

The Effect of Demand Uncertainty on the Decisions and Revenues in the Two Class Revenue Management Model

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Abstract.

We explore the impact of changes in market conditions on optimal allocation decisions and revenues, within the standard two-class revenue management framework, using stochastic dominance relations. We show that an increase in market size leads to higher revenues, and the number of units allocated to the high-end class increases in its market size. The direction of the change in optimal allocation and revenues in response to changes in the variability of the high-end market depends on the relationship between the high and low-end prices. Our structural and numerical results suggest higher variability in the market is generally detrimental to revenues.

Keywords: Revenue Management, Production Management.

JEL Classification: M11.

Özet. Talep Belirsizliğinin İki Sınıflı Gelir Yönetimi Modellerinin Karar ve Gelirleri Üzerindeki Etkisi

Bu makalede talep belirsizliğinin standart iki sınıflı gelir yönetimi modelini üzerindeki etkisi analitik ve sayısal olarak incelenmektedir. Talep belirsizliğinin bu modelden elde edilen optimal tahsis kararı, ve iki sınıftan olan ve toplam gelirler üzerindeki etkisi stokastik ilişkiler kullanılarak incelenmektedir. Talebin stokastik olarak büyümesinin elde edilen optimal gelirleri, ve daha değerli alıcı sınıfına ayrılan kapasite miktarını arttırdığı analitik olarak gösterilmektedir. Talebin varyansındaki değişikliklerin toplam gelirler; her iki dünüştan elde edilen gelirler; ve tahsis kararları üzerindeki etkisinin ise iki alıcı sınıfının satış fiyatları arasındaki ilişkiye bağlı olduğu görülmüştür. Makalede sunulan analitik ve sayısal sonuçlar talep varyansındaki olası bir artışın gelirlere etkisinin genellikle olumsuz olduğunu göstermektedir.

Anahtar Kelimeler: Gelir Yönetimi, Üretim Yönetimi.

JEL Sınıflaması: M11.

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1. Introduction

In this paper, we study the impact of changing market conditions on the firm's allocation decision and revenues within the standard two-class revenue management framework. In this model, a firm sells a fixed resource/capacity (such as seats on a flight or rooms in a hotel) to two market segments with sequential price levels and uncertain demands. The customers with higher willingness to pay arrive later in the sales horizon, and the firm determines the *protection level*, i.e. the number of units to protect for the high-end class, with the objective of maximizing revenues. We focus on this model, because it is the base model for revenue management and the oldest revenue management model still in use. Market factors have been acknowledged as the major source of uncertainty in many operations management models (Davis 1993), and have considerable impact on how operational processes are managed and their results. Hence, it is worthwhile to study how revenue management systems behave under changing market conditions.

We first investigate the impact of a change in the *market size* (modeled with a change in the demand distribution in the sense of *first order stochastic dominance*, see Section 3 for a formal definition) on optimal allocation and revenues. Throughout the paper, "optimal" refers to the expected revenue maximizing level. We show that it is more profitable for the firm to allocate a greater portion of the fixed resource to the customers with higher willingness to pay (henceforth referred to as class 1), i.e., increase the protection level, when the size of its market increases; a stochastically bigger class 1 market also leads to higher class 1 and total revenues if the protection level is chosen optimally.

A change in the size of the low-end market (henceforth referred to as class 2), on the other hand, results in an increase in the revenues obtained from this class, while its impact on the optimal class 1 and total revenues is not clearly determined. Our numerical experiments suggest that although a larger class 2 market generally leads to lower revenues from the high-end market, this relationship is reversed when (1) the values of class 1 and class 2 sales are close, (2) class 2 market is stochastically large, and (3) there is ample capacity. We also observe that total revenues increase in the size of the class 2 market, leading us to conclude that larger markets generally have a positive impact on revenues.

Our analysis of changes in *market variability* (modeled with a *mean preserving spread*, formalized in Section 4) shows that an increase in the high-end market variability always leads to lower revenues given an allocation; however, the behavior of the firm's revenues when the protection level is chosen optimally is not clearly determined. In particular, when the ratio of the class 2 price to the class 1 price is greater than $\frac{1}{2}$ (and hence, the values of class

1 and class 2 sales to the firm are close), optimal class 2 revenues increase as the class 1 market gets more variable, while the optimal protection level, class 1 and total revenues move in the opposite direction. When the price ratio is less than $\frac{1}{2}$, on the other hand, the optimal protection level decreases, and revenues from class 2 sales increase as class 1 market variability increases, while class 1 revenues may increase (because of the increase in the optimal protection level) or decrease (because of higher variability). Our numerical experiments suggest that the protection level effect dominates, leading to higher revenues from class 1 sales, when class 1 sales are more valuable compared to class 2 sales, or when class 2 market variability is low.

We also investigate the impact of a change in the variability of the class 2 market on optimal revenues. We show that a more variable class 2 market leads to lower revenues from this class. Furthermore, although the relationship between class 2 market variability and optimal class 1 revenues is not clearly determined structurally, we observe via our numerical experiments that total and class 1 revenues generally decrease in response to increasing class 2 market variability, leading us to conclude that higher variability is generally detrimental to revenues.

Literature Review. The literature on revenue management is vast; the most comprehensive works to date are the books by Talluri and van Ryzin (2004) and Phillips (2005). McGill and van Ryzin (1999) review the earlier revenue management literature, Elmaghraby and Keskinocak (2003) focus on dynamic pricing, and Shen and Su (2007) discuss customer behavior modeling. Papers most closely related to the model considered in this work are those that study the allocation of a single resource among different customer segments when demand classes arrive sequentially - see e.g. Littlewood (1972), Belobaba (1989), Curry (1990), Wollmer (1992), Brumelle and McGill (1993), Robinson (1995).

There are few papers in the operations management literature that study the impact of stochastically changing demand conditions on an operational model. In the newsvendor context, Gerchak and Mossman (1992) study the magnitude of change in optimal cost and order quantity when there is an increase in demand variability. They show that higher variability leads to higher costs. Song (1994) focuses on the impact of changing leadtime variability on optimal inventory decisions. Ridder et al. (1998) study the behavior of optimal cost with respect to changing market risk under different variability orders. Li and Atkins (2005) focus on the impact of changing demand variability within the price-setting newsvendor problem, and characterize the direction of the change on optimal price and order quantity. Song and Zipkin (1996) consider a more general inventory system than the newsvendor, and study

how increased lead time variability affects the inventory size via numerical experiments; while Song et al. (2010) consider the impact of stochastically larger and more variable lead time on cost and policy parameters. Federgruen and Wang (2012) study when the optimal policy and cost parameters are monotone in demand uncertainty for a similar system. Gupta and Cooper (2005) study stochastic orderings of yield rates that guarantee a coherent ordering of profits. Li and Zheng (2006) compare optimal policies under certain and uncertain yield rates, and show that uncertainty leads to higher prices and lower expected profits.

In the revenue management context, Cooper and Gupta (2006) focus on how simultaneous changes in demand under various stochastic orders impact optimal revenues. Araman and Popescu (2010) show stochastically larger or less variable audiences do not necessarily command lower capacity allocations in the media broadcasting advertising market. Akcay et al. (2009) consider a multi-period revenue management problem where a firm sells a fixed inventory to multiple customer classes, and study the impact of varying problem parameters, including an increase in the arrival probability. Their model can incorporate a set up similar to ours if arrival probabilities are chosen such that customer classes with lower valuations arrive earlier in the sales horizon.

