Likewise, $h_1 = (t-1)^l$, where $l = \deg h_1$. Therefore, we complete the proof. □

For matrices over integers, we have a similar situation as $J(M_n(\mathbb{Z})) = 0$. Hence, Corollary 2.7 still holds if we replace the field F by $\mathbb{Z}$. For instance, choose

$$A = \begin{pmatrix} -2 & 2 & -1 \\ -4 & 4 & -2 \\ -1 & 1 & 0 \end{pmatrix} \in M_3(\mathbb{Z}).$$

Then $\chi(A) = t(t-1)^2$. Thus, A is the sum of an idempotent matrix and a nilpotent matrix that commute. In fact, we have a corresponding factorization

$$A = \begin{pmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 & -1 \\ -2 & 2 & -2 \\ -1 & 1 & -1 \end{pmatrix}.$$

Corollary 2.8. *Let R be a projective-free ring and $\varphi \in M_2(R)$. Then φ is strongly $J^\#$-clean if and only if one of the following holds:*

(1) *$\chi(\varphi) \equiv t^2 \pmod{J(R)}$.*

(2) *$\chi(\varphi) \equiv (t-1)^2 \pmod{J(R)}$.*

(3) *$\chi(\varphi)$ has a root in $J(R)$ and a root in $1 + J(R)$.*

Proof. Suppose that φ is strongly $J^\#$-clean. By virtue of Theorem 2.6, there exists a factorization $\chi(\varphi) = h_0 h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$.

Case I. $\deg(h_0) = 2$ and $\deg(h_1) = 0$. Then $h_0 = \chi(\varphi) = t^2 - \mathrm{tr}(\varphi)t + \det(\varphi)$ and $h_1 = 1$. As $h_0 \in \mathbb{J}_0$, it follows from Lemma 2.3 that $\varphi \in J^\#(M_2(R))$ or $\chi(\varphi) \equiv t^2 \pmod{J(R)}$.

Case II. $\deg(h_0) = 1$ and $\deg(h_1) = 1$. Then $h_0 = t - \alpha$ and $h_1 = t - \beta$. Since R is commutative, $J^\#(R) = J(R)$. As $h_0 \in \mathbb{J}_0$, we see that $h_0 \equiv t \pmod{J(R)}$, and then $\alpha \in J(R)$. As $h_1 \in \mathbb{J}_1$, we see that $h_1 \equiv t-1 \pmod{J(R)}$, and then $\beta \in 1+J(R)$. Therefore, $\chi(\varphi)$ has a root in $J(R)$ and a root in $1+J(R)$.

Case III. $\deg(h_0)=0$ and $\deg(h_1)=2$. Then $h_1(t)=\det(tI_2-\varphi)\equiv(t-1)^2 \pmod{J(R)}$. Set $u=1-t$. Then $\det(uI_2-(I_2-\varphi))\equiv u^2 \pmod{J(R)}$. According to Lemma 2.3, $I_2-\varphi\in J^{\#}(M_2(R))$ or $\chi(\varphi)\equiv(t-1)^2 \pmod{J(R)}$.

Now we show the converse. If $\chi(\varphi)\equiv t^2$ or $\chi(\varphi)\equiv(t-1)^2 \pmod{J(R)}$, then $\varphi\in J^{\#}(M_2(R))$ or $I_2-\varphi\in J^{\#}(M_2(R))$. This implies that φ is strongly $J^{\#}$-clean. Otherwise, $\varphi, I_2-\varphi\notin J(M_2(R))$. In addition, $\chi(\varphi)$ has a root in $J(R)$ and a root in $1+J(R)$. According to [4, Theorem 16.4.31], φ is strongly J-clean, and therefore it is strongly $J^{\#}$-clean. □

Choose $A=\begin{pmatrix}\overline{0} & \overline{2}\\ \overline{1} & \overline{3}\end{pmatrix}\in M_2(\mathbb{Z}_4)$. It is easy to check that $A, I_2-A\in M_2(\mathbb{Z}_4)$ are not nilpotent. But $\chi(A)=t^2+t+2$ has a root $\overline{2}\in J(\mathbb{Z}_4)$ and a root $\overline{1}\in 1+J(\mathbb{Z}_4)$. As $J(\mathbb{Z}_4)=\{\overline{0},\overline{2}\}$ is nil, we know that every matrix in $J^{\#}(M_2(\mathbb{Z}_4))$ is nilpotent. It follows from Corollary 2.8 that A is the sum of an idempotent matrix and a nilpotent matrix that commute. Let $\mathbb{Z}_{(2)}=\{\frac{m}{n}\mid m,n\in\mathbb{Z},\ 2\nmid n\}$, and let $A=\begin{pmatrix}1 & 1\\ \frac{2}{9} & 0\end{pmatrix}\in M_2(\mathbb{Z}_{(2)})$. Then $J(\mathbb{Z}_{(2)})=\{\frac{2m}{n}\mid m,n\in\mathbb{Z},\ 2\nmid n\}$. As $\chi(A)=t^2-t+\frac{2}{9}$ has a root $\frac{1}{3}\in 1+J(\mathbb{Z}_{(2)})$ and a root $\frac{2}{3}\in J(\mathbb{Z}_{(2)})$, by Corollary 2.8, A is strongly J-clean.

