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RINGS OF INVARIANTS FOR MODULAR REPRESENTATIONS OF THE KLEIN FOUR GROUP

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Abstract. We study the rings of invariants for the indecomposable modular representations of the Klein four group. For each such representation we compute the Noether number and give minimal generating sets for the Hilbert ideal and the field of fractions. We observe that, with the exception of the regular representation, the Hilbert ideal for each of these representations is a complete intersection.

Introduction

The modular representation theory of the Klein four group has long attracted attention. The group algebra of Klein four over an infinite field of characteristic 2 is one of the relatively rare examples of a group algebra with domestic representation type (see, for example, [2, §4.4]). If we work over an algebraically closed field, then for each even dimension there is a one parameter family of indecomposable representations and a finite number of exceptional indecomposable representations. For each odd dimension (greater than 1) there are only two indecomposable representations. In this paper we investigate the rings of invariants of the indecomposable representations of the Klein four group over fields of characteristic 2. For each such representation we compute the Noether number and give minimal generating sets for the Hilbert ideal and the field of fractions (definitions are given below). For an indecomposable representation of the Klein four group, say V, our results show that the Noether number is at most $2\dim(V)+1$ (detailed formulae are given later in this introduction) and, with the exception of the regular representation, the Hilbert ideal is generated by a homogeneous system of parameters. We note that the Hilbert ideals are generated by polynomials of degree at most 4, confirming Conjecture 3.8.6(b) of [9] for these representations.

We start with a few definitions and some notation. Suppose that V is a finite dimensional representation of a finite group G over a field $\mathbf{F}$. The induced action on the dual space V^* extends to the symmetric algebra $S(V^*)$ of polynomial functions on V which we denote by $\mathbf{F}[V]$. The action of $g \in G$ on $f \in \mathbf{F}[V]$ is given by $(gf)(v) = f(g^{-1}v)$ for $v \in V$. The ring of invariant polynomials

$$\mathbf{F}[V]^G = \{f \in \mathbf{F}[V] \mid \ g(f) = f \ \forall g \in G\}$$

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is a graded, finitely generated subalgebra of $\mathbf{F}[V]$. The maximal degree of a polynomial in a minimal homogeneous generating set for $\mathbf{F}[V]^G$ is known as the *Noether number* of V. The ideal in $\mathbf{F}[V]$ generated by the homogeneous invariants of positive degree is the *Hilbert ideal* of V. If the characteristic of $\mathbf{F}$ divides $|G|$, then V is called a *modular* representation. Rings of invariants for non-modular representations are reasonably well behaved. For instance, it is well known that if V is non-modular, then $\mathbf{F}[V]^G$ is always Cohen-Macaulay and the Noether number is less than or equal to $|G|$ (see, for example, [9, §3.4, §3.8]). Both of these properties can fail in the modular case. Rings of invariants for modular representations are rarely Cohen-Macaulay, and there is no bound on the degrees of a generating set which depends only on the group order. Computing rings of invariants for modular representations can be difficult even in basic cases. Consider a representation of a cyclic p-group $\mathbf{Z}/p^r$ over a field of characteristic p. The action is easy to describe: up to a change of basis, a generator of the group acts by a sum of Jordan blocks each with eigenvalue 1 and size at most p^r. Despite this, even when $r = 1$, although the Noether numbers are known [12], an explicit generating set has been constructed for only a limited number of cases; see [23] for a summary and recent advances. For $r > 1$, much less is known; see [20] for the study of a specific case and [17] for some partial results on degree bounds. This paper is a part of a programme, initiated in [8], to understand the rings of invariants of modular representations of elementary abelian p-groups. In [8], the rings of invariants of all two dimensional representations and all three dimensional representations for groups of rank at most three were computed; in all cases the rings were shown to be complete intersections.

The results in section 2 apply to an arbitrary group G, but for the rest of the paper $G := \langle \sigma_1, \sigma_2 \rangle \cong \mathbf{Z}/2 \times \mathbf{Z}/2$ denotes the Klein four group. For $\mathbf{F}$ an algebraically closed field of characteristic 2, the indecomposable representations of the Klein four group over $\mathbf{F}$ are the following:

- the trivial representation $\mathbf{F}$;
- the regular representation V_{reg};
- a representation of dimension $2m$ for each $\lambda \in \mathbf{F} \cup \{\infty\}$, which we denote by $V_{m,\lambda}$;
- the representations $\Omega^m(\mathbf{F})$ and $\Omega^{-m}(\mathbf{F})$ of dimension $2m + 1$, where Ω denotes the Heller operator.

See [2, §4.4] for a detailed discussion of this classification. Note that $V_{m,0}$, $V_{m,1}$ and $V_{m,\infty}$, while not equivalent representations, are linked by group automorphisms. Therefore the invariants can be computed using the same matrix group and $\mathbf{F}[V_{m,0}]^G \cong \mathbf{F}[V_{m,1}]^G \cong \mathbf{F}[V_{m,\infty}]^G$. In [10], the depth of $\mathbf{F}[V]^G$ was computed for each of the indecomposable modular representations of the Klein four group. The only indecomposable representations for which the ring of invariants is Cohen-Macaulay are the the trivial representation, the regular representation, $V_{1,\lambda}$, $V_{2,\lambda}$, $\Omega^{-1}(\mathbf{F})$, $\Omega^{-2}(\mathbf{F})$ and $\Omega^1(\mathbf{F})$. Note that, for each of these representations, $\mathbf{F}[V]^G$ is a complete intersection. In [15] separating sets of invariants are given for the indecomposable modular representations of the Klein four group.

We identify $\mathbf{F}[V]$ with the polynomial algebra on the variables x_i and y_j. We use the graded reverse lexicographic order (grevlex) with $x_i < y_j$, $x_i < x_{i+1}$ and $y_j < y_{j+1}$. We adopt the convention that a monomial is a product of variables and a term is a monomial multiplied by a coefficient. For a polynomial $f \in \mathbf{F}[V]$, we denote the leading monomial by $\mathrm{LM}(f)$ and the leading term by $\mathrm{LT}(f)$. We make occasional use of SAGBI bases, the **S**ubalgebra **A**nalog of a **G**röbner **B**asis for **I**deals. For a subset $\mathcal{B} = \{h_1, \ldots, h_\ell\}$ of a subalgebra $A \subset \mathbf{F}[V]$ and a sequence $I = (i_1, \ldots, i_\ell)$ of non-negative integers, denote $\prod_{j=1}^{\ell} h_j^{i_j}$ by h^I. A *tête-a-tête* for $\mathcal{B}$ is a pair (h^I, h^J) with $\mathrm{LM}(h^I) = \mathrm{LM}(h^J)$; we say that a tête-a-tête is non-trivial if the support of I is disjoint from the support of J. The reduction of an S-polynomial is a fundamental calculation in the theory of Gröbner bases. The analogous calculation for SAGBI bases is the *subduction* of a tête-a-tête. $\mathcal{B}$ is a SAGBI basis for A if every non-trivial tête-a-tête subducts to zero. A SAGBI basis is a particularly useful generating set for the subalgebra. For background material on SAGBI bases, see [21, §11] or [19, §3]. For $f \in \mathbf{F}[V]$, we define the *transfer* of f by $\mathrm{Tr}(f) := \sum_{\sigma \in G} \sigma(f)$ and the *norm* of f, which we denote by $N_G(f)$, to be the product over the G-orbit of f. If the coefficient of a monomial M in a polynomial f is non-zero, we say that M *appears* in f.

We conclude the introduction with a summary of the paper. Section 1 contains preliminary results on the invariant theory of $\mathbf{Z}/2$. In section 2, we introduce the concept of a *block hsop*, a particularly nice homogeneous system of parameters, and prove a theorem which we use to compute Noether numbers. A recent result of Peter Symonds [22, Corollary 0.3] is a key ingredient in our proof. The results of this section are valid for any modular representation of a finite group.

In section 3, we consider the even dimensional representations. We include an explicit description of the group actions. We show that for $m > 1$, the Noether number of $V_{m,\lambda}$ is $3m - 2\lfloor m/2 \rfloor$ if $\lambda \in \mathbf{F} \setminus \mathbf{F}_2$ and $3m - 2\lceil m/2 \rceil$ if $\lambda \in \{0, 1, \infty\}$. We also show that the Hilbert ideal of $V_{m,\lambda}$ is generated by a block hsop and is therefore a complete intersection. A transcendence basis for the field of fractions is given; in fact we show $\mathbf{F}[V_{m,\lambda}]^G[x_1]^{-1}$ is a "localised polynomial algebra". For various small dimensional cases, we give generating sets for the rings of invariants and for the other cases we give explicit input sets for the SAGBI/Divide-by-x algorithm introduced in [8, §1].

The odd dimensional representations are considered in sections 4 and 5. We show that the Noether number for $\Omega^{-m}(\mathbf{F})$ is $m+1$ (Corollary 4.2), the Noether number for $\Omega^m(\mathbf{F})$ is $3m$ for $m > 1$ (Corollary 5.2), and that in all cases the Hilbert ideal is generated by a block hsop. We give generating sets for $\mathbf{F}[\Omega^{-m}(\mathbf{F})]^G[x_1^{-1}]$ and for $\mathbf{F}[\Omega^m(\mathbf{F})]^G[(x_1x_2(x_1+x_2))^{-1}]$. We also give explicit input sets for the SAGBI/Divide-by x algorithm.

1. Preliminaries

Let $\mathbf{F}$ denote a field of characteristic 2. Suppose $\langle\sigma\rangle \cong \mathbf{Z}/2$ acts on $S := \mathbf{F}[x_1, \ldots, x_m, y_1, \ldots, y_m]$ by $\sigma(x_j) = x_j$, $\sigma(y_j) = y_j + x_j$. Define $\Delta := \sigma - 1$ and $n_i := y_i^2 + x_iy_i$. We will often write S^σ as shorthand for $S^{\langle\sigma\rangle}$.

Proposition 1.1 ([16], [5], [7])**.** *S^σ is generated by*

$$\{n_1, \ldots, n_m\} \cup \{\Delta(\beta) \mid \beta \text{ divides } y_1 \cdots y_m\}.$$

Corollary 1.2. $\Delta S = ((x_1, \ldots, x_m)\, S)^\sigma$ *and* $S^\sigma / \Delta S \cong \mathbf{F}[n_1, \ldots, n_m]$.

Proof. It is clear from the definition of Δ that $\Delta S \subset (x_1, \ldots, x_m)S$. Since $\Delta^2 = 0$, we have $\Delta S \subseteq ((x_1, \ldots, x_m)\, S)^\sigma$. The result then follows from the definition of n_i and the generating set for S^σ given above. □

Proposition 1.1 and Corollary 1.2 give the following.

