New Functional Characterizations and Optimal Structural Results for Assemble-to-Order Generalized M-Systems

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We consider an assemble-to-order generalized M-system with multiple components and multiple products, batch ordering of components, random lead times, and lost sales. We model the system as an infinite-horizon Markov decision process and seek an optimal control policy, which specifies when a batch of components should be produced and whether an arriving demand for each product should be satisfied. To facilitate our analysis, we introduce new functional characterizations for *convexity* and *submodularity* with respect to certain non-unitary directions. These help us characterize optimal inventory replenishment and allocation policies under a mild condition on component batch sizes via a *new* type of policy: *lattice-dependent base-stock* and *lattice-dependent rationing.*

Key words: assemble-to-order systems; Markov decision processes; optimal control; lattice-dependent policies

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1. Introduction

Assemble-to-order (ATO) production is a popular strategy among manufacturing firms. ATO not only allows companies to reduce their response window by stocking components, but also gives them the flexibility of postponing final assembly until demand is realized (Benjaafar and ElHafsi 2006). Many high-tech firms, facing shorter product life cycles and higher demand for product varieties, use ATO to extend customized product offerings, lower inventory cost, and mitigate the effect of product obsolescence. Besides manufacturing, ATO systems can be observed when customer orders include several items in different quantities (Song 2000). Despite its popularity, however, little is known about the forms of optimal policies for ATO systems. Much of this owes to the considerable difficulty in identifying optimal policies, as ATO systems build upon the features of both assembly and distribution systems (Song and Zipkin 2003). (An assembly system has only one product and aims to coordinate components optimally. A distribution system has only one component and seeks to allocate the component optimally among different products.) Hence, one needs to address both coordination and allocation issues in an ATO system, making them notoriously difficult to analyze.

ATO systems can be categorized according to their product structures (Lu et al. 2010). Figure 1

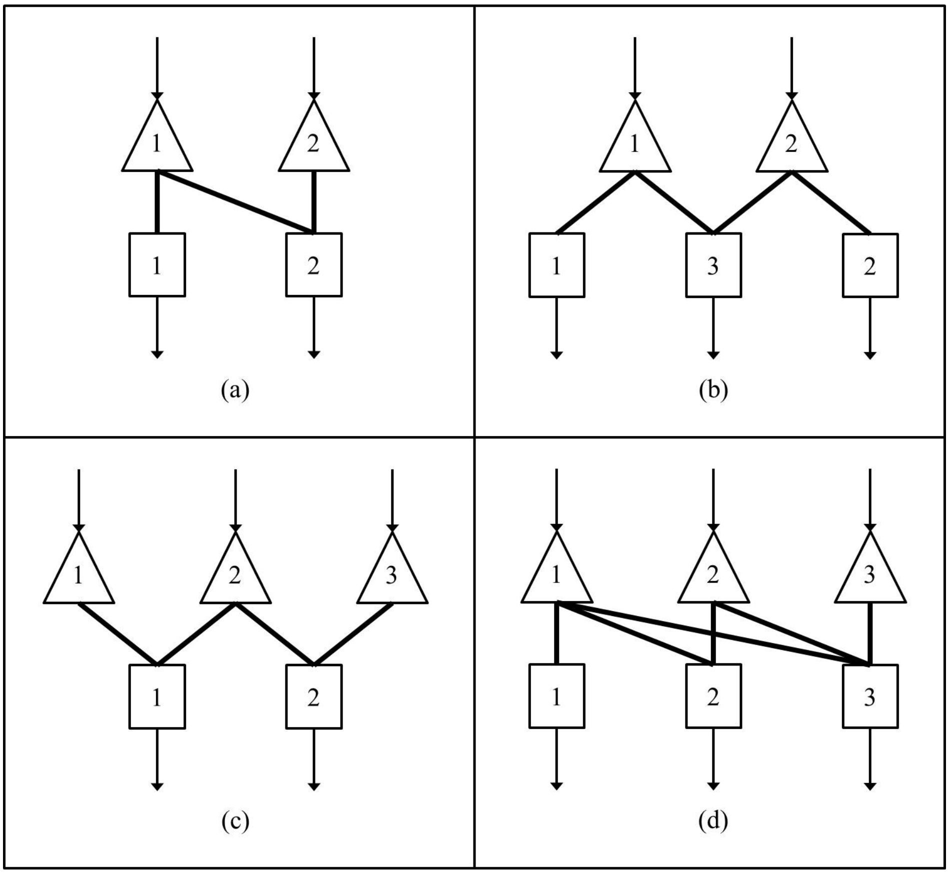

Figure 1 Specific types of ATO product structures: (a) N-system, (b) M-system, (c) W-system, and (d) Nested system with three products.

depicts four such specific types: (a) An N-system, the simplest of the ATO product structures, has two components and two products. One product uses both components while the other product uses only one component. (b) An M-system has two components and three products. One product uses both components while the other two products use different components. (c) A W-system has three components and two products. There is one product-specific component and one common component to each product. (d) A nested system has multiple components and products, where the set of components required by one product is a subset of the set of components needed for the next larger product. Figure 1(d) depicts a nested system with three components.

Several authors have managed to partially or fully characterize optimal policies for specific ATO systems: Dogru et al. (2010) consider a W-system with backordering and identical component lead times. They establish the optimality of a base-stock replenishment policy and a priority-based backorder clearing rule (without reservation) when the "balanced capacity" condition holds, or when both products have the same unit inventory costs. Lu et al. (2010) obtain a similar result for W-systems with backordering, a base-stock replenishment policy, and general component lead times. Specifically, they show that no-holdback component allocation rules are optimal when the "symmetric cost" condition holds. Lu et al. (2010) also extend this optimality result to N-systems and generalized W-systems. Lu et al. (2012) prove the optimality of coordinated base-stock policies and no-holdback rules for N-systems with backordering and symmetric costs, and extend this result to the case with high demand volume and asymmetric costs. The optimal allocation rules in all these papers have the following property: a component is allocated to a demand only if it enables immediate fulfillment of that demand. (Such a property implies a first-come-first-served, FCFS, allocation rule in a lost sales environment, but a non-FCFS rule in a backordering environment.) Lastly, ElHafsi et al. (2008) consider a Markovian nested system with lost sales, proving the optimality of state-dependent base-stock and state-dependent rationing policies. (Stock rationing is a non-FCFS rule; a demand for a particular product is satisfied if and only if the inventory level is higher than a certain threshold.) To our knowledge, there is no extant characterization of the optimal policy for the M-system.

In this paper, we consider the inventory control of a generalized version of the M-system in continuous time. The system involves a single "master" product which requires multiple units from each component, and multiple "individual" products each of which consumes multiple units from a different component. There may be an arbitrary number of individual products; our product structure takes the form of M-system when there are two individual products, cf. Figure 1(b), and includes as a special case the N-system in Figure 1(a) when there is a single individual product.

We formulate the problem as an infinite-horizon Markov decision process (MDP) under the total expected discounted cost criterion. We assume each component is produced in batches of a fixed size in a make-to-stock fashion; production times are independent and exponentially distributed. Demand for each end-product arrives as an independent Poisson process and is lost if not satisfied immediately upon arrival. A control policy specifies when to produce a batch of any component and whether or not to satisfy a demand (upon arrival) from inventory when sufficient inventory exists.

A standard approach for studying the optimal policies of MDPs is to explore the first- and/or second-order properties of the optimal cost function (see Koole 2006). Optimal cost functions for multivariate MDPs (like ours) are typically shown to be *convex* in each dimension of the state space. For examples of such results, see Benjaafar and ElHafsi (2006), ElHafsi et al. (2008), ElHafsi (2009), and Benjaafar et al. (2011). See also Smith and McCardle (2002) for sufficient conditions ensuring convexity in a multivariate Markovian inventory model. However, the existence of counter-examples proves that *convexity* need not hold for our model (see Nadar et al. 2012). Taking an alternative route, we define new functional characterizations of *convexity* and *submodularity* with respect to certain non-unitary directions.

With these new definitions, we characterize the optimal inventory replenishment and allocation policies under a mild condition: If the replenishment batch size for any component equals the number of units needed to make that component's corresponding individual product (Assumption 1), the optimal inventory replenishment policy is a *lattice-dependent base-stock production policy* and the optimal inventory allocation policy is a *lattice-dependent rationing policy* (Theorem 1). This implies that the state space of the problem can be partitioned into disjoint lattices such that on each lattice, (a) it is optimal to produce a batch of a particular component if and only if the state vector is less than the base-stock level associated with that component, and (b) it is optimal to fulfill a demand of a particular product if and only if the state vector is greater than or equal to the rationing level associated with that product. Furthermore, upon replenishment of a particular component, (i) the base-stock level of any other component increases, (ii) the rationing level for any individual product not using that component increases, and (iii) the rationing level for the master product decreases, in a non-strict sense.

Although the optimal policy for the general ATO problem is still unknown, literature on ATO systems is extensive. Song and Zipkin (2003) provide a comprehensive survey of this literature. The paper that is most closely related to ours is Benjaafar and ElHafsi (2006). They consider an ATO *assembly* system with a single end-product which uses one unit of multiple components. The end-product is demanded by multiple customer classes. At any time, there is at most one outstanding order for one unit of each component. They show that, under Markovian assumptions on production and demand, the optimal replenishment is a state-dependent base-stock policy, and the optimal allocation is a state-dependent rationing policy. We extend the model of Benjaafar and ElHafsi (2006) in several directions: (i) We allow our components to be demanded individually as well; (ii) unlike their end-product, our master product may use multiple units from each component; and furthermore (iii) our master product and each of our individual products may require the same component in different quantities. As a result, the state-dependent base-stock and state-dependent rationing (SBSR) policy in Benjaafar and ElHafsi (2006) can be shown to be a special case of our lattice-dependent base-stock and lattice-dependent rationing (LBLR) policy, implying that LBLR is analytically no worse than SBSR for general ATO systems (see Nadar et al. 2012).

We contribute to the ATO literature in several important ways: First, to our knowledge, our study is the first attempt to characterize the optimal replenishment and allocation policies for the generalized M-system. Second, unlike all previous research dealing with the optimal policy characterization for ATO systems, we are the first to allow different products to use the same component in *different* quantities. Third, we define new functional characterizations for *convexity* and *submodularity* with respect to certain non-unitary directions. Fourth, we introduce the notion of a *lattice-dependent* policy, which represents a significant step towards understanding ATO problems, and may aid researchers in developing near-optimal heuristic solutions for general ATO systems.

The rest of this paper is organized as follows: Section 2 formulates the model under the discounted cost criterion. Section 3 introduces our new functional characterizations, establishes the optimal replenishment and allocation policies, and extends our structural results to the average cost case. Section 4 offers several other extensions and Section 5 concludes.

2. Problem Formulation

We consider an ATO system with n components ($j = 1, 2, .., n$) and $n+1$ products ($i = 1, 2, .., n+1$), where each component j is consumed by one *individual* product $i = j$ and also by the *master* product $i = n+1$. Notice that the ATO system we consider reduces to an "M-system" when $n = 2$, cf. Figure 1(b). Define $\mathbf{a} = (a_1, a_2, .., a_n)$ as the vector of component requirements for product $n+1$; a_j is the number of units of component j needed to assemble one unit of the master product $n+1$. Define $\mathbf{b} = (b_1, b_2, .., b_n)$ as the vector of component requirements for all the other products; b_j is the number of units of component j required to make one unit of individual product $i = j$. Each component j is produced in batches of a fixed size q_j in a make-to-stock fashion. Define $\mathbf{q} = (q_1, q_2, .., q_n)$ as the vector of production batch sizes. Production time for component j is independent of the system state and the number of outstanding orders of any type, and exponentially distributed with finite mean $1/\mu_j$. Assembly times are negligible so that assembly operations can be postponed until demand is realized. Demand for each product i arrives as an independent Poisson process with finite rate λ_i. Demand for product i can be fulfilled only if all the required components are available; otherwise, the demand is lost, incurring a unit lost sale cost c_i. Demand may also be rejected in the presence of all the necessary components, again incurring the unit lost sale cost.

The state of the system at time t is the vector $\mathbf{X}(t) = (X_1(t), .., X_n(t))$, where $X_j(t)$ is a non-negative integer denoting the on-hand inventory for component j at time t. Component j held in stock has a holding cost per unit time $h_j(X_j(t))$, which is convex and strictly increasing in the number of available units of component j. Denote by $h(\mathbf{X}(t)) = \sum_j h_j(X_j(t))$ the total inventory holding cost rate at state $\mathbf{X}(t)$. Since both demand interarrival and production times are exponentially distributed, the system retains no memory, and decision epochs can be restricted to times when the state changes. Using the memoryless property, we can formulate the problem as an MDP and confine our analysis to Markovian policies for which actions at each decision epoch depend solely on the current state. A control policy π specifies, for each state $\mathbf{x} = (x_1, .., x_n)$, the action $\mathbf{u}^\pi(\mathbf{x}) = (u^{(1)}, .., u^{(n)}, u_1, .., u_{n+1})$ where $u^{(j)} = 1$ means produce component j, $u^{(j)} = 0$ means do not produce component j, $u_i = 1$ means satisfy demand for product i, and $u_i = 0$ means reject demand for product i. Denote by $\mathbb{U}(\mathbf{x})$ the set of admissible actions at state $\mathbf{x}$. Thus, for any action $\mathbf{u} = (u^{(1)}, .., u^{(n)}, u_1, .., u_{n+1}) \in \mathbb{U}(\mathbf{x})$, the following must hold:

- $u^{(j)} \in \{0, 1\}$, $\forall j$;
- $u_i = 0$ if $x_i < b_i$, and $u_i \in \{0, 1\}$ otherwise, $\forall i \in \{1, 2, .., n\}$; and
- $u_{n+1} = 0$ if $\exists i$ such that $x_i < a_i$, and $u_{n+1} \in \{0, 1\}$ otherwise.

As each ordering decision $u^{(j)}$ specifies only whether or not to produce component j, there is at most one outstanding order for each component at any time. Also, as component orders are not part of our system state, these can in effect be cancelled upon transition to a new state. Both of these assumptions are standard in the literature (see, for example, Ha 1997, Benjaafar and ElHafsi 2006, and ElHafsi et al. 2008).

Let v denote a real-valued function defined on $\mathbb{N}_0^n$ ($\mathbb{N}_0$ is the set of nonnegative integers and $\mathbb{N}_0^n$ is its n-dimensional cross product). Also define $0 < \alpha < 1$ as the discount rate. For a given policy $\pi = \widetilde{\pi}$ and a starting state $\mathbf{X}(0) = \mathbf{x}$, the expected discounted cost over an infinite planning horizon $v^{\widetilde{\pi}}(\mathbf{x})$ can be written as

$$v^{\widetilde{\pi}}(\mathbf{x}) = E\left[\int_0^\infty e^{-\alpha t} h(\mathbf{X}(t))dt + \sum_{i=1}^{n+1} \int_0^\infty e^{-\alpha t} c_i dN_i(t) \;\middle|\; \mathbf{X}(0) = \mathbf{x}, \pi = \widetilde{\pi}\right] \tag{1}$$

where $N_i(t)$ is the cumulative number of demands for product i that have not been fulfilled from on-hand inventory up to time t.