Corollary 2.9. *Let R be a projective-free ring, and $f(t)=t^2+at+b\in R[t]$ with $1+a\in J(R)$ and $b\notin J(R)$. Then the following are equivalent:*
(1) *Every $\varphi\in M_2(R)$ with $\chi(\varphi)=f(t)$ is strongly $J^{\#}$-clean.*
(2) *There exist $r_1\in J(R)$ and $r_2\in 1+J(R)$ such that $f(r_1)=f(r_2)=0$.*
(3) *There exists $r\in J(R)$ such that $f(r)=0$.*

Proof. (1)⇒(2) Since every $\varphi\in M_2(R)$ with $\chi(\varphi)=f(t)$ is strongly $J^{\#}$-clean, it follows by Corollary 2.8 that $f(t)=(t-r_1)(t-r_2)$ with $r_1\in J(R)$ and $r_2\in 1+J(R)$.

(2)⇒(3) is trivial.

(3)⇒(1) As $r^2+ar+b=0$, we see that $f(t)=(t-r)(t+a+r)$. Clearly, $t-r\in\mathbb{J}_0$. As $1+a+r\in J(R)$, we see that $t+a+r\in\mathbb{J}_1$. According to Theorem 2.6, we complete the proof. □

Let φ be a 3×3 matrix over a commutative ring R. Set

$$\operatorname{mid}(\varphi)=\det(I_3-\varphi)-1+\operatorname{tr}(\varphi)+\det(\varphi).$$

Corollary 2.10. *Let R be a projective-free ring and let $\varphi\in M_3(R)$. Then φ is strongly $J^{\#}$-clean if and only if one of the following holds:*
(1) $\chi(\varphi)\equiv t^3 \pmod{J(R)}$.
(2) $\chi(\varphi)\equiv(t-1)^3 \pmod{J(R)}$.
(3) $\chi(\varphi)$ *has a root in* $1+J(R)$, $\operatorname{tr}(\varphi)\in 1+J(R)$, $\operatorname{mid}(\varphi)\in J(R)$ *and* $\det(\varphi)\in J(R)$.
(4) $\chi(\varphi)$ *has a root in* $J(R)$, $\operatorname{tr}(\varphi)\in 2+J(R)$, $\operatorname{mid}(\varphi)\in 1+J(R)$ *and* $\det(\varphi)\in J(R)$.

Proof. Suppose that φ is strongly $J^{\#}$-clean. By virtue of Theorem 2.6, there exists a factorization $\chi(\varphi) = h_0h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$.

Case I. $\deg(h_0) = 3$ and $\deg(h_1) = 0$. Then $h_0 = \chi(\varphi)$ and $h_1 = 1$. As $h_0 \in \mathbb{J}_0$, it follows from Lemma 2.3 that $\varphi \in J^{\#}(M_3(R))$.

Case II. $\deg(h_0) = 0$ and $\deg(h_1) = 3$. Then $h_1(t) = \det(tI_3 - \varphi) \equiv (t-1)^3 \pmod{J(R)}$. Set $u = 1 - t$. Then $\det(uI_3 - (I_3 - \varphi)) \equiv u^3 \pmod{J(R)}$. According to Lemma 2.3, $I_3 - \varphi \in J^{\#}(M_3(R))$.

Case III. $\deg(h_0) = 2$ and $\deg(h_1) = 1$. Then $h_0 = t^2 + at + b$ and $h_1 = t - \alpha$. As $h_0 \in \mathbb{J}_0$, we have $h_0 \equiv t^2 \pmod{J(R)}$, hence $a, b \in J(R)$. As $h_1 \in \mathbb{J}_1$, we have $h_1 \equiv t - 1 \pmod{J(R)}$, hence, $\alpha \in 1 + J(R)$. We see that $a - \alpha = -\mathrm{tr}(\varphi)$, $b - a\alpha = \mathrm{mid}(\varphi)$ and $-b\alpha = -\det(\varphi)$. Therefore, $\mathrm{tr}(\varphi) \in 1 + J(R)$, $\mathrm{mid}(\varphi) \in J(R)$ and $\det(\varphi) \in J(R)$.