Structure. The rest of the paper is organized as follows. Our model, assumptions and notation are presented in Section 2. Section 3 provides structural results and numerical experiments that provide insights regarding the impact of changing market size on optimal allocation decisions and revenues. Similar analysis on changes in market variability is presented in Section 4. Section 5 concludes the paper.

2. The Model

This section sets up our basic model, main assumptions and notation. In the standard two-class revenue management model, the firm determines the allocation of a fixed resource, denoted by C , between two market segments (class 1 and class 2) with sequential price levels (p_1 and p_2 with $p_1 > p_2$ without loss of generality), with the objective of maximizing revenues. In typical revenue management settings, fixed costs are high and variable costs are negligible; hence, earnings are stated in terms of revenues. The firm sets the number of units reserved for the higher priced class, called the *protection level*, denoted by x .

The firm faces uncertain demands D_1 and D_2 from the higher priced class (class 1), and the lower priced class (class 2), respectively; class 2 customers arrive before class 1 customers. The allocation decision is made

before demand from either class is realized. Letting $r_1(x)$ and $r_2(x)$ denote the revenue obtained from class 1, and class 2, respectively, for a given protection level x , the firm's problem is formally modeled as,

$$\max_x R(x) = r_1(x) + r_2(x) = p_1 E[\min(D_1, \max(C - D_2, x))] + p_2 E[\min(D_2, C - x)].$$

Sales to class 2, $\min(D_2, C - x)$, is constrained by the demand for this class, and the *booking limit*, $C - x$, i.e., the units available for sale to class 2. Similarly, sales to class 1 is the minimum of the demand and available units for this class, which may exceed the protection level if class 2 demand falls short of the booking limit. Over-allocating to class 1 leaves the firm with unused capacity, which brings no revenue. Under-allocating to class 1, on the other hand, results in lost class 1 sales; in this case, units that could have been sold to class 1 customers were sold to class 2 customers earlier in the sales horizon at a lower price. Letting $P(D_1 \geq u) = 1 - P(D_1 \leq u)$ denote the survival function of D_1 , the unique optimal protection level is established by Littlewood (1972).

Proposition 1. (Littlewood, 1972) *The revenue function $R(x)$ is quasi-concave in x , and the optimal protection level, x^* is given by*

$$P(D_1 \geq x^*) = p_2 / p_1 \quad (0.1)$$

From (1.1), the optimal protection level is a function of the ratio of class 2 price to class 1 price (p_2 / p_1), and the distribution of class 1 demand D_1 . Intuitively, the expected revenue loss from over- and under-allocating to class 1 is matched at the optimal protection level. When the decision maker over-allocates to class 1, she is left with unused capacity that could have been sold to a class 2 customer earlier in the sales horizon and brought p_2 , if it had not been reserved for class 1. When the decision maker under-allocates to class 1, on the other hand, she loses $p_1 P(D_1 \geq x)$, since she would have earned p_1 from that unit, if it had not been sold to a class 2 customer, provided there was ample demand.

3. The Effect of Larger Demand

This section studies the impact of changes in market size on the firm's optimal allocation decision and revenues. We model an increase in the market size by an increase in D_j , $j=1,2$, in the sense of *first order stochastic dominance*, defined below. Throughout the paper, $E[\cdot]$ denotes the expectation operator.

Definition 1. A random variable X dominates another random variable Y in the sense of first order stochastic dominance, denoted $X \succ_{FSD} Y$, if $P(X \geq u) \geq P(Y \geq u)$ for all u . This is equivalent to $E[h(X)] \geq E[h(Y)]$ for all increasing functions h .

Remark that, if $X \succ_{FSD} Y$, then $E[X] \geq E[Y]$. Hence, X is stochastically larger than Y . For example, for two distributions X and Y , with $X = \delta + Y$, where $\delta \geq 0$, $X \succ_{FSD} Y$. For more on stochastic orders, the reader is referred to Shaked and Shanthikumar (1994), and Müller and Stoyan (2002).

3.1. Larger Class 1 Demand

In this section, we investigate the behavior of the optimal protection level and revenues with respect to changes in the size of the high-end market. We show that an increase in its market size leads to more resources being allocated to class 1, and to higher overall revenues.

We first study the impact of a change in the size of the class 1 market on the optimal allocation decision. Throughout the paper, we use stochastic dominance relations to compare optimal protection levels and revenues. When $D'_1 \succ_{SD} D_1$ where SD is any stochastic order, we denote the revenue under D'_1 as $R'(x) = p_1 E[\min(D'_1, \max(C - D_2, x))] + p_2 E[\min(D_2, C - x)]$, and the corresponding optimal protection level as x^* , i.e. $x^* = \arg \max R'(x)$.

For the newsvendor problem, Song (1994) establishes that a stochastically bigger market (modeled by a shift in the sense of FSD) leads to a higher optimal order quantity. The two-class revenue management and the newsvendor problems are structurally close; in both cases, the firm tries to find the quantity to order/allocate that would minimize the instances of turned away customers and unsold units. Hence, we apply Song's (1994) insight to our problem, and obtain the result below. All proofs are provided in the Appendix.

Proposition 2. The optimal protection level increases in the size of the class 1 market, i.e., $x^* \geq x^*$, whenever $D'_1 \succ_{FSD} D_1$.

This result is quite intuitive; a bigger market size requires more units allocated to the high-end class in order to compensate for the now-higher possibility of turned away customers. Next, we investigate the impact of changing market size on revenues.

Remark 1. An increase in the size of the class 1 market leads to higher total revenues, given a protection level x , i.e., $R'(x) \geq R(x)$ whenever $D'_1 \succ_{FSD} D_1$.

Next, we show that this property is preserved when the protection level is chosen optimally.

Proposition 3. *When there is an increase in the size of the class 1 market ($D_1' \succ_{FSD} D_1$), (a) the optimal class 1 revenue increases, i.e., $r_1'(x^*) \geq r_1(x^*)$, (b) the optimal class 2 revenue decreases, i.e., $r_2'(x^*) \leq r_2(x^*)$, (c) the optimal total revenue increases, i.e., $R(x^*) \geq R(x^*)$.*

The impact of the changing market size on the optimal class 2 revenues follows from Proposition 2; since class 2 revenue is not a function of the class 1 demand, it is impacted only through the change in the optimal protection level, and class 2 revenues decrease in the number of units allocated to class 1. The high-end revenue, on the other hand, is affected both by the increase in demand (which leads to higher revenues for a given protection level; see Remark 1) and the change in the protection level (which increases; see Proposition 2); these two effects lead to higher revenues from class 1 under the optimal allocation. The increase in optimal class 1 revenues offsets the decrease in the class 2 revenues, resulting in higher total revenues.

3.2. Larger Class 2 Demand

This section investigates how the optimal protection level x^* , class 2, class 1, and total revenues change when there is a shift in the low-end demand in the sense of the first order stochastic dominance, i.e., when $D_2' \succ_{FSD} D_2$. We show that the optimal class 2 revenue increases in the size of its market, and present numerical insights on the relationship between the size of the class 2 market, and optimal class 1 and total revenues.

3.2.1. Structural Results

First, remark that, from the optimality condition (1), the optimal protection level is determined by the distribution of class 1 demand and the selling prices of the two market segments; hence, the optimal protection level is not influenced by changes in the class 2 market, i.e., $x^* = x^*$, whenever $D_2' \succ_{FSD} D_2$. This, and the fact that class 2 sales, $\min(D_2, C - x)$ is decreasing in D_2 , leads to the following relationship between the size of the class 2 market and revenues from this class.

