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A noncooperative approach to bankruptcy problems with an endogenous estate

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Abstract We introduce a new class of bankruptcy problems in which the value of the estate is endogenous and depends on agents' investment decisions. There are two investment alternatives: investing in a company (risky asset) and depositing money into a savings account (risk-free asset). Bankruptcy is possible only for the risky asset. We define a game between agents each of which aims to maximize his expected payoff by choosing an investment alternative and a company management which aims to maximize profits by choosing a bankruptcy rule. Our agents are differentiated by their incomes. We consider three most prominent bankruptcy rules in our base model: the proportional rule, the constrained equal awards rule and the constrained equal losses rule. We show that only the proportional rule is a part of any pure strategy subgame perfect Nash equilibrium. This result is robust to changes in income distribution in the economy and can be extended to a larger set of bankruptcy rules and multiple types. However, extension to multiple company framework with competition leads to equilibria where the noncooperative support for the proportional rule disappears.

Keywords Bankruptcy problems · Constrained equal awards rule · Constrained equal losses rule · Noncooperative games · Proportional rule

1 Introduction

As early as 1985, Young argued that incentives of agents should be incorporated into cost-sharing models. He suggested that the absence of *incentives* in standard cost-sharing models makes the analysis of these problems ad hoc. Later, Thomson (2003), in his seminal survey article, addressed the need to combine noncooperative and market-based approaches and

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to investigate the incentive effects of bankruptcy rules on agents' behavior in bankruptcy problems.

Along these lines, we introduce a new class of bankruptcy problems in which the value of the estate is *endogenous* and depends on agents' investment decisions that also determine their claims. Our theoretical framework incorporates important economic factors such as income distribution, stochastic returns on risky investments, and the return on a risk-free outside option. Our model is motivated by the following facts: (i) In investment situations involving bankruptcy, agents may act strategically, (ii) bankruptcy may occur following an investment with stochastic outcomes, (iii) the bankruptcy rule choice and investment decisions may influence each other through the incentive channel, (iv) the claim distribution may have an impact on agents' decisions if there are (positive or negative) *externalities*, and (v) many real-life bankruptcy situations involve payments (to shareholders, lenders, partners etc.), which are not respected by the borrower.

The *bankruptcy problem* was first introduced formally by O'Neill (1982). It describes a situation in which there is a perfectly divisible estate to be allocated to a finite number of agents, whose claims add up to an amount larger than the estate. Many real life situations such as distributing a will to inheritants, liquidating the assets of a bankrupt company, rationing, taxation, and sharing the costs of a public facility can be described using bankruptcy models. The most prominent bankruptcy rules in the literature are the proportional rule (P), the constrained equal awards rule (CEA), the constrained equal losses rule (CEL), and the Talmud rule (T), which will all be used in our study.[1]

We know that *incentives* and *strategic behavior* play a significant role in real-life bankruptcy problems. Hence, the noncooperative game theoretical approach is a natural and potentially fruitful one for studying bankruptcy problems. This approach models the bankruptcy problem as a noncooperative game among the claimants and studies equilibria of the game.[2] The major motivation of studies using this approach is that when the authority does not have a priori preference concerning the rule that will be implemented, he may resort to implementing a noncooperative game form in which the result of agents' strategic interactions determine the rule to be used.

In our model, there are two investment alternatives: investing in a company (risky asset) or depositing money into a savings account (risk-free asset). Bankruptcy is a possible event only for the risky asset. We define a strategic game between agents each of which aims to maximize his expected payoff by choosing an investment alternative and a company management which aims to maximize profits by choosing a bankruptcy rule. This setup is also in line with some recent suggestions in favor of a more liberal bankruptcy law, which would provide a menu of rules and allow companies to choose one among them (see Hart 2000). A social planner who aims to maximize investment in the country by choosing a bankruptcy regime would also fit into our story. There are two types of agents in our base model, who are differentiated by their incomes. We consider three prominent bankruptcy rules in our base model: the proportional rule, the constrained equal awards rule and the constrained equal losses rule.[3] In the game, the company chooses the bankruptcy rule and later all agents simultaneously choose whether

[1] For an extensive survey of the axiomatic literature on bankruptcy problems, see Moulin (2002) and Thomson (2003).

[2] There are only a few papers using this approach: O'Neill (1982), Chun (1989), Dagan et al. (1997), Moreno-Ternero (2002), Herrero (2003), García-Jurado et al. (2006), Chang and Hu (2008), Atlamaz et al. (2011), Ashlagi et al. (2012), and Kıbrıs and Kıbrıs (2013).

[3] The constrained equal awards rule represents a family of rules favoring smaller claimants, the constrained equal losses favoring larger claimants and the proportional rule lies in between the two. The constrained equal awards (losses) rule is *regressive* (*progressive*) and the proportional rule is both *regressive* and *progressive*.

to invest in the risky asset (i.e., the project initiated by the company) or the risk-free asset (savings account in a bank).

Results from our base model provide a strong noncooperative support for the proportional rule.[4] In particular, we show that the proportional rule is a part of any subgame perfect Nash equilibrium. This statement is valid neither for the constrained equal awards rule nor for the constrained equal losses rule. The direct implication of this result is that the proportional rule never leads to an investment volume lower than the one under the constrained equal awards (or losses) rule; and in some cases it leads to an investment in the company strictly higher than the one under the two rules. Moreover, the result supporting the proportional rule is independent of the income distribution and holds even under one-sided uncertainty on the income distribution. We also extend our base model in three dimensions: (i) larger set of rules, (ii) multiple types of agents, and (iii) two companies competing for funds.

Contributions of our paper can be listed as: (a) endogenizing the choice of a bankruptcy rule with a noncooperative procedure, (b) endogenizing the value of claims and the estate, (c) incorporating the well-known bankruptcy model into a context that involves decision-making under uncertainty and mimics a market environment, and (d) providing a noncooperative framework in which the bankruptcy rule choice depends on both borrowers' and lenders incentives. Firstly, endogenously determined bankruptcy rules, claims, and estates are new in the literature. In most of the papers on bankruptcy, the analysis is based on exogenously fixed bankruptcy rules, claims, and estates. In real life, obviously agents' decisions and hence their claims and the estate depend on the bankruptcy rule; and the choice of the bankruptcy rule depends, in turn, on agents' actions. Secondly, many real life instances that involve bankruptcy also involve investment decisions under uncertainty. In our paper, we model the process before the realization of bankruptcy. Incorporation of multiple factors relevant for behavior in situations that may involve bankruptcy makes our approach different than the earlier studies with a noncooperative approach, which used somewhat abstract game forms. Finally, in all bankruptcy papers with noncooperative approaches, the strategic interaction takes place among claimants, whereas the bankruptcy rule decision is influenced by both lenders' and borrower's interests in our paper. It is determined as a result of a sequential game played among the lenders and borrowers.

Kıbrıs and Kıbrıs (2013) is the paper most closely related to ours. These authors also analyzed the investment implications of different bankruptcy rules. Their results are different from ours mostly due to the following differences between our modeling assumptions: (i) we consider standard, constrained versions of equal awards and equal losses rule whereas they considered unconstrained versions and (ii) the agents in our model are risk-neutral whereas Kıbrıs and Kıbrıs (2013) assumed risk aversion.[5] As a result, we find that the proportional rule maximizes the total investment, whereas they found that the equal loss rule maximizes total investment. Finally, some other differences are: (i) the lender (i.e., company) is a strategic player choosing a bankruptcy rule in our model whereas Kıbrıs and Kıbrıs (2013) compared equilibrium investment levels under different rules, (ii) we consider some extensions (e.g., larger family of rules and competition) whereas they did not, and (iii) they conducted a welfare

[4] See Chun (1988), Bergantiños and Sanchez (2002), Gächter and Riedl (2005), Herrero et al. (2010), Ju et al. (2007), Moreno-Ternero (2002, 2006, 2009), and Cetemen et al. (2014) for results supporting the proportional rule.

