LINES GENERATE THE PICARD GROUPS OF CERTAIN FERMAT SURFACES

ALEX DEGTYAREV

Abstract. We answer a question of T. Shioda and show that, for any positive integer m prime to 6, the Picard group of the Fermat surface Φ_m is generated by the classes of lines contained in Φ_m. A few other classes of surfaces are also considered.

1. Introduction

1.1. **Principal results.** All algebraic varieties in the paper are over $\mathbb{C}$. Let m be a positive integer, and let

$$\Phi_m := \{z_0^m + z_1^m + z_2^m + z_3^m = 0\} \subset \mathbb{P}^3$$

be the *Fermat surface*. If $m = 1$ (plane) or $m = 2$ (quadric), then Φ_m contains infinitely many lines (meaning true straight lines in $\mathbb{P}^3$); otherwise, Φ_m is known to contain exactly $3m^2$ lines.

Since Φ_m is simply connected, one can identify its Picard group $\operatorname{Pic}\Phi_m$ and its Néron–Severi lattice $NS(\Phi_m)$. Citing [1], the Néron–Severi group "...is a rather delicate invariant of arithmetic nature. Perhaps for this reason it usually requires some nontrivial work before one can determine the Picard number of a given variety, let alone the full structure of its Néron–Severi group." The Picard groups of Fermat surfaces are related to those of the more general *Delsarte surfaces* (see [14]; they fit into the framework outlined in §2.4). Furthermore, continuing the citation, "Combined with the method based on the inductive structure of Fermat varieties, this might lead to the verification of the Hodge conjecture for all Fermat varieties."

Let $\mathbf{S}_m \subset \operatorname{Pic}\Phi_m$ be the subgroup generated by the classes of the lines contained in Φ_m. Then, according to [13], one has

$$\mathbf{S}_m \otimes \mathbb{Q} = (\operatorname{Pic}\Phi_m) \otimes \mathbb{Q} \quad \text{if and only if } m \leqslant 4 \text{ or g.c.d.}(m, 6) = 1. \tag{1.1}$$

This statement is proved by comparing the dimensions of the two spaces, which are computed independently. In other words, the classes of lines generate $\operatorname{Pic}\Phi_m$ rationally, and a natural question, raised in [1], is whether they also generate the Picard group over the integers. A partial answer to this question was given in [11], almost 30 years later: the equality $\operatorname{Pic}\Phi_m = \mathbf{S}_m$ holds for all integers m prime to 6 in the range $5 \leqslant m \leqslant 100$. This fact is proved by supersingular reduction and a computer aided computation of the discriminants of the lattices involved. (The case $m = 3$ is classical: any nonsingular cubic contains 27 lines, which generate its Picard group. The case $m = 4$, *i.e.*, that of $K3$-surfaces, was settled in [10],

2000 *Mathematics Subject Classification.* Primary: 14J25; Secondary: 14J05, 14H30.
Key words and phrases. Fermat surface, Picard group, Néron–Severi group, Alexander module.

see also [3] for a slight generalization. The proof suggested below works for both cases.)

The principal result of the present paper is the following theorem, answering the above question in the affirmative in the general case.

Theorem 1.2. *Let $m \geqslant 1$ be an integer such that either $m \leqslant 4$ or* g.c.d.$(m, 6) = 1$. *Then* $\operatorname{Pic}\Phi_m = \mathbf{S}_m$, i.e., $\operatorname{Pic}\Phi_m$ *is generated by the classes of lines.*

Since the $3m^2$ lines in Φ_m admit a very explicit description (*cf.* §2.4) and one can easily see how they intersect (see, *e.g.*, Equation (6) in [11]), Theorem 1.2 gives us a complete description of $\operatorname{Pic}\Phi_m = NS(\Phi_m)$, including the intersection form and the action of the automorphism group of Φ_m.

In view of (1.1), Theorem 1.2 is an immediate consequence of the following statement, which is actually proved in the paper, see §4.2. (Throughout the paper, we use $\operatorname{Tors} A$ for the $\mathbb{Z}$-torsion of an abelian group/module A.)

Theorem 1.3. *For any integer $m \geqslant 1$, one has* $\operatorname{Tors}(\operatorname{Pic}\Phi_m/\mathbf{S}_m) = 0$.

In the mean time, an interesting generalization, approaching the problem from a different angle, was suggested in [12]. Briefly, Φ_m can be represented as an m^3-fold ramified covering of the plane, and one can try to study other multiple planes with the same ramification locus (see §2.4 and Problem 2.6 for details). Considered in [12] are cyclic coverings of degree at most 50, and, similar to [11], the proof is also based on comparing the discriminants of the two lattices.

The approach developed in the present paper, including the computation of the Alexander module $A[\alpha]$ (see §3.3), which is crucial for the proof, apply to Delsarte surfaces as well. Here, we make a few first steps towards this generalized problem and work out another special case, see Theorem 4.18. In the forthcoming paper [5], we close the case of *cyclic* Delsarte surfaces started in [12] and modify part of the proof of Theorem 1.3 (see §4.2) to adapt it to slightly more general *diagonal* Delsarte surfaces. On the other hand, numeric experiments show that Theorem 1.3 does not extend literally to all Delsarte surfaces: sometimes, the quotient does have torsion. Next special classes to be studied in more details would probably be the nonsingular Delsarte surfaces and those with **A**–**D**–**E** singularities.

As yet another application, we consider another class of surfaces whose Picard group is rationally generated by lines, see [3]. Let p and q be two square free bivariate homogeneous polynomials of degree m, and denote

$$\Sigma_{p,q} := \{p(z_0, z_1) = q(z_2, z_3)\} \subset \mathbb{P}^3.$$

This nonsingular surface contains an obvious set of m^2 lines, *viz.* those connecting the points $[z_0 : z_1 : 0 : 0]$ and $[0 : 0 : z_2 : z_3]$, where $p(z_0, z_1) = q(z_2, z_3) = 0$, and we denote by $\mathbf{S}_{p,q} \subset \operatorname{Pic}\Sigma_{p,q}$ the subgroup generated by the classes of these lines.

Theorem 1.4 (see §4.4)**.** *For any pair p, q as above,* $\operatorname{Tors}(\operatorname{Pic}\Sigma_{p,q}/\mathbf{S}_{p,q}) = 0$.

Corollary 1.5 (see §4.4)**.** *If m is prime and p, q as above are sufficiently generic, then* $\operatorname{Pic}\Sigma_{p,q}$ *is generated by the classes of the m^2 lines contained in $\Sigma_{p,q}$.*

1.2. **An outline of the proof.** In §2, we reduce the question to the computation of the torsion of the 1-homology of a certain space, see Theorem 2.2. We also recall the classical description of the lines in Φ_m by means of a ramified covering of the plane and, following [12], describe a generalization of the problem to a wider class of surfaces. In §3, we compute the so-called *Alexander module* (or rather Alexander complex) $A[m]$ of the above covering and its reduced version $\bar{A}[m]$. The heart of the proof is a tedious computation of the length $\ell(\bar{A}[m])$, see Lemma 4.4 in §4; then, Theorem 1.3 follows from comparing the result to the expected value given by [1, 13], see §4.2. In §4.3, we work out a toy example, illustrating the suggested line of attack to the generalized problem.

1.3. **Acknowledgements.** I would like to express my gratitude to I. Shimada for bringing the problem to my attention and for many fruitful discussions; it was he who eventually persuaded me to publish these observations. I would also like to thank the anonymous referee of this paper for the elegant proof of (4.12).

2. Preliminaries

2.1. **Prerequisites.** For the reader's convenience, we recall, with references to [7], a few necessary facts from algebraic topology. An ultimate reference would be [8]; unfortunately it is only available in Russian.

By definition, for any topological pair (X, A) we have the following short exact sequence of singular chain complexes:

$$0 \longrightarrow S_*(A) \longrightarrow S_*(X) \longrightarrow S_*(X, A) \longrightarrow 0.$$

All complexes are free; hence, applying $\otimes G$ or $\operatorname{Hom}(\cdot, G)$, we also have short exact sequences of (co-)chain complexes with any coefficient group G. The associated long exact sequences in (co-)homology are called the (co-)homology exact sequences of pair (X, A), *cf.* (3.2) in [7, Chapter III].