Proposition 4. *When there is an increase in the size of the class 2 market ($D_2' \succ_{FSD} D_2$), the optimal class 2 revenue increases, i.e., $r_2'(x^*) \geq r_2(x^*)$.*

The impact of a change in the size of the low-end market on the optimal class 1 revenues, on the other hand, is determined by two, typically opposing, effects. Because sales to class 1, $\min(D_1, \max(C - D_2, x))$, is restricted by the demand and the number of units available for this class (which might be higher than the protection level x , if class 2 demand falls short of the booking limit), a larger class 2 demand implies less units available to class 1 because there is now a lower possibility of class 2 demand falling short of the booking

limit. This leads to a decrease in revenues obtained from class 1. However, an increase in the size of the class 2 demand also decreases the possibility of being left with unsold units because overall demand is now larger, which leads to higher revenues. The direction of the change in optimal class 1 revenues is determined by the stronger effect. This is further explored with our numerical experiments, presented in the next section.

3.2.2. Numerical Experiments

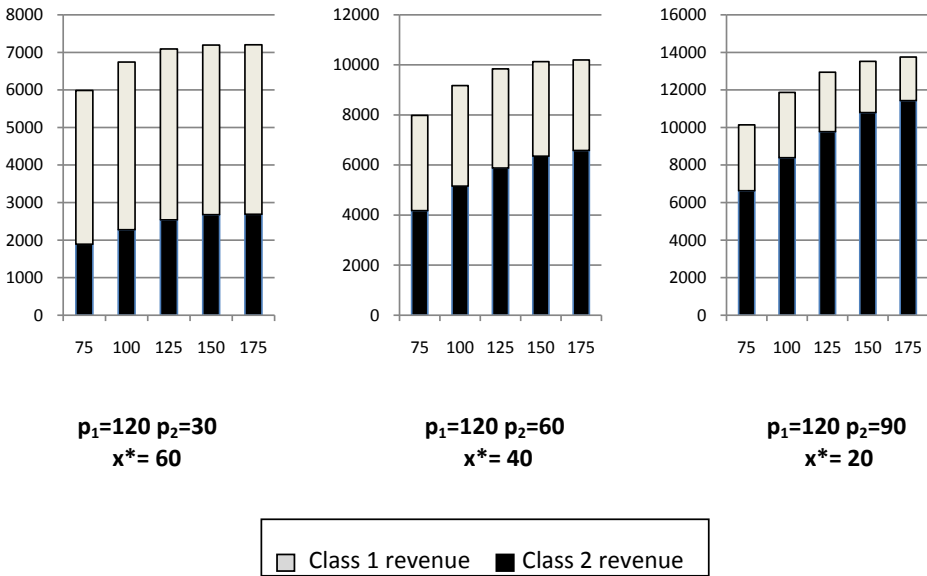
In this section, we evaluate class 2, class 1 and total revenues under the optimal allocation in response to changing class 2 market sizes with respect to problem parameters such as price, capacity and class 1 demand. We observe that, as proved in the previous section, optimal class 2 revenues increase in the size of its market. The direction of the change in revenues from class 1, on the other hand, depends on the problem environment; in particular, we observe that, optimal class 1 revenues increase in the size of the class 2 market when: (1) class 1 sales are much more valuable for the firm compared to class 2 sales, (2) there is ample capacity, and (3) class 1 market is stochastically large. We also observe that total revenues increase in response to increases in the size of the low-end market. We present numerical results for specific problem parameters; extensive numerical experiments with a wide range of parameters suggest that the insights illustrated in this section are robust. Throughout this section, class 2 demand (distributed *Uniform*) is varied consistent with a first order stochastic shift; in particular, we use the following distributions: $D_2 \sim \text{Uniform}(0,150)$, $\text{Uniform}(25,175)$, $\text{Uniform}(50,200)$, $\text{Uniform}(75,225)$, and $\text{Uniform}(100,250)$, corresponding to expected class 2 demands of 75, 100, 125, 150 and 175 respectively ($E[D_2] = (a+b)/2$ for $D_2 \sim \text{Uniform}(a,b)$).

Results with respect to selling prices. Figure 1 presents the optimal class 2, class 1 and total revenues under changing class 2 demands. We kept the class 1 price at $p_1=120$, and varied the class 2 price to obtain price ratios $p_2 / p_1 = \{0.25, 0.5, 0.75\}$. We chose these values because, when $p_2 / p_1 < 1/2$, the value of class 1 sales to the firm is much higher than the class 2 sales, compared to when $p_2 / p_1 > 1/2$. Class 1 demand is distributed $D_1 \sim \text{Uniform}(0,80)$ with capacity set at $C=150$. These particular demand and capacity parameters were chosen because revenue management is most relevant when the capacity is binding, yet ample enough to serve both segments. Also remark that, from the optimality condition (1), when class 1 demand is distributed $\text{Uniform}(a,b)$, the optimal protection level solves $b - (p_2 / p_1)(b - a)$.

We observe that the revenues from class 2 increase in the size of its market, as proved in Proposition 4. The direction of the change in optimal

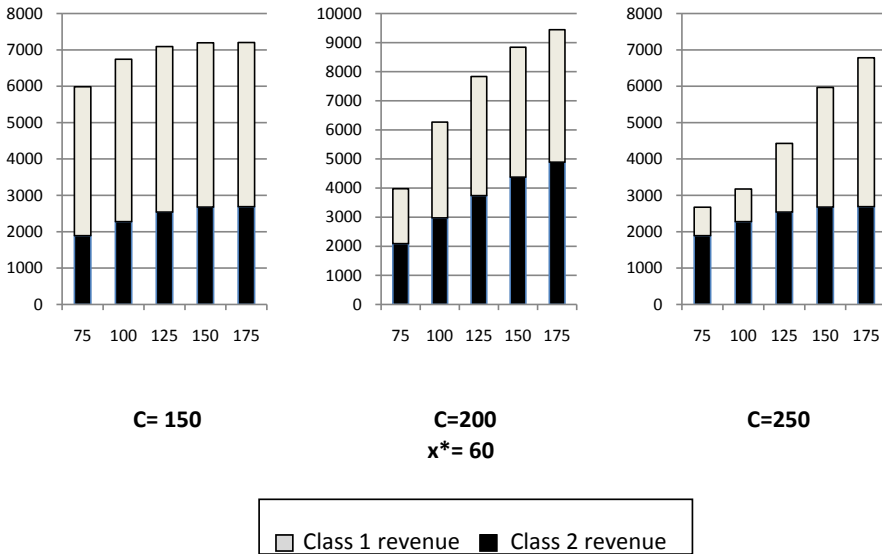
class 1 revenues, however, depends on the problem environment. As argued above, an increase in the size of the class 2 market leads to fewer number of units being available to class 1 (and hence, to lower revenues from this class); however, it also decreases the possibility of unsold units (and hence, increases class 1 revenues). When the size of the class 2 market is small (e.g., when $D_2 \sim \text{Uniform}(0,150)$, or $\text{Uniform}(25,175)$), the total expected demand is less than the available capacity (e.g., when $D_2 \sim \text{Uniform}(0,150)$ with $D_1 \sim \text{Uniform}(0,80)$, total expected demand is equal to $E[D_1] + E[D_2] = 40 + 75 = 115 < 150 = C$), and hence, the possibility of being left with unused capacity is higher. Consequently, when there is an increase in the size of the class 2 market, the decrease in the possibility of unsold units dominates the effect of the available units; hence, revenues from class 1 increase. When the size of the class 2 market is larger however, the expected total demand is close to, and possibly greater than, the available capacity; hence, the possibility of being left with unsold units is already small. In this case, the decrease in the number of units available to class 1 dominates. This effect is particularly more pronounced when the ratio of class 2 price to class 1 price is higher (e.g., $p_2 / p_1 = 0.75$), because in this case, the optimal protection level is low, and hence the possibility of being left with unused capacity due to over-allocation is smaller for all levels of class 2 demand.