Case IV. $\deg(h_0) = 1$ and $\deg(h_1) = 2$. Then $h_0 = t - \alpha$ and $h_1 = t^2 + at + b$. As $h_0 \in \mathbb{J}_0$, we have $h_0 \equiv t \pmod{J(R)}$, hence $\alpha \in J(R)$. As $h_1 \in \mathbb{J}_1$, we have $h_1 \equiv (t-1)^2 \pmod{J(R)}$, and then $a \in -2 + J(R)$ and $b \in 1 + J(R)$. Obviously, $\chi(\varphi) = t^3 - \mathrm{tr}(\varphi)t^2 + \mathrm{mid}(\varphi)t - \det(\varphi)$, and so $a - \alpha = -\mathrm{tr}(\varphi)$, $b - a\alpha = \mathrm{mid}(\varphi)$ and $-b\alpha = -\det(\varphi)$. Therefore, $\mathrm{tr}(\varphi) \in 2 + J(R)$, $\mathrm{mid}(\varphi) \in 1 + J(R)$ and $\det(\varphi) \in J(R)$.

Conversely, if $\chi(\varphi) \equiv t^3$ or $\chi(\varphi) \equiv (t-1)^3 \pmod{J(R)}$, then $\varphi \in J^{\#}(M_3(R))$ or $I_3 - \varphi \in J^{\#}(M_3(R))$. Hence, φ is strongly $J^{\#}$-clean. Suppose that $\chi(\varphi)$ has a root $\alpha \in 1 + J(R)$, $\mathrm{tr}(\varphi) \in 1 + J(R)$ and $\det(\varphi) \in J(R)$. Then $\chi(\varphi) = (t^2 + at + b)(t - \alpha)$ for some $a, b \in R$. This implies that $a - \alpha = -\mathrm{tr}(\varphi)$ and $-b\alpha = -\det(\varphi)$. Hence, $a, b \in J(R)$. Let $h_0 = t^2 + at + b$ and $h_1 = t - \alpha$. Then $\chi(\varphi) = h_0h_1$ where $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$. According to Theorem 2.6, φ is strongly $J^{\#}$-clean.

Suppose that $\chi(\varphi)$ has a root $\alpha \in J(R)$, $\mathrm{tr}(\varphi) \in 2 + J(R)$, $\mathrm{mid}(\varphi) \in 1 + J(R)$ and $\det(\varphi) \in J(R)$. Then $\chi(\varphi) = (t - \alpha)(t^2 + at + b)$ for some $a, b \in R$. This implies that $a - \alpha = -\mathrm{tr}(\varphi)$ and $b - a\alpha = \mathrm{mid}(\varphi)$. Hence, $a \in -2 + J(R)$ and $b \in 1 + J(R)$. Let $h_0 = t - \alpha$ and $h_1 = t^2 + at + b$. Then $\chi(\varphi) = h_0h_1$ where $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$. According to Theorem 2.6, φ is strongly $J^{\#}$-clean, and we are done. □

3 Matrices over Power Series Rings

The purpose of this section is to extend the preceding discussion to matrices over power series rings. We use $R[[x]]$ to stand for the ring of all power series over R. Let $A(x) = (a_{ij}(x)) \in M_n(R[[x]])$. We use $A(0)$ to stand for $(a_{ij}(0)) \in M_n(R)$.

Theorem 3.1. *Let R be a projective-free ring and let $A(x) \in M_2(R[[x]])$. Then the following are equivalent:*

(1) $A(x) \in M_2(R[[x]])$ *is strongly* $J^{\#}$*-clean.*

(2) $A(0) \in M_2(R)$ *is strongly* $J^{\#}$*-clean.*

Proof. (1)⇒(2) Since $A(x)$ is strongly $J^{\#}$-clean in $M_2(R[[x]])$, there exists $E(x) = E^2(x) \in M_2(R[[x]])$ and $U(x) \in J^{\#}(M_2(R[[x]]))$ such that $A(x) = E(x) + U(x)$ and $E(x)U(x) = U(x)E(x)$. This implies that $A(0) = E(0) + U(0)$ and $E(0)U(0) = U(0)E(0)$, where $E(0) = E^2(0) \in M_2(R)$ and $U(0) \in J^{\#}(M_2(R))$. As a result, $A(0)$ is strongly $J^{\#}$-clean in $M_2(R)$.