Unless specified otherwise, all (co-)homology are with coefficients in $\mathbb{Z}$. The other groups can be computed using the so-called universal coefficient formulas (see, *e.g.*, (7.9) and (7.10) in [7, Chapter VI]): for any topological space X, abelian group G, and integer n, there are natural split (not naturally) exact sequences

$$0 \longrightarrow H_n(X) \otimes G \longrightarrow H_n(X; G) \longrightarrow \operatorname{Tor}(H_{n-1}(X), G) \longrightarrow 0,$$
$$0 \longrightarrow \operatorname{Ext}(H_{n-1}(X), G) \longrightarrow H^n(X; G) \longrightarrow \operatorname{Hom}(H_n(X), G) \longrightarrow 0.$$

(Here, $\operatorname{Tor} = \operatorname{Tor}_1$ and $\operatorname{Ext} = \operatorname{Ext}^1$ are the derived functors in the category of $\mathbb{Z}$-modules.) Similar statements hold for the relative groups of pairs (X, A). Assuming all groups finitely generated (*e.g.*, X is a finite *CW*-complex), a consequence of the second exact sequence is the assertion that $H^n(X)$ is free if and only if so is $H_{n-1}(X)$; in this case, $H^n(X) = \operatorname{Hom}(H_n(X), \mathbb{Z})$.

We use the following terminology for various duality isomorphisms in topology of manifolds. Let M be an oriented compact manifold, $\dim M = n$, and $A \subset M$ a 'sufficiently good' (see the end of this paragraph) closed subset. If $\partial M = \varnothing$, the multiplication by the fundamental class $[M]$ establishes canonical isomorphisms

- $H^p(M) = H_{n-p}(M)$ (Poincaré duality) and
- $H^p(M, A) = H_{n-p}(M \smallsetminus A)$ (Poincaré–Lefschetz duality).

In general, the multiplication by $[M, \partial M]$ is an isomorphism

- $H^p(M) = H_{n-p}(M, \partial M)$ (Lefschetz duality).

All statements are classical and well known. For example, they can be derived as special cases of Proposition 7.2 in [7, Chapter VIII], with an extra observation that, in all cases considered in the paper, M and A are at worst compact semialgebraic sets, thus admitting finite triangulations (see, *e.g.*, [9]); hence, they are absolute neighborhood retracts and the Čech cohomology in [7] can be replaced with singular ones. As another consequence of [9], all (co-)homology groups involved are finitely generated.

2.2. **Divisors.** Consider a smooth projective algebraic surface X. By Poincaré duality $H^2(X) = H_2(X)$, we can regard the Néron–Severi lattice $NS(X)$ as a subgroup of the intersection index lattice $H_2(X)/\operatorname{Tors}$, representing a divisor $D \subset X$ by its fundamental class $[D]$, see §2.3 below. (The Néron–Severi *lattice* is the group of divisors modulo numeric equivalence; thus, we ignore the torsion.) Since $\operatorname{Pic} X = H^1(X; \mathcal{O}_X^*)$ and $H^2(X; \mathcal{O}_X)$ is a $\mathbb{C}$-vector space, the exponential exact sequence

$$H^1(X; \mathcal{O}_X) \longrightarrow H^1(X; \mathcal{O}_X^*) \longrightarrow H^2(X) \longrightarrow H^2(X; \mathcal{O}_X) \tag{2.1}$$

implies that $NS(X)$ is a primitive subgroup in $H_2(X)/\operatorname{Tors}$.

If $H_1(X) = 0$, then $H^2(X) = \operatorname{Hom}(H_2(X), \mathbb{Z})$ is torsion free (the universal coefficient formula), and so is $H_2(X) = H^2(X)$. Since also $H^1(X; \mathcal{O}_X) = H^{0,1}(X)$ is trivial in this case, from (2.1) we have $\operatorname{Pic} X = NS(X)$, *i.e.*, we do not need to distinguish between linear, algebraic, or numeric equivalence of divisors.

Consider a reduced curve $D \subset X$. Topologically, the normalization $\tilde{D}$ of D is a closed surface, and the projection $\sigma\colon \tilde{D} \to D$ is a homeomorphism outside a *finite* subset $S \subset \tilde{D}$. Hence,

$$H_2(D) = H_2(D, \sigma(S)) = H_2(\tilde{D}, S) = H_2(\tilde{D}) = \bigoplus \mathbb{Z} \cdot [D_i]$$

is the free abelian group generated by the fundamental classes $[D_i]$ of the irreducible components D_i of D (or, equivalently, the fundamental classes $[\tilde{D}_i]$ of the connected components $\tilde{D}_i$ of $\tilde{D}$). A similar computation in cohomology shows that the group

$$H^2(D) = H^2(\tilde{D}) = \operatorname{Hom}(H_2(D), \mathbb{Z}) = \bigoplus \mathbb{Z} \cdot [D_i]^*$$

is also free (the last identification uses the *canonical* basis $\{[D_i]\}$) and, by the universal coefficient formula, $H_1(D)$ is free. (Essentially, we only use the fact that the singular locus has real codimension at least two.)

2.3. **Imprimitivity *via* homology.** As above, let $D \subset X$ be a reduced curve in a smooth projective surface X. Denoting by $\iota\colon D \hookrightarrow X$ the inclusion, let

$$\mathbf{S}\langle D\rangle = \operatorname{Im}[\iota_*\colon H_2(D) \to H_2(X)/\operatorname{Tors}].$$

As explained in §2.2, $\mathbf{S}\langle D\rangle \subset NS(X)$ is the subgroup generated by the irreducible components of D. For convenience, we retain the notation $\iota\colon D \hookrightarrow X$ and $\mathbf{S}\langle D\rangle$ in the case when $D = \sum n_i D_i$, $n_i \neq 0$, is a divisor in X (thus identifying D with its support $\bigcup D_i$). The *fundamental class* of a divisor D is $[D] := \sum n_i [D_i]$.

Theorem 2.2. *Let $\iota\colon D \hookrightarrow X$ be as above, and assume that $H_1(X) = 0$. Then there are canonical isomorphisms*

$$\operatorname{Tors} H_1(X \smallsetminus D) = \operatorname{Hom}(\mathbf{T}\langle D\rangle, \mathbb{Q}/\mathbb{Z}), \quad H_1(X \smallsetminus D)/\operatorname{Tors} = \operatorname{Hom}(\mathbf{K}\langle D\rangle, \mathbb{Z}),$$

where $\mathbf{T}\langle D\rangle := \operatorname{Tors}(NS(X)/\mathbf{S}\langle D\rangle)$ *and* $\mathbf{K}\langle D\rangle := \operatorname{Ker}[\iota_*\colon H_2(D) \to H_2(X)]$.

Proof. By Poincaré–Lefschetz duality, we have $H_1(X \smallsetminus D) = H^3(X, D)$. Consider the following fragment of the cohomology exact sequence of pair (X, D):

$$H^2(X) \xrightarrow{\iota^*} H^2(D) \xrightarrow{\delta} H^3(X, D) \longrightarrow H^3(X).$$