Results with respect to available capacity. In this section, we vary the capacity available, which essentially determines the level of congestion in the system. Figure 2 presents class 2, class 1 and total revenues under the optimal protection level, for varying sizes of class 2 demand, when capacity is equal to $C=150$, $C=200$, and $C=250$. The class 1 and class 2 prices are $p_1=120$ and $p_2=30$, and the class 1 demand is distributed $D_1 \sim \text{Uniform}(0,80)$. Remark that, from the optimality condition (1), the number of units available for sale does not impact the optimal protection level, x ; hence, it is the same (60) for all capacity levels.

Figure 1. Optimal revenues under varying price ratios with respect to expected class 2 demand

We observe that the optimal class 1 revenues either monotonically increase (when $C=200$, $C=250$), or first increase, and then decrease ($C=150$) as the size of the class 2 market increases. Having more units for sale increases the possibility of unsold units; hence, when there is an increase in the size of the class 2 market, its impact on the number of unsold units is stronger than its impact on the number of units available to class 1. Optimal class 2 revenues increase, as predicted by Proposition 4. We also observe that total revenues increase as the size of the class 2 market increases.

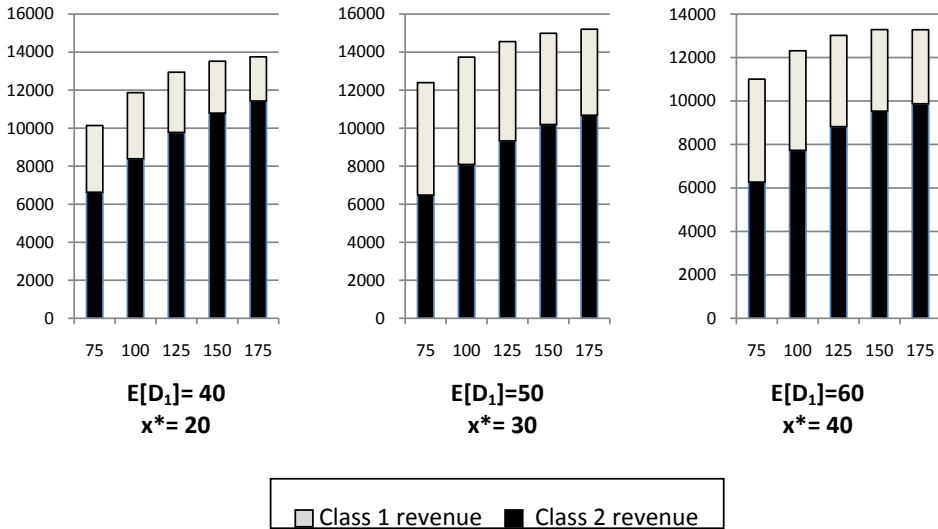
Results with respect to high-end demand. Finally, we vary the class 1 demand, consistent with a first order stochastic shift; in particular, we calculate class 1, class 2 and total optimal revenues under the optimal allocation when class 1 demand is distributed $D_1 \sim \text{Uniform}(0,80)$, $\text{Uniform}(10,90)$, and $\text{Uniform}(20,100)$, corresponding to expected class 1 demands of $E[D_1]=40$, 50, and 60 respectively. Capacity is kept at $C=150$. The selling prices are $p_1=120$ and $p_2=90$ in Figure 3, and $p_1=120$ and $p_2=30$ in Figure 4. We present both cases, because we observe different trends with respect to the change in optimal class 1 revenues in each case.

Figure 2. Optimal revenues under varying capacity levels with respect to expected class 2 demand

When the ratio of class 2 price to class 1 price is equal to $\frac{1}{4}$, i.e., class 1 sales are much more valuable compared to class 2 sales, class 1 revenues increase as the size of the class 2 market gets bigger, when class 1 demand is also stochastically large, e.g., $D_1 \sim \text{Uniform}(20,100)$. This effect is observed because the increase in the size of the class 1 market propagates the decrease in the possibility of unsold units. It also leads to higher total revenues, as class 2 revenue is also increasing in its market size. When the ratio of class 2 price to class 1 price is $\frac{3}{4}$, however, class 1 revenues decrease as class 2 market gets bigger, regardless of the size of the class 1 market. As discussed above, when the selling prices are closer, the optimal protection level is lower, which results in lower possibility of unsold units. Hence, the decrease in the number of units available to class 1 dominates, and class 1 revenues decrease in the size of the class 2 market.

4. The Effect of More Variable Demand

In this section, we consider the impact of changes in market variability on optimal revenues and allocations. In order to model a change in market variability, we employ the concept of *mean preserving spread*, introduced by Rothschild and Stiglitz (1970, 1971).

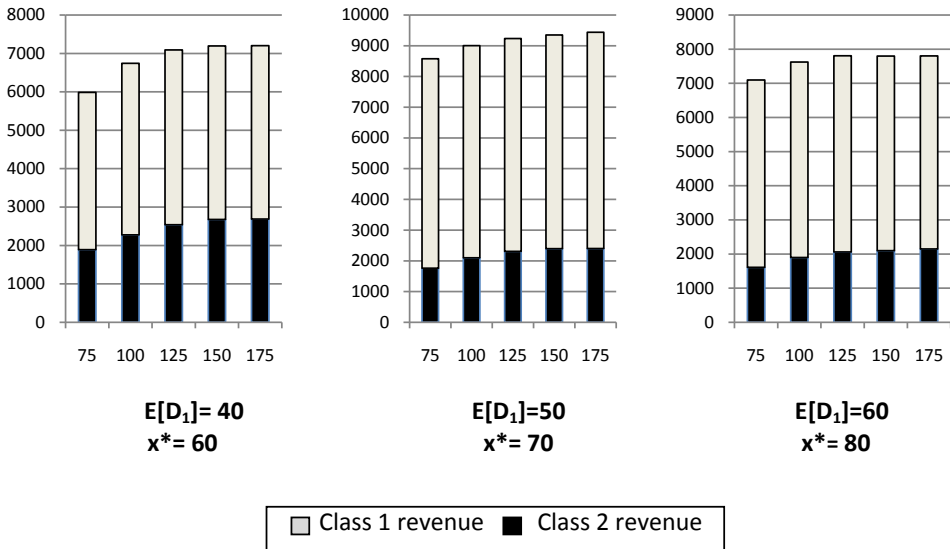
Figure 3. Optimal revenues under varying class 1 market sizes with respect to expected class 2 demand; $p_2 / p_1 = 3/4$ 

Definition 2. A random variable X differs from another random variable Y by a mean preserving spread ($X \succ_{MPS} Y$), if they have the same finite mean and if there is an interval (a,b) such that X assigns no greater probability than Y to any open subinterval of (a,b) , and X assigns no smaller probability than Y to any open interval either to the left or the right of (a,b) .