[5] Another difference is that in our model agents deposit either zero or all of their endowment whereas in Kıbrıs and Kıbrıs (2013) intermediate decisions are also allowed. However, this is not a major reason for differences in results since if we allow agents to optimally choose their deposit amounts, we would have corner solutions due to our risk-neutrality assumption.

analysis whereas we do not. We think that these differences in modeling assumptions and corresponding differences in results make these two papers good complements for readers.

The organization of the paper is as follows: We first introduce the standard bankruptcy problem and the bankruptcy rules employed in the base model and then provide some preparatory results in Sect. 2.1. In Sect. 2.2, we introduce a strategic model of bankruptcy under uncertainty and the bankruptcy problem with an endogenous estate. In Sect. 3, we analyze the equilibria of the bankruptcy game introduced in Sect. 2.1. In Sect. 4, we present three extensions of our model. Finally, Sect. 5 concludes.

2 A model of bankruptcy with an endogenous estate

2.1 Bankruptcy problems and rules

Bankruptcy is defined as a situation in which the sum of individual claims exceeds the size of the available estate. Formally, a bankruptcy problem is represented by a set of claimants $N = \{1, 2, \dots, n\}$, a claims vector $C = (c_1, c_2, \dots, c_n)$ where for all $i \in N$, $c_i \in \mathbb{R}_{++}$, an estate $E \in \mathbb{R}_{++}$ to be divided among the claimants, and the inequality $\sum_{i\in N} c_i > E$. We denote the set of all such bankruptcy problems (C, E) by $\mathcal{B}$.

A bankruptcy rule is a mechanism that allocates the estate to claimants given any bankruptcy problem. Formally, a bankruptcy rule, F, is a function mapping each bankruptcy problem $(C, E) \in \mathcal{B}$ into $\mathbb{R}^n_+$ such that for all $i \in N$, $F_i(C, E) \in [0, c_i]$ and $\sum_{i\in N} F_i(C, E) = E$. Below, we define the bankruptcy rules used in our base model.

The proportional rule allocates the estate proportionally with respect to claims.

Definition 1 (*Proportional Rule*) For all $(C, E) \in \mathcal{B}$, we have $P(C, E) \equiv \lambda_p C$, where λ_p is given by $\lambda_p = (E/\sum_{i\in N} c_i)$.

The constrained equal awards rule allocates the estate as equal as possible, taking claims as upper bounds.

Definition 2 (*Constrained Equal Awards Rule*) For all $(C, E) \in \mathcal{B}$, and all $j \in N$, we have $CEA_j(C, E) \equiv \min\{c_j, \lambda_{cea}\}$, where λ_{cea} solves $\sum_{i\in N} \min\{c_i, \lambda_{cea}\} = E$.

The constrained equal losses rule allocates the shortage of the estate as equal as possible, taking zero as the lower bound.

Definition 3 (*Constrained Equal Losses Rule*) For all $(C, E) \in \mathcal{B}$, and all $j \in N$, we have $CEL_j(C, E) \equiv \max\{0, c_j - \lambda_{cel}\}$, where λ_{cel} solves $\sum_{i\in N} \max\{0, c_i - \lambda_{cel}\} = E$.

Loosely speaking, *CEA* favors small claimants (i.e., it makes *transfers* from bigger claimants to smaller claimants) and *CEL* favors big claimants (i.e., it makes *transfers* from smaller claimants to bigger claimants). The following lemma formalizes the idea of *inter-claimant transfers* under *CEA* and *CEL*, taking P payoffs as a benchmark. Online Appendix A provides closed forms for transfers and some comparative statics.

Lemma 1 *Let $(C, E) \in \mathcal{B}$. Assume without loss of generality that $c_1 \le c_2 \le \cdots \le c_n$. Then,*

(i) there exists a critical level of claims, c^*, such that for all $i \in N$ with $c_i < c^*$, $CEA_i(C, E) > P_i(C, E)$ and for all $i \in N$ with $c_i \ge c^*$, $CEA_i(C, E) \le P_i(C, E)$ and,

(ii) there exists a critical level of claims, $\widetilde{c}$, such that for all $i \in N$ with $c_i < \widetilde{c}$, $CEL_i(C, E) < P_i(C, E)$ and for all $i \in N$ with $c_i \geq \widetilde{c}$, $CEL_i(C, E) \geq P_i(C, E)$.

Proof See online Appendix B. □

Lemma B.1 (see online Appendix B) provides closed form expressions for $F_i(C, E) - P_i(C, E)$ in the model with two types of agents.

2.2 The model

There are n_h agents each with income w_h and n_l agents each with income w_l, such that $0 < w_l < w_h$.[6] Accordingly, N_h is the set of type h agents with $|N_h| = n_h$ and N_l is the set of type l agents with $|N_l| = n_l$. We use t to refer to a generic type i.e., $t \in \{l, h\}$. Therefore, for all agents $i \in N_l \cup N_h$, the individual income $w_i \in \{w_l, w_h\}$. Both types of agents are risk-neutral. Hence, each agent wants to choose the investment alternative that brings the maximum expected return. There are two investment alternatives: agents either invest in a company and become shareholders or deposit their money into a savings account in a bank.

The company runs a *risky* investment project. Depositing money into a bank, on the other hand, brings a *risk-free* return. The state space for the outcome of the risky investment project is $\Omega = \{s, f\}$, where s represents *success* and f represents *failure*. Hence, the outcome of the project is a random variable ω. With probability $\Pr(\omega = s) = \pi_s < 1$, the investment project is successful and brings a payoff of r_s ($0 < r_s \leq 1$) to the company; with probability $\Pr(\omega = f) = 1 - \pi_s$, the investment project fails and brings a payoff of r_f ($r_f < r_s \leq 1$) to the company. The company promises to pay r to depositors, which satisfies $0 \leq r_f < r \leq r_s \leq 1$. However, if the project fails the company cannot honor all claims since $r_f < r$.

On the other hand, the savings account at the bank pays a constant, risk-free return $\overline{r}$. We eliminate two cases that would lead to trivial results: $r_f > \overline{r}$ and $\overline{r} > r$. If $r_f > \overline{r}$ was the case, then no agent would prefer to deposit their money to the bank and if $\overline{r} > r$ was the case, then no agent would prefer to invest in the company. To make the problem interesting, we assume that $r_f < \overline{r} < r$. Thus, the risky asset offers a higher return in the case of success, but a lower return in the case of failure. Having introduced the necessary parameters, now we define the particular class of bankruptcy problems we analyze.

Definition 4 A bankruptcy problem with an *endogenous estate* is a pair (C, E), where C is a claims vector with entries $c_i = (1 + r)w_i$ for all $i \in N$ and $E = (1 + r_f) \sum_{i \in N} w_i$ is the estate. The class of bankruptcy problems with an endogenous estate is denoted by $\widetilde{\mathcal{B}}$.

The endogeneity is due to the fact that the claims vector and the estate are determined by agents' decisions. Moreover, note that C and E in the definition of the bankruptcy problem are derived from w, r_f, and r. Hence, the data of the problem can be written as (w, r_f, r) instead of (C, E). However, to keep the exposition similar to the standard one used in the literature, we use (C, E) notation. All these parameters are common knowledge among players.