Since $H^3(X) = H_1(X) = 0$ (Poincaré duality), we have a canonical isomorphism

$$H_1(X \smallsetminus D) = \operatorname{Coker} \iota^*. \tag{2.3}$$

As explained above, both $H^2(X)$ and $H^2(D)$ are free abelian groups and, for both spaces, we have $H^2(\cdot) = \operatorname{Hom}(H_2(\cdot), \mathbb{Z})$; hence, $\iota^* = \operatorname{Hom}(\iota_*, \mathrm{id}_{\mathbb{Z}})$. The exact sequence

$$0 \longrightarrow \mathbf{K}\langle D\rangle \xrightarrow{\text{in}} H_2(D) \xrightarrow{\iota_*} H_2(X)$$

can be regarded as a free resolution of $\mathbf{Q} := H_2(X)/\mathbf{S}\langle D\rangle$. Applying $\operatorname{Hom}(\cdot, \mathbb{Z})$, we obtain a cochain complex

$$0 \longrightarrow H^2(X) \xrightarrow{\iota^*} H^2(D) \xrightarrow{\text{in}^*} \operatorname{Hom}(\mathbf{K}\langle D\rangle, \mathbb{Z}) \longrightarrow 0$$

computing the derived functors: $H^0 = \operatorname{Hom}(\mathbf{Q}, \mathbb{Z})$, $H^1 = \operatorname{Ext}(\mathbf{Q}, \mathbb{Z})$, $H^i = 0$ for $i \geqslant 2$. By the definition of H^1 and H^2, this gives us a short exact sequence

$$0 \longrightarrow \operatorname{Ext}(\mathbf{Q}, \mathbb{Z}) \longrightarrow \operatorname{Coker} \iota^* \longrightarrow \operatorname{Hom}(\mathbf{K}\langle D\rangle, \mathbb{Z}) \longrightarrow 0.$$

Here, the first group is finite and the last one is free. Hence,

$$\operatorname{Ext}(\mathbf{Q}, \mathbb{Z}) = \operatorname{Tors} \operatorname{Coker} \iota^* \quad \text{and} \quad \operatorname{Hom}(\mathbf{K}\langle D\rangle, \mathbb{Z}) = \operatorname{Coker} \iota^* / \operatorname{Tors}.$$

In view of (2.3), these two isomorphisms prove the two statements of the theorem. For the first statement, one should also observe that $\operatorname{Ext}(\mathbf{Q}, \mathbb{Z}) = \operatorname{Ext}(\operatorname{Tors} \mathbf{Q}, \mathbb{Z})$ (a property of finitely generated abelian groups), $\operatorname{Tors} \mathbf{Q} = \mathbf{T}\langle D\rangle$ (using the fact that $NS(X)$ is primitive in $H_2(S)$), and $\operatorname{Ext}(\mathbf{T}\langle D\rangle, \mathbb{Z}) = \operatorname{Hom}(\mathbf{T}\langle D\rangle, \mathbb{Q}/\mathbb{Z})$ (apply $\operatorname{Hom}(\mathbf{T}\langle D\rangle, \cdot)$ to the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$.) □

2.4. **The covering $\Phi_m \to \Phi_1$.** We make extensive use of the ramified covering $\mathrm{pr}_m \colon \Phi_m \to \Phi := \Phi_1$ given by

$$\mathrm{pr}_m \colon [z_0 : z_1 : z_2 : z_3] \mapsto [z_0^m : z_1^m : z_2^m : z_3^m].$$

Clearly, Φ is the plane $\{z_0 + z_1 + z_2 + z_3 = 0\}$, and pr_m is ramified over the union of four lines $R_i := \Phi \cap \{z_i = 0\}$, $i = 0, 1, 2, 3$. The Galois group of pr_m is $(\mathbb{Z}/m)^3$. Assuming that $m \geqslant 3$, the $3m^2$ lines in Φ_m are the irreducible components of the preimage of the three lines $L_i := \Phi \cap \{z_0 + z_i = 0\}$, $i = 1, 2, 3$. Introduce the divisors $L := L_1 + L_2 + L_3$, $R := R_0 + R_1 + R_2 + R_3$, and $V := L + R$ in Φ.

With a further generalization in mind, redenote $\Phi[m] := \Phi_m$ and consider the pull-backs $L_*[m] := \mathrm{pr}_m^{-1}(L_*)$, $R_*[m] := \mathrm{pr}_m^{-1}(R_*)$, and $V[m] := \mathrm{pr}_m^{-1}(V)$, where $*$ is an appropriate subscript, possibly empty. Each $R_j[m]$ is a plane section of $\Phi[m]$, irreducible and reduced: it is the Fermat curve cut off $\Phi[m]$ by the plane $\{z_j = 0\}$. On the other hand, $L[m]$ also contains a number of plane sections, *e.g.*, those cut off by $\{z_i = \xi z_j\}$, $i \neq j$, $\xi^m = -1$. Thus, for any subset $J \subset \{0, 1, 2, 3\}$, one has

$$\mathbf{S}\langle V[m]\rangle = \mathbf{S}\langle L[m] + R_J[m]\rangle = \mathbf{S}\langle L[m]\rangle = \mathbf{S}_m, \tag{2.4}$$

where $R_J[m] := \sum_{j \in J} R_j[m]$.

Since R is a generic configuration of four lines in the plane Φ, the fundamental group $\mathbb{G} := \pi_1(\Phi \smallsetminus R)$ equals $\mathbb{Z}^3$, see [16, lemma in the proof of Theorem 8]. Since $\mathbb{G}$ is abelian, from the Hurewicz theorem we have $\mathbb{G} = H_1(\Phi \smallsetminus R) = \operatorname{Hom}(\mathbf{K}\langle R\rangle, \mathbb{Z})$, see Theorem 2.2. This group has four canonical generators g_j, $j = 0, 1, 2, 3$, *viz.* the restrictions to $\mathbf{K}\langle R\rangle$ of the four generators of the group $H^2(R) = \bigoplus_j \mathbb{Z} \cdot [R_j]^*$. We have $g_0 + g_1 + g_2 + g_3 = 0$, and $\mathbb{G}$ is freely generated by g_1, g_2, g_3, *cf.*, *e.g.*, (2.3).

An interesting generalization of the original question was suggested in [12]. Given an epimorphism $\alpha \colon \mathbb{G} \twoheadrightarrow G$ to a finite abelian group G, denote by $\mathrm{pr} \colon \Phi[\alpha] \to \Phi$ the minimal resolution of singularities of the ramified covering of Φ defined by α. Let $L_*[\alpha]$, $R_*[\alpha]$, and $V[\alpha]$ be the pull-backs in $\Phi[\alpha]$ of L_*, R_*, and V, respectively. To be consistent with the previous notation, we regard an integer m as the quotient projection $m\colon \mathbb{G} \twoheadrightarrow \mathbb{G}/m\mathbb{G}$. The components of $V[\alpha]$ (including the exceptional divisors) represent some 'obvious' elements of $NS(\Phi[\alpha])$. Using (1.1) and the finite degree map $\Phi[m] \dashrightarrow \Phi[\alpha]$ defined by the inclusion $\operatorname{Ker}\alpha \subset m\mathbb{G}$, $m := |G|$, one has

$$\mathbf{S}\langle V[\alpha]\rangle \otimes \mathbb{Q} = (\operatorname{Pic}\Phi[\alpha]) \otimes \mathbb{Q} \quad \text{whenever g.c.d.}(|G|, 6) = 1. \tag{2.5}$$

Thus, it is natural to ask whether $\mathbf{S}\langle V[\alpha]\rangle = \operatorname{Pic}\Phi[\alpha]$, or, not assuming that $|G|$ is prime to 6, whether $\mathbf{S}\langle V[\alpha]\rangle \subset \operatorname{Pic}\Phi[\alpha]$ is a primitive subgroup.

Problem 2.6 (Shimada–Takahashi [12])**.** When does one have $\mathbf{T}\langle V[\alpha]\rangle = 0$?

According to [12], the answer to this question is in the affirmative if the image G of α is a cyclic group of order $|G| \leqslant 50$. Another example is worked out in §4.3, see Theorem 4.18: the answer is also in the affirmative if $\alpha(g_i) = 0$ for at least one of the standard generators g_i, $i = 0, 1, 2, 3$.

3. The Alexander module

3.1. **The fundamental group.** The line arrangement $L + R \subset \mathbb{P}^2$ is well known; sometimes it is referred to as *Ceva-7*. Its fundamental group has been computed in many ways and in many places; however, since we will work with a particular presentation of this group, we repeat the computation here.