Rothschild and Stiglitz (1970) (and also Landsberger and Meilijson 1990, Pratt and Machina 1997) show that this is equivalent to $E[h(X)] \geq E[h(Y)]$ for all convex functions h . Remark that, if $X \succ_{MPS} Y$, then $Var[X] \geq Var[Y]$. Furthermore, two distributions that differ by a mean preserving spread exhibit single crossing property: if $X \succ_{MPS} Y$, there exists some k such that $P(X \geq u) \leq P(Y \geq u)$ for $u \leq k$ and $P(X \geq u) \geq P(Y \geq u)$ for $u \geq k$. An example for two distributions that differ by a mean preserving spread is $X = \delta Y$, where $\delta \geq 0$, and Y is a random variable with $E[Y] = 0$. In this case $X \succ_{MPS} Y$.

4.1. More Variable Class 1 Demand

This section investigates how optimal allocation decision and revenues behave with respect to changes in the variability of the high-end market. We restrict our analysis to symmetric demand distributions; many traditional demand densities, such as the Normal, Uniform (see Silver and Peterson 1985, Tijms 1994) satisfy this property.

Figure 4. Optimal revenues under varying class 1 market sizes with respect to expected class 2 demand; $p_2 / p_1 = 1/4$ 

4.1.1. Structural Results

For the two class revenue management model, Kocabıykoğlu and Göğüş (2012) establish the relationship between the optimal protection level and high-end demand variability. We present their results below, for completeness.

Proposition 5. (Kocabıykoğlu and Göğüş, 2012) *When there is an increase in the variability of the class 1 market, (a) the optimal protection level decreases if $p_2 / p_1 \geq 1/2$, and (b) the optimal protection level increases if $p_2 / p_1 \leq 1/2$.*

When $p_2 / p_1 \leq 1/2$, class 1 sales are much more valuable compared to the class 2 sales than when $p_2 / p_1 \geq 1/2$; hence, over-allocating to class 1 (i.e. being left with unused capacity) is more detrimental to the firm when $p_2 / p_1 \geq 1/2$, whereas under-allocating is more detrimental when $p_2 / p_1 \leq 1/2$. Consequently, when $p_2 / p_1 \geq 1/2$, in order to avoid being left with unused capacity, which has now a higher probability because of the shift of the probability mass from the center to the lower tail of the class 1 distribution, the firm shifts allocation from class 1 to class 2 customers, resulting in lower protection levels. When $p_2 / p_1 \leq 1/2$, optimal protection level increases, in order to avoid selling to class 2 customers in the expense of the class 1 customers. Note that, in both cases the optimal protection level moves away from the mean (see the proof of Proposition 5 in the Appendix).

Remark 2. *An increase in the variability of the class 1 market leads to lower total revenues, given an allocation level x , i.e., $R(x^*) \leq R(x^*)$ whenever $D_1' \succ_{MPS} D_1$.*

We establish that the above result is preserved when the protection level is chosen optimally, if $p_2 / p_1 \geq 1/2$, with Proposition 6, below.

Proposition 6. *Suppose $p_2 / p_1 \geq 1/2$. When there is an increase in the variability of the class 1 market ($D_1' \succ_{MPS} D_1$) (a) the optimal class 1 revenue decreases, i.e., $r_1'(x^*) \leq r_1(x^*)$, (b) the optimal class 2 revenue increases, i.e., $r_2'(x^*) \geq r_2(x^*)$, and (c) the optimal total revenue decreases, i.e. $R(x^*) \leq R(x^*)$.*

Low-end revenue is influenced by changes in class 1 demand only through the change in the optimal protection level, which is decreasing in the variability of the class 1 market if $p_2 / p_1 \geq 1/2$ by Proposition 5(a). Lower protection levels lead to more available units for class 2, and consequently to higher revenues from this class. Class 1 revenues under the optimal allocation decrease because of increasing demand variability (Remark 2) and the decrease in the optimal protection level (Proposition 5a). The decrease in class 1 revenues dominates the increase in class 2 revenues, resulting in lower total revenues.

The impact a change in the variability of class 1 market on optimal revenues when $p_2 / p_1 \leq 1/2$, however, is not clearly determined. The optimal class 2 revenues decrease, because it is affected only through the change in the optimal protection level, which is higher under more variable demand (Proposition 5b).

Proposition 7. *Suppose $p_2 / p_1 \leq 1/2$. When there is an increase in variability of the class 1 market ($D_1' \succ_{MPS} D_1$), the optimal class 2 revenue decreases, i.e., $r_2'(x^*) \leq r_2(x^*)$.*

The optimal class 1 and total revenues may increase or decrease in response to changing class 1 market variability, when $p_2 / p_1 \leq 1/2$. An increase in the variability of the class 1 market leads to lower revenues (Remark 2), but it also leads to more units being allocated to class 1 (Proposition 5b), and hence to higher revenues. With our numerical experiments, presented in the next section, we provide further insights on the relationship between changing market variability and optimal revenues.

4.1.2. Numerical Experiments

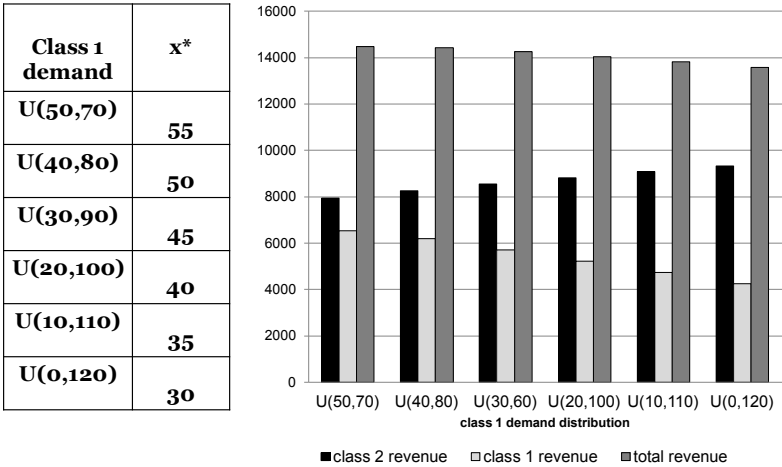
In this section, we first provide an example that illustrates the structural results obtained in Proposition 6. In particular, we evaluate class 1, class 2 and total revenues under varying degrees of class 1 market variability when $p_2 / p_1 \geq 1/2$. Then, we present the results of numerical experiments that

evaluate revenues under changing class 1 market variability with respect to several problem parameters. In our numerical experiments, we focus on examples where $p_2 / p_1 \leq 1/2$, because, as discussed above, the relationship between the variability of the class 1 market and optimal revenues is not clearly determined in this case. We observe that optimal class 1 and total revenues generally decrease in the face of increasing class 1 variability. Throughout this section, class 1 demand is varied in a manner consistent with a mean preserving spread. In particular, the high-end demand is distributed $D_1 \sim \text{Uniform}(50,70)$, $\text{Uniform}(40,80)$, $\text{Uniform}(30,90)$, $\text{Uniform}(20,100)$, $\text{Uniform}(10,110)$, $\text{Uniform}(0,120)$, corresponding to expected class 1 demand $E[D_1] = 60$ and demand variances of 33.33, 133.33, 300, 533.33, 833.33 and 1200, respectively.