The time between the transition to state $\mathbf{x}$ and the transition to the next state is exponentially distributed with rate $\nu_\mathbf{x}(\mathbf{u})$ if action $\mathbf{u} = (u^{(1)}, .., u^{(n)}, u_1, .., u_{n+1}) \in \mathbb{U}(\mathbf{x})$ is selected in state $\mathbf{x}$. Define t_k as the time of occurrence of the kth transition. Also let $t_0 = 0$. The state of the system stays constant between transitions, i.e., $\mathbf{X}(t) = \mathbf{X}(t_k) = (X_1(t_k), .., X_n(t_k))$ for $t_k \le t < t_{k+1}$. Following Lippman (1975), we consider a uniformized version of the problem where the rate of transition ν is an upper bound for all states and controls, i.e., $\nu \ge \nu_\mathbf{x}(\mathbf{u})$, $\forall \mathbf{x}, \mathbf{u}$. Specifically, we will formulate the problem for the choice $\nu = \sum_j \mu_j + \sum_i \lambda_i$. Therefore, the kth transition time interval $(t_{k+1} - t_k)$ is exponentially distributed with rate ν, $\forall k$. The introduction of the uniform transition rate enables us to transform the continuous-time control problem into an equivalent discrete-time control problem.

If action $\mathbf{u} = (u^{(1)}, .., u^{(n)}, u_1, .., u_{n+1}) \in \mathbb{U}(\mathbf{x})$ is selected in state $\mathbf{x}$, the next state is $\mathbf{y}$ with probability $p_{\mathbf{x},\mathbf{y}}(\mathbf{u})$. Thus:

$$p_{\mathbf{x},\mathbf{y}}(\mathbf{u}) = \begin{cases} \frac{\mu_j u^{(j)}}{\nu} & \text{if } \mathbf{y} = \mathbf{x} + q_j e_j, \\ \frac{\lambda_i u_i}{\nu} & \text{if } \mathbf{y} = \mathbf{x} - b_i e_i, \\ \frac{\lambda_{n+1} u_{n+1}}{\nu} & \text{if } \mathbf{y} = \mathbf{x} - \mathbf{a}, \\ \frac{\nu - \sum_{j=1}^n \mu_j u^{(j)} - \sum_{i=1}^{n+1} \lambda_i u_i}{\nu} & \text{if } \mathbf{y} = \mathbf{x}, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

where e_j is the jth unit vector of dimension n. In this discrete-time framework, $N_i(t_k)$ is the cumulative number of unsatisfied demands for product i at the time of the kth transition, and $h(\mathbf{X}(t_k))$ is the total inventory holding cost rate during the time interval $[t_k, t_{k+1})$. Then, $v^{\widetilde{\pi}}(\mathbf{x})$ in (1) can be rewritten as

$$v^{\widetilde{\pi}}(\mathbf{x}) = E\left[\sum_{k=0}^{\infty} \left(\frac{\nu}{\alpha+\nu}\right)^k \frac{h(\mathbf{X}(t_k))}{\alpha+\nu} + \sum_{k=1}^{\infty} \left(\frac{\nu}{\alpha+\nu}\right)^k \cdot \sum_{i=1}^{n+1} c_i (N_i(t_k) - N_i(t_{k-1})) \,\middle|\, \mathbf{X}(0) = \mathbf{x}, \pi = \widetilde{\pi}\right]. \tag{2}$$

Our objective is to identify a policy π^* that minimizes the expected discounted cost. We below formulate the optimality equation that holds for the optimal cost function $v^* = v^{\pi^*}$:

$$v^*(\mathbf{x}) = \min_{\mathbf{u} \in \mathbb{U}(\mathbf{x})} \left\{ \frac{h(\mathbf{x})}{\alpha+\nu} + \left(\frac{\nu}{\alpha+\nu}\right) \sum_{i=1}^{n+1} \frac{\lambda_i c_i (1-u_i)}{\nu} + \left(\frac{\nu}{\alpha+\nu}\right) \sum_{\mathbf{y}} p_{\mathbf{x},\mathbf{y}}(\mathbf{u}) v^*(\mathbf{y}) \right\}. \tag{3}$$

Therefore, our continuous-time control problem is equivalent to a discrete-time control problem with discount factor $\nu/(\alpha+\nu)$ and cost per stage given by

$$\frac{h(\mathbf{x})}{\alpha+\nu} + \left(\frac{\nu}{\alpha+\nu}\right) \sum_{i=1}^{n+1} \frac{\lambda_i c_i (1-u_i)}{\nu}.$$

As it is always possible to redefine the time scale, without loss of generality we assume $\alpha + \nu = 1$. Then the optimality equation in (3) can be simplified as follows:

$$v^*(\mathbf{x}) = h(\mathbf{x}) + \sum_j \mu_j T^{(j)} v^*(\mathbf{x}) + \sum_i \lambda_i T_i v^*(\mathbf{x}), \tag{4}$$

where the operator $T^{(j)}$ for component j is defined as

$$T^{(j)} v(\mathbf{x}) = \min\{v(\mathbf{x} + q_j e_j), v(\mathbf{x})\},$$

the operator T_i for individual product $i \le n$ is given by

$$T_i v(\mathbf{x}) = \begin{cases} \min\{v(\mathbf{x}) + c_i, v(\mathbf{x} - b_i e_i)\} & \text{if } x_i \ge b_i, \\ v(\mathbf{x}) + c_i & \text{otherwise,} \end{cases}$$

and the operator T_{n+1} for the master product $n+1$ is defined as

$$T_{n+1} v(\mathbf{x}) = \begin{cases} \min\{v(\mathbf{x}) + c_{n+1}, v(\mathbf{x} - \mathbf{a})\} & \text{if } \mathbf{x} \ge \mathbf{a}, \\ v(\mathbf{x}) + c_{n+1} & \text{otherwise.} \end{cases}$$

For a given state $\mathbf{x}$, the operator $T^{(j)}$ specifies whether or not to produce a batch of component j; and the operator T_i specifies, upon arrival of a demand for product i, whether or not to fulfill it from inventory, if sufficient inventory exits.

3. Characterization of the Optimal Policy

Define f as a real-valued function on $\mathbb{N}_0^n$, and $\mathbf{p} = (p_1, p_2, .., p_n)$ and $\mathbf{r} = (r_1, r_2, .., r_n)$ as vectors of nonnegative integers. Also let $\Delta_{\mathbf{p}} f = f(\mathbf{x} + \mathbf{p}) - f(\mathbf{x})$.

We introduce the notion of "submodularity (or supermodularity) in the direction of $p_j e_j$ and $p_k e_k$," for distinct j and k, to describe the class of functions f for which $\Delta_{p_j e_j} f$ is nonincreasing (or nondecreasing) with an increase of p_k in the kth dimension. We also introduce the notion of "convexity (or concavity) in the direction of $\mathbf{p}$ and $\mathbf{r}$" to describe the class of functions f for which $\Delta_{\mathbf{p}} f$ is nondecreasing (or nonincreasing) with an increase of $\mathbf{r}$ on $\mathbb{N}_0^n$. We provide a more detailed discussion of our functional characterizations including their relationship to similar concepts in the literature (Veatch and Wein 1982, and Topkis 1978, 1998) in the online appendix.

Denote by $Sub(\mathbf{p})$ the class of functions satisfying the property of "submodularity in the direction of $p_j e_j$ and $p_k e_k$," $\forall j \neq k$. Also, denote by $Cx(\mathbf{p}, \mathbf{r})$ the class of functions satisfying the property of "convexity in the direction of $\mathbf{p}$ and $\mathbf{r}$." Thus:

Definition 1 (Second-Order Properties). Let f be a real-valued function on $\mathbb{N}_0^n$. Also let $\mathbf{p}, \mathbf{r} \in \mathbb{N}_0^n$.

(a) $f \in Sub(\mathbf{p})$, if $f(\mathbf{x} + p_j e_j) - f(\mathbf{x}) \ge f(\mathbf{x} + p_j e_j + p_k e_k) - f(\mathbf{x} + p_k e_k)$, $\forall \mathbf{x} \in \mathbb{N}_0^n$, $\forall j$, and $\forall k \neq j$.

(b) $f \in Cx(\mathbf{p}, \mathbf{r})$, if $f(\mathbf{x} + \mathbf{p} + \mathbf{r}) - f(\mathbf{x} + \mathbf{p}) \ge f(\mathbf{x} + \mathbf{r}) - f(\mathbf{x})$, $\forall \mathbf{x} \in \mathbb{N}_0^n$.

We are able to show, in Lemma 1, that our optimal cost function satisfies the properties of $Sub(\mathbf{b})$, $Cx(\mathbf{a}, b_j e_j)$, $\forall j$, and $Cx(\mathbf{a}, \mathbf{b})$ under the following assumption.

Assumption 1. *$q_j = b_j$, $\forall j$.*

Although we make the above assumption for analytical tractability, this corresponds to systems with replenishment batch sizes which are, reasonably, determined by individual product sizes. Many papers dealing with the optimal policy characterization for Markovian inventory systems assume unitary component usage rates for products and unitary replenishment quantities for components, and therefore Assumption 1 is satisfied in these papers. See, for instance, Ha (1997), Ha (2000), de Véricourt el al. (2002), Benjaafar and ElHafsi (2006), ElHafsi et al. (2008), ElHafsi (2009), Gayon et al. (2009a), and Gayon et al. (2009b). Even when replenishment batch sizes are different from individual product sizes, we believe that batch sizes can still be adjusted to be individual product sizes by negotiating with suppliers. Such adjustments might improve the firm's profitability as we know the optimal policy form in this case (cf. Theorem 1).

Lemma 1 establishes the structural properties of our optimal cost function under Assumption 1. (The proofs of Lemma 1 and all other subsequent results appear in the online appendix.)

LEMMA 1. *Let V^* be the set of real-valued functions satisfying $Sub(\mathbf{b})$, $Cx(\mathbf{a}, b_j e_j)$, $\forall j$, and $Cx(\mathbf{a}, \mathbf{b})$. Under Assumption 1, if $v \in V^*$, then $Tv \in V^*$, where $Tv(\mathbf{x}) = h(\mathbf{x}) + \sum_j \mu_j T^{(j)} v(\mathbf{x}) + \sum_i \lambda_i T_i v(\mathbf{x})$. Furthermore, the optimal cost function v^* is an element of V^*.*

The structural properties of our optimal cost function allow the form of the optimal policy to be specified via certain lattices of the state space, as we show below.

THEOREM 1. *Let $\mathbb{L}(\mathbf{p}, \mathbf{r}) = \{\mathbf{p} + k\mathbf{r} : k \in \mathbb{N}_0\}$ be an n-dimensional lattice with initial vector $\mathbf{p} \in \mathbb{N}_0^n$ and common difference $\mathbf{r} \in \mathbb{N}_0^n$, where $\exists j$ such that $p_j < r_j$. Under Assumption 1, there exists an optimal stationary policy that can be specified as follows:*

(1) *The optimal inventory replenishment policy for each component j is a lattice-dependent base-stock policy with lattice-dependent base-stock levels $S_j^*(\mathbf{p}) \in \mathbb{L}(\mathbf{p}, \mathbf{a})$, $\forall \mathbf{p}$: It is optimal to produce a batch of component j if and only if $\mathbf{x} \in \mathbb{L}(\mathbf{p}, \mathbf{a})$ is less than $S_j^*(\mathbf{p})$.*

(2) *The optimal inventory allocation policy for each individual product $i \leq n$ is a lattice-dependent rationing policy with lattice-dependent rationing levels $R_i^*(\mathbf{p}) \in \mathbb{L}(\mathbf{p}, \mathbf{a})$, $\forall \mathbf{p}$: It is optimal to fulfill a demand for product $i \leq n$ if and only if $\mathbf{x} \in \mathbb{L}(\mathbf{p}, \mathbf{a})$ is greater than or equal to $R_i^*(\mathbf{p})$.*

(3) *The optimal inventory allocation policy for the master product $n+1$ is a lattice-dependent rationing policy with lattice-dependent rationing levels $R_{n+1}^*(\mathbf{p}) \in \mathbb{L}(\mathbf{p}, \mathbf{b})$, $\forall \mathbf{p}$: It is optimal to fulfill a demand for product $n+1$ if and only if $\mathbf{x} \in \mathbb{L}(\mathbf{p}, \mathbf{b})$ is greater than or equal to $R_{n+1}^*(\mathbf{p})$.*

The optimal policy has the following additional properties:

i. *As the system moves to a difference lattice with an increment of b_k in the inventory level of component k, both the optimal base-stock level of component $j \neq k$ and the optimal rationing level for individual product $i \notin \{k, n+1\}$ increase in a non-strict sense, $\forall k$.*

ii. *As the system moves to a difference lattice with an increment of b_k in the inventory level of component k, the optimal rationing level for the master product $n+1$ decreases in a non-strict sense, $\forall k$.*

iii. *It is optimal to fulfill a demand of the master product $n+1$ if $x_j \geq a_j + b_j \left\lfloor \frac{x_j}{b_j} \right\rfloor$, $\forall j$.*

Theorem 1 builds upon the properties of $Cx(\mathbf{a}, b_j e_j)$, $\forall j$, $Cx(\mathbf{a}, \mathbf{b})$, and $Sub(\mathbf{b})$: $Cx(\mathbf{a}, b_j e_j)$ implies that, as the system moves to a higher inventory level on the lattice $\mathbb{L}(\mathbf{p}, \mathbf{a})$, the desirability of producing a batch of component j decreases in a non-strict sense (optimality of base-stock policies, point 1), and the desirability of satisfying a demand for any individual product j increases in a non-strict sense (optimality of rationing policies for each product $j \leq n$, point 2). $Cx(\mathbf{a}, \mathbf{b})$ implies that, as the system moves to a higher inventory level on the lattice $\mathbb{L}(\mathbf{p}, \mathbf{b})$, the incentive to fulfill a demand for the master product $n+1$ increases in a non-strict sense (optimality of a rationing policy for product $n+1$, point 3).