We will use the affine coordinates $x := -z_1/z_0$, $y := -z_3/z_0$ in the plane Φ. In these coordinates, R_0 becomes the line at infinity, and the other components of V are the lines of the form $\{r_x x + r_y y = r\}$ with $r_x, r_y, r \in \{0, 1\}$, see Figure 1. The fundamental group $\pi_1 := \pi_1(\Phi \smallsetminus V)$ is easily computed by the Zariski–van Kampen method [16, 15]. Since we use a modified (or rather intermediate) version of this approach, we outline briefly its proof, using V as a model. (In full detail, the computation using the projection from a singular point is explained, *e.g.*, in [4].) Consider the projection $p\colon \Phi \dashrightarrow \mathbb{P}^1$, $(x, y) \mapsto x$. This projection has four special fibers F_a, *viz.* those over the points $a \in \Delta := \{-1, 0, 1, \infty\}$. (Three of these fibers are components of V, but this fact is irrelevant for the moment.) Let $F_* := \bigcup F_a$, $a \in \Delta$. Then the restriction $p\colon \Phi \smallsetminus (V \cup F_*) \to \mathbb{P}^1 \smallsetminus \Delta$ is a locally trivial fibration and, since $\pi_2(\mathbb{P}^1 \smallsetminus \Delta) = 0$ and the fiber is connected, Serre's exact sequence (*aka* long exact sequence of a fibration) gives us a short exact sequence of fundamental groups

$$\{1\} \longrightarrow \pi_1(F \smallsetminus V) \longrightarrow \pi_1(\Phi \smallsetminus (V \cup F_*)) \longrightarrow \pi_1(\mathbb{P}^1 \smallsetminus \Delta) \longrightarrow \{1\},$$

where F is a typical fiber of p, *e.g.*, the one over $x = \frac{1}{2}$. Choosing $(\frac{1}{2}, -\frac{3}{2})$ for the basepoint, we have $\pi_1(F \smallsetminus V) = \langle v_1, v_2, v_3, v_4\rangle$, see Figure 1. The group $\pi_1(\mathbb{P}^1 \smallsetminus \Delta)$ is free, and the exact sequence splits. A splitting can be constructed geometrically, identifying $\pi_1(\mathbb{P}^1 \smallsetminus \Delta)$ with $\pi_1(S \smallsetminus F_*) = \langle h_1, h_2, h_3\rangle$, where $S \subset \Phi$ is the section $y = -\frac{3}{2}$, the generators h_1, h_2 are as shown in Figure 1, and h_3 is a similar loop about the fiber F_{-1}, not shown in the figure. Thus, one arrives at the presentation

$$\pi_1(\Phi \smallsetminus (V \cup F_*)) = \bigl\langle v_1, v_2, v_3, v_4, h_1, h_2, h_3 \bigm| h_i^{-1} v_j h_i = \beta_i(v_j)\bigr\rangle,$$

where $i = 1, 2, 3$, $j = 1, 2, 3, 4$, and $\beta_i \in \operatorname{Aut}\langle v_1, v_2, v_3, v_4\rangle$ is the so-called *braid monodromy*, *i.e.*, the automorphism of the fundamental group obtained by dragging the fiber along h_i while keeping the basepoint in S. (The formal definition is in

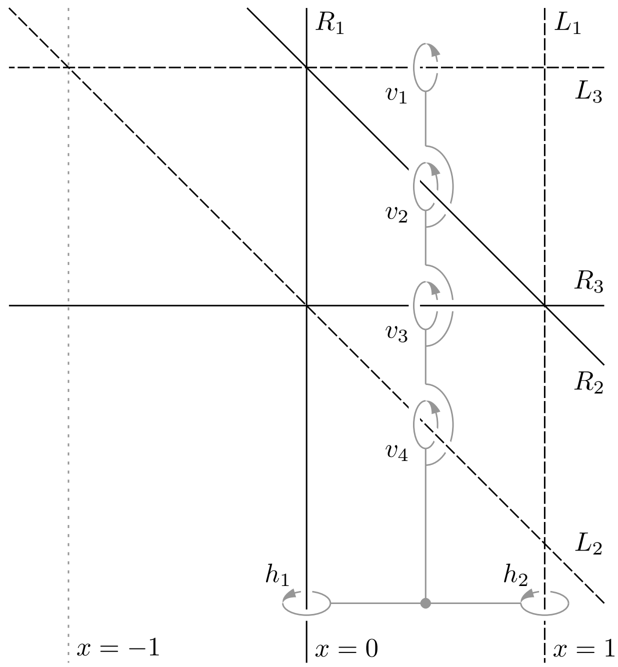

FIGURE 1. The divisor $V := L + R \subset \Phi$

terms of a trivialization of the induced fibration $(p \circ h_i)^* p$ over the segment $[0, 1]$, where $p \circ h_i$ is regarded as a map $[0, 1] \to \mathbb{P}^1 \smallsetminus \Delta$; for all details, see [16, 15].)

Now, in order to pass to $\pi_1(\Phi \smallsetminus V)$, one needs to patch in the only special fiber F_{-1} that is not a component of V. This is done using the Seifert–van Kampen theorem [15]. In fact, the principal application of the theorem in [15] is the following simple observation, which we state in a slightly generalized form.

Lemma 3.1. *Let X be a smooth quasi-projective surface and $D \subset X$ a closed smooth irreducible curve. Then the inclusion homomorphism $\pi_1(X \smallsetminus D) \to \pi_1(X)$ is an epimorphism*; *its kernel is normally generated by the class $[\partial\Gamma]$, where Γ is an analytic disc transversal to D at its center and disjoint from D otherwise.* $\triangleleft$

Since D is assumed irreducible, the conjugacy class of $[\partial\Gamma]$ in the statement does not depend on the choice of Γ or path connecting $\partial\Gamma$ to the basepoint. The proof of the lemma is literally the same as in [15], using a tubular neighborhood of D.

Applying Lemma 3.1 to the curve $F_{-1} \smallsetminus V$ in $\Phi \smallsetminus V$, we obtain an extra relation $h_3 = 1$. In other words, we disregard the generator h_3 and convert the four relations $h_3^{-1} v_j h_3 = \beta_3(v_j)$ into $v_j = \beta_3(v_j)$, $j = 1, 2, 3, 4$.

The computation of the braid monodromy is straightforward and well known, *e.g.*, using equations of the lines; it is left to the reader. (Essentially, it is the braid monodromy of the nodal arrangement $L_1 + L_2 + R_2 + R_3$ of four lines.) Denoting by $\sigma_1, \sigma_2, \sigma_3$ the Artin generators [2] of the braid group $\mathbb{B}_4$ acting on $\langle v_1, v_2, v_3, v_4 \rangle$, we have $\beta_1 = \sigma_1^2 \sigma_3^2$, $\beta_2 = \sigma_2^2$, and $\beta_3 = \sigma_1^{-1} \sigma_3^{-1} \sigma_2^2 \sigma_3 \sigma_1$. (It is worth recalling that, assuming the left action of the automorphism group, the braid monodromy is an *anti*-homomorphism $\pi_1(\mathbb{P}^1 \smallsetminus \Delta) \to \mathbb{B}_4$.) Indeed, β_1 and β_2 are essentially computed in the very first paper on the subject, *viz.* [16]: each is the local monodromy about a simple node (one full twist of a pair of points about their barycenter) or a pair of disjoint nodes. The remaining braid β_3 is the local monodromy about another node, which is translated to the common reference fiber along the real axis; when circumventing the singular fiber at the origin, it gets conjugated by 'one half' of β_1, which is $\sigma_1\sigma_3$.

Putting everything together, after a slight simplification the nontrivial relations for the fundamental group $\pi_1(\Phi \smallsetminus V)$ take the form

$$[h_2, v_1] = [h_2, v_4] = 1, \tag{3.2}$$

$$h_2v_2v_3 = v_2v_3h_2 = v_3h_2v_2 \tag{3.3}$$

(the relations $h_2^{-1}v_jh_2 = \beta_2(v_j)$ from the fiber $x = 1$),

$$h_1v_1v_2 = v_1v_2h_1 = v_2h_1v_1, \tag{3.4}$$

$$h_1v_3v_4 = v_3v_4h_1 = v_4h_1v_3 \tag{3.5}$$

(the relations $h_1^{-1}v_jh_1 = \beta_1(v_j)$ from the fiber $x = 0$), and

$$[v_2^{-1}v_1v_2, v_4] = 1 \tag{3.6}$$

(the relations $v_j = \beta_3(v_j)$ from the fiber $x = -1$).