Lemma 1.3. *Suppose $a_1, \ldots, a_m$ are non-negative integers. Let $f \in S^\sigma$.*

(i) *If $y_1^{a_1} \cdots y_m^{a_m}$ appears in f, then a_i is even for $i \in \{1, \ldots, m\}$.*
(ii) *If $y_1^{a_1} \cdots y_{m-1}^{a_{m-1}} y_m x_m$ appears in f, then a_i is even for $i \in \{1, \ldots, m-1\}$.*

A simple calculation shows that for $a, b \in S$,

$$\Delta(a \cdot b) = \Delta(a)b + a\Delta(b) + \Delta(a)\Delta(b)$$

and $\Delta(a^2) = \Delta(a)^2$. Therefore, if $M = y_1^{a_1} \cdots y_m^{a_m}$ with $a_i > 0$, then the monomial $x_i M / y_i$ appears in $\Delta(M)$ with coefficient 1 if a_i is odd and coefficient 0 if a_i is even. Note that if a monomial M appears (with non-zero coefficient) in $f \in S^\sigma$ and a monomial M' appears in $\Delta(M)$, then there is at least one further monomial, say M'', with $M \neq M''$, such that M'' appears in f and M' appears in $\Delta M''$.

Lemma 1.4. *Suppose M' is a monomial in $\{y_1, \ldots, y_m\}$ and $M = M' x_i y_j$ for some $i, j \in \{1, \ldots, m\}$ with $i \neq j$. Assume further that the degree of y_j in M' is even. If M appears in a polynomial $f \in S^\sigma$, then the degree of y_i in M' is even and $M' x_j y_i$ also appears in f. Moreover, the coefficients in f of these monomials are the same.*

Proof. Since the degree of y_j in M is odd, $M' x_i x_j$ appears in $\Delta(M)$ with coefficient 1. Note that if the degree of y_i in M' is odd, then there is no other monomial in S that produces $M' x_i x_j$ after applying Δ. Therefore, we may assume that the degree of y_i in M' is even. In this case, $M' x_i x_j$ appears in $\Delta(M' y_i x_j)$ and in $\Delta(M' y_i y_j)$. However, the degree of y_j in the monomial $M' y_i y_j$ is odd, so it follows from Lemma 1.3 that $M' y_i y_j$ does not appear in f. Therefore $M' y_i x_j$ appears in f. Since the coefficient of $M' x_i x_i$ in both $\Delta(M' y_i x_j)$ and $\Delta(M' y_j x_i)$ is 1, the coefficients of $M' y_i x_j$ and $M' y_j x_i$ in f must be equal. □

Lemma 1.5. *Suppose that M' is a monomial in $\{y_1, \ldots, y_m\} \setminus \{y_j\}$ for some $j \in \{1, \ldots, m\}$ and $M = M' y_j x_j$. For $f \in S^\sigma$, M appears in f if and only if $M' y_j^2$ appears in f. Moreover, the coefficients in f of these monomials are the same. Finally, $M' y_j^3 x_j$ does not appear in any polynomial in S^σ.*

Proof. Note that $M' x_j^2$ appears in both $\Delta(M)$ and $\Delta(M' y_j^2)$ with coefficient 1. Since these are the only monomials in S that produce $M' x_j^2$ after applying Δ, the result follows. The final statement follows from the fact that $M' y_j^3 x_j$ is the only monomial in S that produces $M' y_j^2 x_j^2$ after applying Δ. □

2. Block HSOPs

In this section, G is an arbitrary finite group, $\mathbf{F}$ is a field of characteristic p for some prime number p dividing the order of G and V is a finite dimensional $\mathbf{F}G$-module. Suppose we have a homogeneous system of parameters $\mathcal{S} = \{h_1, \ldots, h_n\}$ for $\mathbf{F}[V]^G$. Let A denote the algebra generated by $\mathcal{S}$ and let I denote the ideal $(h_1, \ldots, h_n)\mathbf{F}[V]$. Further suppose that there exists a term order for which $\mathcal{S}$ is a Gröbner basis for I and the reduced monomials are the monomial factors of a given monomial, say β. Then the monomial factors of β are a basis for $\mathbf{F}[V]$ as a free A-module; in the language of [6], we have a block basis for $\mathbf{F}[V]$ over A. In this situation, we will refer to $\mathcal{S}$ as a *block hsop* and β as the *top class*. Note that if the elements of $\{\mathrm{LM}(h_1), \ldots, \mathrm{LM}(h_n)\}$ are pair-wise relatively prime, then $\mathcal{S}$ is a block hsop and the top class is the unique maximal reduced monomial.

Theorem 2.1. *Suppose $\mathcal{S} = \{h_1, \ldots, h_n\}$ is a block hsop with top class β. If $\mathrm{Tr}(\beta)$ is indecomposable in $\mathbf{F}[V]^G$, then*

(a) *the Noether number for V is $\deg(\beta)$;*
(b) *the Hilbert ideal of V is generated by $\mathcal{S}$.*

Proof. Proof of (a): The indecomposability of $\mathrm{Tr}(\beta)$ gives a lower bound on the Noether number. The fact that $\deg(\beta)$ is also an upper bound follows from [22, Corollary 0.3].

Proof of (b): Denote the Hilbert ideal of V by $\mathfrak{h}$. Since $\mathcal{S} \subset \mathbf{F}[V]^G$, we have $I \subseteq \mathfrak{h}$. Suppose, by way of contradiction, that there exists $f \in \mathfrak{h} \setminus I$. We may assume that f is homogeneous and that $\mathrm{LM}(f)$ is reduced with respect to I using the chosen term order. Therefore $\mathrm{LM}(f)$ divides β. Reducing β with respect to $\mathcal{S} \cup \{f\}$ produces a polynomial of degree $d := \deg(\beta)$ with lead term less than β. However, $\mathbf{F}[V]/I$ in degree d has dimension one. Thus $\beta \in (h_1, \ldots, h_n, f)\mathbf{F}[V] \subseteq \mathfrak{h}$. Let $\mathcal{C}$ be the reduced monomials with respect to $\mathfrak{h}$ using the chosen term order. Observe that the elements of $\mathcal{C}$ are monomial factors of β with degree less than d. Since $\mathcal{C}$ generates $\mathbf{F}[V]$ as an $\mathbf{F}[V]^G$-module, the transfer ideal, $\mathrm{Tr}(\mathbf{F}[V])$, is generated by $\{\mathrm{Tr}(\gamma) \mid \gamma \in \mathcal{C}\}$ as an $\mathbf{F}[V]^G$-module. Therefore,

$$\mathrm{Tr}(\beta) = \sum_{\gamma \in \mathcal{C}} c_\gamma \, \mathrm{Tr}(\gamma)$$

for some $c_\gamma \in \mathbf{F}[V]^G$. Since the representation is modular, $\mathrm{Tr}(1) = 0$. Furthermore $\deg(\mathrm{Tr}(\gamma)) < d$. Therefore, the equation above gives a decomposition of $\mathrm{Tr}(\beta)$ in terms of invariants of degree less than d, contradicting the indecomposability of $\mathrm{Tr}(\beta)$. $\square$

3. Even dimensional representations

In this section we consider the even dimensional representations $V_{m,\lambda}$. For completeness, we also include a brief discussion of the regular representation in subsection 3.14. For $\lambda \in \mathbf{F}$, the action of $G = \langle \sigma_1, \sigma_2 \rangle$ on $S := \mathbf{F}[V_{m,\lambda}] = \mathbf{F}[x_1, \ldots, x_m, y_1, \ldots, y_m]$ is given by $\sigma_i(x_j) = x_j$, $\sigma_1(y_j) = y_j + x_j$, $\sigma_2(y_1) = y_1 + \lambda x_1$ and $\sigma_2(y_j) = y_j + \lambda x_j + x_{j-1}$ for $j > 1$. Define $n_i := y_i^2 + x_i y_i$ and $u_{ij} = x_i y_j + x_j y_i$. Then $n_i, u_{ij} \in S^{\sigma_1}$. A simple calculation gives $\Delta_2(n_i) = (\lambda^2 + \lambda)x_i^2 + x_{i-1}^2 + x_i x_{i-1}$ and $\Delta_2(u_{ij}) = x_i x_{j-1} + x_{i-1} x_j$ (using the convention that $x_0 = 0$). Define $\ell := \lfloor m/2 \rfloor$ and, for $i \leq \ell$, define

$$N_i := n_i + (\lambda^2 + \lambda) \sum_{j=1}^{i} u_{i-j+1,i+j} + \sum_{j=1}^{i-1} \left(u_{i-j,i+j} + u_{i-j,i+j-1} \right).$$

An explicit calculation, exploiting the fact that $\Delta_2(u_{1j}) = x_1 x_{j-1}$, gives $\Delta_2(N_i) = 0$. Therefore $N_i \in S^G$. Define

$$\mathcal{H} := \{x_1, \ldots, x_m\} \cup \{N_i \mid 1 \leq i \leq m/2\} \cup \{N_G(y_j) \mid m/2 < j \leq m\}.$$

Theorem 3.1. *$\mathcal{H}$ is a block hsop with top class $y_1 \cdots y_\ell y_{\ell+1}^3 \cdots y_m^3$.*

Proof. This follows from the fact that $\mathrm{LT}(N_i) = y_i^2$ and $\mathrm{LT}(N_G(y_j)) = y_j^4$. $\square$

Corollary 3.2. *The image of the transfer, $\mathrm{Tr}(S)$, is the ideal in S^G generated by*

$$\left\{ \mathrm{Tr}(\beta) \mid \beta \text{ divides } y_1 \cdots y_\ell (y_{\ell+1} \cdots y_m)^3 \right\}.$$

Theorem 3.3. *For $\lambda \notin \mathbf{F}_2$ and $m \geq 3$, $\mathrm{Tr}(y_1 \cdots y_\ell y_{\ell+1}^3 \cdots y_m^3)$ is indecomposable.*

See subsection 3.15 for the proof of Theorem 3.3. Combining Theorem 3.3 with Theorem 2.1 gives the following.

Corollary 3.4. *If $\lambda \notin \mathbf{F}_2$ and $m \geq 3$, then the Noether number for $V_{m,\lambda}$ is $3m - 2\lfloor m/2 \rfloor$ and the Hilbert ideal is generated by $\mathcal{H}$.*

Descriptions of S^G for $m \leq 3$ are given in subsection 3.14. The formula given above for the Noether number is valid for $m > 1$.

For $j > 1$, an explicit calculation gives

$$\begin{aligned}
\mathrm{Tr}(y_1 y_2 y_j) &= y_1(x_2 x_{j-1} + x_1 x_j) + y_2 x_1 x_{j-1} + y_j x_1^2 \\
&\quad + x_1 x_2 \left((\lambda^2 + \lambda) x_j + x_{j-1}\right) + x_1^2 (x_j + x_{j-1}) \\
&= u_{12} x_{j-1} + u_{1j} x_1 + \mathrm{Tr}(y_1 y_3) \left((\lambda^2 + \lambda) x_j + x_{j-1}\right) \\
&\quad + \mathrm{Tr}(y_1 y_2)(x_j + x_{j-1}).
\end{aligned}$$

Therefore $t_j := u_{12} x_{j-1} + u_{1j} x_1 \in \mathrm{Tr}(S)$.