Notice that the rationing policy for each product $i \leq n$ in point 2 is defined over lattices with common difference $\mathbf{a}$, while the rationing policy for product $n+1$ in point 3 is defined over lattices with common difference $\mathbf{b}$. The intuition behind these results is as follows: Demands of each product $i \leq n$ compete with those of product $n+1$ for the same component. For a given product $i \leq n$, an increment of $\mathbf{a}$ in the inventory level increases the total demand for its competitor product that can be satisfied, thereby mitigating the competition. Hence, the incentive to fulfill a demand of product $i \leq n$ increases in a non-strict sense (point 2). Likewise, for product $n+1$, an increment of $\mathbf{b}$ in the inventory level mitigates the competition as the total demand for each of its competitors that can be satisfied increases. Hence, the incentive to fulfill a demand of product $n+1$ increases in a non-strict sense (point 3). Note that under the rationing policy described in Theorem 1, for a given product, an increment in the inventory level that does *not* increase the total demand for any of its competitors that can be satisfied, may actually *reduce* the incentive to fulfill a demand of this product (in a non-strict sense).

Theorem 1, using the properties of $Sub(\mathbf{b})$ and $Cx(\mathbf{a}, b_j e_j)$, $\forall j$, proves the following additional properties of the optimal policy: Point (i) says that, based on the property of $Sub(\mathbf{b})$, upon replenishment of a batch of a component k, the desirability of producing a batch of component $j \neq k$

increases while the desirability of satisfying a demand for product $i \notin \{k, n+1\}$ decreases, in a non-strict sense. Therefore, both the base-stock level of component $j \neq k$ and the rationing level for product $i \notin \{k, n+1\}$ increase in a non-strict sense. The intuition is that the presence of the master product $n+1$ requires us to coordinate inventory replenishment and fulfillment decisions across components; it is less beneficial to produce or hold a batch of one component when the inventory level of any other component is significantly smaller. Point (ii) states that, based on the property of $Cx(\mathbf{a}, b_j e_j)$, upon replenishment of a batch of any component j, the incentive to fulfill a demand for product $n+1$ increases in a non-strict sense since the total demand for one of its competitors that can be satisfied increases. Lastly, point (iii) shows that it is optimal to fulfill a demand of product $n+1$ as long as the total demand for any other product that can be satisfied stays the same.

As far as we aware, we are the first to characterize the optimal policy for the generalized M-system. We refer to this optimal policy as a *lattice-dependent base-stock and lattice-dependent rationing* (LBLR) policy. In Section 4.2, we will generalize our optimality results by allowing our products to be requested by multiple demand classes.

Benjaafar and ElHafsi (2006) study an assembly system, which is a special case of our generalized M-system, and show the optimality of a state-dependent base-stock and state-dependent rationing (SBSR) policy. An LBLR policy differs from an SBSR policy in the following ways: There may be inventory levels $\mathbf{x}_1 \in \mathbb{L}(\mathbf{p}_1, \mathbf{a})$ and $\mathbf{x}_2 \in \mathbb{L}(\mathbf{p}_2, \mathbf{a})$, $\mathbf{x}_1 \geq \mathbf{x}_2$, $\mathbf{p}_1 \neq \mathbf{p}_2$, such that an LBLR policy allows a particular component to be produced at $\mathbf{x}_1$ even if it is not produced at $\mathbf{x}_2$, but an SBSR policy does *not*. Likewise, there may be inventory levels $\mathbf{x}_1 \in \mathbb{L}(\mathbf{p}_1, \mathbf{b})$ and $\mathbf{x}_2 \in \mathbb{L}(\mathbf{p}_2, \mathbf{b})$, $\mathbf{x}_1 \geq \mathbf{x}_2$, $\mathbf{p}_1 \neq \mathbf{p}_2$, such that an LBLR policy allows a demand for product $n+1$ to be rejected at $\mathbf{x}_1$ even if it is satisfied at $\mathbf{x}_2$, but again an SBSR policy does *not*. Conversely, if $\mathbf{a} \neq \sum_j ze_j$ for $z \in \mathbb{N}_0$, then there also may exist inventory levels $\mathbf{x}_1 \geq \mathbf{x}_2$, such that an SBSR policy allows a particular component to be produced at $\mathbf{x}_1$ even if it is not produced at $\mathbf{x}_2$, but an LBLR policy does *not*. But if $\mathbf{a}$ is chosen optimally, then it can be shown that an SBSR policy is a subclass of LBLR policies (see Nadar et al. 2012).

To our knowledge, we are also the first to establish the optimal policy for an ATO system in which different products use different quantities of the same component. For the simplest example of such a system, consider a single-component model with two products (denoted by 1 and 2). This is a special case of our generalized M-system; products 1 and 2 can be viewed as the individual and master products of the M-system, respectively. Suppose that products 1 and 2 consume 1 and 2 units of the component, respectively, and the replenishment batch size is 1, satisfying Assumption 1. (Products 1 and 2 can also be viewed as the master and individual products, respectively; if the replenishment batch size is 2, Assumption 1 is again satisfied.) As far as we know, there is no optimality result in the literature for such a system. (If both products required one unit from the component, the optimal policy would be a fixed base-stock and fixed rationing, FBFR, policy with single base-stock level for the component and single rationing level for each product; see Ha 1997.) Theorem 1 establishes the optimality of an LBLR policy for this problem.

Now, suppose that $\mu = 1$, $\lambda_1 = 1$, $\lambda_2 = 10$, $c_1 = 20$, $c_2 = 100$, $h = 40$, and $\alpha = 0.5$. (We assumed linear holding cost rates, i.e., $h(x) = hx$.) Then:

- A base-stock policy is optimal on each of the following two lattices: $\{0, 2, 4, ..\}$ and $\{1, 3, 5, ..\}$. The base-stock levels are 18 and 21, respectively.
- For product 1, a rationing policy is optimal on each of the following two lattices: $\{0, 2, 4, ..\}$ and $\{1, 3, 5, ..\}$. The rationing levels for product 1 are 14 and 1, respectively.
- For product 2, however, a rationing policy is optimal on the entire state space, i.e., $\{0, 1, 2, ..\}$, since product 1 uses one unit of the component. The rationing level for product 2 is 2.

Notice that base-stock levels and/or rationing levels on different lattices in general need not be adjacent. When they are, an LBLR policy reduces to an FBFR policy.

The Case of Average Cost. As our optimization criterion, we now take the average cost per unit time over an infinite planning horizon. Given a policy $\pi = \widetilde{\pi}$, the average cost rate is given by

$$v^{\widetilde{\pi}}(\mathbf{x}) = \limsup_{T \to \infty} \frac{1}{T} \left\{ \int_0^T h(\mathbf{X}(t))dt + \sum_{i=1}^{n+1} \int_0^T c_i dN_i(t) \right\}. \tag{5}$$

The objective is to identify a policy π^* that yields $v^*(\mathbf{x}) = \inf_\pi v^\pi(\mathbf{x})$ for all states $\mathbf{x}$. The following proposition shows that our structural results carry over to the average cost case:

Proposition 1. *Suppose that Assumption 1 holds and the Markov chain governing the system is irreducible. Then there exists a stationary policy that is optimal under the average cost criterion. This policy retains all the properties of the optimal policy under the discounted cost criterion, as introduced in Theorem 1. Also, the optimal average cost is finite and independent of the initial state; there exists a finite constant v^* such that $v^*(\mathbf{x}) = v^*$, $\forall \mathbf{x}$.*

4. Extensions

In this section we discuss several extensions of the optimality results in Section 3.

4.1. Generalized N-Systems

Our analysis can be extended to systems in which a nonempty subset of the components is *not* demanded individually. Define A_1 as the set of components used by product $n+1$ only, and A_2 as the set of components j used by products $i=j$ and $i=n+1$. Thus, $A_1=\{1,2,..,n\}\setminus A_2$. We label such systems as generalized N-systems, since the product structure in this case takes the form of N-system when $n=2$ and $A_2=\{1\}$, or $n=2$ and $A_2=\{2\}$, cf. Figure 1(a). Generalized N-systems are a special case of our generalized M-systems when the demand rate for each individual product $i\in A_1$ is zero, and therefore an LBLR policy is optimal for these systems under Assumption 1. However, Assumption 1 is no longer restrictive for the replenishment batch size of component $j\in A_1$: As the demand rate for individual product $i\in A_1$ is zero, q_j may be chosen arbitrarily for component $j=i$, $\forall i\in A_1$.

We are the first to show the optimality of an LBLR policy for such general N-systems. Different more restricted versions of the N-system have been studied in the literature: Lu et al. (2010) prove that no-holdback rules are optimal among all allocation rules for N-systems with backordering, a base-stock replenishment policy, and a symmetric cost structure. In a recent paper, Lu et al. (2012) establish the optimality of coordinated base-stock policies and no-holdback rules for N-systems with backordering and symmetric costs. Lu et al. (2012) also extend this result to the case with high demand volume and asymmetric costs. Lastly, in a lost sales environment, ElHafsi et al. (2008) consider a nested product structure with unitary component usage rates and unitary replenishment quantities. The nested system of ElHafsi et al. (2008) reduces to an N-system when there are two components. Under Markovian assumptions on production and demand, ElHafsi et al. (2008) show the optimality of an SBSR policy.

4.2. The Case with Multiple Demand Classes

In this subsection, we extend our generalized M-system by allowing each product to be requested by multiple demand classes with different lost sale costs. Denote by $D^{(i)}$ the number of different demand classes for product i, and let $d^{(i)}=1,2,..,D^{(i)}$. A demand for one unit of product i from class $d^{(i)}$ arrives as an independent Poisson process with rate $\lambda_{i,d^{(i)}}$ and has a lost sale cost $c_{i,d^{(i)}}$, $\forall i$. Without loss of generality, we assume $c_{i,1}\geq c_{i,2}\geq\cdots\geq c_{i,D^{(i)}}$, $\forall i$. We therefore modify our optimality equation in (4) as follows:

$$v^*(\mathbf{x})=h(\mathbf{x})+\sum_{j=1}^{n}\mu_j T^{(j)}v^*(\mathbf{x})+\sum_{i=1}^{n+1}\sum_{d^{(i)}=1}^{D^{(i)}}\lambda_{i,d^{(i)}}T_{i,d^{(i)}}v^*(\mathbf{x}),$$

where the replenishment operator $T^{(j)}$ for component j stays the same as in (4), the operator $T_{i,d^{(i)}}$ for demand class $d^{(i)}$ of individual product i is defined as

$$T_{i,d^{(i)}}v(\mathbf{x}) = \begin{cases} \min\{v(\mathbf{x}) + c_{i,d^{(i)}}, v(\mathbf{x} - b_i e_i)\} & \text{if } x_i \geq b_i, \\ v(\mathbf{x}) + c_{i,d^{(i)}} & \text{otherwise,} \end{cases}$$

and the operator $T_{n+1,d^{(n+1)}}$ for demand class $d^{(n+1)}$ of the master product $n+1$ is defined as

$$T_{n+1,d^{(n+1)}}v(\mathbf{x}) = \begin{cases} \min\{v(\mathbf{x}) + c_{n+1,d^{(n+1)}}, v(\mathbf{x} - \mathbf{a})\} & \text{if } \mathbf{x} \geq \mathbf{a}, \\ v(\mathbf{x}) + c_{n+1,d^{(n+1)}} & \text{otherwise.} \end{cases}$$

The operator $T_{i,d^{(i)}}$ is associated with the decision to fulfill a demand for individual product $i \leq n$ from class $d^{(i)}$. Likewise, the operator T_{n+1} is associated with the decision to fulfill a demand for the master product $n+1$ from class $d^{(n+1)}$.

In this case, if Assumption 1 holds, it can be shown that an LBLR policy is optimal under the following modifications: (i) The optimal inventory allocation for demand class $d^{(i)}$ of each product $i \leq n$ is a lattice-dependent rationing policy with rationing levels $R^*_{i,d^{(i)}}(\mathbf{p}) \in \mathbb{L}(\mathbf{p}, \mathbf{a})$, $\forall \mathbf{p}$, (ii) the optimal inventory allocation for demand class $d^{(n+1)}$ of product $n+1$ is a lattice-dependent rationing policy with rationing levels $R^*_{n+1,d^{(n+1)}}(\mathbf{p}) \in \mathbb{L}(\mathbf{p}, \mathbf{b})$, $\forall \mathbf{p}$, and (iii) it is optimal to fulfill a demand of product $n+1$ from class 1 as long as the total demand for any other product that can be satisfied stays the same. Furthermore, $R^*_{i,1}(\mathbf{p}) \leq R^*_{i,2}(\mathbf{p}) \leq \cdots \leq R^*_{i,D^{(i)}}(\mathbf{p})$, $\forall \mathbf{p}$, $\forall i$.

4.3. The Case with Variable Replenishment Quantities

We next allow the replenishment quantity of each component j to be integral multiples of the batch size q_j. For this extension, we modify the replenishment control operator $T^{(j)}$ in (4) as follows:

$$T^{(j)}v(\mathbf{x}) = \min_{z \in \mathbb{N}_0} \{v(\mathbf{x} + z q_j e_j)\}.$$

The operator $T^{(j)}$ is associated with the decision to produce z batches of component j. (If z is restricted to be either one or zero at each of these control operators, the problem reduces to the one described in Section 2.)

Under this modification, again if $q_j = b_j$, $\forall j$, it can be shown that our propagation results continue to hold: The optimal cost function satisfies the properties of $Sub(\mathbf{b})$, $Cx(\mathbf{a}, b_j e_j)$, $\forall j$, and $Cx(\mathbf{a}, \mathbf{b})$. Thus the optimal allocation policy is a lattice-dependent *rationing* policy. But the optimal *replenishment* policy has no clear structure: Consider two different system states $\mathbf{x}_1$ and $\mathbf{x}_2$ such that $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{L}(\mathbf{p}, \mathbf{a})$. The original system, where $z \in \{0, 1\}$ at each replenishment operator, moves from the lattice $\mathbb{L}(\mathbf{p}, \mathbf{a})$ to the lattice $\mathbb{L}(\mathbf{p} + q_j e_j, \mathbf{a})$ upon replenishment of component j at both states $\mathbf{x}_1$ and $\mathbf{x}_2$. Such transitions are governed by the structural properties of the optimal cost function, implying the optimality of a lattice-dependent base-stock policy. However, the revised system, where $z \in \mathbb{N}_0$, may move from the lattice $\mathbb{L}(\mathbf{p}, \mathbf{a})$ to different lattices upon replenishment of component j since different replenishment quantities might be chosen at states $\mathbf{x}_1$ and $\mathbf{x}_2$. But then the structural properties of the optimal cost function may not apply.

Nevertheless, we can characterize the optimal replenishment policy for generalized M-systems with unitary component usage rates for products (i.e., $\mathbf{a} = \mathbf{e}$ and $\mathbf{b} = \mathbf{e}$) and unitary replenishment batch sizes for components (i.e., $\mathbf{q} = \mathbf{e}$) (as is standard in the ATO literature). In this special case of generalized M-systems, the optimal cost function satisfies the properties of $Sub(\mathbf{e})$ and $Cx(\mathbf{e}, e_j)$, $\forall j$. Then, it can be shown that the optimal cost function is convex in the inventory level of each component, and the optimal replenishment policy is a state-dependent base-stock policy with state-dependent base-stock levels at each component.