Example for Proposition 6. Figure 5 plots the optimal class 1, class 2 and total revenues under varying degrees of class 1 demand variability, and price levels $p_1=120$ and $p_2=90$ (i.e., $p_2 / p_1 = 3/4 > 1/2$). Class 2 demand is distributed $D_2 \sim \text{Uniform}(50,200)$, with capacity $C=150$. Remark that, the optimal protection level decreases as the class 1 market gets more variable. This decrease in the number of units allocated to class 1 leads to higher class 2 revenues (Proposition 6b). For example, when the class 1 demand distribution changes from $D_1 \sim \text{Uniform}(40,80)$ to $D_1 \sim \text{Uniform}(30,90)$ (hence leading to higher variability), this change impacts the optimal class 2 revenues through the change in the protection level (which decreases from 50 to 45). The corresponding class 2 revenue levels are $r_2'(45) = 8543 > 8250 = r_2(50)$. The optimal class 1 revenues, on the other hand, are affected by both the decrease in the protection level (which leads to a decrease in revenues of $r_1(50) - r_1(45) = 6183 - 5866 = 317$) and the increase in variability (which leads to a decrease in revenues of $r_1(45) - r_1'(45) = 5866 - 5713 = 153$); resulting in an overall decrease of 470.

Results with respect to selling prices when $p_2 / p_1 \leq 1/2$. In this section, we keep the class 1 price at $p_1=120$, and vary the class 2 price to obtain price ratios $p_2 / p_1 = \{0.125, 0.25, 0.375\}$. Class 2 demand is distributed $D_2 \sim \text{Uniform}(50,200)$, with capacity $C=150$. Figure 6 presents class 1, class 2 and total revenues under the optimal allocation (the optimal protection levels for each demand and price ratio pair are given in Table 1 in the Appendix).

Figure 5. Optimal revenues under varying degrees of class 1 market variability; $p_2 / p_1 = 3 / 4$



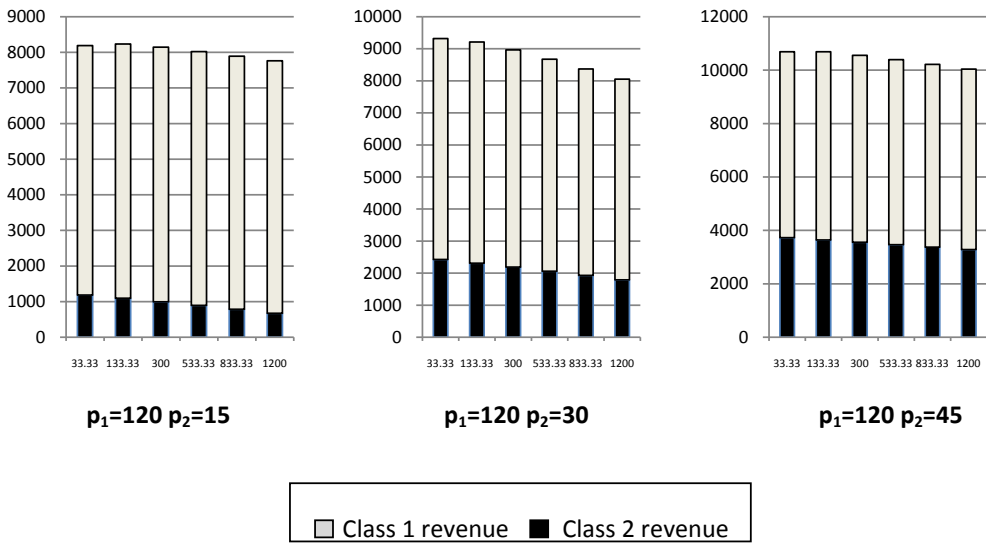
Note: $p_1=120$, $p_2=90$, $D_2 \sim \text{Uniform}(50,200)$, $C=150$.

We observe that the optimal class 2 revenue decreases as the class 1 market gets more variable, as proved in Proposition 7. The optimal class 1 revenue first increases, and then decreases in response to increasing variability in its market; the increasing trend lasts longer when the ratio of class 2 price to class 1 price is small (e.g., when $p_2 / p_1 = 0.125$). This is observed because in this case, class 1 sales are much more valuable for the firm than class 2 sales and hence the increase in the number of units available to class 1 dominates the decrease in revenues due to higher demand variability. Total revenues exhibit a similar trend.

Results with respect to low-end demand when $p_2 / p_1 \leq 1 / 2$. In this section, we vary the class 2 demand consistent with a mean preserving spread. In particular, we evaluate class 1, class 2 and total revenues when class 2 demand is distributed $D_2 \sim \text{Uniform}(100,150)$, $\text{Uniform}(50,200)$ and $\text{Uniform}(0,250)$, corresponding to expected class 2 demand $E[D_2]=125$, and variances of 208.33, 1875, and 5208.33, respectively. Capacity is set at $C=150$, and class 1 and class 2 selling prices are $p_1=120$ and $p_2=30$. Figure 7 presents our results (the optimal protection levels under the demand distributions and price levels considered are given in Table 1 in the Appendix). When class 2 variability is higher, (e.g., when $D_2 \sim \text{Uniform}(50,200)$ and $\text{Uniform}(0,250)$), class 1 revenues first increase, and then decrease in class 1 market variability, whereas class 1 revenues increase monotonically in the face of increasing variability in its market, when class 2 market variability is low (e.g., when $D_2 \sim$

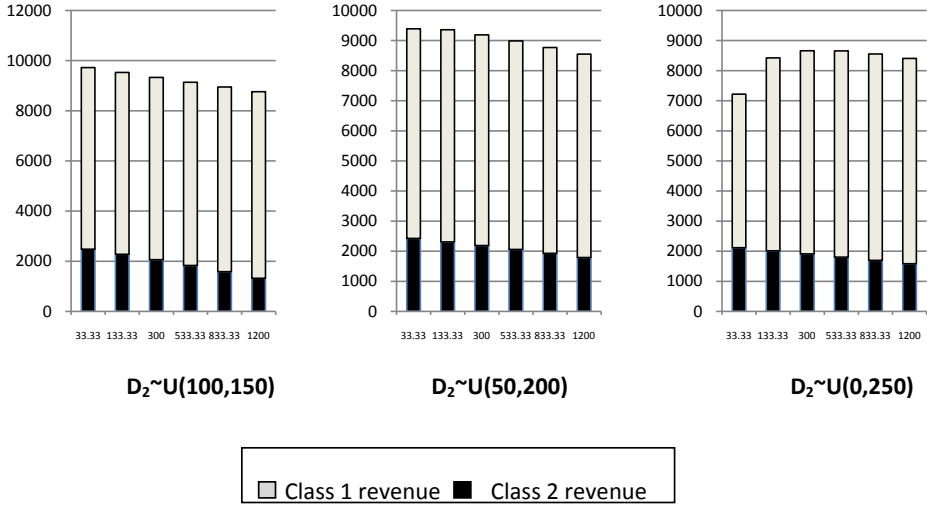
$Uniform(0,150)$); the low variability of class 2 demand absorbs the negative impact of higher class 1 market variability on revenues. Total revenues, on the other hand, either decrease monotonically, or first increase, then decrease, suggesting higher class 1 variability is generally detrimental to revenues.

Figure 6. Optimal revenues under varying price ratios with respect to class 1 variance



4.2. More Variable Class 2 Demand

This section presents our results regarding the direction of the change in optimal revenues in response to a change in the class 2 market consistent with a mean preserving spread, i.e., when $D_2' \succ_{MPS} D_2$. We show that the optimal revenues from class 2 decrease as its market gets more variable. Furthermore, our numerical experiments suggest that higher class 2 variability leads to lower revenues from the high-end market, except when the number of units available for sale is low. Total revenues decrease monotonically in response to an increase in the variability of the low-end market.

Figure 7. Optimal revenues under varying class 2 market variability with respect to class 1 variance

4.2.1. Structural Results

As argued in Section 3.2, the optimal protection level is not influenced by changes in the class 2 demand. Hence, an increase in the variability of the class 2 market impacts the revenues from this class only through the change in demand. We show that higher variability leads to lower revenues, with Proposition 8, below.