Theorem 3.5. *For $m > 3$ and $\lambda \notin \mathbf{F}_2$,*

$$\mathbf{F}[V_{m,\lambda}]^G [x_1^{-1}] = \mathbf{F}[x_1, \ldots, x_m, N_1, N_2, t_3, \ldots, t_m][x_1^{-1}].$$

Proof. We use [4, Theorem 2.4]. $\mathbf{F}[x_1, \ldots, x_m, y_1]^G$ is the polynomial algebra generated by $\{x_1, \ldots, x_m, N_G(y_1)\}$. Since $N_1 = y_1^2 + x_1 y_1 + (\lambda^2 + \lambda)(x_1 y_2 + x_2 y_1)$, we see that $N_1 \in \mathbf{F}[x_1, x_2, y_1, y_2]$ is degree 1 in y_2 with coefficient $(\lambda^2 + \lambda) x_1$. Using the equation above, $t_j \in \mathbf{F}[x_1, \ldots, x_m, y_1, y_2, y_j]$ is degree 1 in y_j with coefficient x_1^2. Thus $S^G[x_1^{-1}] = \mathbf{F}[x_1, \ldots, x_m, N_G(y_1), N_1, t_3, \ldots, t_m][x_1^{-1}]$. To complete the proof, we need only rewrite $N_G(y_1)$ in terms of N_2 and the other generators. An explicit calculation gives

$$N_G(y_1) = y_1^4 + x_1^2 y_1^2 (\lambda^2 + \lambda + 1) + x_1^3 y_1 (\lambda^2 + \lambda).$$

Define $c := \lambda^2 + \lambda$. Subduction gives

$$N_G(y_1) = N_1^2 + ((c x_2)^2 + c x_1^2) N_1 + (c x_1)^2 N_2 + (c^3 x_2 + c^2 x_1) t_3 + c^3 x_1 t_4,$$

as required. $\square$

Remark 3.6. For $m > 3$ and $\lambda \notin \mathbf{F}_2$, it follows from Theorem 3.5 and Theorem 3.1 that S^G is the normalisation of the algebra generated by $\mathcal{B} := \mathcal{H} \cup \{t_3, \ldots, t_m\}$. Furthermore, applying the SAGBI/Divide-by-x algorithm of [8] with $x = x_1$ to $\mathcal{B}$ computes a SAGBI basis for S^G.

Using the familiar formula for the group cohomology of a cyclic group, we have

$$H^1\left(\langle \sigma_2 \rangle, \Delta_1 S\right) \cong \left(\Delta_1 S\right)^{\sigma_2} / \Delta_2 \Delta_1 S = \left(\Delta_1 S\right)^{\sigma_2} / \mathrm{Tr}\, S$$

and $H^1\left(\langle \sigma_1 \rangle, \Delta_2 S\right) \cong \left(\Delta_2 S\right)^{\sigma_1} / \mathrm{Tr}\, S$. Note that $H^1\left(\langle \sigma_1 \rangle, \Delta_2 S\right)$ and $H^1\left(\langle \sigma_2 \rangle, \Delta_1 S\right)$ are both finitely generated S^G-modules and, therefore, are also finitely generated over the algebra generated by $\mathcal{H}$. In the following $\sqrt{\mathrm{Tr}\, S}$ denotes the radical of the image of the transfer.

Proposition 3.7. *For $\lambda \notin \mathbf{F}_2$, $(\Delta_2 S)^{\sigma_1} = (\Delta_1 S)^{\sigma_2} = \left((x_1, \ldots, x_m)\, S\right)^G = \sqrt{\mathrm{Tr}\, S}$ and*

$$\sqrt{\mathrm{Tr}\, S} / \mathrm{Tr}\, S \cong H^1(\langle \sigma_2 \rangle, \Delta_1 S) \cong H^1(\langle \sigma_1 \rangle, \Delta_2 S).$$

Furthermore $S^G / \sqrt{\mathrm{Tr}\, S} \cong \mathbf{F}[N_1, \ldots, N_\ell, N_G(y_{\ell+1}), \ldots, N_G(y_m)]$.

Proof. For $\lambda \notin \mathbf{F}_2$,

$$\Delta_1 V^*_{m,\lambda} = \Delta_2 V^*_{m,\lambda} = (\sigma_1\sigma_2 + 1)V^*_{m,\lambda} = \mathrm{Span}_{\mathbf{F}}\{x_1, \ldots, x_m\}.$$

Using [18, Theorem 2.4] (see also [11, Theorem 2.4]), $\sqrt{\operatorname{Tr} S} = ((x_1, \ldots, x_m)S)^G$. Applying Proposition 1.1 with $\sigma = \sigma_1$ gives $\Delta_1 S = ((x_1, \ldots, x_m)\,S)^{\sigma_1}$. Thus $(\Delta_1 S)^{\sigma_2} = ((x_1, \ldots, x_m)S)^G$. Applying Proposition 1.1 with $\sigma = \sigma_2$ gives $(\Delta_2 S)^{\sigma_1} = ((x_1, \ldots, x_m)S)^G$.

To prove the final statement, first observe that

$$\mathcal{N} := \{N_1, \ldots, N_\ell, N_G(y_{\ell+1}), \ldots, N_G(y_m)\}$$

is algebraically independent modulo $\sqrt{\operatorname{Tr} S}$. Therefore, there is a subalgebra of $S^G/\sqrt{\operatorname{Tr} S}$ isomorphic to $A := \mathbf{F}[N_1, \ldots, N_\ell, N_G(y_{\ell+1}), \ldots, N_G(y_m)]$. We will show that for every $f \in S^G$, there exists $F \in A$ with $f - F \in \sqrt{\operatorname{Tr} S}$. We proceed with a minimal counterexample. Without loss of generality, we may assume f is homogeneous of positive degree. Since $\mathrm{LM}(g(y_i)) = y_i$ for all $g \in G$, using [19, Theorem 3.2], there exists a finite SAGBI basis for S^G and therefore a finite SAGBI-Gröbner basis for the ideal $\sqrt{\operatorname{Tr} S}$. We may assume that f is reduced, i.e., equal to its normal form. Therefore $\mathrm{LM}(f) = \prod_{i=1}^m y^{a_i}$. Using Lemma 1.3, each a_i is even. It follows from Proposition 3.15.2 that $\mathrm{LM}(f)$ does not divide $\prod_{i=\ell+1}^m y_i^2$. Since $\mathrm{LT}(N_i) = y_i^2$ and $\mathrm{LT}(N_G(y_j)) = y_j^4$, there exits $N \in \mathcal{N}$ with $\mathrm{LT}(N) = y_k^{b_k}$ dividing $\mathrm{LM}(f)$. Note that $N = y_k^{b_k} + \widetilde{N}$ for some $\widetilde{N} \in (x_1, \ldots, x_m)S$. Since N is monic as a polynomial in y_k, we can divide f by N to get $f = qN + r$ for unique $q, r \in S$ with $\deg_{y_k}(r) < \deg_{y_k}(N) = b_k$. Furthermore, since we are using grevlex with $x_i < y_k$, we have $\mathrm{LM}(r) < \mathrm{LM}(f)$. Applying $g \in G$ gives $f = g(f) = g(q)N + g(r)$. However, $\deg_{y_k}(g(r)) \leq \deg_{y_k}(r)$. Therefore, by the uniqueness of the remainder, $g(r) = r$ and $g(q) = q$. Thus $q, r \in S^G$ with $q < f$ and $r < f$. By the minimality of f, there exists $F_1, F_2 \in A$ with $q - F_1, r - F_2 \in \sqrt{\operatorname{Tr} S}$. Therefore $F := NF_1 - F_2 \in A$ and $f - F \in \sqrt{\operatorname{Tr} S}$, giving the required contradiction. ☐

While $V_{m,0}$ and $V_{m,1}$ are not equivalent representations, the automorphism of G which fixes σ_1 and exchanges σ_2 and $\sigma_1\sigma_2$, takes $V_{m,0}$ to $V_{m,1}$. Therefore $\mathbf{F}[V_{m,0}]^G \cong \mathbf{F}[V_{m,1}]^G$. Hence, to compute $\mathbf{F}[V_{m,\lambda}]^G$ with $\lambda \in \mathbf{F}_2$, it is sufficient to take $\lambda = 0$.

Substituting $\lambda = 0$ into the expression for N_i given above gives an element in $\mathbf{F}[V_{m,0}]^G$ with lead term y_i^2 for $i \leq \lceil m/2 \rceil$. Define $\ell' := \lceil m/2 \rceil$ and

$$\mathcal{H}' := \{x_1, \ldots, x_m\} \cup \{N_i \mid 1 \leq i \leq (m+1)/2\} \cup \{N_G(y_j) \mid (m+1)/2 < j \leq m\}.$$

Looking at lead terms gives the following.

Theorem 3.8. *For $\lambda \in \mathbf{F}_2$, $\mathcal{H}'$ is a block hsop with top class $y_1 \cdots y_{\ell'} y_{\ell'+1}^3 \cdots y_m^3$.*

Theorem 3.9. *For $\lambda \in \mathbf{F}_2$ and $m > 3$, $\mathrm{Tr}(y_1 \cdots y_{\ell'} y_{\ell'+1}^3 \cdots y_m^3)$ is indecomposable.*

See subsection 3.16 for the proof of Theorem 3.9. Combining Theorem 3.9 with Theorem 2.1 gives the following.

Corollary 3.10. *For $m > 3$, the Noether number for $V_{m,0}$ is $3m - 2\lceil m/2 \rceil$ and the Hilbert ideal is generated by $\mathcal{H}'$.*

Descriptions of $\mathbf{F}[V_{m,0}]^G$ for $m \leq 3$ are given in subsection 3.14. The above formula for the Noether number is valid for $m > 1$.

Theorem 3.11. *For $m > 2$,*

$$\mathbf{F}[V_{m,0}]^G[x_1^{-1}] = \mathbf{F}[x_1, \ldots, x_m, N_1, N_2, t_3, \ldots, t_m][x_1^{-1}].$$

Proof. We construct the field of fractions for an upper-triangular action as in [4] or [14]. From Remark 3.14.3 we see that $\mathbf{F}[x_1, x_2, y_1, y_2]^G[x_1^{-1}] = \mathbf{F}[x_1, x_2, N_1, \widetilde{w}][x_1^{-1}]$, where $\widetilde{w} := (x_1 + x_2)u_{12} + x_1 n_2$. Since $t_j \in \mathbf{F}[x_1, \ldots, x_m, y_1, \ldots, y_j]^G$ has degree one as a polynomial in y_j with coefficient x_1^2, we have

$$\mathbf{F}[V_{m,0}]^G[x_1^{-1}] = \mathbf{F}[x_1, \ldots, x_m, N_1, \widetilde{w}, t_3, \ldots, t_m][x_1^{-1}].$$

The result then follows from the relation $\widetilde{w} = x_1 N_2 + t_3$. □

Remark 3.12. For $m > 2$ it follows from Theorem 3.11 and Theorem 3.8 that $\mathbf{F}[V_{m,0}]^G$ is the normalisation of the algebra generated by $\mathcal{B}' := \mathcal{H}' \cup \{t_3, \ldots, t_m\}$. Furthermore, applying the SAGBI/Divide-by-x algorithm of [8] with $x = x_1$ to $\mathcal{B}'$ computes a SAGBI basis for $\mathbf{F}[V_{m,0}]^G$.