(2)⇒(1) Construct a ring morphism $\varphi : R[[x]] \to R$ given by $f(x) \mapsto f(0)$. Then $R \cong R[[x]]/\ker f$, where $\ker f = \{f(x) \mid f(0) = 0\} \subseteq J(R[[x]])$. For any finitely generated projective $R[[x]]$-module P, $P \otimes_R (R[[x]]/\ker f)$ is a finitely generated projective $R[[x]]/\ker f$-module, hence it is free. Write $P \otimes_R (R[[x]]/\ker f) \cong (R[[x]]/\ker f)^m$ for some $m \in \mathbb{N}$. Then

$$P \otimes_R (R[[x]]/\ker f) \cong (R[[x]])^m \otimes_R (R[[x]]/\ker f).$$

That is, $P/P(\ker f) \cong (R[[x]])^m/(R[[x]])^m(\ker f)$ with $\ker f \subseteq J(R[[x]])$. By the Nakayama theorem, $P \cong (R[[x]])^m$ is free. Thus, $R[[x]]$ is projective-free. Since $A(0)$ is strongly $J^{\#}$-clean in $M_2(R)$, it follows from Corollary 2.8 that $A(0) \in J^{\#}(M_2(R))$, or $I_2 - A(0) \in J^{\#}(M_2(R))$, or the characteristic polynomial $\chi(A(0)) = y^2 + \mu y + \lambda$ has a root $\alpha \in 1 + J(R)$ and a root $\beta \in J(R)$. If $A(0) \in J^{\#}(M_2(R))$, then $A(x) \in J^{\#}(M_2(R[[x]]))$. If $I_2 - A(0) \in J^{\#}(M_2(R))$, then $I_2 - A(x) \in J^{\#}(M_2(R[[x]]))$. Otherwise, write $y = \sum_{i=0}^{\infty} b_i x^i$ and $\chi(A(x)) = y^2 - \mu(x)y - \lambda(x)$. Then $y^2 = \sum_{i=0}^{\infty} c_i x^i$, where $c_i = \sum_{k=0}^{i} b_k b_{i-k}$. Let $\mu(x) = \sum_{i=0}^{\infty} \mu_i x^i$ and $\lambda(x) = \sum_{i=0}^{\infty} \lambda_i x^i \in R[[x]]$, where $\mu_0 = \mu$ and $\lambda_0 = \lambda$. Then $y^2 - \mu(x)y - \lambda(x) = 0$ holds in $R[[x]]$ if the following equations are satisfied:

$$\begin{gathered} b_0^2 - b_0\mu_0 - \lambda_0 = 0, \\ (b_0b_1 + b_1b_0) - (b_0\mu_1 + b_1\mu_0) - \lambda_1 = 0, \\ (b_0b_2 + b_1^2 + b_2b_0) - (b_0\mu_2 + b_1\mu_1 + b_2\mu_0) - \lambda_2 = 0, \\ \cdots \end{gathered}$$

Obviously, $\mu_0 = \alpha + \beta \in U(R)$ and $\alpha - \beta \in U(R)$. Let $b_0 = \alpha$. Since R is commutative, there exists some $b_1 \in R$ such that $b_0b_1 + b_1(b_0 - \mu_0) = \lambda_1 + b_0\mu_1$. Further, there exists some $b_2 \in R$ such that

$$b_0b_2 + b_2(b_0 - \mu_0) = \lambda_2 - b_1^2 + b_0\mu_2 + b_1\mu_1.$$

By iteration of this process, we get $b_3, b_4, \ldots$, and so on. Then $y^2 - \mu(x)y - \lambda(x) = 0$ has a root $y_0(x) \in 1 + J(R[[x]])$. If $b_0 = \beta \in J(R)$, analogously, we can show that $y^2 - \mu(x)y - \lambda(x) = 0$ has a root $y_1(x) \in J(R[[x]])$. In light of Corollary 2.8, the result follows. □

Corollary 3.2. *Let R be a projective-free ring and let $A(x) \in M_2(R[[x]]/(x^m))$ $(m \geq 1)$. Then the following are equivalent:*

(1) *$A(x) \in M_2(R[[x]]/(x^m))$ is strongly $J^{\#}$-clean.*

(2) *$A(0) \in M_2(R)$ is strongly $J^{\#}$-clean.*

Proof. (1)⇒(2) is obvious.