4.4. The Case with Compound Poisson Demand

Lastly, we allow customer orders for each product to arrive according to an independent compound Poisson process. Specifically, in this case, customers for product i arrive as an independent Poisson process with a finite rate λ_i, but an arriving customer for product i requests δ_i units from product i. We assume the random variables δ_i are independent across different products and across different customers for the same product. The requested amounts are bounded above for each product i by the quantity D_i. The probability that the size of a customer order for product i will be d is $\mathbf{Pr}\{\delta_i = d\} = p_i(d)$, $i = 1, 2, .., n+1$ and $d = 1, 2, .., D_i$. Any unsatisfied part of the demand for each product i is lost, incurring a unit lost sale cost c_i. Thus our optimality equation in (4) can be modified as follows:

$$v^*(\mathbf{x}) = h(\mathbf{x}) + \sum_j \mu_j T^{(j)} v^*(\mathbf{x}) + \sum_i \lambda_i \left(\sum_{d=1}^{D_i} p_i(d) T_{i,d} v^*(\mathbf{x}) \right),$$

where the replenishment operator $T^{(j)}$ for component j stays the same as in (4), the operator $T_{i,d}$ for a customer order for d units of individual product $i \leq n$ is defined as

$$T_{i,d} v(\mathbf{x}) = \min_{z \in \{0,1,..,d\} \ s.t. \ x_i \geq z b_i} \{ v(\mathbf{x} - z b_i e_i) + (d - z) c_i \},$$

and the operator $T_{n+1,d}$ for a customer order for d units of the master product $n+1$ is defined as

$$T_{n+1,d} v(\mathbf{x}) = \min_{z \in \{0,1,..,d\} \ s.t. \ \mathbf{x} \geq z \mathbf{a}} \{ v(\mathbf{x} - z \mathbf{a}) + (d - z) c_{n+1} \}.$$

The operator $T_{i,d}$ is associated with the decision to fulfill z units (if sufficient inventory exists) out of d requested units for individual product $i \leq n$. Likewise, the operator $T_{n+1,d}$ is associated with the decision to fulfill z units (if sufficient inventory exists) out of d requested units for the master product $n+1$. (The problem reduces to the one described in Section 2 when $\mathbf{Pr}\{\delta_i = 1\} = 1$, $\forall i \in \{1, 2, .., n+1\}$.)

In this case, once again if $q_j = b_j$, $\forall j$, it can be shown that the structural properties of our optimal cost function in (4) continue to hold: The optimal replenishment policy is a lattice-dependent *base-stock* policy. But the optimal *allocation* policy has no clear structure. Consider two different system states $\mathbf{x}_1$ and $\mathbf{x}_2$ such that $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{L}(\mathbf{p}, \mathbf{a})$. The original system with unitary Poisson demand moves from the lattice $\mathbb{L}(\mathbf{p}, \mathbf{a})$ to the lattice $\mathbb{L}(\mathbf{p} - b_i e_i, \mathbf{a})$ if a demand for individual product i is satisfied at both states $\mathbf{x}_1$ and $\mathbf{x}_2$. Such transitions are governed by the structural properties of the optimal cost function, implying the optimality of a lattice-dependent rationing policy. However, the revised system with compound Poisson demand may move from the lattice $\mathbb{L}(\mathbf{p}, \mathbf{a})$ to different lattices upon arrival of a customer order for d units of individual product i, since different quantities from the d requested units might be satisfied at states $\mathbf{x}_1$ and $\mathbf{x}_2$. (A similar argument can be made for the master product.) But then the structural properties of the optimal cost function do not apply.

Again, we can characterize the optimal allocation policy for generalized M-systems with compound Poisson demand, unitary component usage rates for products, and unitary replenishment batch sizes for components. In this case, since the optimal cost function is convex in the inventory level of each component, the optimal allocation policy is a state-dependent rationing policy with state-dependent rationing levels for each product. Furthermore, our generalized M-systems with unitary component requirements, unitary replenishment batch sizes, and compound Poisson demand become equivalent to the assembly system in ElHafsi (2009) when the demand rates for our individual products are zero. ElHafsi (2009) proves the optimality of a state-dependent rationing policy for the end-product. Thus we extend the optimality result in ElHafsi (2009) by allowing the components to also be demanded individually.

5. Concluding Remarks

We have studied the inventory replenishment and allocation problem for generalized ATO M-systems. We significantly extend the existing literature by characterizing the optimal policy when different products use different quantities of the same component. When replenishment batch sizes are determined by the individual product sizes, a lattice-dependent base-stock and lattice-dependent rationing (LBLR) policy is optimal for both the discounted cost and average cost cases. An LBLR policy is optimal also when (i) some components are not demanded individually and their replenishment batch sizes are chosen arbitrarily, and/or (ii) each product is requested by multiple demand classes. A lattice-dependent rationing policy remains optimal when the possible replenishment quantities for any component are integral multiples of the size of the corresponding individual product. A lattice-dependent base-stock policy remains optimal when customer orders for any product arrive as an independent compound Poisson process.

In a companion paper (Nadar et al. 2012), we conduct numerical experiments to evaluate the use of an LBLR policy as a heuristic for general ATO systems (which may *not* satisfy Assumption 1, or even our generalized M-system product structure), comparing it with two other heuristics: a state-dependent base-stock and rationing policy (SBSR), and a fixed base-stock and rationing policy (FBFR), both adapted from Benjaafar and ElHafsi (2006). In the average cost case, we numerically show that LBLR *always* yields the optimal cost in over 1800 examples, while SBSR (or FBFR) provides solutions within 2.7% (or 4.8%) of the optimal cost. We are also able to show analytically that LBLR outperforms the other heuristics. Based on these results, future research could investigate whether an LBLR policy is indeed optimal for general ATO systems and if so, how the state space should be partitioned into disjoint lattices. However, one may need a different methodology to prove the optimality of LBLR, because in Nadar et al. (2012) we also provide counter-examples which show that the second-order properties of our optimal cost function, which are *sufficient* to ensure the optimality of LBLR, may fail to hold for general ATO systems.

Future extensions of the current paper could also consider ATO systems with backorders. In this case, one needs to include the number of backordered demands for each product in the state space, and investigate the optimal backorder clearing mechanism upon replenishment of any component. However, both the state and action spaces become extremely large as a result. Also, as our products will differ in their both backordering costs and component requirements, it is unclear which products will have fulfillment priority at different inventory levels, adding significant complexity to the backorder clearing problem. Another direction for future research is to extend our model to phase-type or even general component production and demand interarrival times. Also, it would be more realistic to allow for dependent demand across products and over time. Lastly, extending our model to include nonzero assembly times is an interesting problem to pursue. However, with today's manufacturing technology, assembly times are usually small and our model is likely to provide a good approximation in general.

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Online Technical Appendix

EC.1. New Functional Characterizations

Define f as a real-valued function on $\mathbb{N}_0^n$, and $\mathbf{p} = (p_1, p_2, .., p_n)$ and $\mathbf{r} = (r_1, r_2, .., r_n)$ as vectors of nonnegative integers. Also let $\Delta_{\mathbf{p}} f = f(\mathbf{x}+\mathbf{p}) - f(\mathbf{x})$.

Section 3 introduces the notion of "submodularity (or supermodularity) in the direction of $p_j e_j$ and $p_k e_k$," for distinct j and k, to describe the class of functions f for which $\Delta_{p_j e_j} f$ is nonincreasing (or nondecreasing) with an increase of p_k in the kth dimension. This terminology is inspired by that of Veatch and Wein (1982), who define submodularity of a function with respect to control directions in its domain. However, unlike the examples in Veatch and Wein (1982), our direction sets may involve non-unitary vectors.

Our notion of submodularity (or supermodularity) concurs with the theory of submodular (or supermodular) functions on a lattice developed by Topkis (1978, 1998), with some adjustments. It can be shown that if a function on $\mathbb{N}_0^n$ is "submodular in the direction of $p_j e_j$ and $p_k e_k$," $\forall j \neq k$, then it satisfies Topkis' submodularity property on the lattice

$$\left\{ \mathbf{r} + \sum_{i=1}^{n} z_i p_i e_i : z_i \in \mathbb{N}_0 \right\},$$

$\forall \mathbf{r}$ such that $r_i < p_i$, $\forall i \in \{1, 2, .., n\}$, and vice versa. Notice that if $\exists i$ such that $p_i > 1$, then $\mathbf{r}$ can be chosen in multiple ways, and thus "submodularity in the direction of $p_j e_j$ and $p_k e_k$," $\forall j \neq k$, implies Topkis' submodularity concept on multiple disjoint subspaces of $\mathbb{N}_0^n$ which is, strictly speaking, a generalization of Topkis' submodularity. But, if a function on $\mathbb{N}_0^n$ is "submodular in the direction of e_j and e_k," $\forall j \neq k$ (i.e., $p_i = 1$, $\forall i$), then it is submodular on the lattice $\mathbb{N}_0^n$, and vice versa. In this case our definition is equivalent to that in Topkis. (The same arguments can be extended to our notion of supermodularity.)

Consider the M-system depicted in Figure 1(b). Suppose that products 1 and 2 use 2 units from components 1 and 2, respectively, and product 3 uses 1 unit from each component. Also, suppose that the replenishment batch sizes are 2, satisfying Assumption 1. Assume that the system is initially on the lattice $\mathbb{L}_0 \triangleq \{z_1(2,0) + z_2(0,2) : z_1, z_2 \in \mathbb{N}_0\}$. If the demand rate for product 3 is zero, then the state space of the problem can be restricted to this lattice without loss of generality. In this case, Topkis' submodularity property is equivalent to our notion of submodularity (satisfied by the optimal cost function). But if the demand rate for product 3 is nonzero, the system moves to the lattice $\mathbb{L}_1 \triangleq \{(1,1) + z_1(2,0) + z_2(0,2) : z_1, z_2 \in \mathbb{N}_0\}$ when a demand for product 3 occurs and is satisfied. The system stays on $\mathbb{L}_1$ until fulfillment of a future demand for product 3, at which time it returns to $\mathbb{L}_0$. Topkis' submodularity property need not hold on the join of the two lattices, while ours does. However, in view of our notion of submodularity, Topkis' submodularity property holds on either lattice individually.

Several authors dealing with the optimal policy characterization for Markovian inventory systems prove that the optimal cost function is "submodular in the direction of e_j and e_k," enabling them to explore the comparative statics of the optimal policy parameters with respect to the state space. See, for instance, Benjaafar and ElHafsi (2006), ElHafsi et al. (2008), ElHafsi (2009), and Gayon et al. (2009). Unlike these authors, through our notion of "submodularity in the direction p_je_j and p_ke_k," we provide the comparative statics of the optimal policy parameters with respect to certain *subspaces* of the state space; see point (i) of Theorem 1. This specialization (along with our notion of convexity) enables us to prove structural properties for ATO problems that have non-unitary component demands for the first time in the literature.

Section 3 also introduces the notion of "convexity (or concavity) in the direction of $\mathbf{p}$ and $\mathbf{r}$" to describe the class of functions f for which $\Delta_{\mathbf{p}}f$ is nondecreasing (or nonincreasing) with an increase of $\mathbf{r}$ in the domain. This terminology is again inspired by that of Veatch and Wein (1982), who use the term "convexity in the direction $\mathbf{p}$" to describe the class of functions f with $\Delta_{\mathbf{p}}f$ nondecreasing with an increase of $\mathbf{p}$ in the domain. Thus, our notion of convexity reduces to that of Veatch and Wein (1982) when $\mathbf{p}=\mathbf{r}$. It further reduces to convexity in the jth dimension when $\mathbf{p}=\mathbf{r}=e_j$.

Our notion of convexity might be usefully employed in determining the optimal policy for complex inventory models in which the optimal cost function need not be convex. In the literature on optimal policy characterization for inventory systems with batch ordering, several authors use the property of "convexity in the direction of p and 1," where p is a positive integer, to establish the optimality of a threshold ordering policy. See, for instance, Gallego and Toktay (2004), and Huh and Janakiraman (2012). Apart from this literature, Ha (2000) uses the same property for a single-item inventory system with lost sales, Erlang replenishment lead times, and Poisson demand. Ha (2000) then proves that the optimal replenishment policy is a critical work level policy. (All of these papers use different terminologies to describe what we refer to as "convexity in the direction of p and 1.") In our study, however, we show that the optimal cost function is "convex in the direction of $\mathbf{p}=\mathbf{a}$ and $\mathbf{r}=\mathbf{b}$," (which is weaker than "convexity in the direction of p and 1" in one dimensional problems) for generalized M-systems. This result continues to hold even for (i) generalized N-systems, (ii) the case with multiple demand classes, (iii) the case with variable replenishment quantities, and (iv) the case with compound Poisson demand.

EC.2. Proofs of the Results in Section 3

We need the following two auxiliary lemmas in the proof of Lemma 1.

LEMMA EC.1. *$Cx(\mathbf{p}, r_1e_1) \cap \cdots \cap Cx(\mathbf{p}, r_ne_n) \subseteq Cx(\mathbf{p}, \mathbf{r})$, $\forall \mathbf{p}, \mathbf{r} \in \mathbb{N}_0^n$.*

Proof of Lemma EC.1. $f \in Cx(\mathbf{p}, r_je_j)$, $\forall j$, implies the following inequalities:

$$\begin{aligned}
f(\mathbf{x}+\mathbf{p}+r_1e_1) - f(\mathbf{x}+r_1e_1) &\geq f(\mathbf{x}+\mathbf{p}) - f(\mathbf{x}),\\
f(\mathbf{x}+\mathbf{p}+r_1e_1+r_2e_2) - f(\mathbf{x}+r_1e_1+r_2e_2) &\geq f(\mathbf{x}+\mathbf{p}+r_1e_1) - f(\mathbf{x}+r_1e_1),\\
&\vdots\\
f(\mathbf{x}+\mathbf{p}+\sum_{j\leq n} r_je_j) - f(\mathbf{x}+\sum_{j\leq n} r_je_j) &\geq f(\mathbf{x}+\mathbf{p}+\sum_{j<n} r_je_j) - f(\mathbf{x}+\sum_{j<n} r_je_j)
\end{aligned}$$

Summation of these inequalities implies $f(\mathbf{x}+\mathbf{p}+\mathbf{r}) - f(\mathbf{x}+\mathbf{r}) \geq f(\mathbf{x}+\mathbf{p}) - f(\mathbf{x})$, i.e., $f \in Cx(\mathbf{p}, \mathbf{r})$. $\square$

LEMMA EC.2. *For operators $T^{(j)}$ and T_i, and holding cost rate h, it holds that*

(a) $T^{(j)} : Sub(\mathbf{q}) \cap Cx(\mathbf{a}, q_1e_1) \cap \cdots \cap Cx(\mathbf{a}, q_ne_n) \to Sub(\mathbf{q}) \cap Cx(\mathbf{a}, q_1e_1) \cap \cdots \cap Cx(\mathbf{a}, q_ne_n)$, $\forall j$,

(b) $T_i : Sub(\mathbf{b}) \cap Cx(\mathbf{a}, b_1e_1) \cap \cdots \cap Cx(\mathbf{a}, b_ne_n) \to Sub(\mathbf{b}) \cap Cx(\mathbf{a}, b_1e_1) \cap \cdots \cap Cx(\mathbf{a}, b_ne_n)$, $\forall i$, *and*

(c) $h \in Sub(\mathbf{q}) \cap Cx(\mathbf{a}, q_1e_1) \cap \cdots \cap Cx(\mathbf{a}, q_ne_n) \cap Sub(\mathbf{b}) \cap Cx(\mathbf{a}, b_1e_1) \cap \cdots \cap Cx(\mathbf{a}, b_ne_n)$.