Proposition 3.13. *For $\lambda = 0$:*

$$\begin{array}{rcl} \sqrt{\operatorname{Tr} S} & = & \left((x_1, \ldots, x_{m-1})\, S\right)^G, \\ H^1(\langle \sigma_1 \rangle, \Delta_2 S) & \cong & \left((x_1, \ldots, x_{m-1})\, S\right)^G / \operatorname{Tr} S, \\ H^1(\langle \sigma_2 \rangle, \Delta_1 S) & \cong & \left((x_1, \ldots, x_m)\, S\right)^G / \operatorname{Tr} S, \\ S^G / \left((x_1, \ldots, x_m)\, S\right)^G & \cong & \mathbf{F}[N_1, \ldots, N_{\ell'}, N_G(y_{\ell'+1}), \ldots, N_G(y_m)]. \end{array}$$

Proof. Direct calculation gives $\Delta_1 V_{m,0}^* = (\sigma_1\sigma_2 + 1)V_{m,0}^* = \operatorname{Span}_{\mathbf{F}}\{x_1, \ldots, x_m\}$ and $\Delta_2 V_{m,0}^* = \operatorname{Span}_{\mathbf{F}}\{x_1, \ldots, x_{m-1}\}$. Using [18, Theorem 2.4],

$$\sqrt{\operatorname{Tr} S} = \bigcap_{g \in G,\, |g|=2} (((g-1)V_{m,0}^*)S)^G = \left((x_1, \ldots, x_{m-1})\, S\right)^G.$$

The rest of the proof is analogous to the proof of Proposition 3.7. □

3.14. **Even dimensional examples.**

Remark 3.14.1. It follows from [9, Theorem 3.75] that $\mathbf{F}[V_{1,\lambda}]^G$ is the polynomial ring generated by x_1 and $N_G(y_1)$.

Define $w := \Delta_2(n_2)u_{12} + x_1^2 n_2$. Note that $N_G(y_2) = n_2^2 + n_2\Delta_2(n_2)$ and recall that $\Delta_2(n_2) = (\lambda^2 + \lambda)x_2^2 + x_1x_2 + x_1^2$. A simple calculation shows that $\operatorname{LT}(w) = (\lambda^2 + \lambda)y_1x_2^3$. Subduction gives

$$w^2 = \Delta_2(n_2)^2 x_2^2 N_1 + x_1^4 N_G(y_2) + w\Delta_2(n_2)\left(\Delta_2(n_2) + x_1^2\right). \tag{3.1}$$

Theorem 3.14.2. *If $\lambda \notin \mathbf{F}_2$, then $\mathbf{F}[V_{2,\lambda}]^G$ is the hypersurface generated by x_1, x_2, N_1, w and $N_G(y_2)$, subject to the above relation.*

Proof. Since N_1 has degree 1 in y_2 with coefficient $(\lambda^2+\lambda)x_1^2$, using [4, Theorem 2.4], we have $\mathbf{F}[V_{2,\lambda}]^G[x_1^{-1}] = \mathbf{F}[x_1, x_2, N_G(y_1), N_1][x_1^{-1}]$. Subduction gives

$$N_G(y_1) = N_1^2 + (\lambda^2 + \lambda)^2(x_2^2 N_1 + w) + x_1^2(w^2 + w)N_1.$$

Therefore $\mathbf{F}[V_{2,\lambda}]^G[x_1^{-1}] = \mathbf{F}[x_1, x_2, N_1, w][x_1^{-1}]$. Furthermore $\{x_1, x_2, N_1, N_G(y_2)\}$ is a block hsop. Taking $\mathcal{B} := \{x_1, x_2, N_1, w, N_G(y_2)\}$, we see that there is a single non-trivial tête-a-tête, which subducts to 0 using equation (3.1). Therefore, using [8, Theorem 1.1], $\mathcal{B}$ is a SAGBI basis for $\mathbf{F}[V_{2,\lambda}]^G$. □

It follows from Theorem 3.14.2 that the Noether number for $V_{2,\lambda}$ is 4 and the Hilbert ideal is generated by $\{x_1, x_2, N_1, N_G(y_2)\}$.

Remark 3.14.3. A Magma [3] calculation shows that $\mathbf{F}[V_{2,0}]^G$ is a hypersurface with generators $x_1, x_2, n_1, \widetilde{w} := (x_1+x_2)u_{12} + x_1n_2, \widetilde{N}_2 := n_2^2 + n_2(x_1^2 + x_1x_2)$ and relation $\widetilde{w}^2 + x_2^2(x_2+x_1)^2n_1 + x_1x_2(x_1+x_2)\widetilde{w} = x_1^2\widetilde{N}_2$. Therefore the Noether number for $V_{2,0}$ is 4 and the Hilbert ideal is generated by $x_1, x_2, n_1, \widetilde{N}_2$. Using the relation to eliminate $\widetilde{N}_2$ gives $\mathbf{F}[V_{2,0}]^G[x_1^{-1}] = \mathbf{F}[x_1, x_2, n_1, \widetilde{w}][x_1^{-1}]$.

Define $u_{123} := x_1(n_2 + u_{12} + u_{13}) + (\lambda^2+\lambda)x_2u_{13}$. Simple calculations give $\mathrm{LM}(u_{123}) = y_1x_2x_3$ and $\Delta_2(u_{123}) = 0$.

Theorem 3.14.4. *If $\lambda \notin \mathbf{F}_2$, then $\mathbf{F}[V_{3,\lambda}]^G[x_1^{-1}] = \mathbf{F}[x_1, x_2, x_3, N_1, u_{123}, t_3][x_1^{-1}]$.*

Proof. From the proof of Theorem 3.14.2, $\mathbf{F}[V_{2,\lambda}]^G[x_1^{-1}] = \mathbf{F}[x_1, x_2, N_1, w][x_1^{-1}]$. Since t_3 is degree 1 in y_3 with coefficient x_1^2, using [4, Theorem 2.4], we have

$$\mathbf{F}[V_{3,\lambda}]^G[x_1^{-1}] = \mathbf{F}[x_1, x_2, x_3, N_1, w, t_3][x_1^{-1}].$$

An explicit calculation gives $w = (\lambda^2+\lambda)x_2t_3 + x_1u_{123} + x_1t_3$, and the result follows. ∎

With $c := \lambda^2 + \lambda$, define

$$n_{23} := (n_2 + u_{12} + u_{13})(cx_3 + x_2 + x_1) + c(x_1n_3 + x_2u_{23} + cx_3u_{23}),$$

$u_{133} := x_1^{-1}(cx_3t_3 + x_2u_{123})$, $u_{2333} := x_1^{-1}((cx_3+x_2)n_{222} + n_{23}x_2^2 + x_2^2(u_{123}+t_3))$ and $n_{222} := x_1^{-2}(t_3^2 + N_1(x_2^4 + x_1^2x_3^2) + (c(x_2^3 + x_1x_2x_3) + x_1x_2^2)t_3)$.

A straightforward calculation gives $n_{23}, u_{133}, n_{222}, u_{2333} \in \mathbf{F}[V_{3,\lambda}]^G$ and $\mathrm{LT}(n_{23}) = cy_2^2x_3$, $\mathrm{LT}(u_{133}) = cy_1x_3^2$, $\mathrm{LT}(n_{222}) = y_2^2x_2^2$, $\mathrm{LT}(u_{2333}) = c^2y_2x_3^3$. Define

$$\begin{aligned}\mathcal{B}_{3,\lambda} \;:=\; & \{x_1, x_2, x_3, N_1, t_3, u_{123}, u_{133}, n_{23}, n_{222}, u_{2333}, N_G(y_2), N_G(y_3)\} \\ \cup\; & \left\{\mathrm{Tr}(y_1y_2y_3^3), \mathrm{Tr}(y_1y_2^3y_3), \mathrm{Tr}(y_2^3y_3^3), \mathrm{Tr}(y_1y_2^3y_3^3)\right\}.\end{aligned}$$

Further calculation gives $\mathrm{LT}(\mathrm{Tr}(y_1y_2y_3^3)) = cy_2y_1x_3^3$, $\mathrm{LT}(\mathrm{Tr}(y_1y_2^3y_3)) = y_2^2y_1x_2^2$, $\mathrm{LT}(\mathrm{Tr}(y_2^3y_3^3)) = cy_2^3x_3^3$, $\mathrm{LT}(\mathrm{Tr}(y_1y_2^3y_3^3)) = cy_1y_2^3x_3^3$.

Remark 3.14.5. Suppose $\lambda \notin \mathbf{F}_2$, i.e., $c \neq 0$. Applying the SAGBI/Divide-by-x algorithm to $\{x_1, x_2, x_3, N_1, u_{123}, t_3, N_G(y_2), N_G(y_3)\}$ produces a SAGBI basis for $\mathbf{F}[V_{3,\lambda}]^G$. A Magma calculation over the rational function field $\mathbf{F}_2(\lambda)$ shows that for generic λ, $\mathcal{B}_{3,\lambda}$ is a SAGBI basis for $\mathbf{F}_2(\lambda)[V_{3,\lambda}]^G$. Since the lead coefficients of the elements of $\mathcal{B}_{3,\lambda}$ lie in $\{1, c, c^2\}$, the calculations could have been performed over $\mathbf{F}_2[\lambda, c^{-1}]$. Therefore $\mathcal{B}_{3,\lambda}$ is a SAGBI basis for $\mathbf{F}[V_{3,\lambda}]^G$, as long as $c \neq 0$. It follows from this that, for $\lambda \notin \mathbf{F}_2$, the Hilbert ideal is generated by $x_1, x_2, x_3, N_1, N_G(y_2), N_G(y_3)$. Although a SAGBI basis need not be a minimal generating set, running a SAGBI basis test on $\mathcal{B}_{3,\lambda} \setminus \{\mathrm{Tr}(y_1y_2^3y_3^3)\}$ shows that $\mathrm{Tr}(y_1y_2^3y_3^3)$ is indecomposable and hence the Noether number is 7.

Remark 3.14.6. A Magma calculation shows that $\mathbf{F}[V_{3,0}]^G$ is generated by

$$\{x_1, x_2, x_3, n_1, n_2+u_{13}+u_{12}, t_3, (x_3+x_2)u_{13}+n_3x_1, N_G(y_3), \mathrm{Tr}(y_2y_3^3), \mathrm{Tr}(y_1y_2y_3^3)\}.$$

Furthermore, this is a SAGBI basis and $\mathrm{Tr}(y_1y_2y_3^3)$ is indecomposable. Therefore the Hilbert ideal is generated by $\{x_1, x_2, x_3, n_1, n_2 + u_{13} + u_{12}, N_G(y_3)\}$ and the Noether number is 5.