The company, denoted as m, chooses a bankruptcy rule F. The company's objective is to maximize its profit. Note that, given $r, r_s, r_f, \overline{r}$, and π_s, maximizing the profit is equivalent to maximizing the investment attracted. The bankruptcy rule chosen affects agents' investment decisions since it affects their returns in case of a bankruptcy. Hence, the company takes into account the possible actions of agents while choosing the bankruptcy rule. We use P, *CEA*, and

[6] In fact, what we mean by w_l (w_h) is the part of the income that is reserved for investment by a type l (h) agent.

CEL in our base model. Accordingly, the company's strategy space is $\psi_m = \{P, CEA, CEL\}$. The company's decision is observed by all agents. Hence, each decision of the company starts a proper subgame to be played by agents. We denote these three subgames by Γ^P, Γ^{CEA}, and Γ^{CEL}.

Knowing the bankruptcy rule, F, chosen by m, all agents $i \in N_l \cup N_h$ simultaneously choose whether to invest their money in the risky asset (i.e., playing in) or to invest in the risk-free asset (i.e., playing *out*). For all $i \in N_l \cup N_h$, we denote agent i's actions by $a_i \in \{in, out\}$ and the actions taken by agent i under rule $F \in \{P, CEA, CEL\}$ as $a_{i,F}$. We describe what each agent i would do in each subgame Γ^F by agent i's strategy, which is denoted by $s_i \in \psi_i$. Agent i's strategy space, ψ_i, can be written as

$$\psi_i = \{(a_{i,P}, a_{i,CEA}, a_{i,CEL}) \mid a_{i,F} \in \{in, out\} \text{ and } F \in \{P, CEA, CEL\}\}. \tag{1}$$

The company's payoff function is denoted as $V_m(F) = \sum_{t\in\{l,h\}} n_{t,in}(F)w_t$, where $n_{t,in}(F)$ stands for the number of type t agents who play in in the subgame Γ^F. Therefore, we can write the company's objective to maximize the investment it attracts as

$$\max_{F\in\{P,CEA,CEL\}} V_m(F). \tag{2}$$

Note that once r and r_f are fixed, E and C are both determined by agents' actions. When writing agents' payoffs under bankruptcy, we employ the notation, $V_{i,in}(F, n_{h,in}, n_{l,in})$ since agent i's payoff in case of bankruptcy is determined by F, $n_{h,in}$,and $n_{l,in}$. Similarly, agent i's *transfer* under rule F can be written as $S_i(F, n_{h,in}, n_{l,in})$.

Now, given agent i's action in Γ^F, the payoff of agent $i \in N_l \cup N_h$ can be written as

$$V_i(F, a_{i,F}) = \begin{cases} V_{i,out} = (1+\bar{r})w_i, & \text{if } a_{i,F} = out \\ V^e_{i,in}(F, n_{h,in}, n_{l,in}) = \pi_s(1+r)w_i + (1-\pi_s)[V_{i,in}(P, n_{h,in}, n_{l,in}) \\ \qquad + S_i(F, n_{h,in}, n_{l,in})], & \text{if } a_{i,F} = in \end{cases} \tag{3}$$

where the superscript e refers to the expected value. Notice that the first part of $V^e_{i,in}(F, n_{h,in}, n_{l,in})$ is agent i's payoff in case of successful completion of the project and the second part is his payoff in case of a bankruptcy. Also note that for all $i \in N_l \cup N_h$, $S_i(P, n_{h,in}, n_{l,in}) = 0$ under P.

The following lemma enables us to simplify the notation $V_{j,in}(P, n_{h,in}, n_{l,in})$, since it shows that the payoff of each agent under P is independent from other agents' types, actions, etc. Consequently, we can write agent j's payoff under P as P_j.

Lemma 2 *Assume that for all $j \in N_l \cup N_h$, the claim structure is $c_j = (1+r)w_j$ and the estate is $E = (1+r_f)\sum_{i\in N_l\cup N_h} w_i$. Then $V_{j,in}(P, n_{h,in}, n_{l,in}) \equiv P_j = (1+r_f)w_j$.*

Proof See online Appendix B. □

This result is valid for any finite number of types. By Lemma 2, if agent i is of type t, then $V_{i,in}(P, n_{h,in}, n_{l,in}) = (1+r_f)w_t$. We rewrite agent i's expected payoff under P as

$$P^e_i = \pi_s(1+r)w_i + (1-\pi_s)P_i. \tag{4}$$

Using the expected payoff under P, we can rewrite agent i's expected payoff under *CEA* as

$$\begin{aligned} V^e_{i,in}(CEA, n_{h,in}, n_{l,in}) &= \pi_s(1+r)w_i + (1-\pi_s)[P_i + S_i(CEA, n_{h,in}, n_{l,in})] \\ &= P^e_i + (1-\pi_s)S_i(CEA, n_{h,in}, n_{l,in}), \end{aligned} \tag{5}$$

Table 1 Sequential game Γ

Players	$\{m\} \cup N_l \cup N_h$
Strategies	$\psi_m \times \prod_{i \in N_l \cup N_h} \psi_i$
Payoffs	$(V_m(F), (V_{i,s_i}(F, s_{-i})), i \in N_l \cup N_h$

where P_i^e stands for the expected payoff that agent i would get under P and $S_i(CEA, n_{h,in}, n_{l,in})$ is the *transfer* that agent i makes/receives under *CEA*, when $n_{h,in}$ type h agents and $n_{l,in}$ type l agents play in. Similarly, under *CEL*, agent i's expected payoff can be rewritten as

$$\begin{aligned} V^e_{i,in}(CEL, n_{h,in}, n_{l,in}) &= \pi_s(1+r)w_i + (1-\pi_s)[P_i + S_i(CEL, n_{h,in}, n_{l,in})] \\ &= P_i^e + (1-\pi_s)S_i(CEL, n_{h,in}, n_{l,in}), \end{aligned} \tag{6}$$

where $S_i(CEL, n_{h,in}, n_{l,in})$ is the transfer that agent i makes/receives under *CEL*, when $n_{h,in}$ type h agents and $n_{l,in}$ type l agents play in.

Table 1 and the sequence of actions described before define the sequential game Γ with three proper subgames Γ^P, Γ^{CEA}, and Γ^{CEL}, where s_{-i} denotes all agents' strategies except agent i. We look for *pure* strategy equilibria of this game. The equilibrium concept we employ is subgame perfect Nash equilibrium.

3 Analysis of equilibrium decisions

We analyze the equilibria of the game defined in Table 1. We first analyze the subgames $\Gamma^F \in \{\Gamma^P, \Gamma^{CEA}, \Gamma^{CEL}\}$ played among all agents $i \in N_l \cup N_h$. Therefore, in the following, when we use the term equilibrium, it refers to the agents' equilibrium actions in the corresponding subgame. After analyzing agents' behavior in each subgame, we analyze the company's action in equilibrium. This is followed by the description of the subgame perfect Nash equilibria of the game along with the resulting investment in the company.

3.1 Some preparatory results

Before the analysis of agents' equilibrium investment decisions, we prove some preparatory lemmas and corollaries. Corollary B.1 (see online Appendix B) provides closed form expressions for c^* and $\widetilde{c}$ in $\widetilde{\mathcal{B}}$ and Corollary B.2 (see online Appendix B) derives closed form transfers under *CEA* and *CEL*. The following lemma shows that c^* and $\widetilde{c}$ are always between c_l and c_h.

Lemma 3 *Let* $(C, E) \in \widetilde{\mathcal{B}}$. *Assume that* $n_{h,in} > 0$ *and* $n_{l,in} > 0$. *Then, for all* $n_{h,in}$ *and* $n_{l,in}$, *(i)* $c_l \le c^* \le c_h$ *and (ii)* $c_l \le \widetilde{c} \le c_h$.