By Lemma 3.1, the inclusion $\mathrm{in}\colon \Phi \smallsetminus V \hookrightarrow \Phi \smallsetminus R$ induces the map

$$\mathrm{in}_*\colon \pi_1 \twoheadrightarrow \mathbb{G}: \qquad h_1 \mapsto g_1, \quad v_2 \mapsto g_2, \quad v_3 \mapsto g_3, \quad h_2, v_1, v_4 \mapsto 0.$$

3.2. **The 'universal' covering.** Throughout the paper we use freely the following well-known fact, often referred to as *theory of covering spaces*: for any connected, locally path connected, and micro-simply connected topological space X (*e.g.*, for any connected simplicial complex) with a basepoint $x_0 \in X$, there is a natural equivalence between the category of coverings $(\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and covering maps (identical on X) and that of subgroups of $\pi_1(X, x_0)$ and inclusions. If the subgroup is normal (*regular*, or *Galois* coverings), it can be described as the kernel of an epimorphism $\alpha\colon \pi_1(X, x_0) \twoheadrightarrow G$; the image G serves then as the group of the deck translations of the covering.

Consider an epimorphism $\alpha\colon \mathbb{G} \twoheadrightarrow G$. In this section, we do not assume G finite; in fact, we start with a study of the 'universal' $\mathbb{G}$-covering, corresponding to the identity map $0\colon \mathbb{G} \twoheadrightarrow \mathbb{G}/0\mathbb{G} = \mathbb{G}$. (Admittedly awkward, this notation is compliant with $m\colon \mathbb{G} \twoheadrightarrow \mathbb{G}/m\mathbb{G}$ introduced earlier.)

Consider the composition

$$\tilde{\alpha}\colon \pi_1 \xrightarrow{\mathrm{in}_*} \pi_1(\Phi \smallsetminus R) = \mathbb{G} \xrightarrow{\alpha} G$$

and denote by $\Phi^\circ[\alpha]$ the G-covering of $\Phi \smallsetminus V$ defined by $\tilde{\alpha}$. By the Hurewicz theorem, $H_1(\Phi^\circ[\alpha])$ is the abelianization of $\pi_1(\Phi^\circ[\alpha]) = \operatorname{Ker}\tilde{\alpha}$. The action of the deck translations of the covering makes this group a $\mathbb{Z}[G]$-module; regarded as such, it is often referred to as the *Alexander module* of $\tilde{\alpha}$.

The construction of the Alexander module fits into a more general framework and admits a purely algebraic description. Consider a group π and an epimorphism $\tilde{\alpha}\colon \pi \twoheadrightarrow G$ to an abelian group G. Then the Alexander module of $\tilde{\alpha}$ is the abelian group $A := \operatorname{Ker}\tilde{\alpha}/[\operatorname{Ker}\tilde{\alpha}, \operatorname{Ker}\tilde{\alpha}]$ regarded as a $\mathbb{Z}[G]$-module *via* the G-action defined as follows: given $a \in A$ and $g \in G$, the image $g(a)$ is the class in A of the element $\tilde{g}\tilde{a}\tilde{g}^{-1} \in \operatorname{Ker}\tilde{\alpha}$, where $\tilde{a}, \tilde{g} \in \pi$ are some lifts of a, g, respectively. This class does not depend on the choice of the lifts, and the action is well defined.

Crucial is the fact that $H_1(\Phi^\circ[\alpha])$ depends on the epimorphism $\tilde\alpha\colon \pi_1 \twoheadrightarrow G$ only. Hence, we can replace $\Phi \smallsetminus V$ with any CW-complex X with $\pi_1(X) = \pi_1$. We take for X a space with a single 0-cell e^0, one 1-cell $e^1_i \in \{a_1, a_2, a_3, c_1, c_2, c_3\}$ for each of the six generators $h_1, v_2, v_3, h_2, v_4, v_1$ of π_1 (in the order listed), and one 2-cell e^2_j for each relation (3.2)–(3.6). In the $\mathbb{G}$-covering $X[0]$, each cell e gives rise to a whole $\mathbb{G}$-orbit $\{g \otimes e \,|\, g \in \mathbb{G}\}$. (For the moment, the symbols $g \otimes e$ are merely cell labels; we only assume that the labelling is compatible with the $\mathbb{G}$-action, *i.e.*, for any cell e in X and pair $h, g \in \mathbb{G}$ we have $h(g \otimes e) = (h + g) \otimes e$.)

Following the tradition, let us identify $\mathbb{Z}[\mathbb{G}]$ with the ring

$$\Lambda := \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]$$

of Laurent polynomials, where the variables t_1, t_2, t_3 correspond to the generators $h_1 \mapsto g_1$, $v_2 \mapsto g_2$, $v_3 \mapsto g_3$ about R_1, R_2, R_3, respectively. In other words, we identify $\mathbb{G}$ with the multiplicative abelian group generated by t_1, t_2, t_3; we will also use this multiplicative notation in the cell labels. We can assume, in addition, that the labelling is chosen so that the left end of each 'initial' 1-cell $1 \otimes e$ is attached to $1 \otimes e^0$, *i.e.*, $(1 \otimes e)(0) = 1 \otimes e^0$. (Here, we regard an oriented 1-cell as a path $[0, 1] \to X[0]$.) Then, from the definition of the covering it follows that the right ends are attached as follows:

$$(1 \otimes a_i)(1) = t_i \otimes e^0, \quad (1 \otimes c_j)(1) = 1 \otimes e^0, \quad i, j = 1, 2, 3, \tag{3.7}$$

i.e., the generators h_1, v_2, v_3 are 'unwrapped', whereas h_2, v_1, v_4 remain 'latent'. The other ends are determined by the $\mathbb{G}$-action: for a 1-cell e in X, a monomial t in t_1, t_2, t_3, and $\epsilon = 0, 1$ we have $(t \otimes e)(\epsilon) = t((1 \otimes e)(\epsilon))$.

Recall that the member C_n of the cellular chain complex associated to a CW-complex Y is the free abelian group generated by the n-cells of Y. Thus, each cell e of X gives rise to a direct summand $\bigoplus \mathbb{Z}(g \otimes e)$, $g \in \mathbb{G}$, in the complex of $X[0]$; this summand is naturally identified with the free Λ-module Λe. (It is this identification that explains the usage of $\otimes$ in the labels.) Furthermore, since the CW-structure on $X[0]$ is $\mathbb{G}$-invariant, the boundary homomorphisms are Λ-linear. Thus, the chain complex $C_* := C_*[0]$ of $X[0]$ is a complex of free Λ-modules of the form

$$0 \longrightarrow C_2 \xrightarrow{\partial_2} C_1 = \Lambda a_1 \oplus \Lambda a_2 \oplus \Lambda a_3 \oplus \Lambda c_1 \oplus \Lambda c_2 \oplus \Lambda c_3 \xrightarrow{\partial_1} C_0 = \Lambda \longrightarrow 0$$

(we omit the generator e^0 of C_0), where ∂_1 is given by (3.7):

$$\partial_1 a_i = (t_i - 1), \quad \partial_1 c_j = 0, \quad i, j = 1, 2, 3. \tag{3.8}$$

The module C_2 has nine generators, of which six have non-trivial images under ∂_2:

$$(t_2 t_3 - 1)c_1, \tag{3.9}$$

$$(t_3 - 1)c_1 + (t_3 - 1)a_2 - (t_2 - 1)a_3 \tag{3.10}$$

from (3.3),

$$(t_1 t_3 - 1)c_2, \tag{3.11}$$

$$(t_3 - 1)c_2 + (t_3 - 1)a_1 - (t_1 - 1)a_3 \tag{3.12}$$

from (3.5), and

$$(t_1 t_2 - 1)c_3, \tag{3.13}$$

$$(t_1 - 1)c_3 + (t_1 - 1)a_2 - (t_2 - 1)a_1 \tag{3.14}$$

from (3.4). Relations (3.2) and (3.6) contribute 0 to $\operatorname{Im}\partial_2$.