The ring of invariants for the regular representation was computed in [1, Corollary 1.8] and [10, Lemma 5.2]. We include an alternate calculation here for completeness. Choose a basis $\{x, y_1, y_2, z\}$ for V_{reg}^* so that $\Delta_i(z) = y_i$ and $\mathrm{Tr}(z) = x$. Define $u := y_1y_2 + xz$ and $h := (u^2 + N_G(y_1)N_G(y_2))/x = y_1^2y_2 + y_2^2y_1 + x(z^2 + y_1y_2)$.

Theorem 3.14.7. $\mathbf{F}[V_{reg}]^G$ *is the complete intersection generated by*

$$\mathcal{C} = \{x, u, N_G(y_1), N_G(y_2), h, N_G(z)\}$$

subject to the relations

$$u^2 = N_G(y_1)N_G(y_2) + xh$$

and

$$h^2 = N_G(y_1)^2 N_G(y_2) + N_G(y_1)N_G(y_2)^2 + x\left(hN_G(y1) + uh + hN_G(y_2) + xN_G(z)\right).$$

Proof. It follows from [9, Theorem 3.75] that $\mathbf{F}[x, y_1, y_2]^G$ is the polynomial ring generated by x, $N_G(y_1)$ and $N_G(y_2)$. Since u is degree 1 in z with coefficient x, using [4, Theorem 2.4] we have $\mathbf{F}[V_{reg}]^G[x^{-1}] = \mathbf{F}[x, N_G(y_1), N_G(y_2), u][x^{-1}]$. Using the graded reverse lexicographic order with $z > y_1 > y_2 > x$, there are two non-trivial tête-a-têtes among the elements of $\mathcal{C}$. These two tête-a-têtes subduct to zero using the given relations. Therefore $\mathcal{C}$ is a SAGBI basis for the subalgebra it generates. Since $\{x, N_G(y_1), N_G(y_2), N_G(z)\}$ is a block hsop, applying [8, Theorem 1.1] shows that $\mathcal{C}$ is a SAGBI basis for $\mathbf{F}[V_{reg}]^G$. Since all relations come from subducting tête-a-têtes, the ring of invariants is the given complete intersection. ☐

It follows from the above theorem that for V_{reg} the Noether number is 4 and the Hilbert ideal is generated by $\{x, u, N_G(y_1), N_G(y_2), N_G(z)\}$. We note that V_{reg} is the only indecomposable modular representation of G whose Hilbert ideal is not generated by a block hsop.

3.15. **The proof of Theorem 3.3.** Suppose, by way of contradiction, that $\mathrm{Tr}(y_1 \cdots y_\ell y_{\ell+1}^3 \cdots y_m^3)$ is decomposable. Working modulo the G-stable ideal $(x_1, \ldots, x_{m-1})S$, it is easy to see that

$$\mathrm{LT}(\mathrm{Tr}(y_1 \cdots y_\ell y_{\ell+1}^3 \cdots y_m^3)) = (\lambda^2 + \lambda) y_1 \cdots y_\ell y_{\ell+1}^3 \cdots y_{m-1}^3 x_m^3.$$

Thus there are two monomials of positive degree, say M_1 and M_2, such that $M_1 M_2 = y_1 \cdots y_\ell y_{\ell+1}^3 \cdots y_{m-1}^3 x_m^3$, and both M_1 and M_2 appear in G-invariant polynomials. We use the following results to rule out possible factorisations.

Lemma 3.15.1. *Suppose $f \in S^G$, M' is a monomial in $y_1, \ldots, y_m$, and $i > 1$. If the degree of y_i in M' is even, then $M'y_i x_m$ does not appear in f. Further suppose $j < m$. Then the degree of y_i in M' is even and $M'y_i x_j$ appears in f if and only if the degree of y_{j+1} in M' is even and $M'y_{j+1}x_{i-1}$ appears in f.*

Proof. We list the monomials in S that produce $M'x_{i-1}x_j$ after applying Δ_2:

(1) $M'y_i x_j$ if the degree of y_i in M' is even;
(2) $M'x_{i-1}y_{j+1}$ if $j < m$ and the degree of y_{j+1} in M' is even;
(3) $M'x_{i-1}y_j$ if the degree of y_j in M' is even and $\lambda \neq 0$;
(4) $M'y_{i-1}x_j$ if the degree of y_{i-1} in M' is even and $\lambda \neq 0$;
(5) $M'y_{i-1}y_j$ if the degree of y_{i-1} and y_j in M' is even and $\lambda \neq 0$;
(6) $M'y_{i-1}y_{j+1}$ if $j < m$ and the degree of y_{i-1} and y_{j+1} in M' is even and $\lambda \neq 0$;
(7) $M'y_i y_{j+1}$ if $j < m$ and the degree of y_i and y_{j+1} in M' is even;
(8) $M'y_i y_j$ if $i \neq j$ and the degree of y_i and y_j in M' is even and $\lambda \neq 0$.

Note that the monomials in (5)–(8) do not appear in f by Lemma 1.3 because the degree of either y_i or y_{i-1} is odd. On the other hand, by Lemma 1.4 the monomials in (3) and (4) appear in f with the same coefficient (which is possibly zero). Call this coefficient α. Then the coefficient of $M'x_{i-1}x_j$ in $\Delta_2(\alpha M'x_{i-1}y_j + \alpha M'y_{i-1}x_j)$

is $2\lambda\alpha = 0$. It follows that the monomial in (1) appears in f if and only if the monomial in (2) appears in f. $\square$

Proposition 3.15.2. *Let $M = \prod_{i\in I} y_i^2$ for some non-empty subset $I \subseteq \{1,\ldots,m\}$ and assume that M appears in a polynomial $f \in S^G$. Let j denote the maximum integer in I. Then $2j \le m+1$. Furthermore, if $\lambda \in \mathbf{F}\setminus\mathbf{F}_2$, then $2j \le m$.*

Proof. If $j=1$, then $2j \le m+1$ implies $m \ge 1$ and $2j \le m$ gives $m > 1$. For $m = 1$, we have $S^G = \mathbf{F}[x_1, N_G(y)]$ and, if $\lambda \in \mathbf{F}\setminus\mathbf{F}_2$, then $\mathrm{LT}(N_G(y_1)) = y_1^4$. Thus the assertion holds for $j = 1$.

Suppose $j > 1$ and assume that M is maximal among all such monomials that appear in f. Let M' denote the monomial $\prod_{i\in I\setminus\{j\}} y_i^2$. Using Lemma 1.5 (with $\sigma = \sigma_1$), we see that $M'x_jy_j$ appears in f. Since $j > 1$, by Lemma 3.15.1, $j < m$ and $M'x_{j-1}y_{j+1}$ appears in f. Applying Lemma 1.4 shows that $M'x_{j+1}y_{j-1}$ appears in f. If $j-1 > 1$, then, again using Lemma 3.15.1, we have $j+1 < m$ and $M'x_{j-2}y_{j+2}$ appears in f. In this case, by applying Lemma 1.4, we see that $M'x_{j+2}y_{j-2}$ appears in f. Continue alternating Lemma 3.15.1 and Lemma 1.4 until $j-k=1$. This shows that $M'y_{j-k}x_{j+k} = M'y_1x_{2j-1}$ appears in f. Thus $2j-1 \le m$, as required.

Suppose that $\lambda \in \mathbf{F}\setminus\mathbf{F}_2$. Note that $M'x_j^2$ appears in $\Delta_2(M + M'x_jy_j)$ with coefficient $\lambda+\lambda^2 \ne 0$. Since $\Delta_2(f) = 0$, there must be other monomials in f that produce $M'x_j^2$ after applying Δ_2. The monomials $M'y_jy_{j+1}$, $M'x_jy_{j+1}$ and $M'y_{j+1}^2$ are the only such monomials. However, $M'y_jy_{j+1}$ does not appear in f by Lemma 1.3, and the maximality of j implies that $M'y_{j+1}^2$ does not appear in f either. It follows that $M'x_jy_{j+1}$ appears in f. Applying Lemma 1.4 and Lemma 3.15.1 repeatedly we see that $M'x_1y_{2j}$ appears in f. Hence $2j \le m$. $\square$

Write $M_1 = y_1^{a_1}\cdots y_{m-1}^{a_{m-1}}x_m^{a_m}$ and $M_2 = y_1^{b_1}\cdots y_{m-1}^{b_{m-1}}x_m^{b_m}$, where a_i and b_i are non-negative integers. We have $a_i + b_i = 1$ for $i \le \ell$ and $a_i + b_i = 3$ for $i > \ell$.

Suppose $a_m = 0$. Then, using Lemma 1.3 (with $\sigma = \sigma_1$), a_i is even for all i. Thus $a_i = 0$ for $i \le \ell$. Hence Proposition 3.15.2 applies, forcing $a_i = 0$ for $i > \ell \ge m/2$. Therefore, if $a_m = 0$, we have $M_1 = 1$ and the factorisation is trivial. Hence $a_m > 0$. Similarly, $b_m > 0$. Without loss of generality, we assume $a_m = 1$ and $b_m = 2$.

Lemma 3.15.3. *If $m \ge 3$, then a_{m-1} is even. If $m \ge 4$, then a_{m-2} is even.*

Proof. Both statements follow from Lemma 3.15.1. $\square$

Lemma 3.15.4. *If $m \ge 3$, then b_{m-1} and b_{m-2} are not both odd.*

Proof. Assume on the contrary that both b_{m-1} and b_{m-2} are odd and that M_2 appears in $f_2 \in S^G$. Define $M = y_1^{b_1}\cdots y_{m-3}^{b_{m-3}}y_{m-2}^{b_{m-2}-1}y_{m-1}^{b_{m-1}-1}$ so that $M_2 = My_{m-2}y_{m-1}x_m^2$. Then $Mx_{m-2}y_{m-1}x_m^2$ appears in $\Delta_1(My_{m-2}y_{m-1}x_m^2)$. Since $\Delta_1(f_2) = 0$, there must be other monomials in f_2 that produce $Mx_{m-2}y_{m-1}x_m^2$ after applying Δ_1. The only monomials with this property are $My_{m-2}y_{m-1}y_m^2$, $My_{m-2}y_{m-1}x_my_m$, $Mx_{m-2}y_{m-1}y_m^2$ and $Mx_{m-2}y_{m-1}x_my_m$. However $My_{m-2}y_{m-1}y_m^2$ does not appear in f_2 by Lemma 1.3 because the degree of y_{m-1} in this monomial is odd. Also, $My_{m-2}y_{m-1}x_my_m$ does not appear in f_2 by Lemma 3.15.1. If $Mx_{m-2}y_{m-1}y_m^2$ appears in f_2, then, since the degree of y_{m-2} in this monomial is odd, $Mx_{m-2}^2y_m^2$ appears in $\Delta_2(Mx_{m-2}y_{m-1}y_m^2)$. So there must be another monomial in f_2 that produces $Mx_{m-2}^2y_m^2$ after applying Δ_2. The only monomials in S with this property are $My_{m-1}^2y_m^2$ if $b_{m-1} = 1$, $My_{m-2}^2y_m^2$ if $b_{m-2} = 1$,

$My_{m-2}y_{m-1}y_m^2$ and $Mx_{m-2}y_{m-2}y_m^2$. The first three monomials do not appear in f_2 by Lemma 1.3 and Proposition 3.15.2. On the other hand $Mx_{m-2}y_{m-2}y_m^2$ does not appear in f_2 if $b_{m-2}=3$ by Lemma 3.15.1. If $b_{m-2}=1$, then $Mx_{m-2}y_{m-2}y_m^2$ appears in f_2 if and only if $My_{m-2}^2y_m^2$ appears in f_2. However the latter monomial does not appear in f_2 by Lemma 1.3 and Proposition 3.15.2. Therefore $Mx_{m-2}y_{m-1}y_m^2$ does not appear in f_2.