Proof See online Appendix B. □

Since if one type is making *transfers* the other type should be receiving *transfers* in the case with two types, the result mentioned in Lemma 3 is trivial. This result is required for the comparative static analyses we conduct in Lemma B.2 (see online Appendix B). It ensures that when the number of type $t \in \{l, h\}$ agents changes, the identity of the types (i.e, *transfer-maker* or *transfer-receiver*) stays the same.

Lemma B.2 provided in online Appendix B presents comparative statics with simple intuitions. It analyzes the changes in *per-capita* transfer with respect to changes in the number of type h and type l agents. We see that if the number of agents of types who are making *transfers* increases, per-capita *transfers* they make decrease and per-capita *transfers* other types receive increase. On the other hand, if the number of agents of types who are receiving *transfers* increases, per-capita *transfers* they receive decrease per-capita *transfers* other types make increase.

Tie-Breaking Assumption *Every agent plays "in" when he is indifferent between "in" and "out".*

This tie-breaking assumption is employed in the rest of the paper. The following lemma states that each agent's decision in equilibrium is determined by his *type only*.

Lemma 4 (Symmetry) *If agents i and j are of the same type $t \in \{l, h\}$, their strategies are the same in equilibrium.*

Proof See online Appendix B. □

Lemma 4 has three important implications. First of all, it shows that if there exists an equilibrium it will be symmetric, i.e., same types play the same strategy in equilibrium. The tie-breaking assumption is important for the validity of Lemma 4. If agents of the same type play strategies that are different from each other when these agents are indifferent, the statement in Lemma 4 is not valid anymore. However, breaking the ties in favor of playing in is not crucial for the proofs. Assuming that every agent plays *out* when he is indifferent would work equally well. We stick to one of these (i.e., playing in) since extending the analysis by allowing both possibilities does not bring any additional insights. Second, this symmetry result enables us to employ a more compact notation for equilibrium actions in subgames Γ^F (in game Γ): $(a_{h,F}, a_{l,F})$ means that all type h agents play $a_{h,F}$ and all type l agents play $a_{l,F}$ in Γ^F. Third, this result also enables us to use a simpler notation when writing agents' expected payoffs. Since we know that agents of the same type act identically, we can write the expected payoff of a representative type t agent who plays in under F as $V_t^e(F, s_{-t})$ instead of writing individual expected payoff as $V_{i,in}^e(F, n_{h,in}, n_{l,in})$. We will employ this notation in the remainder of the analysis.

The following corollary relates the symmetry result to equilibrium values of $V_m(F)$: since we show that agents of the same type play the same strategies in equilibrium, this reduces the number of possible values the equilibrium investment volume can take.

Corollary 1 *In equilibrium, $V_m(F)$ can take four values: 0, $n_h w_h$, $n_l w_l$ and $n_h w_h + n_l w_l$.*

Proof See online Appendix B. □

The following lemma shows that certain strategy profiles cannot be a part of any equilibrium under *CEA* and *CEL*.

Lemma 5 *The following statements about strategy profiles are valid.*

(i) In Γ^{CEA}, the strategy profile (for all $i \in N_h$, $s_i = (., in, .)$ and for all $j \in N_l$, $s_j = (., out, .)$) cannot be an equilibrium,
(ii) In Γ^{CEL}, the strategy profile (for all $i \in N_h$, $s_i = (., ., out)$ and for all $j \in N_l$, $s_j = (., ., in)$) cannot be an equilibrium.

Proof See online Appendix B. □

Table 2 Payoff matrix under P

$h \setminus l$	in	out
in	P_h^e, P_l^e	$P_h^e, V_{l,out}$
out	$V_{h,out}, P_l^e$	$V_{h,out}, V_{l,out}$

The result in this lemma has a simple intuition: if, in equilibrium, the parameter values are such that even the type of agents who are making (receiving) transfers find playing in (out) optimal, the type of agents who are receiving (making) transfers also find it optimal to play in (out).

3.2 Characterization of all nash equilibria in subgames

In this subsection, we describe agents' investment behavior and characterize all Nash equilibria in Γ^P, Γ^{CEA}, and Γ^{CEL}. Recall that Lemma 4 enables us to use type best responses instead of agent best responses. Hence, in this section, we use type t's best response to a strategy played by the other type. We denote the best response of type t agents to action a_{-t} played by the other type of agents in the subgame Γ^F by $BR_t(F, a_{-t})$.[7]

Proposition 1 *Equilibria in Γ^P can be described as follows:*

(i) *If for all $t \in \{l,h\}$, $P_t^e < V_{t,out}$, then the unique equilibrium strategy profile is* $(a_{h,P}, a_{l,P}) = (out, out)$.
(ii) *If for all $t \in \{l,h\}$, $P_t^e \geq V_{t,out}$, then the unique equilibrium strategy profile is* $(a_{h,P}, a_{l,P}) = (in, in)$.

Proof In Γ^P, the payoff matrix given in Table 2 can be used to show representative type h and type l agent's expected payoffs. The first (second) item in each cell represents each type h (type l) agent's expected payoff.

Recall that by Lemma 2, the expected payoff of each agent is independent of other agents' strategies under P. This implies that all equilibria are *dominant strategy* equilibria. Also, note that by Lemma B.3 (see online Appendix B), $P_l^e < V_{l,out}$ if and only if $P_h^e < V_{h,out}$. Therefore, if $P_h^e \geq V_{h,out}$, then $BR_h(P, in) = BR_h(P, out) = in$, and similarly if $P_l^e \geq V_{l,out}$, then $BR_l(P, in) = BR_l(P, out) = in$. If $P_h^e < V_{h,out}$, then $BR_h(P, in) = BR_h(P, out) = out$, and similarly if $P_l^e < V_{l,out}$, then $BR_l(P, in) = BR_l(P, out) = out$. □

Note that neither (in, out) nor (out, in) equilibria are possible in Γ^P. This is due to proportionality, which implies that $P_h^e \geq V_{h,out}$ if and only if $P_l^e \geq V_{l,out}$.

Proposition 2 *Equilibria in Γ^{CEA} can be described as follows:*

(i) *If for all $t \in \{l,h\}$, $P_t^e < V_{t,out}$, then the unique equilibrium strategy profile is* $(a_{h,CEA}, a_{l,CEA}) = (out, out)$.
(ii) *If for all $t \in \{l,h\}$, $P_t^e \geq V_{t,out}$ and $V_h^e(CEA, in) = P_h^e + (1-\pi_s)S_h(CEA, in) < V_{h,out}$, then the unique equilibrium strategy profile is $(a_{h,CEA}, a_{l,CEA}) = (out, in)$.*
(iii) *If for all $t \in \{l,h\}$, $P_t^e \geq V_{t,out}$ and $V_h^e(CEA, in) = P_h^e + (1-\pi_s)S_h(CEA, in) \geq V_{h,out}$, then the unique equilibrium strategy profile is $(a_{h,CEA}, a_{l,CEA}) = (in, in)$.*

[7] Also note that each agent has one information set in each subgame and two actions. Therefore, the terms *strategy* and *action* refer to same objects in subgames Γ^P, Γ^{CEA}, and Γ^{CEL}.

Table 3 Payoff matrix under *CEA*

$h \setminus l$	in	out
in	$P_h^e + (1-\pi_s)S_h(CEA, in),\ P_l^e + (1-\pi_s)S_l(CEA, in)$	$P_h^e, V_{l,out}$
out	$V_{h,out}, P_l^e$	$V_{h,out}, V_{l,out}$

Proof In Γ^{CEA}, the payoff matrix given in Table 3 can be used.