Proposition 8. *When there is an increase in the variability of the class 2 market ($D_2' \succ_{MPS} D_2$), the optimal class 2 revenue decreases, i.e., $r_2'(x^*) \leq r_2(x^*)$.*

The impact of higher class 2 variability on class 1 revenues under the optimal allocation, however, is not clearly determined. As noted above, a change in class 2 demand impacts the number of units available for class 1 ($\max(C - D_2, x)$), and consequently, class 1 sales. A mean preserving shift in the class 2 demand implies a shift of probability mass from the center to the tails. The shift to the lower tail might result in more units being available to class 1 (and hence, higher revenues), but also in a higher possibility of unsold units (and hence, lower revenues). The shift to the upper tail, on the other hand, leads to higher class 2 demand, and consequently to fewer units being available to class 1 (and lower revenues), and a lower possibility of unsold units (and higher revenues). This tradeoff is further explored in the next section with numerical experiments.

4.2.2. Numerical Experiments

In this section, we present the results of numerical experiments that evaluate class 2, class 1 and total revenues under varying degrees of class 2 market variability with respect to problem parameters such as capacity and class 1 demand. Throughout this section, we consider the following class 2 demand distributions, obtained through mean preserving shifts: $D_2 \sim \text{Uniform}(75,125)$, $\text{Uniform}(50,150)$, $\text{Uniform}(25,175)$, $\text{Uniform}(0,200)$, corresponding to variances of 208.33, 833.33, 1875 and 3333.33, respectively. Note that under all four distributions, expected class 2 demand is equal to 100.

Results with respect to available capacity. In this section, we evaluate revenues under the optimal allocation for capacity levels $C=100$, $C=150$ and $C=200$. Class 1 demand is distributed $D_1 \sim \text{Uniform}(0,80)$, and prices are given by $p_1=120$ and $p_2=90$ (implying an optimal protection level of 20). Note that, since class 1 demand remains the same, and the expected class 2 demand does not change when the distribution shifts by a mean preserving spread, lower available capacity implies a busier, more congested system. Figure 8 presents our results.

We observe that the optimal class 2 revenues decrease as the variability of its market increases, as predicted by Proposition 8. The optimal class 1 revenues either monotonically increase (e.g., when $C=100$), or decrease (e.g., $C=150$, $C=200$). As argued above, increasing variability is generally detrimental to revenues; however, when the number of units available for sale is low, the system absorbs the negative impact of the shift of probability mass from the center to the tails, leading to higher class 1 revenues. Total revenues decrease as class 2 market variability increases.

Results with respect to high-end demand. In this section, we vary the high-end demand consistent with a mean preserving spread. In particular, Figure 9 presents class 2, class 1 and total revenues under the optimal protection level, when class 1 demand is distributed $D_1 \sim \text{Uniform}(20,60)$, $\text{Uniform}(10,70)$, $\text{Uniform}(0,80)$, corresponding to expected class 1 demand $E[D_1]=40$, and variances 133.33, 300, 533.33, respectively. Capacity is set at $C=150$, and prices are $p_1=120$ and $p_2=90$.

From Figure 9, class 2, class 1 and total revenues under the optimal allocation decrease as the variability of the class 2 market increases, leading us to conclude that higher variance in the low-end market is generally detrimental to revenues.

Figure 8. Optimal revenues under varying capacity levels with respect to class 2 variance

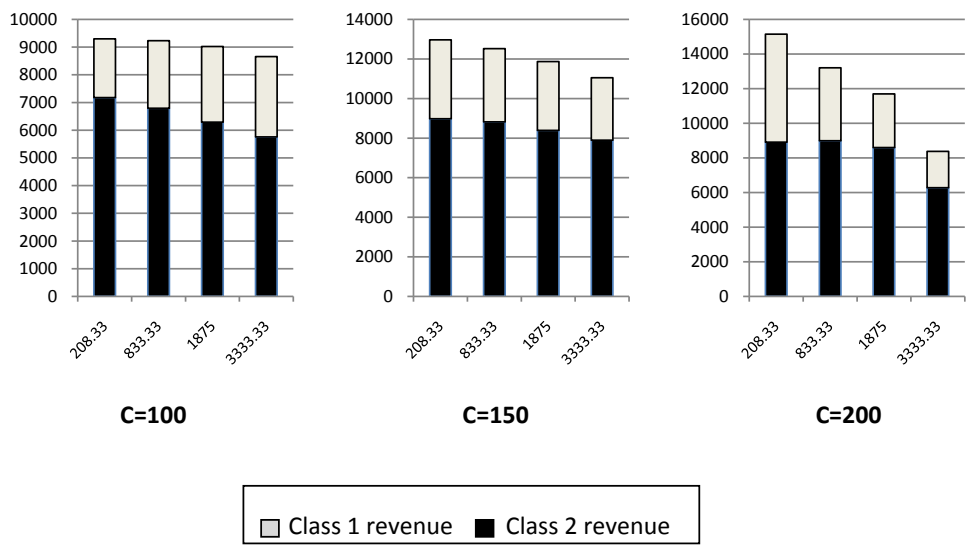
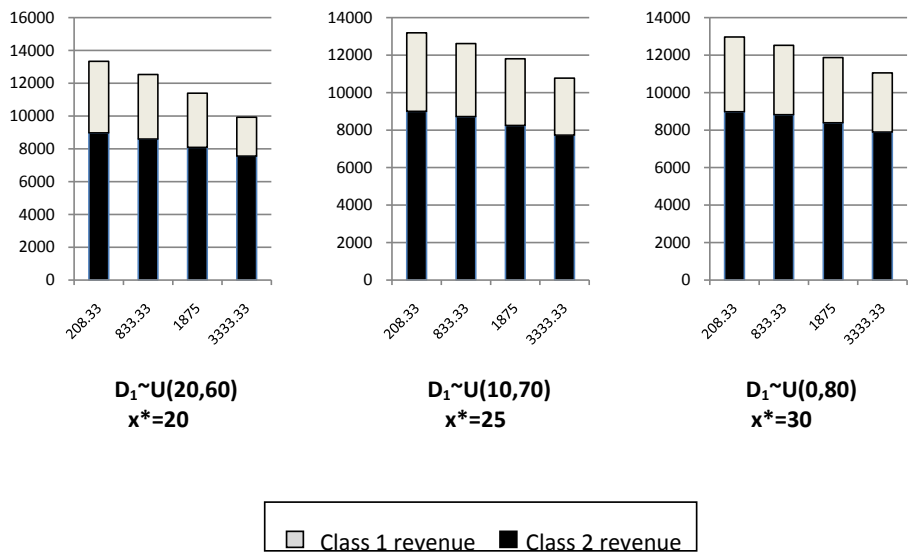


Figure 9. Optimal revenues under varying class 1 market variability with respect to class 2 variance



5. Conclusion

In this paper, we study the effect of demand uncertainty on optimal decisions and revenues within the two class revenue management model. We first study the impact of changing market size. We show that it is more profitable for the firm to allocate more to the high-end class as its market gets bigger in the sense of first order stochastic dominance; a bigger market also leads to higher overall revenues. While an increase in the size of the low-end market does not impact optimal decisions, it leads to higher revenues from this class, and, as our numerical experiments suggest, higher total revenues, leading us to conclude that bigger markets are consistently more beneficial for the firm in terms of revenues.