We finish the proof by showing that $Mx_{m-2}y_{m-1}x_my_m$ does not appear in f_2. Note that $Mx_{m-2}^2x_my_m$ appears in $\Delta_2(Mx_{m-2}y_{m-1}x_my_m)$. The other monomials that produce $Mx_{m-2}^2x_my_m$ after applying Δ_2 are $My_{m-1}^2x_my_m$ if $b_{m-1}=1$, $My_{m-2}^2x_my_m$ if $b_{m-2}=1$, $My_{m-2}y_{m-1}x_my_m$ and $Mx_{m-2}y_{m-2}x_my_m$. The first two monomials appear in f_2 if and only if $My_{m-1}^2y_m^2$ and $My_{m-2}^2y_m^2$ appear in f_2, respectively, by Lemma 1.5. However neither of the latter monomials appear in f_2 by Lemma 1.3 and Proposition 3.15.2. The third monomial does not appear in f_2 by Lemma 3.15.1. Finally, $Mx_{m-2}y_{m-2}x_my_m$ appears in f_2 if and only if $My_{m-2}^2x_my_m$ appears in f_2 because these are the only monomials in S that produce $Mx_{m-2}^2x_my_m$ after applying Δ_1. However $My_{m-2}^2x_my_m$ appears in f_2 if and only if $My_{m-2}^2y_m^2$ appears in f_2 by Lemma 1.5, and the latter monomial does not appear in f_2 by Proposition 3.15.2. ☐

Returning to the proof of Theorem 3.3, first assume that $m \geq 4$. Then by Lemma 3.15.3, a_{m-2} and a_{m-1} are both even. Therefore b_{m-2} and b_{m-1} are both odd, contradicting Lemma 3.15.4.

Suppose $m=3$ and M_1 appears in $f_1 \in S^G$. By Lemma 3.15.3, a_2 is even. Thus b_2 is odd and, by Lemma 3.15.4, b_1 is even. Therefore $b_1=0$, $a_1=1$ and $M_1=y_1y_2^{a_2}x_3$. By Lemma 1.4, $x_1y_2^{a_2}y_3$ also appears in f_1. Thus $y_2^{a_2+1}x_2$ appears in f_1 as well by Lemma 3.15.1. This contradicts Lemma 1.5 if $a_2=2$ and Proposition 3.15.2 if $a_2=0$.

3.16. **The proof of Theorem 3.9.** Suppose, by way of contradiction, that $\mathrm{Tr}(y_1\cdots y_{\ell'}y_{\ell'+1}^3\cdots y_m^3)$ is decomposable. Working modulo the G-stable ideal $(x_1,\dots,x_{m-2},x_{m-1}^2)S$, a straightforward calculation gives

$$\mathrm{LT}(\mathrm{Tr}(y_1\cdots y_{\ell'}y_{\ell'+1}^3\cdots y_m^3)) = y_1\cdots y_{\ell'}y_{\ell'+1}^3\cdots y_{m-1}^3x_{m-1}x_m^2.$$

Thus there are two monomials of positive degree, say M_1 and M_2, such that $M_1M_2 = y_1\cdots y_{\ell'}y_{\ell'+1}^3\cdots y_{m-1}^3x_{m-1}x_m^2$, and both M_1 and M_2 appear in G-invariant polynomials, say f_1 and f_2. Without loss of generality, we may assume $M_1 = y_1^{a_1}\cdots y_{m-1}^{a_{m-1}}x_{m-1}x_m^{a_m}$ and $M_2 = y_1^{b_1}\cdots y_{m-1}^{b_{m-1}}x_m^{b_m}$. It follows from Lemma 1.3 and Proposition 3.15.2 that $b_m>0$.

Lemma 3.16.1. *If $m>i>1$, then b_i is even and a_i is odd.*

Proof. Note that $V_{m,0}^*$ and $(m-1)V_2\oplus 2V_1$ are isomorphic σ_2-modules, where the two copies of V_1 are generated by x_m and y_1 and where each pair x_{i-1},y_i for $2\leq i\leq m$ generate a copy of V_2. Therefore we have $S^{\sigma_2}\cong \mathbf{F}[x_1,\dots,x_{m-1},y_2,\dots,y_m]^{\sigma_2}\otimes \mathbf{F}[x_m,y_1]$. Hence the fact that b_i is even follows from Lemma 1.3 (with $\sigma=\sigma_2$). Since b_i is even and a_i+b_i is odd, a_i is odd. ☐

We have $b_m>0$ and $a_m+b_m=2$. Therefore, there are two cases, $a_m=0$ and $a_m=1$. First assume that $a_m=0$. If $a_{m-1}=3$, then M_1 does not appear in f_1 by Lemma 1.5. On the other hand, if $a_{m-1}=1$, then by Lemma 1.5, $y_1^{a_1}\cdots y_{m-1}^{a_{m-1}+1}$ appears in f_1, contradicting Lemma 1.3 because a_{m-2} is odd.

Suppose that $a_m = 1$. Set $M = y_1^{a_1} \cdots y_{m-1}^{a_{m-1}-1}$ so that $M_1 = My_{m-1}x_{m-1}x_m$. Then $Mx_{m-2}x_{m-1}x_m$ appears in $\Delta_2(M_1)$. The only other monomials in S that produce $Mx_{m-2}x_{m-1}x_m$ after applying Δ_2 are $My_{m-2}y_mx_m$ and $Mx_{m-2}y_mx_m$. However by Lemma 1.5 $My_{m-2}y_mx_m$ appears in f_1 if and only if $My_{m-2}y_m^2$ does, but the latter monomial does not appear in f_1 by Lemma 1.3 and Proposition 3.15.2. Finally, if $Mx_{m-2}y_mx_m$ appears in f_1, there must be another monomial in f_1 that produces $Mx_{m-2}x_m^2$ after applying Δ_1. Since a_{m-2} is odd, $Mx_{m-2}y_m^2$ is the only such monomial. However if $a_{m-2} = 3$, then $Mx_{m-2}y_m^2$ does not appear in f_1. If $a_{m-2} = 1$, then again by Lemma 1.5, $My_{m-2}y_m^2$ also appears in f_1, contradicting Proposition 3.15.2.

4. The easy odd case

In this section we consider the odd dimensional representations $\Omega^{-m}(\mathbf{F})$. The action of G on $S := \mathbf{F}[\Omega^{-m}(\mathbf{F})] = \mathbf{F}[x_1, \ldots, x_m, y_1, \ldots, y_{m+1}]$ is given by $\sigma_i(x_j) = x_j$, $\sigma_1(y_j) = y_j + x_j$ and $\sigma_2(y_j) = y_j + x_{j-1}$, using the convention that $x_0 = 0$ and $x_{m+1} = 0$. As in section 3, define $n_i := y_i^2 + x_iy_i$ and $u_{ij} = x_iy_j + x_jy_i$. Then $n_i, u_{ij} \in S^{\sigma_1}$. A simple calculation gives $\Delta_2(n_i) = x_{i-1}^2 + x_ix_{i-1}$ and $\Delta_2(u_{ij}) = x_ix_{j-1} + x_{i-1}x_j$. For $i \in \{1, \ldots, m+1\}$ define

$$N_i := n_i + \sum_{j=1}^{i-1} (u_{i-j,i+j} + u_{i-j,i+j-1})$$

so that $N_1 = n_1$ and $N_2 = n_2 + u_{12} + u_{13}$. An explicit calculation, exploiting the fact that $\Delta_2(u_{1j}) = x_1x_{j-1}$, gives $\Delta_2(N_i) = 0$. Therefore $N_i \in S^G$. Define $\mathcal{H}_{-m} := \{x_1, \ldots, x_m, N_1, \ldots, N_{m+1}\}$. Since $\mathrm{LM}(N_i) = y_i^2$, $\mathcal{H}_{-m}$ is a block hsop with top class $y_1 \cdots y_{m+1}$, and the image of the transfer is generated by $\mathrm{Tr}(\beta)$ for β dividing $y_1 \cdots y_{m+1}$.

Theorem 4.1. *For $m > 3$, $\mathrm{Tr}(y_1 \cdots y_{m+1})$ is indecomposable.*

See subsection 4.8 for the proof of Theorem 4.1. Combining Theorem 4.1 with Theorem 2.1 gives the following.

Corollary 4.2. *If $m > 3$, then the Noether number for $\Omega^{-m}(\mathbf{F})$ is $m+1$ and the Hilbert ideal is generated by $\mathcal{H}_{-m}$.*

Remarks 4.4 and 4.6 show that the above formula for the Noether number is valid for $m \geq 1$.

As in section 3, define $t_j := u_{12}x_{j-1} + u_{1j}x_1$.

Theorem 4.3. *For $m > 2$,*

$$\mathbf{F}[\Omega^{-m}(\mathbf{F})]^G[x_1^{-1}] = \mathbf{F}[x_1, \ldots, x_m, N_1, N_2, t_3, \ldots, t_{m+1}][x_1^{-1}].$$

Proof. We construct the field of fractions for an upper-triangular action as in [4] or [14]. The restriction of the action of G to the span of $\{x_1, x_2, y_1, y_2\}$ is $V_{2,0}^*$. Therefore, using Remark 3.14.3, $\mathbf{F}[x_1, x_2, y_1, y_2]^G[x_1^{-1}] = \mathbf{F}[x_1, x_2, n_1, \widetilde{w}]^G[x_1^{-1}]$. Since $t_j \in \mathbf{F}[x_1, \ldots, x_m, y_1, \ldots, y_j]^G$ has degree one as a polynomial in y_j with coefficient x_1^2, we have $\mathbf{F}[\Omega^{-m}(\mathbf{F})]^G[x_1^{-1}] = \mathbf{F}[x_1, \ldots, x_m, n_1, \widetilde{w}, t_3, \ldots, t_{m+1}][x_1^{-1}]$. The result then follows from the fact that $\widetilde{w} = x_1N_2 + t_3$ and $N_1 = n_1$. $\square$

Remark 4.4. It is easy to see that $\mathbf{F}[\Omega^{-1}(\mathbf{F})]^G = \mathbf{F}[x_1, n_1, y_2^2 + x_1y_2]$. A Magma calculation shows that $\mathbf{F}[\Omega^{-2}(\mathbf{F})]^G$ is the hypersurface with generators x_1, x_2, N_1, N_2, N_3, t_3 and relation $t_3^2 + x_2^4N_1 + x_1x_2(x_1+x_2)t_3 + x_1^2x_2^2N_2 = x_1^4N_3$. Therefore, the Noether number for this representation is $m+1=3$.