By the definition of *CEA* and Lemma 1, $S_h(CEA, in) < 0$ and $S_l(CEA, in) > 0$. If the risk-free asset pays more than the best possible expected payoff that type l agents can get, the analysis is trivial since type l agents would never play in. Hence, we assume that $V_l^e(CEA, in) = P_l^e + (1-\pi_s)S_l(CEA, in) > V_{l,out}$. This assumption implies that $BR_l(CEA, in) = in$. The relationship between P_l^e and $V_{l,out}$ determines type l agents' best response to type h agents playing out. If $P_l^e \geq V_{l,out}$, then $BR_l(CEA, out) = in$; if $P_l^e < V_{l,out}$, then $BR_l(CEA, out) = out$. On the other hand, type h's best response against in depends on the relationship between $V_h^e(CEA, in) = P_h^e + (1-\pi_s)S_h(CEA, in)$ and $V_{h,out}$. If

$$V_h^e(CEA, in) = P_h^e + (1-\pi_s)S_h(CEA, in) \geq V_{h,out},$$

then $BR_h(CEA, in) = in$; if

$$V_h^e(CEA, in) = P_h^e + (1-\pi_s)S_h(CEA, in) < V_{h,out},$$

then $BR_h(CEA, in) = out$. Therefore, these inequalities characterize agents' equilibrium behavior in Γ^{CEA}. □

Proposition 3 *Equilibria in* Γ^{CEL} *can be described as follows:*

- (i) *If for all* $t \in \{l, h\}$, $P_t^e < V_{t,out}$, *then the unique equilibrium strategy profile is* $(a_{h,CEL}, a_{l,CEL}) = (out, out)$.
- (ii) *If for all* $t \in \{l, h\}$, $P_t^e \geq V_{t,out}$ *and* $V_l^e(CEL, in) = P_l^e + (1-\pi_s)S_l(CEL, in) < V_{l,out}$, *then the unique equilibrium strategy profile is* $(a_{h,CEL}, a_{l,CEL}) = (in, out)$.
- (iii) *If for all* $t \in \{l, h\}$, $P_t^e \geq V_{t,out}$ *and* $V_l^e(CEL, in) = P_l^e + (1-\pi_s)S_l(CEL, in) \geq V_{l,out}$, *then the unique equilibrium strategy profile is* $(a_{h,CEL}, a_{l,CEL}) = (in, in)$.

Proof In Γ^{CEL}, the payoff matrix given in Table 4 can be used.

By the definition of *CEL* and Lemma 1, $S_h(CEL, in) > 0$ and $S_l(CEL, in) < 0$. If the outside asset pays more than the best possible expected payoff that type h agents can get, the analysis is trivial since then type h agents would never play in. Hence, we assume that $V_h^e(CEL, in) = P_h^e + (1-\pi_s)S_h(CEL, in) > V_{h,out}$. This assumption implies that $BR_h(CEL, in) = in$. The relationship between P_h^e and $V_{h,out}$ determines type h agents' best response to type l agents playing out. If $P_h^e \geq V_h$, then $BR_h(CEL, out) = in$; if $P_h < V_{h,out}$, then $BR_h(CEL, out) = out$. On the other hand, type l's best response to type h agents playing in depends on the relationship between $V_l^e(CEL, in) = P_l^e + (1-\pi_s)S_l(CEL, in)$ and

Table 4 Payoff matrix under *CEL*

$h \setminus l$	in	out
in	$P_h^e + (1-\pi_s)S_h(CEL, in),\ P_l^e + (1-\pi_s)S_l(CEL, in)$	$P_h^e, V_{l,out}$
out	$V_{h,out}, P_l^e$	$V_{h,out}, V_{l,out}$

$V_{l,out}$. If

$$V_l^e(CEL, in) = P_l^e + (1 - \pi_s) S_l(CEL, in) \geq V_{l,out},$$

then $BR_l(CEL, in) = in$; if

$$V_l^e(CEL, in) = P_l^e + (1 - \pi_s) S_l(CEL, in) < V_{l,out},$$

then $BR_l(CEL, in) = out$. Therefore, these inequalities characterize agents' equilibrium actions in Γ^{CEL}. □

Note that in a Nash equilibrium of Γ^{CEA}, if type h agents play in, type l agents also play in. Similarly, in a Nash equilibrium of Γ^{CEL}, if type l agents play in, type h agents also play in . Also note that, if the equilibrium of Γ^P is the strategy profile $(a_{h,P}, a_{l,P}) = (out, out)$, then the equilibrium strategy profiles of Γ^{CEA} and Γ^{CEL} are also $(a_{h,CEA}, a_{l,CEA}) = (a_{h,CEL}, a_{l,CEL}) = (out, out)$.

3.3 Characterization of all subgame perfect nash equilibria

In this subsection, we analyze the company's behavior and characterize all subgame perfect Nash equilibria in Γ. As we mentioned in Sect. 2, the company's payoff function is $V_m(F) = \sum_{t \in \{l,h\}} n_{t,in}(F) w_t$, where $n_{t,in}(F)$ is the number of type t agents played in under F. Therefore, the company's decision depends on the equilibrium strategies of agents in each subgame. In the previous subsection, we analyzed the equilibrium strategies of agents in all three subgames. Below, we list different combinations of inequalities and the subgame perfect Nash equilibrium strategy profiles along with the equilibrium investment in the company. In the strategy profile (s_m, s_h, s_l), the first entry refers to the company's strategy (i.e., $s_m \in \psi_m$) , second to type h agents' (i.e., $s_h \in \psi_h$), and third to type l agents' (i.e., $s_l \in \psi_l$). Moreover, the first entry in a representative type t agent's strategy profile refers to his equilibrium action in Γ^P, the second to his equilibrium action in Γ^{CEA}, and the third to his equilibrium action in Γ^{CEL}.

C1. If for all $t \in \{l, h\}$

$$\begin{aligned} P_t^e &\geq V_{t,out}, \\ P_h^e + (1 - \pi_s) S_h(CEA, in) &< V_{h,out}, \text{ and} \\ P_l^e + (1 - \pi_s) S_l(CEL, in) &< V_{l,out}, \end{aligned}$$

then given agents' equilibrium actions in Γ^P, Γ^{CEA}, and Γ^{CEL} presented in the previous subsection, the company prefers P and the equilibrium investment in the company is $V_m(F) = \sum_{t \in \{l,h\}} n_t w_t$. As we showed in the previous subsection, under these parameter restrictions, neither *CEA* nor *CEL* can attract all types to invest in the company, whereas P can. Hence, the *unique* subgame perfect Nash equilibrium strategy profile is

$$(s_m, s_h, s_l) = (P, (in; out; in), (in; in; out)).$$

C2. If for all $t \in \{l, h\}$,

$$\begin{aligned} P_t^e &\geq V_{t,out}, \\ P_h^e + (1 - \pi_s) S_h(CEA, in) &\geq V_{h,out}, \text{ and} \\ P_l^e + (1 - \pi_s) S_l(CEL, in) &< V_{l,out}, \end{aligned}$$

then given agents' equilibrium actions in Γ^P, Γ^{CEA}, and Γ^{CEL}, the company prefers P or *CEA* to *CEL* and the equilibrium investment in the company is $V_m(F) =$

$\sum_{t\in\{l,h\}} n_t w_t$. Both *CEA* and P can attract all types to invest in the company whereas the *CEL* can only attract h types. Hence, the subgame perfect Nash equilibrium strategy profiles are

$$(s_m, s_h, s_l) = (P, (in; in; in), (in; in; out)) \text{ and}$$
$$(s_m, s_h, s_l) = (CEA, (in; in; in), (in; in; out)).$$