(2)⇒(1) Let $\psi : R[[x]] \to R[[x]]/(x^m)$ be given by $\psi(f) = \overline{f}$. Then it reduces a surjective ring homomorphism $\psi^* : M_2(R[[x]]) \to M_2(R[[x]]/(x^m))$. Hence, we have $B \in M_2(R[[x]])$ such that $\psi^*(B(x)) = A(x)$. According to Theorem 3.1, we complete the proof. □

Example 3.3. Let $R = \mathbb{Z}_4[x]/(x^2)$ and $A(x) = \begin{pmatrix} \overline{2} & \overline{2}+\overline{2}x \\ \overline{2}+x & \overline{3}+\overline{3}x \end{pmatrix} \in M_2(R)$. Clearly, $\mathbb{Z}_4$ is a projective-free ring, and $R = \mathbb{Z}_4[[x]]/(x^2)$. Since we have the strongly $J^\#$-clean decomposition $A(0) = \begin{pmatrix} \overline{0} & \overline{2} \\ \overline{2} & \overline{1} \end{pmatrix} + \begin{pmatrix} \overline{2} & \overline{0} \\ \overline{0} & \overline{2} \end{pmatrix}$ in $M_2(\mathbb{Z}_4)$, it follows by Corollary 3.2 that $A(x) \in M_2(R)$ is strongly $J^\#$-clean.

Theorem 3.4. *Let R be a projective-free ring and let $A(x) \in M_3(R[[x]])$. Then the following are equivalent:*

(1) *$A(x) \in M_3(R[[x]])$ is strongly $J^\#$-clean.*
(2) *$A(x) \in M_3(R[[x]]/(x^m))$ $(m \geq 1)$ is strongly $J^\#$-clean.*
(3) *$A(0) \in M_3(R)$ is strongly $J^\#$-clean.*

Proof. (1)⇒(2) and (2)⇒(3) are clear.

(3)⇒(1) As $A(0)$ is strongly $J^\#$-clean in $M_3(R)$, it follows from Corollary 2.10 that $A(0) \in J^\#(M_3(R))$; or $I_3 - A(0) \in J^\#(M_3(R))$; or $\chi(A(0))$ has a root in $J(R)$, $\mathrm{tr}(A(0)) \in 2 + J(R)$, $\mathrm{mid}(A(0)) \in 1 + J(R)$ and $\det(A(0)) \in J(R)$; or $\chi(A(0))$ has a root in $1 + J(R)$, $\mathrm{tr}(A(0)) \in 1 + J(R)$, $\mathrm{mid}(A(0)) \in J(R)$ and $\det(A(0)) \in J(R)$. If $A(0) \in J^\#(M_3(R))$ or $I_3 - A(0) \in J^\#(M_3(R))$, then $A(x) \in J^\#(M_3(R[[x]]))$ or $I_3 - A(x) \in J^\#(M_3(R[[x]]))$. Hence, $A(x) \in M_3(R[[x]])$ is strongly $J^\#$-clean. Assume that $\chi(A(0)) = t^3 - \mu t^2 - \lambda t - \gamma$ has a root $\alpha \in J(R)$, $\mathrm{tr}(A(0)) \in 2 + J(R)$, $\mathrm{mid}(A(0)) \in 1 + J(R)$ and $\det(A(0)) \in J(R)$. Write $y = \sum_{i=0}^{\infty} b_i x^i$. Then $y^2 = \sum_{i=0}^{\infty} c_i x^i$, where $c_i = \sum_{k=0}^{i} b_k b_{i-k}$. Furthermore, $y^3 = \sum_{i=0}^{\infty} d_i x^i$, where $d_i = \sum_{k=0}^{i} b_k c_{i-k}$. Let $\mu(x) = \sum_{i=0}^{\infty} \mu_i x^i$, $\lambda(x) = \sum_{i=0}^{\infty} \lambda_i x^i$, $\gamma(x) = \sum_{i=0}^{\infty} \gamma_i x^i \in R[[x]]$, where $\mu_0 = \mu$, $\lambda_0 = \lambda$ and $\gamma_0 = \gamma$. Then $y^3 - \mu(x)y^2 - \lambda(x)y - \gamma(x) = 0$ holds in $R[[x]]$ if the following equations are satisfied:

$$\begin{aligned} b_0^3 - b_0^2\mu_0 - b_0\lambda_0 - \gamma_0 &= 0, \\ (3b_0^2 - 2b_0\mu_0 - \lambda_0)b_1 &= \gamma_1 + b_0^2\mu_1 + b_0\lambda_1, \\ (3b_0^2 - 2b_0\mu_0 - \lambda_0)b_2 &= \gamma_2 + b_0^2\mu_2 + b_1^2\mu_0 + 2b_0b_1\mu_1 + b_0\lambda_2 + b_1\lambda_0 - 3b_0b_1^2, \\ &\cdots \end{aligned}$$