Example 3.15. The proof of (3.9)–(3.14) is a straightforward computation. As an example, consider (3.3), which can be written in the form of two relations

$$h_2v_2v_3h_2^{-1}v_3^{-1}v_2^{-1}=1,\quad h_2v_2v_3v_2^{-1}h_2^{-1}v_3^{-1}=1.$$

The word in the left hand side of the first relation corresponds to the sequence c_1, a_2, a_3, c_1^{-1}, a_3^{-1}, a_2^{-1} of 1-cells in X along which a 2-cell e_1^2 is attached. (The inverse for a 1-cell means the reversion of the orientation.) Lift this sequence to $X[0]$, *starting each cell at the end of the previous one*, see (3.7):

$$1\otimes c_1,\ 1\otimes a_2,\ t_2\otimes a_3,\ (t_2t_3\otimes c_1)^{-1},\ (t_2\otimes a_3)^{-1},\ (1\otimes a_2)^{-1}.$$

(Observe that, for example, $t_2\otimes a_3$ connects $t_2\otimes e^0$ to $t_2t_3\otimes e^0$, see (3.7); hence, the lift of a_3^{-1} *starting* at $t_2t_3\otimes e^0$ is $(t_2\otimes a_3)^{-1}$; it *ends* at $t_2\otimes e^0$. Note also that $(1\otimes a_2)^{-1}$ ends at $1\otimes e^0$, *i.e.*, the lift is a loop, as expected.) We obtain a sequence of 1-cells along which a 2-cell in $X[0]$, *viz.* one of the lifts of e_1^2, is attached; writing this sequence as a chain, we get $\partial_2e_1^2=(1-t_2t_3)c_1\in C_1$, which is (3.9) up to sign. Similarly, the second relation lifts to the sequence

$$1\otimes c_1,\ 1\otimes a_2,\ t_2\otimes a_3,\ (t_3\otimes a_2)^{-1},\ (t_3\otimes c_1)^{-1},\ (1\otimes a_3)^{-1},$$

which gives us (3.10).

3.3. **Other coverings.** Now, given an epimorphism $\alpha\colon\mathbb{G}\twoheadrightarrow G$, it induces a ring homomorphism $\alpha_*\colon\Lambda\twoheadrightarrow\mathbb{Z}[G]$, making $\mathbb{Z}[G]$ a Λ-module. Clearly, the G-covering $X[\alpha]$ is the quotient space $X[0]/\operatorname{Ker}\alpha$, the cells in $X[\alpha]$ being the $\operatorname{Ker}\alpha$-orbits of those in $X[0]$. The chain homomorphism $C_*\to C_*(X[\alpha])$ induced by the quotient projection merely identifies the basis elements (which are the cells) within each orbit of $\operatorname{Ker}\alpha$; algebraically, it can be expressed as the tensor product

$$\operatorname{id}\otimes\alpha_*\colon C_*=C_*\otimes_\Lambda\Lambda\longrightarrow C_*\otimes_\Lambda\mathbb{Z}[G]=C_*(X[\alpha]).$$

Recall, see the beginning of §3.2, that the 1-homology of the covering spaces depend only on the homomorphism $\tilde\alpha\colon\pi_1\twoheadrightarrow G$. Hence, the group $H_1(\Phi^\circ[\alpha])=H_1(X[\alpha])$ is computed by the complex $C_*[\alpha]:=C_*\otimes_\Lambda\mathbb{Z}[G]$. In view of the right exactness $\operatorname{Coker}(\partial_2\otimes_\Lambda\alpha_*)=(\operatorname{Coker}\partial_2)\otimes_\Lambda\mathbb{Z}[G]$, our primary interest is the quotient $A[\alpha]:=C_1[\alpha]/\operatorname{Im}\partial_2$. Explicitly, $A[\alpha]$ can be described as the Λ-module generated by the six elements a_1,a_2,a_3,c_1,c_2,c_3 that are subject to relations (3.9)–(3.14) and the extra relation

$$t_1^{r_1}t_2^{r_2}t_3^{r_3}=1\quad\text{whenever }\alpha(r_1g_1+r_2g_2+r_3g_3)=0.\tag{3.16}$$

Summarizing, after the identification $C_0[\alpha]=\mathbb{Z}[G]$ and $H_0(X[\alpha])=\mathbb{Z}$, we have an exact sequence

$$0\longrightarrow H_1(\Phi^\circ[\alpha])\hookrightarrow A[\alpha]\xrightarrow{\partial_1}\mathbb{Z}[G]\longrightarrow\mathbb{Z}\longrightarrow 0,\tag{3.17}$$

where the last homomorphism is the augmentation $g\mapsto 1$, $g\in G$.

Recall that the *rank* $\operatorname{rk}A$ of a finitely generated abelian group A is the maximal number of linearly independent elements of A, whereas its *length* $\ell(A)$ is the minimal number of elements generating A. One has $\operatorname{rk}A=\ell(A)$ if and only if A is free.

Lemma 3.18. *For any epimorphism $\alpha\colon\mathbb{G}\twoheadrightarrow G$, there is a natural isomorphism* $\operatorname{Tors}H_1(\Phi^\circ[\alpha])=\operatorname{Tors}A[\alpha]$. *If G is finite, then* $\ell(H_1(\Phi^\circ[\alpha]))=\ell(A[\alpha])-|G|+1$ *and* $\operatorname{rk}H_1(\Phi^\circ[\alpha])=\operatorname{rk}A[\alpha]-|G|+1$.

Proof. Since $\operatorname{Im}\partial_1 \subset \mathbb{Z}[G]$ is a free abelian group, the inclusion in (3.17) induces an isomorphism of the torsion parts. This isomorphism and the obvious fact that $\ell(A) = \operatorname{rk} A + \ell(\operatorname{Tors} A)$ for any finitely generated abelian group A imply that the length and rank identities in the statement are equivalent to each other. The rank identity follows from the additivity of rank in (3.17) and the observation that $\operatorname{rk}\mathbb{Z}[G] = |G|$. □

3.4. **Fermat surfaces.** If the image G of $\alpha\colon \mathbb{G} \twoheadrightarrow G$ is finite, one obviously has $\Phi^\circ[\alpha] = \Phi[\alpha] \smallsetminus V[\alpha]$. If $\alpha = m \in \mathbb{Z}$, *i.e.*, in the case of a classical Fermat surface $\Phi[m]$, it is more convenient to consider a smaller divisor $\bar{L}[m] := L[m] + R_0[m]$, see (2.4). The fundamental group $\pi_1(\Phi[m] \smallsetminus \bar{L}[m])$ is given by Lemma 3.1: it is the quotient of $\operatorname{Ker}\tilde{\alpha} = \pi_1(\Phi^\circ[\alpha])$ by the extra relations $h_1^m = v_2^m = v_3^m = 1$ (as the ramification index at each component of $R[m]$ is obviously m). Hence, the homology group $H_1(\Phi[m] \smallsetminus \bar{L}[m])$ can be computed using the complex $C_*[m]$ with three extra 2-cells e_i^2, $i = 1, 2, 3$, mapped by ∂_2 to $\varphi_m(t_i)a_i$, where

$$\varphi_n(t) := (t^n - 1)/(t - 1), \quad n \in \mathbb{Z}.$$

This computation is similar to Example 3.15: for example, the loop h_1^m lifts to the sequence $1 \otimes a_1, t_1 \otimes a_1, t_1^2 \otimes a_1, \ldots, t_1^{m-1} \otimes a_1$ of 1-cells, which results in the chain $(1 + t_1 + t_1^2 + \ldots + t_1^{m-1})a_1 = \varphi_m(t_1)a_1 \in C_1[m]$. Note that this chain is a cycle, as in $C_1[m]$ we have the relation $t_1^m = 1$.