We also consider changes in market variability; we summarize previous results that show it is more profitable to increase protection levels when the ratio of prices is less than $\frac{1}{2}$, and decrease protection levels when it is greater than $\frac{1}{2}$. Higher class 1 market variability generally leads to lower revenues, except when the variability in the class 2 market is low. An increase in the variability of the class 2 market, on the other hand, does not impact optimal allocations, but, as our numerical experiments suggest, leads to lower total revenues, leading us to conclude that higher variability is detrimental to the firm's revenues.

Many possibilities exist for future work to build on results presented in this paper. It would be of potential interest to study the impact of changing market factors on multiple-class systems, or settings where the firm jointly determines allocation and market prices, as well as an extension of the current work to asymmetric distributions.

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APPENDIX

Proof of Proposition 2. We write $P(D_1' \geq x^*) \geq P(D_1 \geq x^*) = p_2 / p_1$, where the equality follows from the optimality condition (1), and the inequality from the definition of *FSD*. This relationship, by the quasi-concavity of the revenue function, implies $x^* \geq x^*$.

Proof of Remark 1. Since class 2 revenue is not a function of the high-end demand, an increase in the size of the class 1 market impacts total revenues for a given protection level x only through the class 1 revenue $r_1(x)$. Class 1 revenue increases in its market size, because $h(u) = E_{D_2} [\min(u, \max(C - D_2, x))]$ is increasing in u (where E_Y denotes that the expectation is taken over the random variable Y), and from the definition *FSD*, $E_{D_1'} [h(D_1')] \geq E_{D_1} [h(D_1)]$, for all increasing functions h , whenever $D_1' \succ_{FSD} D_1$. This implies $r_1'(x) \geq r_1(x)$. Since $r_2'(x) = r_2(x)$, the result follows.

Proof of Proposition 3. (a) We write $r_1'(x^*) \geq p_1 E [\min(D_1', \max(C - D_2, x^*))] \geq r_1(x^*)$, where the first inequality follows because $r_1(x)$ is increasing in x , and from Proposition 2, $x^* \geq x^*$. The second inequality holds because $h(u) = E_{D_2} [\min(u, \max(C - D_2, x))]$ is increasing in u , and from the definition of *FSD*, $E_{D_1'} [h(D_1')] \geq E_{D_1} [h(D_1)]$, for all increasing functions h , whenever $D_1' \succ_{FSD} D_1$. (b) We write $r_2'(x^*) \leq r_2(x^*)$, where the inequality follows because $r_2(x)$ is decreasing in x , and from Proposition 2, $x^* \geq x^*$. (c) We write $R(x^*) \geq r_1'(x^*) + r_2'(x^*) = r_1'(x^*) + r_2(x^*) \geq R(x^*)$, where the first inequality follows because $x^* = \arg \max R(x)$, hence $R'(x^*) \geq R'(x^*)$ for any $x^* = x^*$. The equality follows because $r_2(x)$ is not a function of the class 1 demand D_1 , hence $r_2'(x) = r_2(x)$. The last inequality follows because $h(u) = E_{D_2} [\min(u, \max(C - D_2, x))]$ is increasing in u , and from the definition of *FSD*, $E_{D_1'} [h(D_1')] \geq E_{D_1} [h(D_1)]$, for all increasing functions h , whenever $D_1' \succ_{FSD} D_1$.

Proof of Proposition 4. We write $r_2'(x^*) \geq p_2 E [\min(D_2, C - x^*)] = r_2(x^*)$, where the inequality follows because $g(u) = \min(u, C - x)$ is increasing in u , and from the definition of *FSD*, $E [g(D_2')] \geq E [g(D_2)]$, for all increasing functions g , whenever $D_2' \succ_{FSD} D_2$. The equality follows because $x^* = x^*$.

Proof of Proposition 5. (a) First remark that for two symmetric distributions, if $X \succ_{MPS} Y$, then $E[X] = E[Y]$, and $P(X \geq E[X]) \leq P(Y \geq E[Y]) = 1/2$. This, alongside the single crossing property of distributions that differ by a mean preserving spread, imply, for $u \leq E[X] = E[Y]$, $P(X \geq u) \leq P(Y \geq u)$, and for $u \geq E[X] = E[Y]$, $P(X \geq u) \geq P(Y \geq u)$. This implies $P(D_1' \geq u) \geq P(D_1 \geq u)$ whenever $D_1' \succ_{MPS} D_1$ and $u \leq E[D_1'] = E[D_1]$. When $p_2 / p_1 \geq 1/2$, from the optimality condition (1) and the quasi-concavity of $R(x)$, $x^* \leq E[D_1]$ (because $P(D_1 \geq E[D_1]) = 1/2 \leq p_2 / p_1 = P(D_1 \geq x^*)$). Hence, we can write $P(D_1' \geq x^*) \geq P(D_1 \geq x^*) = p_2 / p_1$, which from the quasi-concavity of $R(x)$ implies $x^{*'} \geq x^*$. The second part is proved analogously.

Proof of Remark 2. Since class 2 revenue, $r_2(x)$, is not a function of class 1 demand, an increase in the variability of the class 1 demand impacts total revenues only through the class 1 revenue. This is lower whenever $D_1' \succ_{MPS} D_1$, as $h(u) = E_{D_2}[\min(u, \max(C - D_2, x))]$ is concave in u , and from the definition of MPS, $E_{D_1'}[h(D_1')] \leq E_{D_1}[h(D_1)]$ for all concave functions h . This implies $r_1'(x) \leq r_1(x)$. Since $r_2'(x) = r_2(x)$, the result follows.

Proof of Proposition 6. (a) We write $r_1'(x^*) \leq p_1 E[\min(D_1', \max(C - D_2, x^*))] \leq r_1(x^*)$, where the first inequality follows because $E[\min(D_1, \max(C - D_2, u))]$ is increasing in u , and from Proposition 5(a), $x^* \leq x^{*}$. The second inequality holds because of the same arguments as the proof of Remark 2, above. (b) We write $r_2'(x^*) \geq r_2(x^*)$, where the inequality follows because $r_2(x)$ is decreasing in x , and from Proposition 5(a), $x^* \leq x^{*}$. (c) We write $R(x^*) \leq r_1(x^*) + r_2'(x^*) = r_1(x^*) + r_2(x^*) \leq R(x^*)$ where the first inequality follows because $x^* = \arg \max R'(x)$, hence $R'(x) \leq R(x)$ for any $x^* \neq x^{*}$. The equality follows because $r_2(x)$ is not a function of the class 1 demand D_1 . The last inequality follows because of the same arguments as the proof of Remark 2, above.

Proof of Proposition 7. The proof is analogous to the proof of Proposition 6(b).

Proof of Proposition 8. We write $r_2'(x^*) \leq p_2 E[\min(D_2, C - x^*)] \leq r_2(x^*)$, where the inequality follows because $g(u) = \min(u, C - x)$ is concave in u , and from the definition of MPS, $E[g(D_2')] \leq E[g(D_2)]$, for all concave functions g , whenever $D_2' \succ_{MPS} D_2$. The equality follows because $x^* = x^{*}$.

Table 1. Optimal protection levels for Figures 6 and 7

Class 1 Demand	Price ratio			Class 1 Demand	Price ratio		
	0.125	0.25	0.375		0.125	0.25	0.375
Uniform(50,70)	67.5	65	62.5	Uniform(20,100)	90	80	70
Uniform(40,80)	75	70	65	Uniform(10,110)	97.5	75	72.5
Uniform(30,90)	82.5	75	67.5	Uniform(0,120)	105	90	75