Let $b_0 = \alpha \in J(R)$. Obviously, $\mu_0 = \mathrm{tr}(A(0)) \in 2 + J(R)$ and $\lambda_0 = -\mathrm{mid}(A(0)) \in U(R)$. Hence, $3b_0^2 - 2b_0\mu_0 - \lambda_0 \in U(R)$. Thus, we see that

$$\begin{aligned} b_1 &= (3b_0^2 - 2b_0\mu_0 - \lambda_0)^{-1}(\gamma_1 + b_0^2\mu_1 + b_0\lambda_1), \\ b_2 &= (3b_0^2 - 2b_0\mu_0 - \lambda_0)^{-1}(\gamma_2 + b_0^2\mu_2 + b_1^2\mu_0 + 2b_0b_1\mu_1 + b_0\lambda_2 + b_1\lambda_0 - 3b_0b_1^2). \end{aligned}$$

By iteration of this process, we get $b_3, b_4, \ldots,$ and so on. Then the polynomial $y^3 - \mu(x)y^2 - \lambda(x)y - \gamma(x) = 0$ has a root $y_0(x) \in J(R[[x]])$. It follows from $\mathrm{tr}(A(0)) \in 2 + J(R)$ that $\mathrm{tr}(A(x)) \in 2 + J(R[[x]])$. Likewise, $\mathrm{mid}(A(x)) \in 1 + J(R[[x]])$. According to Corollary 2.10, $A(x) \in M_3(R[[x]])$ is strongly $J^\#$-clean.

Assume that $\chi(A(0))$ has a root $1+\alpha \in J(R)$, $\mathrm{tr}(A(0)) \in 1+J(R)$, $\mathrm{mid}(A(0)) \in J(R)$ and $\det(A(0)) \in J(R)$. Then

$$\det(I_3 - A(0)) = 1 - \mathrm{tr}(A(0)) + \mathrm{mid}(A(0)) - \det(A(0)) \in J(R).$$

Set $B(x) = I_3 - A(x)$. Then $\chi(B(0))$ has a root $\alpha \in J(R)$, $\mathrm{tr}(B(0)) \in 2 + J(R)$ and $\det(B(0)) \in J(R)$. Hence, $\mathrm{mid}(B(0)) = \det(A(0)) - 1 + \mathrm{tr}(B(0)) + \det(B(0)) \in 1 + J(R)$. By the preceding discussion, we see that $B(x) \in M_3(R[[x]])$ is strongly $J^{\#}$-clean, and then we are done. □

From the evidence above, we end this paper by asking the following question: Let R be a projective-free ring and let $A(x) \in M_n(R[[x]])$ $(n \geq 4)$. Does the strong $J^{\#}$-cleanness of $A(x) \in M_n(R[[x]])$ coincide with that of $A(0) \in M_n(R)$?

References

[1] G. Borooah, A.J. Diesl, T.J. Dorsey, Strongly clean triangular matrix rings over local rings, *J. Algebra* **312** (2007) 773–797.

[2] G. Borooah, A.J. Diesl, T.J. Dorsey, Strongly clean matrix rings over projective-free rings, *J. Pure Appl. Algebra* **212** (2008) 281–296.

[3] H. Chen, On strongly J-clean rings, *Comm. Algebra* **38** (2010) 3790–3804.

[4] H. Chen, *Rings Related to Stable Range Conditions*, Series in Algebra 11, World Scientific, Hackensack, NJ, 2011.

[5] A.J. Diesl, Classes of strongly clean rings, Ph.D. Thesis, University of California, Berkeley, 2006.

[6] T.J. Dorsey, Cleanness and strong cleanness of rings of matrices, Ph.D. Thesis, University of California, Berkeley, 2006.

[7] W.K. Nicholson, Clean rings: a survey, in: *Advances in Ring Theory*, World Scientific, Hackensack, NJ, 2005, pp. 181–198.

[8] A.A. Tuganbaev, *Rings Close to Regular*, Kluwer Academic Publishers, Dordrecht-Boston-London, 2002.