Remark 3.19. Strictly speaking, the new complex is that of abelian groups rather than Λ-modules, as we add three 2-cells only, *i.e.*, three summands $\mathbb{Z}e_i^2$ in $C_2[m]$. However, in the presence of the relations $t_i^m = 1$, $i = 1, 2, 3$, *cf.* (3.16), one can use (3.9)–(3.14) to show that all three images $\varphi_m(t_i)a_i$ are $\mathbb{G}$-invariant. Hence, without changing the 1-homology of the complex, we can formally replace each summand $\mathbb{Z}e_i^2$ with Λe_i^2, extending ∂_2 by Λ-linearity. Geometrically, we replace a single disk Γ as in Lemma 3.1 with a G-orbit consisting of m^3 disks. Since the curve $R_i[m]$ patched in is irreducible (all disks intersecting the same component), this change does not affect the fundamental group.

Now, as in §3.3, instead of extending the C_2-term of the complex, we can add extra relations to C_1. Summarizing, we have

$$H_1(\Phi[m] \smallsetminus \bar{L}[m]) = \operatorname{Ker}[\partial_1\colon \bar{A}[m] \to C_0[m]],$$

where $\bar{A}[m]$ is the quotient of $A[0]$ by the extra relations

$$t_i^m = 1, \quad \varphi_m(t_i)a_i = 0, \quad i = 1, 2, 3. \tag{3.20}$$

Arguing as in the proof of Lemma 3.18, we obtain the identity

$$\ell(H_1(\Phi[m] \smallsetminus \bar{L}[m])) = \ell(\bar{A}[m]) - m^3 + 1. \tag{3.21}$$

3.5. **Other Delsarte surfaces.** In the generalized case, the first question that arises is whether Theorem 2.2 is applicable, *i.e.*, whether $H_1(\Phi[\alpha]) = 0$. To state the result, introduce the following notation: given a pair of integers $0 \leqslant i, j \leqslant 3$, let $\mathbb{G}_{ij} := \mathbb{Z}g_i \oplus \mathbb{Z}g_j \subset \mathbb{G}$, where $g_i \in \mathbb{G}$ are the canonical generators, see §2.4.

Recall that the blow-up $\sigma\colon \tilde{X} \to X$ of a *smooth* point of a surface X induces an isomorphism of both the fundamental group π_1 and first homology group H_1 of the surface. Hence, up to canonical isomorphism, the groups π_1 and H_1 do not depend on the resolution of singularities.

Proposition 3.22. *For an epimorphism* $\alpha\colon \mathbb{G} \twoheadrightarrow G$, $|G| < \infty$, *one has*

$$\pi_1(\Phi[\alpha]) = H_1(\Phi[\alpha]) = \operatorname{Ker}\alpha / \sum \mathbb{G}_{ij} \cap \operatorname{Ker}\alpha,$$

the summation running over all pairs $0 \leqslant i, j \leqslant 3$ *of integers.*

Proof. We start with the abelian group $\pi_1(\Phi \smallsetminus R) = \mathbb{G}$ generated by $h_1 \mapsto g_1$, $v_2 \mapsto g_2$, $v_3 \mapsto g_3$, see §3.1. Clearly, $\pi_1(\Phi[\alpha] \smallsetminus R[\alpha]) = H_1(\Phi[\alpha] \smallsetminus R[\alpha]) = \operatorname{Ker}\alpha$. (This group can also be regarded as a Λ-module, but the module structure is trivial: $t_1 = t_2 = t_3 = 1$.) For the rest of the proof, we use the additive notation for the fundamental group (as the groups of interest are subquotients of $\mathbb{G}$).

Let $\Phi'[\alpha]$ be the manifold obtained from $\Phi[\alpha] \smallsetminus R[\alpha]$ by patching the components of the proper transform of $R[\alpha]$ *away from the exceptional divisor*. At a generic point of R_i, the ramification index m_i of the ramified covering $\Phi[\alpha] \to \Phi$ equals the index $[\mathbb{G}_{ii} : \mathbb{G}_{ii} \cap \operatorname{Ker}\alpha]$, $i = 0, 1, 2, 3$. Hence, by Lemma 3.1, the inclusion induces an epimorphism $\operatorname{Ker}\alpha \twoheadrightarrow \pi_1(\Phi'[\alpha])$ whose kernel is generated by the elements $m_i g_i$. Thus, we have an isomorphism

$$\pi_1(\Phi'[\alpha]) = \operatorname{Ker}\alpha / \textstyle\sum_i \mathbb{G}_{ii} \cap \operatorname{Ker}\alpha, \quad i = 0, 1, 2, 3. \tag{3.23}$$

(Strictly speaking, unlike the case of the Fermat surfaces, the curve $R_i[\alpha]$ may be reducible, so that we need to attach a separate disk Γ as in Lemma 3.1 for each component of this curve. However, since the G-action is trivial in the 1-homology $H_1 = \pi_1$, all disks result in the same relation $m_i g_i = 0$, *cf.* Remark 3.19.)

What remains is patching the exceptional divisors. Fix a pair $0 \leqslant i < j \leqslant 3$ and let $\tilde{S}$ be a singular point of the normalized, but yet unresolved ramified covering over the point $S := R_i \cap R_j$. Fix a resolution of singularities and let E be the exceptional divisor over $\tilde{S}$. Pick a sufficiently small ball $U \subset \Phi$ about S and denote by $\tilde{U}$ the connected component of the preimage of U containing E. With respect to an appropriate smooth triangulation, $\tilde{U}$ is a regular neighborhood of E; hence, E is a strict deformation retract of $\tilde{U}$, $\tilde{U} \sim E$. On the other hand, $\tilde{U}$ is a 4-manifold with boundary $\partial\tilde{U}$, and the latter is a covering of the 3-sphere ∂U ramified over the Hopf link $R \cap \partial U$.

Note also that the contraction of E gives us the space $\tilde{U}/E$ which is the cone over $\partial\tilde{U}$ (with the vertex $\tilde{S} = E/E$); hence, we have a homotopy equivalence (strict deformation retraction) $\tilde{U} \smallsetminus E = (\tilde{U}/E) \smallsetminus \tilde{S} \sim \partial\tilde{U}$.

We have $\pi_1(\partial U \smallsetminus R) = \mathbb{G}_{ij}$ and, hence, $\pi_1(\partial\tilde{U} \smallsetminus R[\alpha]) = \mathbb{G}_{ij} \cap \operatorname{Ker}\alpha$. As above, similar to Lemma 3.1, patching the union of circles $\partial\tilde{U} \cap R[\alpha]$ results in the pair of relations $m_i g_i = m_j g_j = 0$. Thus,

$$\pi_1(\partial\tilde{U}) = (\mathbb{G}_{ij} \cap \operatorname{Ker}\alpha)/(\mathbb{G}_{ii} \cap \operatorname{Ker}\alpha + \mathbb{G}_{jj} \cap \operatorname{Ker}\alpha) \tag{3.24}$$

is a finite group. Then $H_1(\partial\tilde{U}; \mathbb{Q}) = 0$, *i.e.*, $\partial\tilde{U}$ is a rational homology sphere and $\tilde{S}$ is a rational singular point. For us, important is the fact that $\pi_1(\tilde{U}) = \pi_1(E) = 0$, which can easily be proved directly. Indeed, since $\tilde{U} \sim E$ and $\dim_{\mathbb{R}} E = 2$, we have $H^3(\tilde{U}; \mathbb{Q}) = 0$; then also $H_1(\tilde{U}, \partial\tilde{U}; \mathbb{Q}) = 0$ (Lefschetz duality), and the fragment

$$H_1(\partial\tilde{U}; \mathbb{Q}) \longrightarrow H_1(\tilde{U}; \mathbb{Q}) \longrightarrow H_1(\tilde{U}, \partial\tilde{U}; \mathbb{Q})$$

of the homology exact sequence of pair $(\tilde{U}, \partial\tilde{U})$ implies $H_1(\tilde{U}; \mathbb{Q}) = H_1(E; \mathbb{Q}) = 0$. On the other hand, E is a connected projective algebraic curve, and it is easily seen that E is homotopy equivalent to the wedge of closed topological surfaces (the components of the normalization of E) and a number of circles. (Roughly, we can 'blow-up' the locally reducible singular points of E to line segments, separating the analytic branches and replacing E with a disjoint union of topologically nonsingular closed surfaces with a number of segments attached. Then, within each surface, move the ends of the segments to a single point. Finally, contract several segments to make the surfaces share a common point; the result is a wedge as stated.) For such a wedge $E \sim \bigvee_i E_i$, all groups are easily computed (*e.g.*, using iteratedly the Mayer–Vietoris exact sequence (8.8) in [7, Chapter III] and Seifert–van Kampen theorem [15], or just decomposing the wedge into cells):

$$H_1(E;\mathbb{Q}) = \bigoplus\nolimits_i H_1(E_i;\mathbb{Q}), \quad \pi_1(E) = *_i\, \pi_1(E_i).$$

Clearly, $H_1(E;\mathbb{Q}) = 0$ if and only if all surface components are 2-spheres and there are no circles present. Then obviously $\pi_1(E) = 0$.