Proof of Lemma EC.2. Recall that $T^{(j)}v(\mathbf{x}) = \min\{v(\mathbf{x}+q_je_j), v(\mathbf{x})\}$, $T_iv(\mathbf{x}) = \min\{v(\mathbf{x}) + c_i, v(\mathbf{x}-b_ie_i)\}$ if $x_i \geq b_i$, and $T_iv(\mathbf{x}) = v(\mathbf{x}) + c_i$ otherwise, for $i \leq n$; and $T_{n+1}v(\mathbf{x}) = \min\{v(\mathbf{x}) + c_{n+1}, v(\mathbf{x}-\mathbf{a})\}$ if $x_j \geq a_j$ for all j, and $T_{n+1}v(\mathbf{x}) = v(\mathbf{x}) + c_{n+1}$ otherwise.

(a) Assume that $v \in Sub(\mathbf{q}) \cap Cx(\mathbf{a}, q_1e_1) \cap \cdots \cap Cx(\mathbf{a}, q_ne_n)$. We will show $T^{(j)}v \in Sub(\mathbf{q}) \cap Cx(\mathbf{a}, q_1e_1) \cap \cdots \cap Cx(\mathbf{a}, q_ne_n)$.

- First we show $T^{(j)}v \in Sub(\mathbf{q})$, i.e., $T^{(j)}v(\mathbf{x}+q_ie_i) - T^{(j)}v(\mathbf{x}) \geq T^{(j)}v(\mathbf{x}+q_ie_i+q_ke_k) - T^{(j)}v(\mathbf{x}+q_ke_k)$, $\forall k \neq i$. Pick arbitrary $k \in \{1, 2, .., n\}$. There are four different scenarios we need to consider depending on the optimal actions at $T^{(j)}v(\mathbf{x}+q_ie_i)$ and $T^{(j)}v(\mathbf{x}+q_ke_k)$ (if this inequality holds under suboptimal actions of $T^{(j)}v(\mathbf{x})$ and/or $T^{(j)}v(\mathbf{x}+q_ie_i+q_ke_k)$, it also holds under optimal actions of these operators, and thus we do not enforce the optimal actions at these operators). These four scenarios are as follows:

(1) Suppose that $T^{(j)}v(\mathbf{x}+q_ie_i)=v(\mathbf{x}+q_ie_i)<v(\mathbf{x}+q_je_j+q_ie_i)$ and $T^{(j)}v(\mathbf{x}+q_ke_k)=v(\mathbf{x}+q_ke_k)<v(\mathbf{x}+q_je_j+q_ke_k)$. As we assume $v\in Sub(\mathbf{q})$, the following inequalities hold:

$$\begin{aligned}T^{(j)}v(\mathbf{x}+q_ie_i)-T^{(j)}v(\mathbf{x}) &\geq v(\mathbf{x}+q_ie_i)-v(\mathbf{x})\\ &\geq v(\mathbf{x}+q_ie_i+q_ke_k)-v(\mathbf{x}+q_ke_k)\\ &\geq T^{(j)}v(\mathbf{x}+q_ie_i+q_ke_k)-T^{(j)}v(\mathbf{x}+q_ke_k)\end{aligned}$$

(2) Suppose that $T^{(j)}v(\mathbf{x}+q_ie_i)=v(\mathbf{x}+q_je_j+q_ie_i)<v(\mathbf{x}+q_ie_i)$ and $T^{(j)}v(\mathbf{x}+q_ke_k)=v(\mathbf{x}+q_ke_k)<v(\mathbf{x}+q_je_j+q_ke_k)$. If $j=k$, then it is easy to verify that

$$\begin{aligned}T^{(j)}v(\mathbf{x}+q_ie_i)-T^{(j)}v(\mathbf{x}) &\geq v(\mathbf{x}+q_ie_i+q_je_j)-v(\mathbf{x}+q_je_j)\\ &\geq T^{(j)}v(\mathbf{x}+q_ie_i+q_je_j)-T^{(j)}v(\mathbf{x}+q_je_j)\end{aligned}$$

If $j\neq k$, as we assume $v\in Sub(\mathbf{q})$, the following inequalities hold:

$$\begin{aligned}T^{(j)}v(\mathbf{x}+q_ie_i)-T^{(j)}v(\mathbf{x}) &\geq v(\mathbf{x}+q_je_j+q_ie_i)-v(\mathbf{x})\\ &\geq v(\mathbf{x}+q_je_j)-v(\mathbf{x}+q_je_j+q_ke_k)\\ &\quad +v(\mathbf{x}+q_je_j+q_ie_i+q_ke_k)-v(\mathbf{x})\\ &\geq v(\mathbf{x}+q_je_j+q_ie_i+q_ke_k)-v(\mathbf{x}+q_ke_k)\\ &\geq T^{(j)}v(\mathbf{x}+q_ie_i+q_ke_k)-T^{(j)}v(\mathbf{x}+q_ke_k)\end{aligned}$$

(3) Suppose that $T^{(j)}v(\mathbf{x}+q_ie_i)=v(\mathbf{x}+q_ie_i)<v(\mathbf{x}+q_je_j+q_ie_i)$ and $T^{(j)}v(\mathbf{x}+q_ke_k)=v(\mathbf{x}+q_je_j+q_ke_k)<v(\mathbf{x}+q_ke_k)$. If $j=i$, then it is easy to verify that

$$\begin{aligned}T^{(j)}v(\mathbf{x}+q_je_j)-T^{(j)}v(\mathbf{x}) &\geq v(\mathbf{x}+q_je_j)-v(\mathbf{x}+q_je_j)\\ &= v(\mathbf{x}+q_je_j+q_ke_k)-v(\mathbf{x}+q_je_j+q_ke_k)\\ &\geq T^{(j)}v(\mathbf{x}+q_je_j+q_ke_k)-T^{(j)}v(\mathbf{x}+q_ke_k)\end{aligned}$$

If $j\neq i$, as we assume $v\in Sub(\mathbf{q})$, the following inequalities hold:

$$\begin{aligned}T^{(j)}v(\mathbf{x}+q_ie_i)-T^{(j)}v(\mathbf{x}) &\geq v(\mathbf{x}+q_ie_i)-v(\mathbf{x})\\ &\geq v(\mathbf{x}+q_je_j+q_ie_i)-v(\mathbf{x}+q_je_j)\\ &\geq v(\mathbf{x}+q_je_j+q_ie_i+q_ke_k)-v(\mathbf{x}+q_je_j+q_ke_k)\\ &\geq T^{(j)}v(\mathbf{x}+q_ie_i+q_ke_k)-T^{(j)}v(\mathbf{x}+q_ke_k)\end{aligned}$$

(4) Suppose that $T^{(j)}v(\mathbf{x}+q_ie_i)=v(\mathbf{x}+q_je_j+q_ie_i)<v(\mathbf{x}+q_ie_i)$ and $T^{(j)}v(\mathbf{x}+q_ke_k)=v(\mathbf{x}+q_je_j+q_ke_k)<v(\mathbf{x}+q_ke_k)$. As we assume $v\in Sub(\mathbf{q})$, the following inequalities hold:

$$\begin{aligned}
T^{(j)}v(\mathbf{x}+q_ie_i)-T^{(j)}v(\mathbf{x}) &\geq v(\mathbf{x}+q_je_j+q_ie_i)-v(\mathbf{x}+q_je_j)\\
&\geq v(\mathbf{x}+q_je_j+q_ie_i+q_ke_k)-v(\mathbf{x}+q_je_j+q_ke_k)\\
&\geq T^{(j)}v(\mathbf{x}+q_ie_i+q_ke_k)-T^{(j)}v(\mathbf{x}+q_ke_k)
\end{aligned}$$

Hence our inequality holds in each of the possible scenarios. Therefore, $T^{(j)}v\in Sub(\mathbf{q})$.

- Next we show $T^{(j)}v\in Cx(\mathbf{a},q_ie_i)$, $\forall i\in\{1,2,..,n\}$, i.e., $T^{(j)}v(\mathbf{x}+q_ie_i+\mathbf{a})-T^{(j)}v(\mathbf{x}+\mathbf{a})\geq T^{(j)}v(\mathbf{x}+q_ie_i)-T^{(j)}v(\mathbf{x})$, $\forall i$. Again, there are four different scenarios depending on the optimal actions at $T^{(j)}v(\mathbf{x}+q_ie_i+\mathbf{a})$ and $T^{(j)}v(\mathbf{x})$:

(1) Suppose that $T^{(j)}v(\mathbf{x}+q_ie_i+\mathbf{a})=v(\mathbf{x}+q_ie_i+\mathbf{a})<v(\mathbf{x}+q_je_j+q_ie_i+\mathbf{a})$ and $T^{(j)}v(\mathbf{x})=v(\mathbf{x})<v(\mathbf{x}+q_je_j)$. As we assume $v\in Cx(\mathbf{a},q_ie_i)$, the following inequalities hold:

$$\begin{aligned}
T^{(j)}v(\mathbf{x}+q_ie_i+\mathbf{a})-T^{(j)}v(\mathbf{x}+\mathbf{a}) &\geq v(\mathbf{x}+q_ie_i+\mathbf{a})-v(\mathbf{x}+\mathbf{a})\\
&\geq v(\mathbf{x}+q_ie_i)-v(\mathbf{x})\\
&\geq T^{(j)}v(\mathbf{x}+q_ie_i)-T^{(j)}v(\mathbf{x})
\end{aligned}$$

(2) Suppose that $T^{(j)}v(\mathbf{x}+q_ie_i+\mathbf{a})=v(\mathbf{x}+q_je_j+q_ie_i+\mathbf{a})<v(\mathbf{x}+q_ie_i+\mathbf{a})$ and $T^{(j)}v(\mathbf{x})=v(\mathbf{x})<v(\mathbf{x}+q_je_j)$. As we assume $v\in Cx(\mathbf{a},q_ie_i)$ and $v\in Cx(\mathbf{a},q_je_j)$, the following inequalities hold:

$$\begin{aligned}
T^{(j)}v(\mathbf{x}+q_ie_i+\mathbf{a})-T^{(j)}v(\mathbf{x}+\mathbf{a}) &\geq v(\mathbf{x}+q_je_j+q_ie_i+\mathbf{a})-v(\mathbf{x}+\mathbf{a})\\
&\geq v(\mathbf{x}+q_je_j+q_ie_i)+v(\mathbf{x}+q_je_j+\mathbf{a})\\
&\quad -v(\mathbf{x}+q_je_j)-v(\mathbf{x}+\mathbf{a})\\
&\geq v(\mathbf{x}+q_je_j+q_ie_i)-v(\mathbf{x})\\
&\geq T^{(j)}v(\mathbf{x}+q_ie_i)-T^{(j)}v(\mathbf{x})
\end{aligned}$$

(3) Suppose that $T^{(j)}v(\mathbf{x}+q_ie_i+\mathbf{a})=v(\mathbf{x}+q_ie_i+\mathbf{a})<v(\mathbf{x}+q_je_j+q_ie_i+\mathbf{a})$ and $T^{(j)}v(\mathbf{x})=v(\mathbf{x}+q_je_j)<v(\mathbf{x})$. If $j=i$, then it is easy to verify that

$$\begin{aligned}
T^{(j)}v(\mathbf{x}+q_je_j+\mathbf{a})-T^{(j)}v(\mathbf{x}+\mathbf{a}) &\geq v(\mathbf{x}+q_je_j+\mathbf{a})-v(\mathbf{x}+q_je_j+\mathbf{a})\\
&= v(\mathbf{x}+q_je_j)-v(\mathbf{x}+q_je_j)\\
&\geq T^{(j)}v(\mathbf{x}+q_je_j)-T^{(j)}v(\mathbf{x})
\end{aligned}$$

If $j \neq i$, as we assume $v \in Cx(\mathbf{a}, q_i e_i)$ and $v \in Sub(\mathbf{q})$, the following inequalities hold:

$$\begin{aligned} T^{(j)}v(\mathbf{x}+q_ie_i+\mathbf{a}) - T^{(j)}v(\mathbf{x}+\mathbf{a}) &\geq v(\mathbf{x}+q_ie_i+\mathbf{a}) - v(\mathbf{x}+\mathbf{a}) \\ &\geq v(\mathbf{x}+q_ie_i) - v(\mathbf{x}) \\ &\geq v(\mathbf{x}+q_je_j+q_ie_i) - v(\mathbf{x}+q_je_j) \\ &\geq T^{(j)}v(\mathbf{x}+q_ie_i) - T^{(j)}v(\mathbf{x}) \end{aligned}$$

(4) Suppose that $T^{(j)}v(\mathbf{x}+q_ie_i+\mathbf{a}) = v(\mathbf{x}+q_je_j+q_ie_i+\mathbf{a}) < v(\mathbf{x}+q_ie_i+\mathbf{a})$ and $T^{(j)}v(\mathbf{x}) = v(\mathbf{x}+q_je_j) < v(\mathbf{x})$. As we assume $v \in Cx(\mathbf{a}, q_ie_i)$, the following inequalities hold:

$$\begin{aligned} T^{(j)}v(\mathbf{x}+q_ie_i+\mathbf{a}) - T^{(j)}v(\mathbf{x}+\mathbf{a}) &\geq v(\mathbf{x}+q_je_j+q_ie_i+\mathbf{a}) - v(\mathbf{x}+q_je_j+\mathbf{a}) \\ &\geq v(\mathbf{x}+q_je_j+q_ie_i) - v(\mathbf{x}+q_je_j) \\ &\geq T^{(j)}v(\mathbf{x}+q_ie_i) - T^{(j)}v(\mathbf{x}) \end{aligned}$$

Hence our inequality holds in all the possible scenarios. Therefore, $T^{(j)}v \in Cx(\mathbf{a}, q_ie_i)$, $\forall i$.