C3. If for all $t \in \{l, h\}$,

$$\begin{aligned} P_t^e &\geq V_{t,out}, \\ P_h^e + (1-\pi_s)S_h(CEA, in) &< V_{h,out}, \text{ and} \\ P_l^e + (1-\pi_s)S_l(CEL, in) &\geq V_{l,out}, \end{aligned}$$

then given agents' equilibrium actions in Γ^P, Γ^{CEA}, and Γ^{CEL}, the company prefers P or *CEL* to *CEA* and the equilibrium investment in the company is $V_m(F) = \sum_{t\in\{l,h\}} n_t w_t$. Both *CEL* and P can attract all types to invest in the company whereas *CEA* can only attract l types. Hence, the subgame perfect Nash equilibrium strategy profiles are

$$(s_m, s_h, s_l) = (P, (in; out; in), (in; in; in)) \text{ and}$$
$$(s_m, s_h, s_l) = (CEL, (in; out; in), (in; in; in)).$$

C4. If for all $t \in \{l, h\}$,

$$\begin{aligned} P_t^e &\geq V_{t,out}, \\ P_h^e + (1-\pi_s)S_h(CEA, in) &\geq V_{h,out}, \text{ and} \\ P_l^e + (1-\pi_s)S_l(CEL, in) &\geq V_{l,out}, \end{aligned}$$

then given agents' equilibrium actions in Γ^P, Γ^{CEA}, and Γ^{CEL}, the company is indifferent between all three rules, and the equilibrium investment in the company is $V_m(F) = \sum_{t\in\{l,h\}} n_t w_t$. All rules are equally able to attract all types to invest in the company. Hence, the subgame perfect Nash equilibrium strategy profiles are

$$(s_m, s_h, s_l) = (P, (in; in; in), (in; in; in)),$$
$$(s_m, s_h, s_l) = (CEA, (in; in; in), (in; in; in)), \text{ and}$$
$$(s_m, s_h, s_l) = (CEL, (in; in; in), (in; in; in)).$$

C5. If for all $t \in \{l, h\}$,

$$P_t^e < V_{t,out},$$

then given agents' equilibrium actions in Γ^P, Γ^{CEA}, and Γ^{CEL}, the company is indifferent between all three rules, and the equilibrium investment in the company is $V_m(F) = 0$. Since $P_t^e \geq V_{t,out}$ is a necessary condition for both types of agents' equilibrium decisions to be in, none of the rules can attract neither of the two types to invest in the company. Hence, the subgame perfect Nash equilibrium strategy profiles are

$$(s_m, s_h, s_l) = (P, (out; out; out), (out; out; out)),$$
$$(s_m, s_h, s_l) = (CEA, (out; out; out), (out; out; out)), \text{ and}$$
$$(s_m, s_h, s_l) = (CEL, (out; out; out), (out; out; out)).$$

Note that in C4 and C5, the company's decision does not really matter. Basically, *anything goes* in these cases. As we have shown above, besides $P_t^e \geq V_{t,out}$,

$$P_h^e + (1 - \pi_s) S_h(CEA, in) \geq V_{h,out} \text{ and}$$
$$P_l^e + (1 - \pi_s) S_l(CEL, in) \geq V_{l,out}$$

should be satisfied in C4. The interpretation of this is that neither under *CEA* nor under *CEL*, per-capita transfers from disadvantaged type of agents to advantaged type of agents are significantly high. This, intuitively, can be due to (i) a small difference between w_l and w_h, (ii) a low probability of bankruptcy (i.e., $1 - \pi_s$), or (iii) a small outside asset payoff ($\overline{r}$). C5 shows another situation in which the decision will not make a difference. No matter which rule the company chooses, the investment in the company will be 0. However, this has nothing to do with the income distribution in the society. We show in Lemma B.3 (see online Appendix B) that $P_t^e \geq V_{t,out}$ does not contain any income distribution parameters (e.g., n_l, n_h, w_l and w_h). Hence, the validity of this condition depends only on the risk-return characteristics of investment alternatives. Intuitively, if the payoff from the risk-free asset is sufficiently high, or the probability of bankruptcy is sufficiently high (or more generally the expected return from the risky investment is sufficiently low) then $P_t^e < V_{t,out}$ will hold.

3.4 Equilibrium and results

In this subsection, we present some results which are implied by the analyses of equilibria conducted in Sects. 3.2 and 3.3. First of all, in the light of the analyses carried out in the previous section it is easy to see that a pure strategy subgame perfect Nash equilibrium of Γ exists. Below, we present our main result of the base model.

Theorem 1 *For any bankruptcy problem $(C, E) \in \widetilde{\mathcal{B}}$, only P is a part of any subgame perfect Nash equilibrium of Γ.*

Proof Notice that in each of five cases analyzed in Sect. 3.3, P appears in subgame perfect Nash equilibrium. Since, we characterized all equilibria in Sect. 3.3, the result immediately follows. Note that in C3, *CEA* is not a part of the subgame perfect Nash Equilibria; in C2, *CEL* is not a part of the subgame perfect Nash equilibria and in C1, neither of these rules belong to the subgame perfect Nash Equilibria. Hence, the statement in the theorem is valid only for P. □

In Corollary B.3 (see online Appendix B), we show that our main result is robust with respect to changes in the income distribution.

Remark 1 By Corollary B.3, even if there is an uncertainty about the income distribution (i.e., the company does not know the income distribution for sure) the statement in Theorem 1 is still valid. In fact, for probability distributions that assign non-zero probability to all possible income distributions, P would be the unique optimal strategy for an expected-profit maximizing company.

Remark 2 For certain real-life circumstances, one may suggest that an increase in the investment volume can lead to an increase in the return rate (e.g., the investment project involves increasing returns to capital) and/or a decrease in the rate of risk (e.g., a higher level of capital increases the likelihood of success). If we incorporate such possibilities, we expect that our results may quantitatively change. The relative (with respect to P) positions of *CEA* and *CEL* would improve. However, we still expect P to outperform other rules.

Remark 3 For certain real-life circumstances, one may suggest that investors move sequentially rather than simultaneously. The result in Theorem 1 would still be valid even if agents' investment decisions are sequential instead of simultaneous. It follows from the fact that sequencing (independent of which type of agents moves first) cannot increase the equilibrium investment volume under *CEA* or *CEL*; and it does not have any effect on the investment volume under P.

4 Extensions

4.1 Extension to a larger set of rules

In this subsection, we show that our results remain valid if we enlarge the set of rules we use in our model. Our first candidate is the Talmud rule (TAL) (see Aumann and Maschler 1985). Like the ones we used in our base model, the Talmud rule is also a prominent bankruptcy rule satisfying a large set of *desirable* properties (see Herrero and Villar 2001 and Thomson 2003).

Definition 5 (*Talmud Rule*) For all $(C, E) \in \widetilde{\mathcal{B}}$, and all $j \in N$,

$$TAL_j(C, E) \equiv \begin{cases} \min\{\frac{c_j}{2}, \lambda_t\} & \text{if } E \leq \sum_{i \in N} \frac{c_i}{2} \\ c_j - \min\{\frac{c_j}{2}, \lambda_t\} & \text{if } E \geq \sum_{i \in N} \frac{c_i}{2} \end{cases}$$

where λ_t solves $\sum_{i \in N} TAL_i(C, E) = E$.

Proposition 4 *Denote the extended game for which* $F \in \{P, CEA, CEL, TAL\}$ *by* $\widehat{\Gamma}$ *and let* $(E, C) \in \widetilde{\mathcal{B}}$,

- (i) P *is the only rule that is always a part of the subgame perfect Nash equilibrium of* $\widehat{\Gamma}$, *when* $E \neq \sum_{i \in N} \frac{c_i}{2}$ *and*
- (ii) P *and* TAL *are the only rules that are always parts of subgame perfect Nash equilibrium of* $\widehat{\Gamma}$, *when* $E = \sum_{i \in N} \frac{c_i}{2}$ *(i.e.,* $r = 1$ *and* $r_f = 0$*).*

Proof See online Appendix B. □

Note that instead of adding TAL to $\{P, CEA, CEL\}$, if we replaced P with TAL, then TAL would outperform *CEA* and *CEL* in attracting investment into the company. More generally, we conjecture that if we replace P with any other order-preserving rule F in between *CEA* and *CEL* (in the regressivity–progressivity spectrum), F will outperform *CEA* and *CEL* in attracting investment. Nevertheless, as soon as P is added back to $\{F, CEA, CEL\}$, P will again outperform others.