Remark 4.5. It follows from Theorem 4.3 that applying the SAGBI/Divide-by-x algorithm of [8] with $x = x_1$ to

$$\{x_1, \dots, x_m, N_1, N_2, \dots, N_{m+1}, t_3, \dots, t_{m+1}\}$$

produces a SAGBI basis for $\mathbf{F}[\Omega^{-m}(\mathbf{F})]^G$.

Remark 4.6. A Magma calculation shows that $\mathbf{F}[\Omega^{-3}(\mathbf{F})]^G$ is generated by

$$\{x_1, x_2, x_3, n_1, N_2, N_3, n_4, t_3, t_4, u_{233}, u_{133}, \mathrm{Tr}(y_1y_2y_3y_4)\},$$

where $u_{133} := x_3u_{13} + x_1u_{24}$ and $u_{233} := x_3u_{23} + x_2u_{24} + x_3u_{14}$. Furthermore, this set is a SAGBI basis, and running a SAGBI test with $\mathrm{Tr}(y_1y_2y_3y_4)$ omitted shows that $\mathrm{Tr}(y_1y_2y_3y_4)$ is indecomposable. Therefore the Noether number for this representation is $m+1=4$ and the Hilbert ideal is generated by the block hsop $x_1, x_2, x_3, n_1, N_2, N_3, n_4$. From [10], we know $\mathrm{depth}(\mathbf{F}[\Omega^{-3}(\mathbf{F})]^G) = 6$. The relation $x_2t_4 + x_3t_3 + x_1u_{133} = 0$ shows that the partial hsop $\{x_1, x_2, x_3\}$ is not a regular sequence, giving an alternate proof of the fact that the ring is not Cohen-Macaulay.

Proposition 4.7. *For* $S = \mathbf{F}[\Omega^{-m}]$, $(\Delta_2 S)^{\sigma_1} = (\Delta_1 S)^{\sigma_2} = ((x_1, \dots, x_m)\,S)^G = \sqrt{\mathrm{Tr}\,S}$ *and*

$$\sqrt{\mathrm{Tr}\,S}/\,\mathrm{Tr}\,S \cong H^1(\langle\sigma_2\rangle, \Delta_1 S) = H^1(\langle\sigma_1\rangle, \Delta_2 S).$$

Furthermore $S^G/\sqrt{\mathrm{Tr}\,S} \cong \mathbf{F}[N_1, \dots, N_m]$.

Proof. The proof is analogous to the proof of Proposition 3.7. (Note that $\mathrm{LT}(N_i) = y_i^2$ and so an analogue of Proposition 3.15.2 is unnecessary.) □

4.8. **Proof of Theorem 4.1.** Suppose by way of contradiction that $\mathrm{Tr}(y_1 \cdots y_{m+1})$ is decomposable. Working modulo the G-stable ideal $(x_1, \dots, x_{m-1})S$, it is easy to see that

$$\mathrm{LT}(\mathrm{Tr}(y_1 \cdots y_{m+1})) = y_1 \cdots y_{m-1}x_m^2.$$

Thus there are two monomials, say M_1 and M_2, such that $M_1M_2 = y_1 \cdots y_{m-1}x_m^2$, $\deg(M_2) \le \deg(M_1) < m+1$ and both M_1 and M_2 appear in G-invariant polynomials. Since a G-invariant is also a σ_1-invariant, it follows from Lemma 1.3 that both M_1 and M_2 are divisible by x_m. Since $m+1 \ge 5$, we have $\deg(M_1) \ge 3$. The required contradiction is then a consequence of the following lemma.

Lemma 4.8.1. *Let* $M = (\prod_{j\in J} y_j)x_k$ *for some* $k \le m$ *and set* $J \subseteq \{1, \dots, k-1\}$ *with* $|J| > 1$. *Then* M *does not appear with a non-zero coefficient in a* G*-invariant polynomial.*

Proof. Let d denote the maximum integer in J. We proceed by induction on $k-d$. Assume on the contrary that M appears in a G-invariant polynomial f. Set $M' = \prod_{j\in J,\, j\ne d} y_j$. Then we have $M = M'y_dx_k$. From Lemma 1.4 we get that $M'x_dy_k$ also appears in f. Furthermore, since $M'x_dx_{k-1}$ appears in $\Delta_2(M'x_dy_k)$, there must be another monomial in f that produces $M'x_dx_{k-1}$ after applying Δ_2. If $k-d=1$, then the only other monomial that produces $M'x_dx_{k-1} = M'x_d^2$ after applying Δ_2 is $M'y_k^2$. However, this monomial cannot appear in f by Lemma 1.3. This establishes the basis case for the induction. If $k-d>1$, the only monomials (other than $M'x_dy_k$) that produce $M'x_dx_{k-1}$ after applying Δ_2 are $M'y_{d+1}y_k$ and $M'y_{d+1}x_{k-1}$. Again by Lemma 1.3, $M'y_{d+1}y_k$ cannot appear in f. Moreover, if $d+1 < k-1$, then $M'y_{d+1}x_{k-1}$ does not appear in f by induction. On the other hand, if $d+1 = k-1$, then $M'y_{d+1}x_{k-1}$ does not appear in f by Lemma 1.3. $\square$

5. The hard odd case

In this section we consider the odd dimensional representations $\Omega^m(\mathbf{F})$. The action of G on $S := \mathbf{F}[\Omega^m(\mathbf{F})] = \mathbf{F}[x_1, \ldots, x_{m+1}, y_1, \ldots, y_m]$ is given by $\sigma_i(x_j) = x_j$, $\sigma_1(y_j) = y_j + x_j$ and $\sigma_2(y_j) = y_j + x_{j+1}$. Define

$$\mathcal{H}_m := \{x_1, \ldots, x_{m+1}, N_G(y_1), \ldots, N_G(y_m)\}.$$

Since $\mathrm{LM}(N_G(y_i)) = y_i^4$, $\mathcal{H}_m$ is a block hsop with top class $(y_1 \cdots y_m)^3$ and the image of the transfer is generated by $\mathrm{Tr}(\beta)$ for β dividing $(y_1 \cdots y_m)^3$.

Theorem 5.1. *For $m > 2$, $\mathrm{Tr}(y_1^3 \cdots y_m^3)$ is indecomposable.*

See subsection 5.8 for the proof of Theorem 5.1. Combining Theorem 5.1 with Theorem 2.1 gives the following.

Corollary 5.2. *If $m > 2$, then the Noether number for $\Omega^m(\mathbf{F})$ is $3m$ and the Hilbert ideal is generated by $\mathcal{H}_m$.*

From Remark 5.4, the Noether number for $\Omega^2(\mathbf{F})$ is 6.

For $j > 1$, define $v_j := u_{1j}(x_2^2 + x_1x_2) + n_1(x_jx_2 + x_1x_{j+1})$.

Theorem 5.3. *For $m > 1$,*

$$\mathbf{F}[\Omega^m]^G[(x_1x_2(x_1+x_2))^{-1}] = \mathbf{F}[x_1, \ldots, x_{m+1}, N_G(y_1), v_2, \ldots, v_m][(x_1x_2(x_1+x_2))^{-1}].$$

Proof. We use [4, Theorem 2.4]. $\mathbf{F}[x_1, \ldots, x_m, y_1]^G$ is the polynomial algebra generated by $\{x_1, \ldots, x_m, N_G(y_1)\}$. The invariant $v_j \in \mathbf{F}[x_1, x_2, x_j, x_{j+1}, y_1, y_j]$ has degree one as a polynomial in y_j and the coefficient of y_j is $x_1x_2(x_1+x_2)$. $\square$

It is easy to see that $\mathbf{F}[\Omega^1(\mathbf{F})]^G = \mathbf{F}[x_1, x_2, N_G(y_1)]$, and, therefore, the Noether number is 4.

Remark 5.4. A Magma calculation shows that $\mathbf{F}[\Omega^2(\mathbf{F})]^G$ is generated by

$$\mathcal{B}_2 := \{x_1, x_2, x_3, N_G(y_1), N_G(y_2), v_2, n_{13}, u_{1233}, \mathrm{Tr}(y_1^3y_2^3)\},$$

where $n_{13} = x_3n_1 + x_3u_{12} + x_1n_2$ and $u_{1233} = (x_3^2 + x_2x_3)u_{12} + (x_2^2 + x_1x_3)n_2$. Therefore the Hilbert ideal for $\Omega^2(\mathbf{F})$ is generated by $x_1, x_2, x_3, N_G(y_1), N_G(y_2)$. In fact, $\mathcal{B}_2$ is a SAGBI basis using grevlex with $y_2 > y_1 > x_3 > x_2 > x_1$. Although a SAGBI basis need not be a minimal generating set, running a SAGBI basis test on $\mathcal{B}_2 \setminus \{\mathrm{Tr}(y_2^3y_3^3)\}$ shows that $\mathrm{Tr}(y_2^3y_3^3)$ is indecomposable and hence the Noether number is 6. From [10], we know $\mathrm{depth}(\mathbf{F}[\Omega^2(\mathbf{F})]^G) = 4$. The relation $x_3v_2 + (x_2^2 + x_1x_3)n_{13} + x_1u_{1233} = 0$ shows that the partial hsop $\{x_1, x_2, x_3\}$ is not a regular sequence, giving an alternate proof of the fact that the ring is not Cohen-Macaulay.

Remark 5.5. We have been unable to find "polynomial generators" for the ring $\mathbf{F}[\Omega^m(\mathbf{F})]^G[x_1^{-1}]$. We note that x_1 is not in the radical of the image of the transfer for these representations but that $x_1x_2(x_1+x_2)$ is. Furthermore, x_1 is in the radical of the image of the transfer for $\Omega^{-m}(\mathbf{F})$ and $V_{m,\lambda}$. Hence $\mathbf{F}[\Omega^{-m}]^G[x_1^{-1}]$ and $\mathbf{F}[V_{m,\lambda}]^G[x_1^{-1}]$ are "trace-surjective" in the sense of [13].