(b) Assume that $v \in Sub(\mathbf{b}) \cap Cx(\mathbf{a}, b_1e_1) \cap \cdots \cap Cx(\mathbf{a}, b_ne_n)$. We will show $T_iv \in Sub(\mathbf{b}) \cap Cx(\mathbf{a}, b_1e_1) \cap \cdots \cap Cx(\mathbf{a}, b_ne_n)$.

Case I: Suppose that $i \leq n$.

- First we show $T_iv \in Sub(\mathbf{b})$, i.e., $T_iv(\mathbf{x}+b_je_j) - T_iv(\mathbf{x}) \geq T_iv(\mathbf{x}+b_je_j+b_ke_k) - T_iv(\mathbf{x}+b_ke_k)$, $\forall k \neq j$. Pick arbitrary $k \in \{1, 2, .., n\}$. There are four different scenarios we need to consider depending on the optimal actions at $T_iv(\mathbf{x}+b_je_j)$ and $T_iv(\mathbf{x}+b_ke_k)$ (if this inequality holds under suboptimal actions of $T_iv(\mathbf{x})$ and/or $T_iv(\mathbf{x}+b_je_j+b_ke_k)$, it also holds under optimal actions of these operators, and thus we do not enforce the optimal actions at these operators). These four scenarios are as follows:

  (1) Suppose that $T_iv(\mathbf{x}+b_je_j) = v(\mathbf{x}+b_je_j) + c_i$ and $T_iv(\mathbf{x}+b_ke_k) = v(\mathbf{x}+b_ke_k) + c_i$. As we assume $v \in Sub(\mathbf{b})$, the following inequalities hold:

$$\begin{aligned} T_iv(\mathbf{x}+b_je_j) - T_iv(\mathbf{x}) &\geq v(\mathbf{x}+b_je_j) + c_i - v(\mathbf{x}) - c_i \\ &\geq v(\mathbf{x}+b_je_j+b_ke_k) + c_i - v(\mathbf{x}+b_ke_k) - c_i \\ &\geq T_iv(\mathbf{x}+b_je_j+b_ke_k) - T_iv(\mathbf{x}+b_ke_k) \end{aligned}$$

  (2) Suppose that $x_i \geq b_i$, $T_iv(\mathbf{x}+b_je_j) = v(\mathbf{x}+b_je_j) + c_i$ and $T_iv(\mathbf{x}+b_ke_k) = v(\mathbf{x}+b_ke_k - b_ie_i)$. If $i = k$, then it is easy to verify that

$$T_iv(\mathbf{x}+b_je_j) - T_iv(\mathbf{x}) \geq v(\mathbf{x}+b_je_j) + c_i - v(\mathbf{x}) - c_i$$

$$\geq T_i v(\mathbf{x}+b_je_j+b_ie_i) - T_i v(\mathbf{x}+b_ie_i)$$

If $i \neq k$, as we assume $v \in Sub(\mathbf{b})$, the following inequalities hold:

$$\begin{aligned}
T_i v(\mathbf{x}+b_je_j) - T_i v(\mathbf{x}) &\geq v(\mathbf{x}+b_je_j) + c_i - v(\mathbf{x}-b_ie_i) \\
&\geq v(\mathbf{x}) - v(\mathbf{x}+b_ke_k) \\
&\quad + v(\mathbf{x}+b_je_j+b_ke_k) + c_i - v(\mathbf{x}-b_ie_i) \\
&\geq v(\mathbf{x}+b_je_j+b_ke_k) + c_i - v(\mathbf{x}+b_ke_k-b_ie_i) \\
&\geq T_i v(\mathbf{x}+b_je_j+b_ke_k) - T_i v(\mathbf{x}+b_ke_k)
\end{aligned}$$

(3) Suppose that $x_i \geq b_i$ if $i \neq j$, $T_i v(\mathbf{x}+b_je_j) = v(\mathbf{x}+b_je_j-b_ie_i)$ and $T_i v(\mathbf{x}+b_ke_k) = v(\mathbf{x}+b_ke_k)+c_i$. If $i=j$, then it is easy to verify that

$$\begin{aligned}
T_i v(\mathbf{x}+b_ie_i) - T_i v(\mathbf{x}) &\geq v(\mathbf{x}) - v(\mathbf{x}) - c_i \\
&= v(\mathbf{x}+b_ke_k) - v(\mathbf{x}+b_ke_k) - c_i \\
&\geq T_i v(\mathbf{x}+b_ie_i+b_ke_k) - T_i v(\mathbf{x}+b_ke_k)
\end{aligned}$$

If $i \neq j$, as we assume $v \in Sub(\mathbf{b})$, the following inequalities hold:

$$\begin{aligned}
T_i v(\mathbf{x}+b_je_j) - T_i v(\mathbf{x}) &\geq v(\mathbf{x}+b_je_j-b_ie_i) - v(\mathbf{x}-b_ie_i) \\
&\geq v(\mathbf{x}+b_je_j) - v(\mathbf{x}) \\
&\geq v(\mathbf{x}+b_je_j+b_ke_k) + c_i - v(\mathbf{x}+b_ke_k) - c_i \\
&\geq T_i v(\mathbf{x}+b_je_j+b_ke_k) - T_i v(\mathbf{x}+b_ke_k)
\end{aligned}$$

(4) Suppose that $x_i \geq b_i$, $T_i v(\mathbf{x}+b_je_j) = v(\mathbf{x}+b_je_j-b_ie_i)$ and $T_i v(\mathbf{x}+b_ke_k) = v(\mathbf{x}+b_ke_k-b_ie_i)$. As we assume $v \in Sub(\mathbf{b})$, the following inequalities hold:

$$\begin{aligned}
T_i v(\mathbf{x}+b_je_j) - T_i v(\mathbf{x}) &\geq v(\mathbf{x}+b_je_j-b_ie_i) - v(\mathbf{x}-b_ie_i) \\
&\geq v(\mathbf{x}+b_je_j+b_ke_k-b_ie_i) - v(\mathbf{x}+b_ke_k-b_ie_i) \\
&\geq T_i v(\mathbf{x}+b_je_j+b_ke_k) - T_i v(\mathbf{x}+b_ke_k)
\end{aligned}$$

Hence our inequality holds in all the possible scenarios. Therefore, $T_i v \in Sub(\mathbf{b})$.

- Next we show $T_i v \in Cx(\mathbf{a}, b_je_j)$, $\forall j \in \{1,2,..,n\}$, i.e., $T_i v(\mathbf{x}+b_je_j+\mathbf{a}) - T_i v(\mathbf{x}+\mathbf{a}) \geq T_i v(\mathbf{x}+b_je_j) - T_i v(\mathbf{x})$, $\forall j$. Again, there are four different scenarios depending on the optimal actions at $T_i v(\mathbf{x}+b_je_j+\mathbf{a})$ and $T_i v(\mathbf{x})$:

(1) Suppose that $x_i \geq b_i$, $T_i v(\mathbf{x}+b_j e_j+\mathbf{a}) = v(\mathbf{x}+b_j e_j+\mathbf{a}-b_i e_i)$ and $T_i v(\mathbf{x}) = v(\mathbf{x}-b_i e_i)$. As we assume $v \in Cx(\mathbf{a}, b_j e_j)$, the following inequalities hold:

$$\begin{aligned} T_i v(\mathbf{x}+b_j e_j+\mathbf{a}) - T_i v(\mathbf{x}+\mathbf{a}) &\geq v(\mathbf{x}+b_j e_j+\mathbf{a}-b_i e_i) - v(\mathbf{x}+\mathbf{a}-b_i e_i) \\ &\geq v(\mathbf{x}+b_j e_j-b_i e_i) - v(\mathbf{x}-b_i e_i) \\ &\geq T_i v(\mathbf{x}+b_j e_j) - T_i v(\mathbf{x}) \end{aligned}$$

(2) Suppose that $x_i \geq b_i$, $T_i v(\mathbf{x}+b_j e_j+\mathbf{a}) = v(\mathbf{x}+b_j e_j+\mathbf{a})+c_i$ and $T_i v(\mathbf{x}) = v(\mathbf{x}-b_i e_i)$. As we assume $v \in Cx(\mathbf{a}, b_j e_j)$ and $v \in Cx(\mathbf{a}, b_i e_i)$, the following inequalities hold:

$$\begin{aligned} T_i v(\mathbf{x}+b_j e_j+\mathbf{a}) - T_i v(\mathbf{x}+\mathbf{a}) &\geq v(\mathbf{x}+b_j e_j+\mathbf{a})+c_i - v(\mathbf{x}+\mathbf{a}-b_i e_i) \\ &\geq v(\mathbf{x}+b_j e_j) + v(\mathbf{x}+\mathbf{a}) \\ &\quad -v(\mathbf{x}) + c_i - v(\mathbf{x}+\mathbf{a}-b_i e_i) \\ &\geq v(\mathbf{x}+b_j e_j) + c_i - v(\mathbf{x}-b_i e_i) \\ &\geq T_i v(\mathbf{x}+b_j e_j) - T_i v(\mathbf{x}) \end{aligned}$$

(3) Suppose that $x_i + a_i \geq b_i$ if $i \neq j$, $T_i v(\mathbf{x}+b_j e_j+\mathbf{a}) = v(\mathbf{x}+b_j e_j+\mathbf{a}-b_i e_i)$ and $T_i v(\mathbf{x}) = v(\mathbf{x})+c_i$. If $i = j$, it is easy to verify that

$$\begin{aligned} T_i v(\mathbf{x}+b_i e_i+\mathbf{a}) - T_i v(\mathbf{x}+\mathbf{a}) &\geq v(\mathbf{x}+\mathbf{a}) - v(\mathbf{x}+\mathbf{a}) - c_i \\ &= v(\mathbf{x}) - v(\mathbf{x}) - c_i \\ &\geq T_i v(\mathbf{x}+b_i e_i) - T_i v(\mathbf{x}) \end{aligned}$$

If $i \neq j$, as we assume $v \in Sub(\mathbf{b})$ and $v \in Cx(\mathbf{a}, b_j e_j)$, the following inequalities hold:

$$\begin{aligned} T_i v(\mathbf{x}+b_j e_j+\mathbf{a}) - T_i v(\mathbf{x}+\mathbf{a}) &\geq v(\mathbf{x}+b_j e_j+\mathbf{a}-b_i e_i) - v(\mathbf{x}+\mathbf{a}-b_i e_i) \\ &\geq v(\mathbf{x}+b_j e_j+\mathbf{a}) - v(\mathbf{x}+\mathbf{a}) \\ &\geq v(\mathbf{x}+b_j e_j) + c_i - v(\mathbf{x}) - c_i \\ &\geq T_i v(\mathbf{x}+b_j e_j) - T_i v(\mathbf{x}) \end{aligned}$$

(4) Suppose that $T_i v(\mathbf{x}+b_j e_j+\mathbf{a}) = v(\mathbf{x}+b_j e_j+\mathbf{a})+c_i$ and $T_i v(\mathbf{x}) = v(\mathbf{x})+c_i$. As we assume $v \in Cx(\mathbf{a}, b_j e_j)$, the following inequalities hold:

$$\begin{aligned} T_i v(\mathbf{x}+b_j e_j+\mathbf{a}) - T_i v(\mathbf{x}+\mathbf{a}) &\geq v(\mathbf{x}+b_j e_j+\mathbf{a})+c_i - v(\mathbf{x}+\mathbf{a}) - c_i \\ &\geq v(\mathbf{x}+b_j e_j) + c_i - v(\mathbf{x}) - c_i \\ &\geq T_i v(\mathbf{x}+b_j e_j) - T_i v(\mathbf{x}) \end{aligned}$$

Hence our inequality holds in all the possible scenarios. Therefore, $T_i v \in Cx(\mathbf{a}, b_j e_j)$, $\forall j$.

Case II: Suppose that $i = n+1$.

- First we show $T_{n+1}v \in Sub(\mathbf{b})$, i.e., $T_{n+1}v(\mathbf{x}+b_je_j) - T_{n+1}v(\mathbf{x}) \geq T_{n+1}v(\mathbf{x}+b_je_j+b_ke_k) - T_{n+1}v(\mathbf{x}+b_ke_k)$, $\forall k \neq j$. Pick arbitrary $k, j \in \{1,2,..,n\}$. There are four possible scenarios depending on the optimal actions at $T_{n+1}v(\mathbf{x}+b_je_j)$ and $T_{n+1}v(\mathbf{x}+b_ke_k)$:

  (1) Suppose that $T_{n+1}v(\mathbf{x}+b_je_j) = v(\mathbf{x}+b_je_j)+c_{n+1}$ and $T_{n+1}v(\mathbf{x}+b_ke_k) = v(\mathbf{x}+b_ke_k)+c_{n+1}$. As we assume $v \in Sub(\mathbf{b})$, the following inequalities hold:

$$\begin{aligned}
T_{n+1}v(\mathbf{x}+b_je_j) - T_{n+1}v(\mathbf{x}) &\geq v(\mathbf{x}+b_je_j)+c_{n+1} - v(\mathbf{x}) - c_{n+1} \\
&\geq v(\mathbf{x}+b_je_j+b_ke_k)+c_{n+1} - v(\mathbf{x}+b_ke_k) - c_{n+1} \\
&\geq T_{n+1}v(\mathbf{x}+b_je_j+b_ke_k) - T_{n+1}v(\mathbf{x}+b_ke_k)
\end{aligned}$$

  (2) Suppose that $\mathbf{x}+b_ke_k \geq \mathbf{a}$, $T_{n+1}v(\mathbf{x}+b_je_j) = v(\mathbf{x}+b_je_j)+c_{n+1}$ and $T_{n+1}v(\mathbf{x}+b_ke_k) = v(\mathbf{x}+b_ke_k-\mathbf{a})$. As we assume $v \in Sub(\mathbf{b})$ and $v \in Cx(\mathbf{a}, b_je_j)$, the following inequalities hold:

$$\begin{aligned}
T_{n+1}v(\mathbf{x}+b_je_j) - T_{n+1}v(\mathbf{x}) &\geq v(\mathbf{x}+b_je_j)+c_{n+1} - v(\mathbf{x}) - c_{n+1} \\
&\geq v(\mathbf{x}+b_je_j+b_ke_k) - v(\mathbf{x}+b_ke_k) \\
&\geq v(\mathbf{x}+b_je_j+b_ke_k-\mathbf{a}) - v(\mathbf{x}+b_ke_k-\mathbf{a}) \\
&\geq T_{n+1}v(\mathbf{x}+b_je_j+b_ke_k) - T_{n+1}v(\mathbf{x}+b_ke_k)
\end{aligned}$$