Moreno-Ternero and Villar (2006) generalized the Talmud rule by introducing the TAL-family of rules, which contains the Talmud, constrained equal awards and the constrained equal losses rules as special cases.

Definition 6 (*TAL-family*) The TAL-family consists of all rules of the following form: For some $\theta \in [0, 1]$, for all $(E, C) \in \widetilde{\mathcal{B}}$ and for all $i \in N$,

$$F_i^\theta(C, E) = \begin{cases} \min\{\theta c_i, \lambda\} & \text{if } E \leq \sum_{i \in N} \theta c_i \\ \max\{\theta c_i, c_i - \mu\} & \text{if } E \geq \sum_{i \in N} \theta c_i \end{cases}$$

where μ and λ solve $\sum_{i\in N} F_i^{\theta}(C, E) = E$.[8]

For the extended game including all members of the TAL-family along with P in the strategy space of the company, we can argue—along the same lines as the proof of Proposition 4—that for any bankruptcy problem $(E, C) \in \widetilde{\mathcal{B}}$, there always exists a subgame perfect Nash equilibrium of $\widetilde{\Gamma}$ with P and the member of the TAL-family parametrized by $\theta^* = \frac{1+r_f}{1+r}$. Given r and r_f, $\frac{E}{\sum_{i\in N} c_i} = \frac{1+r_f}{1+r} \in [\frac{1}{2}, 1)$. Therefore, a result similar to the one in Proposition 4 would follow. We leave the extension to a larger set of rules as an open question for future work.

4.2 Extension to multiple types

In this subsection, we show that our main result extends to a model with more than two types of agents. We do not present a full characterization of all subgame perfect equilibria as we did in previous sections. Instead, using some of the results from the analyses we earlier conducted, we present a short proof showing that the result in Theorem 1 holds true in a model with multiple types of agents.

Proposition 5 *For any $(C, E) \in \widetilde{\mathcal{B}}$ with more than two types of agents, P is a part of any subgame perfect Nash equilibrium of Γ and it is the only one to do so among P, CEA, and CEL.*

Proof First of all, observe from Lemma B.3 (see online Appendix B) that $P_t^e \geq V_{t,out}$ does not contain any income distribution parameters. Thus, the conditions under which P attracts all types into company remain the same. This implies that P is still a part of any subgame perfect Nash equilibrium of Γ, in a game with multiple types of agents (see the analysis in Sect. 3.3). To show that it is the only rule among the three to do so, finding a case where P still attracts all types but neither *CEA* nor *CEL* can do so is sufficient. To do this, take the case where $P_t^e = V_{t,out}$. Under this condition, P still attracts all types in to investment in the company. However, if *CEA* (or *CEL*) is used, the company cannot attract all types since if one type of agents is receiving transfers, then some others should be making transfers, which implies that transfer-making agents receive a payoff less than P_t^e. But since $P_t^e = V_{t,out}$, this implies that transfer-making type of agents receive an expected payoff less than the risk-free return. Therefore, they do not invest in the company. Finally, when $P_t^e < V_{t,out}$, still none of the rules can attract any investment. Hence, the result follows. □

4.3 Competition for investment

So far we assume a single company in our analysis. However, in many cases there are more than one company competing for the same group of investors in real life. In this subsection, we introduce a competition between two companies. Below, we first set up the model with two companies and then present our results.

In this subsection, we assume that there are two companies ($\overline{m}$ and $\underline{m}$) competing for the same set of potential investors in $N_l \cup N_h$. We model the competition between these two companies à la Stackelberg. Accordingly, $\overline{m}$ is the first-mover and $\underline{m}$ is the second-mover. Risk-return characteristics (π_s, r, and r_f) are assumed to be identical for both companies

[8] Note that $F = CEL$ for $\theta = 0$, $F = CEA$ for $\theta = 1$, and $F = T$ for $\theta = 1/2$. Moreover, for all $\theta \in [0, 1]$, F^{θ} coincides with *CEA* (on adjusted θ-claims) if $E \leq \sum_{i\in N} \theta c_i$ and *CEL* (on adjusted $(1-\theta)$ -claims) if $E \geq \sum_{i\in N} \theta c_i$. Finally, note that P is not a member of the TAL-family.

and there is, again, a single risk-free asset (with return $\overline{r}$) available for agents. As in our base model, each company chooses a bankruptcy rule out of three (P, CEA, and CEL) rules to maximize its investment volume. However, now $\overline{m}$ moves first and chooses a bankruptcy rule and then $\underline{m}$, observing $\overline{m}$'s action, moves and also chooses a bankruptcy rule. Accordingly, $\overline{m}$'s strategy space is denoted by $\psi_{\overline{m}} = \{P, CEA, CEL\}$ and $\underline{m}$'s strategy space is denoted by $\psi_{\underline{m}} = \{\, (P, P), (CEA, P), (CEL, P), (P, CEA), (CEA, CEA), (CEL, CEA), (P, CEL), (CEA, CEL), (CEL, CEL)\, \}$ where first entries refer to $\overline{m}$'s actions and second entries refer to $\underline{m}$'s actions. Each action couple (consisting of actions of both companies) starts a subgame to be played by agents. We denote the subgame started by $\overline{m}$ playing $F_{\overline{m}}$ and $\underline{m}$ playing $F_{\underline{m}}$ by $\Gamma^{F_{\overline{m}}, F_{\underline{m}}}$. After observing companies' actions, agents simultaneously choose whether to invest in one of the companies or in the risk-free outside asset. Since there are two companies now, agents i's action set is $a_i = (\overline{m}, \underline{m}, out)$. For all $i \in N_l \cup N_h$, we denote the actions taken by agent i in subgame $\Gamma^{F_{\overline{m}}, F_{\underline{m}}}$ as $a_{i,\Gamma^{F_{\overline{m}} F_{\underline{m}}}}$. We describe what each agent i does in each subgame $\Gamma^{F_{\overline{m}}, F_{\underline{m}}}$ by agent i's strategy, which is again denoted by $s_i \in \psi_i$. Agent i's strategy space, ψ_i, can be written as

$$\psi_i = \{(a_{i,\Gamma^{F_{\overline{m}} F_{\underline{m}}}}) \mid F_{\overline{m}}, F_{\underline{m}} \in \{P, CEA, CEL\} \text{ and } a_{i,\Gamma^{F_{\overline{m}}, F_{\underline{m}}}} \in \{\overline{m}, \underline{m}, out\}\}. \tag{7}$$

We denote $\overline{m}$'s payoff function as $V_{\overline{m}}(F_{\overline{m}}, F_{\underline{m}}) = \sum_{t\in\{l,h\}} n_{t,\overline{m}}(F_{\overline{m}}, F_{\underline{m}}) w_t$, where $n_{t,\overline{m}}$ stands for the number of type t agents who invest in $\overline{m}$ in the subgame $\Gamma^{F_{\overline{m}}, F_{\underline{m}}}$ and $\underline{m}$'s payoff function as $V_{\underline{m}}(F_{\overline{m}}, F_{\underline{m}}) = \sum_{t\in\{l,h\}} n_{t,\underline{m}}(F_{\overline{m}}, F_{\underline{m}}) w_t$, where $n_{t,\underline{m}}$ stands for the number of type t agents who invest in $\underline{m}$ in the subgame $\Gamma^{F_{\overline{m}}, F_{\underline{m}}}$. Therefore, we can write the objective function for company $m \in \{\underline{m}, \overline{m}\}$ as

$$\max_{F_m \in \{P, CEA, CEL\}} V_m(F_m, .). \tag{8}$$

We, again, focus on pure strategy subgame perfect Nash equilibria of this game. The only difference is we add the following refinement assumptions:

Assumption 1 Each agent plays his weakly dominant strategy in case there is one.