Proposition 5.6. *For $S=\mathbf{F}[\Omega^m(\mathbf{F})]$ and $m\geq 3$,*

$$\sqrt{\operatorname{Tr} S}=((x_2x_{m+1}+x_2x_1,x_1x_{m+1}+x_1x_2,x_2^2+x_2x_1,x_3+x_2,\ldots,x_m+x_2)S)^G.$$

Proof. Direct calculation gives $\Delta_1(\Omega^m(F)^*)=\operatorname{Span}_{\mathbf{F}}\{x_1,\ldots,x_m\}$, $\Delta_2(\Omega^m(F)^*)=\operatorname{Span}_{\mathbf{F}}\{x_2,\ldots,x_{m+1}\}$, and $(\sigma_1\sigma_2+1)(\Omega^m(F)^*)=\operatorname{Span}_{\mathbf{F}}\{x_1+x_2,\ldots,x_m+x_{m+1}\}$. Using [18, Theorem 2.4] and computing intersections of ideals gives

$$\begin{aligned}\sqrt{\operatorname{Tr} S} &= \bigcap_{g\in G,\,|g|=2}(((g-1)\Omega^m(\mathbf{F})^*)S)^G\\ &= ((x_2x_{m+1}+x_2x_1,x_1x_{m+1}+x_2x_1,x_2^2+x_2x_1,x_3+x_2,\ldots,x_m+x_2)S)^G.\end{aligned}$$

□

Remark 5.7. The above shows that for $m\geq 3$, we have $x_2+x_3\in\sqrt{\operatorname{Tr} S}$. In fact, for

$$\alpha:=(x_1+x_2+x_3)y_2y_3+(x_1+x_2+x_3+x_4)y_1y_3+(x_2+x_3+x_4)y_1y_2+y_1^2y_3+y_1y_3^2,$$

$\operatorname{Tr}(\alpha)=(x_2+x_3)^3$. Define $x:=x_2+x_3$ and use the variables $x<x_1<x_3<x_4<\cdots<x_{m+1}<y_1<\cdots<y_m$ with the grevlex order. Define $\rho:\mathbf{F}[\Omega^m(\mathbf{F})][x^{-1}]\to\mathbf{F}[\Omega^m(\mathbf{F})]^G[x^{-1}]$ by $\rho(f)=x^{-3}\operatorname{Tr}(f\alpha)$. Then ρ restricts to the identity on $\mathbf{F}[\Omega^m(\mathbf{F})]^G$ and $\mathbf{F}[\Omega^m(\mathbf{F})]^G[x^{-1}]$ is "trace-surjective". Define

$$\mathcal{B}_m:=\mathcal{H}_m\cup\{\operatorname{Tr}(\beta)\mid\beta\,\text{divides}\,(y_1\cdots y_m)^3\}.$$

Since $\{\beta\mid\beta\,\text{divides}\,(y_1\cdots y_m)^3\}$ generates $\mathbf{F}[\Omega^m(\mathbf{F})][x^{-1}]$ as a module over the ring $\mathbf{F}[\mathcal{H}_m][x^{-1}]$ and ρ is surjective, we see that $\mathcal{B}_m\cup\{x^{-1}\}$ generates $\mathbf{F}[\Omega^m(\mathbf{F})]^G[x^{-1}]$. Thus, since $\mathcal{H}_m$ is an hsop, applying the SAGBI/Divide-by-x algorithm to $\mathcal{B}_m$ produces a generating set, in fact a SAGBI basis, for $\mathbf{F}[\Omega^m(\mathbf{F})]^G$.

5.8. **Proof of Theorem 5.1.** Suppose, by way of contradiction, that $\operatorname{Tr}(y_1^3\cdots y_m^3)$ is decomposable. Working modulo the G-stable ideal $(x_1,\ldots,x_{m-1})S$, it is not difficult to see that

$$\operatorname{LT}(\operatorname{Tr}(y_1^3\cdots y_m^3))=y_1^3\cdots y_{m-1}^3x_mx_{m+1}^2.$$

Write $y_1^3\cdots y_{m-1}^3x_mx_{m+1}^2=M_1M_2$, where M_1 and M_2 are monomials of positive degree which appear in G-invariant polynomials. We use the following results to eliminate possible factorisations.

Lemma 5.8.1. *Suppose $1\leq i\leq m$, $2\leq k\leq m+1$, $k\neq i+1$ and M is a monomial in $y_1,\ldots,y_m$. Further suppose that the degree of y_i in M is even and y_ix_kM appears in a G-invariant polynomial f. Then the degree of y_{k-1} in M is even and $x_{i+1}y_{k-1}M$ appears in f.*

Proof. Since the degree of y_i in M is even, $x_{i+1}x_kM$ appears in $\Delta_2(y_ix_kM)$. Since $\Delta_2(f)=0$, f must contain another monomial that produces $x_{i+1}x_kM$ after applying Δ_2. If the degree of y_{k-1} in M is odd, then there is no such monomial. Thus the degree of y_{k-1} in M is even and applying Δ_2 to either $y_iy_{k-1}M$ or $x_{i+1}y_{k-1}M$ produces $x_{i+1}x_kM$. However, by Lemma 1.3, $y_iy_{k-1}M$ does not appear in f. Thus $x_{i+1}y_{k-1}M$ appears in f. □

Proposition 5.8.2. *Suppose $M=y_1^{e_1}\cdots y_m^{e_m}$. If k is a positive integer and Mx_1^k or Mx_{m+1}^k appears in a G-invariant polynomial, then e_j is even for $1\leq j\leq m$.*

Proof. Note that $S^{\sigma_1} \cong \mathbf{F}[x_i, y_i \mid i \le m]^{\sigma_1} \otimes \mathbf{F}[x_{m+1}]$ and $S^{\sigma_2} \cong \mathbf{F}[x_{i+1}, y_i \mid i \le m]^{\sigma_2} \otimes \mathbf{F}[x_1]$. If Mx_{m+1}^k appears in a G-invariant polynomial, then M appears in a σ_1-invariant polynomial, and the result follows from applying Lemma 1.3 with $\sigma = \sigma_1$. If Mx_1^k appears in a G-invariant polynomial, then M appears in a σ_2-invariant polynomial, and the result follows from applying Lemma 1.3 with $\sigma = \sigma_2$. □

Proposition 5.8.3. *Suppose $M = \prod_{j \in J} y_j^2$ for a non-empty index set $J \subseteq \{1, \ldots, m\}$. Then M does not appear in a G-invariant polynomial.*

Proof. Suppose, by way of contradiction, that M appears in a G-invariant polynomial f. Let ℓ denote the largest integer in J and set $M' = M/y_\ell^2$. Note that $M'x_{\ell+1}^2$ appears in $\Delta_2(M)$, and since $\Delta_2(f) = 0$, there must be another monomial in f that produces $M'x_{\ell+1}^2$ after applying Δ_2. The only other monomial in S with this property is $M'y_\ell x_{\ell+1}$. Therefore, this monomial also appears in f. If $\ell = m$, then the degree of y_m in $M'y_\ell x_{\ell+1} = M'y_m x_{m+1}$ is odd, and we have a contradiction by Proposition 5.8.2. Otherwise, using Lemma 1.4, $M'x_\ell y_{\ell+1}$ appears in f. If $\ell = 1$, this also gives a contradiction using Proposition 5.8.2. Otherwise, we apply Lemma 5.8.1 and conclude that $M'y_{\ell-1}x_{\ell+2}$ appears in f. This gives a contradiction if $\ell + 1 = m$. Continuing in this fashion, the process terminates with either $M'y_{2\ell-m}x_{m+1}$ or $M'y_{2\ell}x_1$ appearing in f, again contradicting Proposition 5.8.2. □

Returning to the proof of Theorem 5.1, first suppose that M_1 is a factor of $y_1^3 \cdots y_{m-1}^3$. Since M_1 appears in a σ_1-invariant, we have from Lemma 1.3 that the degree of each y_i in M_1 is even. However, since these degrees are at most two, we get a contradiction using Proposition 5.8.3. Similarly, M_2 is a not factor of $y_1^3 \cdots y_{m-1}^3$. Therefore we may assume x_m divides M_1 and x_{m+1} divides M_2. By Proposition 5.8.2, the degrees of the variables $y_1, \ldots, y_{m-1}$ in M_2 are even. Hence the degrees of these variables in M_1 are odd. Therefore we have either $M_1 = y_1^{a_1} \cdots y_{m-1}^{a_{m-1}} x_m$ or $M_1 = y_1^{a_1} \cdots y_{m-1}^{a_{m-1}} x_m x_{m+1}$, where $a_1, \ldots, a_{m-1}$ are odd. Let f denote the G-invariant polynomial in which M_1 appears. Suppose that $M_1 = y_1^{a_1} \cdots y_{m-1}^{a_{m-1}} x_m$. Since $y_1^{a_1} \cdots y_{m-1}^{a_{m-1}-1} x_m^2$ appears in $\Delta_2(M_1)$ and $\Delta_2(f) = 0$, there must be another monomial in f that produces $y_1^{a_1} \cdots y_{m-1}^{a_{m-1}-1} x_m^2$ after applying Δ_2. However, $y_1^{a_1} \cdots y_{m-1}^{a_{m-1}+1}$ is the only other monomial in S with this property. Since f is also σ_1-invariant and a_1 is odd, we get a contradiction by Lemma 1.3. Therefore, we may assume that $M_1 = y_1^{a_1} \cdots y_{m-1}^{a_{m-1}} x_m x_{m+1}$. Then $y_1^{a_1} \cdots y_{m-1}^{a_{m-1}-1} x_m^2 x_{m+1}$ appears in $\Delta_2(M_1)$. Since $\Delta_2(f) = 0$, there must be another monomial in f that produces $y_1^{a_1} \cdots y_{m-1}^{a_{m-1}-1} x_m^2 x_{m+1}$ after applying Δ_2. The monomials in S with this property are $y_1^{a_1} \cdots y_{m-1}^{a_{m-1}+1} y_m$, $y_1^{a_1} \cdots y_{m-1}^{a_{m-1}} x_m y_m$, $y_1^{a_1} \cdots y_{m-1}^{a_{m-1}+1} x_{m+1}$, $y_1^{a_1} \cdots y_{m-1}^{a_{m-1}-1} x_m^2 y_m$. The first two monomials do not appear in f by Lemma 1.3 because the degree of y_1 is odd. For the same reason the third monomial does not appear in f by Proposition 5.8.2. Finally, if $y_1^{a_1} \cdots y_{m-1}^{a_{m-1}-1} x_m^2 y_m$ appears in f, then there must be another monomial in f that produces $y_1^{a_1-1} x_1 \cdots y_{m-1}^{a_{m-1}-1} x_m^2 y_m$ after applying Δ_1. However, $y_1^{a_1} \cdots y_{m-1}^{a_{m-1}-1} y_m^3$ and $y_1^{a_1-1} x_1 \cdots y_{m-1}^{a_{m-1}-1} y_m^3$ are the only monomials in S with this property. Since neither of these monomials can appear in f, by Lemma 1.3 and Proposition 5.8.2 respectively, we have ruled out all possible factorisations, proving Theorem 5.1.

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