  (3) Suppose that $\mathbf{x}+b_je_j \geq \mathbf{a}$, $T_{n+1}v(\mathbf{x}+b_je_j) = v(\mathbf{x}+b_je_j-\mathbf{a})$ and $T_{n+1}v(\mathbf{x}+b_ke_k) = v(\mathbf{x}+b_ke_k)+c_{n+1}$. As we assume $v \in Cx(\mathbf{a}, b_ke_k)$ and $v \in Sub(\mathbf{b})$, the following inequalities hold:

$$\begin{aligned}
T_{n+1}v(\mathbf{x}+b_je_j) - T_{n+1}v(\mathbf{x}) &\geq v(\mathbf{x}+b_je_j-\mathbf{a}) - v(\mathbf{x}) - c_{n+1} \\
&\geq v(\mathbf{x}+b_je_j) - v(\mathbf{x}+b_je_j+b_ke_k) \\
&\quad + v(\mathbf{x}+b_je_j+b_ke_k-\mathbf{a}) - v(\mathbf{x}) - c_{n+1} \\
&\geq v(\mathbf{x}+b_je_j+b_ke_k-\mathbf{a}) - v(\mathbf{x}+b_ke_k) - c_{n+1} \\
&\geq T_{n+1}v(\mathbf{x}+b_je_j+b_ke_k) - T_{n+1}v(\mathbf{x}+b_ke_k)
\end{aligned}$$

  (4) Suppose that $\mathbf{x}+b_je_j \geq \mathbf{a}$, $\mathbf{x}+b_ke_k \geq \mathbf{a}$, $T_{n+1}v(\mathbf{x}+b_je_j) = v(\mathbf{x}+b_je_j-\mathbf{a})$ and $T_{n+1}v(\mathbf{x}+b_ke_k) = v(\mathbf{x}+b_ke_k-\mathbf{a})$. Notice that, for $j \neq k$, $\mathbf{x}+b_je_j \geq \mathbf{a}$ and $\mathbf{x}+b_ke_k \geq \mathbf{a}$ imply,

respectively, $x_t \geq a_t$ for all $t \neq j$ and $x_t \geq a_t$ for all $t \neq k$, and therefore $\mathbf{x} \geq \mathbf{a}$. As we assume $v \in Sub(\mathbf{b})$, the following inequalities hold:

$$\begin{aligned} T_{n+1}v(\mathbf{x}+b_je_j) - T_{n+1}v(\mathbf{x}) &\geq v(\mathbf{x}+b_je_j-\mathbf{a}) - v(\mathbf{x}-\mathbf{a}) \\ &\geq v(\mathbf{x}+b_je_j+b_ke_k-\mathbf{a}) - v(\mathbf{x}+b_ke_k-\mathbf{a}) \\ &\geq T_{n+1}v(\mathbf{x}+b_je_j+b_ke_k) - T_{n+1}v(\mathbf{x}+b_ke_k) \end{aligned}$$

Hence our inequality holds in all the possible scenarios. Therefore, $T_{n+1}v \in Sub(\mathbf{b})$.

- Next we show $T_{n+1}v \in Cx(\mathbf{a}, b_je_j)$, $\forall j \in \{1,2,..,n\}$, i.e., $T_{n+1}v(\mathbf{x}+b_je_j+\mathbf{a}) - T_{n+1}v(\mathbf{x}+\mathbf{a}) \geq T_{n+1}v(\mathbf{x}+b_je_j) - T_{n+1}v(\mathbf{x})$, $\forall j$. Again, there are four different scenarios depending on the optimal actions at $T_{n+1}v(\mathbf{x}+b_je_j+\mathbf{a})$ and $T_{n+1}v(\mathbf{x})$:

(1) Suppose that $\mathbf{x} \geq \mathbf{a}$, $T_{n+1}v(\mathbf{x}+b_je_j+\mathbf{a}) = v(\mathbf{x}+b_je_j)$ and $T_{n+1}v(\mathbf{x}) = v(\mathbf{x}-\mathbf{a})$. As we assume $v \in Cx(\mathbf{a}, b_je_j)$, the following inequalities hold:

$$\begin{aligned} T_{n+1}v(\mathbf{x}+b_je_j+\mathbf{a}) - T_{n+1}v(\mathbf{x}+\mathbf{a}) &\geq v(\mathbf{x}+b_je_j) - v(\mathbf{x}) \\ &\geq v(\mathbf{x}+b_je_j-\mathbf{a}) - v(\mathbf{x}-\mathbf{a}) \\ &\geq T_{n+1}v(\mathbf{x}+b_je_j) - T_{n+1}v(\mathbf{x}) \end{aligned}$$

(2) Suppose that $\mathbf{x} \geq \mathbf{a}$, $T_{n+1}v(\mathbf{x}+b_je_j+\mathbf{a}) = v(\mathbf{x}+b_je_j+\mathbf{a}) + c_{n+1}$ and $T_{n+1}v(\mathbf{x}) = v(\mathbf{x}-\mathbf{a})$. As we assume $v \in Cx(\mathbf{a}, b_je_j)$, the following inequalities hold:

$$\begin{aligned} T_{n+1}v(\mathbf{x}+b_je_j+\mathbf{a}) - T_{n+1}v(\mathbf{x}+\mathbf{a}) &\geq v(\mathbf{x}+b_je_j+\mathbf{a}) + c_{n+1} - v(\mathbf{x}+\mathbf{a}) - c_{n+1} \\ &\geq v(\mathbf{x}+b_je_j) - v(\mathbf{x}) \\ &\geq v(\mathbf{x}+b_je_j-\mathbf{a}) - v(\mathbf{x}-\mathbf{a}) \\ &\geq T_{n+1}v(\mathbf{x}+b_je_j) - T_{n+1}v(\mathbf{x}) \end{aligned}$$

(3) Suppose that $T_{n+1}v(\mathbf{x}+b_je_j+\mathbf{a}) = v(\mathbf{x}+b_je_j)$ and $T_{n+1}v(\mathbf{x}) = v(\mathbf{x}) + c_{n+1}$. Then it is easy to verify that

$$\begin{aligned} T_{n+1}v(\mathbf{x}+b_je_j+\mathbf{a}) - T_{n+1}v(\mathbf{x}+\mathbf{a}) &\geq v(\mathbf{x}+b_je_j) - v(\mathbf{x}) \\ &= v(\mathbf{x}+b_je_j) + c_{n+1} - v(\mathbf{x}) - c_{n+1} \\ &\geq T_{n+1}v(\mathbf{x}+b_je_j) - T_{n+1}v(\mathbf{x}) \end{aligned}$$

(4) Suppose that $T_{n+1}v(\mathbf{x}+b_je_j+\mathbf{a}) = v(\mathbf{x}+b_je_j+\mathbf{a})+c_{n+1}$ and $T_{n+1}v(\mathbf{x}) = v(\mathbf{x})+c_{n+1}$. As we assume $v \in Cx(\mathbf{a}, b_je_j)$, the following inequalities hold:

$$\begin{aligned}T_{n+1}v(\mathbf{x}+b_je_j+\mathbf{a}) - T_{n+1}v(\mathbf{x}+\mathbf{a}) &\geq v(\mathbf{x}+b_je_j+\mathbf{a})+c_{n+1}-v(\mathbf{x}+\mathbf{a})-c_{n+1}\\ &\geq v(\mathbf{x}+b_je_j)+c_{n+1}-v(\mathbf{x})-c_{n+1}\\ &\geq T_{n+1}v(\mathbf{x}+b_je_j)-T_{n+1}v(\mathbf{x})\end{aligned}$$

Hence our inequality holds in all the possible scenarios. Therefore, $T_{n+1}v \in Cx(\mathbf{a}, b_je_j)$, $\forall j$.

(c) We below show $h \in Sub(\mathbf{p}) \cap Cx(\mathbf{r}, p_1e_1) \cap \cdots \cap Cx(\mathbf{r}, p_ne_n)$, for any $\mathbf{r}$ and $\mathbf{p}$.

- First we prove $h \in Sub(\mathbf{p})$ (i.e., $h(\mathbf{x}+p_je_j) - h(\mathbf{x}) \geq h(\mathbf{x}+p_je_j+p_ke_k) - h(\mathbf{x}+p_ke_k)$, $\forall k \neq j$): $h(\mathbf{x}+p_je_j) - h(\mathbf{x}) = \sum_{i\neq j} h_i(x_i) + h_j(x_j+p_j) - \sum_{i\neq j} h_i(x_i) - h_j(x_j) = h_j(x_j+p_j) - h_j(x_j) = \sum_{i\notin\{j,k\}} h_i(x_i) + h_j(x_j+p_j) + h_k(x_k+p_k) - \sum_{i\notin\{j,k\}} h_i(x_i) - h_j(x_j) - h_k(x_k+p_k) = h(\mathbf{x}+p_je_j+p_ke_k) - h(\mathbf{x}+p_ke_k)$, $\forall k \neq j$.
- Second we prove $h \in Cx(\mathbf{r}, p_je_j)$ (i.e., $h(\mathbf{x}+p_je_j+\mathbf{r}) - h(\mathbf{x}+\mathbf{r}) \geq h(\mathbf{x}+p_je_j) - h(\mathbf{x})$, $\forall j$): $h(\mathbf{x}+p_je_j+\mathbf{r}) - h(\mathbf{x}+\mathbf{r}) = \sum_{i\neq j} h_i(x_i+r_i) + h_j(x_j+p_j+r_j) - \sum_{i\neq j} h_i(x_i+r_i) - h_j(x_j+r_j) = h_j(x_j+p_j+r_j) - h_j(x_j+r_j) \geq h_j(x_j+p_j) - h_j(x_j) = \sum_{i\neq j} h_i(x_i) + h_j(x_j+p_j) - \sum_{i\neq j} h_i(x_i) - h_j(x_j) = h(\mathbf{x}+p_je_j) - h(\mathbf{x})$, $\forall j$. The inequality above follows from the assumption that h_j is a convex function, $\forall j$.

Since $h \in Sub(\mathbf{p}) \cap Cx(\mathbf{r}, p_1e_1) \cap \cdots \cap Cx(\mathbf{r}, p_ne_n)$, for any $\mathbf{r}$ and $\mathbf{p}$, we have $h \in Sub(\mathbf{q}) \cap Cx(\mathbf{a}, q_1e_1) \cap \cdots \cap Cx(\mathbf{a}, q_ne_n) \cap Sub(\mathbf{b}) \cap Cx(\mathbf{a}, b_1e_1) \cap \cdots \cap Cx(\mathbf{a}, b_ne_n)$.

□

We are now ready to prove Lemma 1.

Proof of Lemma 1. Define V^* as the set of functions satisfying the properties of $Sub(\mathbf{b})$, $Cx(\mathbf{a}, b_je_j)$, $\forall j$, and $Cx(\mathbf{a}, \mathbf{b})$. Also, define the operator T on the set of real-valued functions v: $Tv(\mathbf{x}) = h(\mathbf{x}) + \sum_j \mu_j T^{(j)}v(\mathbf{x}) + \sum_i \lambda_i T_i v(\mathbf{x})$. First we show $T: V^* \to V^*$. By Lemma EC.1, $Cx(\mathbf{r}, p_1e_1) \cap \cdots \cap Cx(\mathbf{r}, p_ne_n) \subseteq Cx(\mathbf{r}, \mathbf{p})$, and therefore $Sub(\mathbf{p}) \cap Cx(\mathbf{r}, p_1e_1) \cap \cdots \cap Cx(\mathbf{r}, p_ne_n) \subseteq Cx(\mathbf{r}, \mathbf{p})$. This, combined with Lemma EC.2, yields $T^{(j)}: Sub(\mathbf{q}) \cap Cx(\mathbf{a}, q_1e_1) \cap \cdots \cap Cx(\mathbf{a}, q_ne_n) \cap Cx(\mathbf{a}, \mathbf{q}) \to Sub(\mathbf{q}) \cap Cx(\mathbf{a}, q_1e_1) \cap \cdots \cap Cx(\mathbf{a}, q_ne_n) \cap Cx(\mathbf{a}, \mathbf{q})$, and $T_i: Sub(\mathbf{b}) \cap Cx(\mathbf{a}, b_1e_1) \cap \cdots \cap Cx(\mathbf{a}, b_ne_n) \cap Cx(\mathbf{a}, \mathbf{b}) \to Sub(\mathbf{b}) \cap Cx(\mathbf{a}, b_1e_1) \cap \cdots \cap Cx(\mathbf{a}, b_ne_n) \cap Cx(\mathbf{a}, \mathbf{b})$. By Assumption 1, $\mathbf{q} = \mathbf{b}$; and therefore $T^{(j)}, T_i: Sub(\mathbf{b}) \cap Cx(\mathbf{a}, b_1e_1) \cap \cdots \cap Cx(\mathbf{a}, b_ne_n) \cap Cx(\mathbf{a}, \mathbf{b}) \to Sub(\mathbf{b}) \cap Cx(\mathbf{a}, b_1e_1) \cap \cdots \cap Cx(\mathbf{a}, b_ne_n) \cap Cx(\mathbf{a}, \mathbf{b})$. That is, $T^{(j)}: V^* \to V^*$ and $T_i: V^* \to V^*$. By Lemmas EC.1 and EC.2, we also know $h \in V^*$. Now let $v \in V^*$. Since $T^{(j)}v \in V^*$, $T_iv \in V^*$, and $h \in V^*$, and our second-order properties are preserved by linear transformations, $Tv \in V^*$. Hence, $T: V^* \to V^*$. Following Propositions 3.1.5 and 3.1.6 in Bertsekas (2007), we verify that $\lim_{k\to\infty}(T^k v_0)(\mathbf{x}) = v^*(\mathbf{x})$ where v_0 is the zero function, v^* is the optimal cost function, and T^k refers to k compositions of operator T. Since $v_0 \in V^*$ and $T: V^* \to V^*$, we have $T^k v_0 \in V^*$, and therefore $v^* \in V^*$.

□

Proof of Theorem 1. By Lemma 1, we know $v^* \in V^*$. Define, for $v^* \in V^*$,

$$
\begin{aligned}
S_j^*(\mathbf{p}) &= \min\{\mathbf{p} + z\mathbf{a} : v^*(\mathbf{p} + z\mathbf{a} + q_j e_j) - v^*(\mathbf{p} + z\mathbf{a}) > 0,\ z \in \mathbb{N}_0\},\ \forall j,\\
R_i^*(\mathbf{p}) &= \min\{\mathbf{p} + z\mathbf{a} : v^*(\mathbf{p} + z\mathbf{a}) - v^*(\mathbf{p} + z\mathbf{a} - b_i e_i) > -c_i,\ z \in \mathbb{N}_0,\ \text{and}\ p_i + z a_i \geq b_i\},\ \forall i \leq n,\\
R_{n+1}^*(\mathbf{p}) &= \min\{\mathbf{p} + z\mathbf{b} : v^*(\mathbf{p} + z\mathbf{b}) - v^*(\mathbf{p} + z\mathbf{b} - \mathbf{a}) > -c_{n+1},\ z \in \mathbb{N}_0,\ \text{and}\ \mathbf{p} + z\mathbf{b} \geq \mathbf{a}\}.
\end{aligned}
$$

(1) Since $v^* \in Cx(\mathbf{a}, b_j e_j)$ and $\mathbf{q} = \mathbf{b}$, $v^*(\mathbf{p} + z\mathbf{a} + q_j e_j) - v^*(\mathbf{p} + z\mathbf{a})$ is increasing in z. As z increases, since the holding cost rate h is strictly increasing, this difference will eventually cross 0. Therefore, the lattice-dependent base-stock policy is optimal.