Assumption 2 When a certain type of agents are indifferent between two companies, half of them invest in $\overline{m}$ and the other half in $\underline{m}$.[9]

Table 5 defines the sequential game $\overline{\Gamma}$, where s_{-i} denotes all agents' strategies except agent i.

Below, we list our main findings. Formal proofs for these results can be found in online Appendix C where we characterize all Nash equilibria in each subgame and all subgame perfect Nash equlbria of the extended model. In the following, we denote the total amount of money held by type l (h) agents as W_l (W_h).

(i) The income distribution is very influential on companies' decisions and hence the bankruptcy rules chosen in equilibrium. In all subgame perfect Nash equilibria (except the ones that involve all two-combinations of three rules), $\overline{m}$, i.e., the first-mover, chooses a bankruptcy rule favoring type of agents with a larger W_t .

(ii) In all subgame perfect Nash equilibria (except the ones that involve all two-combinations of three rules), $\underline{m}$, i.e., the second-mover, follows the first-mover by choosing the same bankruptcy rule when $W_h \geq 2W_l$ (or $W_l \geq 2W_l$). On the other hand, when $2W_l \geq W_h \geq W_l$ (or $2W_h \geq W_l \geq W_h$), the first-mover and the second-mover choose different

[9] For this assumption, we further need to assume that there are even number of agents of each type.

Table 5 Sequential game $\overline{\Gamma}$

Players	$\{\overline{m}, \underline{m}\} \cup N_l \cup N_h$
Strategies	$\psi_{\overline{m}} \times \psi_{\underline{m}} \prod_{i \in N_l \cup N_h} \psi_i$
Payoffs	$(V_{\overline{m}}(F_{\overline{m}}, F_{\underline{m}}), V_{\underline{m}}(F_{\overline{m}}, F_{\underline{m}})(V_{i,s_i}(F_{\overline{m}}, F_{\underline{m}}, s_{-i})), i \in N_l \cup N_h$

bankruptcy rules in which case the first-mover has an advantage in terms of the amount of investment.

(iii) Under some circumstances (e.g., $W_h \geq 2W_l$ or $W_l \geq 2W_l$), the competition between two companies leads to a smaller amount of total investment compared to the single-company case. In such circumstances, $\underline{m}$ follows $\overline{m}$ by choosing the same rule that favors a certain type of agents. The other type of agents do not invest in a company leading to a lower total investment compared to the investment volume in the one-company case.

(iv) The type of agents with a larger W_t cannot obtain a payoff advantage over the other type of agents, even though the first-mover chooses a bankruptcy rule that favors them in all generic cases. When $2W_l \geq W_h \geq W_l$ (or $2W_h \geq W_l \geq W_h$), companies choose different bankruptcy rules and each type of agent invests in the company whose bankruptcy rule favors his type (if they invest in a company at all). Hence, there are no transfers between two types of agents. When $W_h \geq 2W_l$ (or $W_l \geq 2W_l$), both companies choose the same bankruptcy rule (the one that favors the type of agents with a larger W_t) and only the type of agents who are favored by this rule invest in companies. Hence, again, there are no transfers between two types of agents.

Our results in this subsection show that incorporating the competition for funds between borrowers (companies) has a critical influence on bankruptcy rules chosen in equilibrium. Interestingly, we do not find any noncooperative support for the proportional rule here. In fact, the results imply that the use of the proportional rule in the one-company case can attract more investment compared to the two-company case, which involves other rules (see iii above). Similarly, if a central authority implements the proportional rule in the two-company case, this can attract more investment to these two companies than a liberal regime that allows companies to choose their bankruptcy rules strategically (as modeled above).

5 Conclusion

In this paper, we introduce a new class of bankruptcy problems, which has an empirical and a strategic appeal. In these bankruptcy problems, the value of the estate to be allocated is endogenous and depends on agents' investment decisions. This is what we observe in many real life situations. For instance, the amount to be allocated by a firm to its shareholders/stockholders may depend on the initial amount of money borrowed from them. Similarly, a bank offers a return rate on deposits. Moreover, in line with some recent suggestions in favor of a more liberal bankruptcy law, which provides a menu of bankruptcy procedures and allows companies to select among them (see Hart 2000), we allow the company in our base model to choose from a menu of bankruptcy rules that consists of three most well-known rules. The company's objective in choosing a bankruptcy rule is to maximize its profits. Agents, in an attempt to maximize their expected payoffs, observe the choice made by the company and decide whether to lend money to the company (risky investment) or deposit their money in a savings account (risk-free investment).

Our results from the base model show that the proportional rule receives a strong support in this strategic setting: the proportional rule is the only rule among the three that is always a part of subgame perfect Nash equilibrium. A direct implication is that there is no equilibrium in which the proportional rule leads to a lower investment volume than the other rules. This result is independent of the income distribution in the economy and holds even under one-sided uncertainty on income distribution (i.e., the company does not know the income distribution perfectly). In fact, for probability distributions that assign non-zero probability to all possible income distributions, the proportional rule would be the unique optimal strategy for an expected profit maximizing company. We also extend our base model to a setup that includes a larger set of bankruptcy rules (e.g., Talmud rule or TAL-family), more than two types of agents, and competition between two firms for investment funds. Interestingly, the results of the extended model with competition do not lend much strategic support to the proportional rule.

Our results provide (i) another potential reason for the centrality of the proportional rule in the bankruptcy literature and (ii) a possible, interesting context (i.e., competition between firms) where this support fails to prevail. To the best of our knowledge, this is the first paper which models the bankruptcy rule determination as a sequential game between lenders (agents) and borrowers (companies). It is also the first paper, which embeds the classical bankruptcy problem in a decision-making under uncertainty environment and uses bankruptcy rules as a tool for competition in attracting investors.

We should emphasize here that we introduce *a* model where incentives, strategic interaction, and uncertainty are incorporated in to a bankruptcy framework. We do not claim that this is *the only* way to do it. For instance, we just think that a stochastic return on investment is a natural assumption. There are surely other possible ways to introduce uncertainty. One such modeling alternative would be making claims stochastic.[10] Ertemel and Kumar (2013) characterize ex-ante and ex-post proportional rules in such a setting. There may also be alternative game forms that can be used in incorporating incentives and strategic interaction. We firmly believe that future research on bankruptcy games will explore some of these alternative ways and arrive at *better* models. In that respect, this study should be considered as one of the first attempts along these lines.

Another point worth mentioning is the proportional return structure embedded in our model. That is, the company promises a percentage return to investors, which leads to a proportional relationship between the endogenous estate and the aggregate claim. This type of a return structure is very common in many real life circumstances. That is why we made this assumption. On the other hand, this assumption can be seen as a factor leading towards the support found for the proportional rule in the base model. Nevertheless, the same assumption is made in the extended model with competition as well, where we did not find support for the proportional rule. We believe that investigating the effects of alternative—may be more sophisticated, state-contingent—return structures on equilibrium behavior is a worthy goal, left for future work.

Finally, we believe that our framework can be developed in many other dimensions. In particular, future work on the topic may endogenize market forces that govern interest rates and incorporate network structures that are present in markets for funds.

[10] I would like to thank an anonymous reviewer for raising this question.

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