(2) Since $v^* \in Cx(\mathbf{a}, b_i e_i)$, $\forall i \leq n$, $v^*(\mathbf{p} + z\mathbf{a}) - v^*(\mathbf{p} + z\mathbf{a} - b_i e_i)$ is increasing in z. We know that, as z increases, this difference will eventually cross 0. Therefore, as z increases, this difference should also cross $-c_i$. Hence, the lattice-dependent rationing policy is optimal.

(3) Since $v^* \in Cx(\mathbf{a}, \mathbf{b})$, $v^*(\mathbf{p} + z\mathbf{b}) - v^*(\mathbf{p} + z\mathbf{b} - \mathbf{a})$ is increasing in z. As z increases, since the holding cost rate h is strictly increasing, this difference will eventually cross $-c_{n+1}$. Therefore, the lattice-dependent rationing policy is optimal.

Next we will prove properties (i)-(iii):

i. To prove property (i), first, we show that the optimal base-stock levels for each component j obey $S_j^*(\mathbf{p} + b_k e_k) \geq S_j^*(\mathbf{p}) + b_k e_k$, $\forall k \neq j$. Let $S_j^*(\mathbf{p}) = \mathbf{p} + z_1\mathbf{a}$ and $S_j^*(\mathbf{p} + b_k e_k) = \mathbf{p} + b_k e_k + z_2\mathbf{a}$. Then, it is not optimal to produce a batch of component j at $\mathbf{x} = \mathbf{p} + z_1\mathbf{a}$ and $\mathbf{x} = \mathbf{p} + b_k e_k + z_2\mathbf{a}$. Since $v^* \in Sub(\mathbf{b})$, it is not optimal to produce a batch of component j at $\mathbf{x} = \mathbf{p} + z_2\mathbf{a}$, implying $z_2 \geq z_1$. Therefore, we must have $S_j^*(\mathbf{p} + b_k e_k) \geq S_j^*(\mathbf{p}) + b_k e_k$.

Second, we show that the optimal rationing levels for each product $i \leq n$ obey $R_i^*(\mathbf{p} + b_k e_k) \geq R_i^*(\mathbf{p}) + b_k e_k$, $\forall k \neq i$. Let $R_i^*(\mathbf{p}) = \mathbf{p} + z_1\mathbf{a}$ and $R_i^*(\mathbf{p} + b_k e_k) = \mathbf{p} + b_k e_k + z_2\mathbf{a}$. Then, it is optimal to fulfill a demand for product i at $\mathbf{x} = \mathbf{p} + z_1\mathbf{a}$ and $\mathbf{x} = \mathbf{p} + b_k e_k + z_2\mathbf{a}$. Since $v^* \in Sub(\mathbf{b})$, it is also optimal to fulfill a demand for product i at $\mathbf{x} = \mathbf{p} + z_2\mathbf{a}$, implying $z_2 \geq z_1$. Therefore, we must have $R_i^*(\mathbf{p} + b_k e_k) \geq R_i^*(\mathbf{p}) + b_k e_k$.

ii. To prove (ii), we will show that the optimal rationing levels for product $n+1$ obey $R^*_{n+1}(\mathbf{p}+b_k e_k) \le R^*_{n+1}(\mathbf{p}) + b_k e_k$, $\forall k$. Let $R^*_{n+1}(\mathbf{p}) = \mathbf{p} + z_1 \mathbf{b}$ and $R^*_{n+1}(\mathbf{p}+b_k e_k) = \mathbf{p} + b_k e_k + z_2 \mathbf{b}$. Then, it is optimal to fulfill a demand for product $n+1$ at $\mathbf{x} = \mathbf{p} + z_1 \mathbf{b}$ and $\mathbf{x} = \mathbf{p} + b_k e_k + z_2 \mathbf{b}$. Since $v^* \in Cx(\mathbf{a}, b_k e_k)$, it is also optimal to fulfill a demand for product $n+1$ at $\mathbf{x} = \mathbf{p} + z_1 \mathbf{b} + b_k e_k$, implying $z_1 \ge z_2$. Therefore, we must have $R^*_{n+1}(\mathbf{p}) + b_k e_k \ge R^*_{n+1}(\mathbf{p}+b_k e_k)$.

iii. Lastly, we will prove that it is optimal to fulfill a demand of product $n+1$ if $x_j \ge a_j + b_j \left\lfloor \frac{x_j}{b_j} \right\rfloor$, $\forall j$. Define $\widetilde{V}$ as the set of real-valued functions f defined on $\mathbb{N}_0^n$ such that $f(\mathbf{x}) - f(\mathbf{x}-\mathbf{a}) + c_{n+1} \ge 0$, for $x_j \ge a_j + b_j \left\lfloor \frac{x_j}{b_j} \right\rfloor$, $\forall j$. Recall $Tv(\mathbf{x}) = h(\mathbf{x}) + \sum_j \mu_j T^{(j)} v(\mathbf{x}) + \sum_i \lambda_i T_i v(\mathbf{x})$. We show below $T: \widetilde{V} \to \widetilde{V}$.

Assume that $v \in \widetilde{V}$. We want to prove $Tv \in \widetilde{V}$. Since h is an increasing convex function and $\sum_j \mu_j + \sum_i \lambda_i \le 1$, the following inequality holds:

$$
\begin{aligned}
& Tv(\mathbf{x}) - Tv(\mathbf{x}-\mathbf{a}) + c_{n+1} \\
& = h(\mathbf{x}) - h(\mathbf{x}-\mathbf{a}) + \sum_j \mu_j (T^{(j)} v(\mathbf{x}) - T^{(j)} v(\mathbf{x}-\mathbf{a})) + \sum_i \lambda_i (T_i v(\mathbf{x}) - T_i v(\mathbf{x}-\mathbf{a})) + c_{n+1} \\
& \ge \sum_j \mu_j (T^{(j)} v(\mathbf{x}) - T^{(j)} v(\mathbf{x}-\mathbf{a}) + c_{n+1}) + \sum_i \lambda_i (T_i v(\mathbf{x}) - T_i v(\mathbf{x}-\mathbf{a}) + c_{n+1})
\end{aligned}
$$

To prove $Tv \in \widetilde{V}$, it suffices to show $T^{(j)} v(\mathbf{x}) - T^{(j)} v(\mathbf{x}-\mathbf{a}) + c_{n+1} \ge 0$, $\forall j$, and $T_i v(\mathbf{x}) - T_i v(\mathbf{x}-\mathbf{a}) + c_{n+1} \ge 0$, $\forall i$, where $x_k \ge a_k + b_k \left\lfloor \frac{x_k}{b_k} \right\rfloor$, $\forall k$. We prove these inequalities as follows:

- First we show $T^{(j)} v(\mathbf{x}) - T^{(j)} v(\mathbf{x}-\mathbf{a}) + c_{n+1} \ge 0$. There are two possible scenarios depending on the optimal action at $T^{(j)} v(\mathbf{x})$:
  (1) Suppose that $T^{(j)} v(\mathbf{x}) = v(\mathbf{x} + q_j e_j) < v(\mathbf{x})$: $T^{(j)} v(\mathbf{x}) - T^{(j)} v(\mathbf{x}-\mathbf{a}) + c_{n+1} \ge v(\mathbf{x} + q_j e_j) - v(\mathbf{x} + q_j e_j - \mathbf{a}) + c_{n+1} \ge 0$. The second inequality follows from the fact that $v \in \widetilde{V}$ and $x_j + q_j \ge a_j + b_j \left\lfloor \frac{x_j}{b_j} \right\rfloor + q_j = a_j + b_j \left\lfloor \frac{x_j + q_j}{b_j} \right\rfloor$. (By Assumption 1, $q_j = b_j$.)
  (2) Suppose that $T^{(j)} v(\mathbf{x}) = v(\mathbf{x}) \le v(\mathbf{x} + q_j e_j)$: $T^{(j)} v(\mathbf{x}) - T^{(j)} v(\mathbf{x}-\mathbf{a}) + c_{n+1} \ge v(\mathbf{x}) - v(\mathbf{x}-\mathbf{a}) + c_{n+1} \ge 0$. The second inequality follows from the assumption of $v \in \widetilde{V}$.
- Second we show $T_i v(\mathbf{x}) - T_i v(\mathbf{x}-\mathbf{a}) + c_{n+1} \ge 0$, for $i \le n$. There are two possible scenarios depending on the optimal action at $T_i v(\mathbf{x})$:
  (1) Suppose that $T_i v(\mathbf{x}) = v(\mathbf{x}) + c_i$: $T_i v(\mathbf{x}) - T_i v(\mathbf{x}-\mathbf{a}) + c_{n+1} \ge v(\mathbf{x}) + c_i - v(\mathbf{x}-\mathbf{a}) - c_i + c_{n+1} \ge 0$. The second inequality follows from the assumption of $v \in \widetilde{V}$.
  (2) Suppose that $x_i \ge b_i$ and $T_i v(\mathbf{x}) = v(\mathbf{x} - b_i e_i)$: $T_i v(\mathbf{x}) - T_i v(\mathbf{x}-\mathbf{a}) + c_{n+1} \ge v(\mathbf{x} - b_i e_i) - v(\mathbf{x} - \mathbf{a} - b_i e_i) + c_{n+1} \ge 0$. The second inequality follows from the fact that $v \in \widetilde{V}$ and $x_i - b_i \ge a_i + b_i \left\lfloor \frac{x_i}{b_i} \right\rfloor - b_i = a_i + b_i \left\lfloor \frac{x_i - b_i}{b_i} \right\rfloor$. Here notice that, as we assume $x_i \ge a_i + b_i \left\lfloor \frac{x_i}{b_i} \right\rfloor$ and $x_i \ge b_i$, we should have $x_i \ge a_i + b_i$, implying $\mathbf{x} \ge \mathbf{a} + b_i e_i$.

- Lastly we show $T_{n+1}v(\mathbf{x}) - T_{n+1}v(\mathbf{x}-\mathbf{a}) + c_{n+1} \geq 0$. There are two possible scenarios depending on the optimal action at $T_{n+1}v(\mathbf{x})$:
  (1) Suppose that $T_{n+1}v(\mathbf{x}) = v(\mathbf{x}) + c_{n+1} < v(\mathbf{x}-\mathbf{a})$: $T_{n+1}v(\mathbf{x}) - T_{n+1}v(\mathbf{x}-\mathbf{a}) + c_{n+1} \geq v(\mathbf{x}) + c_{n+1} - v(\mathbf{x}-\mathbf{a}) - c_{n+1} + c_{n+1} \geq 0$. The second inequality follows from the assumption of $v \in \widetilde{V}$.
  (2) Suppose that $T_{n+1}v(\mathbf{x}) = v(\mathbf{x}-\mathbf{a}) \leq v(\mathbf{x}) + c_{n+1}$: $T_{n+1}v(\mathbf{x}) - T_{n+1}v(\mathbf{x}-\mathbf{a}) + c_{n+1} \geq v(\mathbf{x}-\mathbf{a}) - v(\mathbf{x}-\mathbf{a}) - c_{n+1} + c_{n+1} = 0$.

Since $\sum_j \mu_j (T^{(j)}v(\mathbf{x}) - T^{(j)}v(\mathbf{x}-\mathbf{a}) + c_{n+1}) + \sum_i \lambda_i (T_i v(\mathbf{x}) - T_i v(\mathbf{x}-\mathbf{a}) + c_{n+1}) \geq 0$, we have $Tv(\mathbf{x}) - Tv(\mathbf{x}-\mathbf{a}_j) + c_j \geq 0$. Hence, $T : \widetilde{V} \to \widetilde{V}$. Following Propositions 3.1.5 and 3.1.6 in Bertsekas (2007), we verify that $\lim_{k\to\infty}(T^k v_0)(\mathbf{x}) = v^*(\mathbf{x})$ where v_0 is the zero function, v^* is the optimal cost function, and T^k refers to k compositions of operator T. Since $v_0 \in \widetilde{V}$ and $T : \widetilde{V} \to \widetilde{V}$, we have $T^k v_0 \in \widetilde{V}$, and therefore $v^* \in \widetilde{V}$. Since $v^*(\mathbf{x}) - v^*(\mathbf{x}-\mathbf{a}) + c_{n+1} \geq 0$, for $x_j \geq a_j + b_j \left\lfloor \frac{x_j}{b_j} \right\rfloor, \forall j$, it is optimal to fulfill a demand of product $n+1$ if $x_j \geq a_j + b_j \left\lfloor \frac{x_j}{b_j} \right\rfloor, \forall j$. □

Proof of Proposition 1. We first prove the following conditions: (i) There exists a stationary policy π that induces an irreducible positive recurrent Markov chain with finite average cost v^π, and (ii) the number of states for which $h(\mathbf{x}) \leq v^\pi$ is finite. To prove condition (i), consider a policy where the production of each component is controlled by a base-stock policy with an independent and fixed critical level, and inventory allocation follows a first-come-first-served policy. Notice that we have a finite-state Markov chain under this policy. Hence, this policy yields a finite average cost. It is easy to prove condition (ii) as the inventory holding cost rate for each component is increasing convex in its inventory level. Thus, for any positive value γ, the number of states for which $h(\mathbf{x}) \leq \gamma$ is always finite. Under conditions (i) and (ii), there exists a constant v^* and a function $f(\mathbf{x})$ such that $f(\mathbf{x}) + v^* = \inf\{h(\mathbf{x}) + \sum_j \mu_j T^{(j)} f(\mathbf{x}) + \sum_i \lambda_i T_i f(\mathbf{x})\}$ (Weber and Stidham 1987). The stationary policy that minimizes the righthand side of the above equation for each state $\mathbf{x}$ is an optimal policy for the average cost criterion and yields a constant average cost v^*. Hence, properties of the optimal policy for the average cost are determined through the function $f(\mathbf{x})$. Recall that properties of the optimal policy for the discounted costs are determined through $v^*(\mathbf{x})$. Since the same event operators are applied to $f(\mathbf{x})$, the optimal policy for the average cost retains the same structure as in the discounted cost case.

□

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