Now, start with $\Phi'[\alpha]$ and proceed patching the exceptional divisors one by one. Let Φ'' be an intermediate space, not yet containing E. Applying the Seifert–van Kampen theorem [15] to the union $\Phi'' \cup \tilde{U}$ and using the homotopy equivalence $\Phi'' \cap \tilde{U} = \tilde{U} \smallsetminus E \sim \partial\tilde{U}$, we obtain the amalgamated free product

$$\pi_1(\Phi'' \cup \tilde{U}) = \bigl(\pi_1(\Phi'') * \pi_1(\tilde{U})\bigr)/\pi_1(\partial\tilde{U}) = \pi_1(\Phi'')/(\mathbb{G}_{ij} \cap \operatorname{Ker}\alpha).$$

(For the last isomorphism, we use (3.24) and the identity $\pi_1(\tilde{U}) = 0$.) The group $\pi_1(\Phi'[\alpha])$ is given by (3.23) and, after all the exceptional divisors have been patched, we arrive at the expression in the statement. □

If $H_1(\Phi[\alpha]) = 0$, Theorem 2.2 and Lemma 3.18 imply that

$$\mathbf{T}\langle V[\alpha]\rangle \cong \operatorname{Tors} A[\alpha]. \tag{3.25}$$

Unfortunately, as a $\mathbb{Z}[G]$-module, $A[\alpha]$ is far from free and it is difficult to control its $\mathbb{Z}$-torsion. (Experiments show that, at least, the intermediate quotients similar to those considered in Lemma 4.4 do often have torsion.) An attempt of a direct computation is made in §4.3, whereas in the case of the classical Fermat surfaces we have to take a detour and estimate the length instead. The following two exact sequences may prove useful:

$$A[\alpha] \xrightarrow{\partial_1} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0,$$

where ϵ is the augmentation, see (3.17), and

$$0 \longrightarrow A^\circ[\alpha] \longrightarrow \operatorname{Ker}\partial_1 \longrightarrow \operatorname{Ker}\alpha \longrightarrow 0,$$

where $A^\circ[\alpha] \subset A[\alpha]$ is the submodule generated by c_1, c_2, c_3. The former sequence merely states that $H_0(C_*[\alpha]) = H_0(\Phi^\circ[\alpha]) = \mathbb{Z}$. For the latter, we patch $L[\alpha]$ (by using Lemma 3.1 or merely forgetting the generators h_2, v_1, v_4, hence c_1, c_2, c_3 in the first place) to compute the group $H_1(\Phi[\alpha] \smallsetminus R[\alpha]) = \pi_1(\Phi[\alpha] \smallsetminus R[\alpha]) = \operatorname{Ker}\alpha$; the resulting complex is $0 \to A[\alpha]/A^\circ[\alpha] \to \mathbb{Z}[G] \to 0$. Both sequences split, and we can extend (3.25) to

$$\mathbf{T}\langle V[\alpha]\rangle \cong \operatorname{Tors} A^\circ[\alpha] = \operatorname{Tors} A[\alpha], \tag{3.26}$$

still under the assumption that $H_1(\Phi[\alpha]) = 0$.

4. Proof of the main theorem

4.1. **The length of $\bar{A}[m]$.** Fix an integer $m \geqslant 1$ and consider the Λ-module $\bar{A}[m]$ introduced in §3.4. Recall that $\bar{A}[m]$ is generated by six elements a_i, c_j, $i, j = 1, 2, 3$, subject to the relations (3.9)–(3.14) and (3.20). Observe that relations (3.9), (3.11), and (3.13) can be recast in the form

$$t_i c_k = t_j^{-1} c_k \quad \text{whenever } \{i, j, k\} = \{1, 2, 3\}. \tag{4.1}$$

We introduce a few *ad hoc* notations. Given $i = 1, 2, 3$, let

$$\Lambda_i := \mathbb{Z}[t_i]/(t_i^m - 1), \qquad \bar{\Lambda}_i := \mathbb{Z}[t_i]/\varphi_m(t_i).$$

These rings can be regarded as Λ-modules, but we usually do not specify the action of the other two variables: it varies from case to case. In fact, we repeatedly use the following simple observation, which is an immediate consequence of (4.1).

Lemma 4.2. *Let $i, j, k \in \{1, 2, 3\}$, $k \neq i$, and $p \in \Lambda$, and let A be a subquotient of $\bar{A}[m]$ generated by a single element $x := pc_i$. Assume that either $t_j = 1$ or $t_i = t_k^{\pm 1}$ on A. Then A is a quotient of $\Lambda_s x$ for an appropriate index $s \in \{1, 2, 3\}$.*

If x is also annihilated by $\varphi_m(t_s)$, then A is a quotient of $\bar{\Lambda}_s x$. ◁

The precise description of the 'appropriate' index s (not necessarily unique) is left to the reader. Clearly, $\ell(\Lambda_s) = m$ and $\ell(\bar{\Lambda}_s) = m - 1$.

For a generator $x \in \{a_1, a_2, a_3, c_1, c_2, c_3\}$, let

$$x' := (t_1 - 1)x, \quad \tilde{x} := (t_3 - 1)x, \quad \tilde{x}' := (t_1 - 1)\tilde{x}.$$

Observe that always

$$\varphi_m(t_1)x' = \varphi_m(t_3)\tilde{x} = \varphi_m(t_1)\tilde{x}' = \varphi_m(t_3)\tilde{x}' = 0. \tag{4.3}$$

We will use a filtration $0 = A_0 \subset A_1 \subset \ldots \subset A_7 = \bar{A}[m]$, where $A_k \subset \bar{A}[m]$ are the submodules defined in Lemma 4.4 below.

Let $\delta_m := 1$ if m is even and $\delta_m := 0$ if m is odd.

Lemma 4.4. *One has the following equations and inequalities*:

1. $\ell(A_1/A_0) = m^3 - m^2$, *where A_1 is the submodule generated by a_3*;
2. $\ell(A_2/A_1) \leqslant 3(m - 1) - \delta_m$, *where $A_2 := A_1 + \Lambda\tilde{a}_2' + \Lambda\tilde{c}_3'$*;
3. $\ell(A_3/A_2) \leqslant 3(m - 1)$, *where $A_3 := A_1 + (t_3 - 1)\bar{A}[m]$*;
4. $\ell(A_4/A_3) = m^2 - m$, *where $A_4 := A_3 + \Lambda a_1$*;
5. $\ell(A_5/A_4) \leqslant m - 1$, *where $A_5 := A_4 + \Lambda a_2' + \Lambda c_3'$*;
6. $\ell(A_6/A_5) = m - 1$, *where $A_6 := A_5 + \Lambda a_2$*;
7. $\ell(A_7/A_6) \leqslant 2m + 1$, *where $A_7 := \bar{A}[m]$.*

Hence, $\ell(A) \leqslant m^3 + 9m - 7 - \delta_m$.

Proof. One has $\ell(A_1) \leqslant m^2(m-1)$ due to (3.20). On the other hand, the boundary homomorphism ∂_1 maps A_1 onto $(t_3 - 1)C_0[m]$. Hence, there are no other relations in A_1, and Statement (1) holds. Furthermore, ∂_1 factors to a homomorphism

$$\bar{A}[m]/A_3 \to C_0' := C_0[m]/(t_3 - 1)$$

which maps A_4/A_3 isomorphically onto $(t_1 - 1)C_0'$, proving Statement (4). Then, ∂_1 factors to

$$\bar{A}[m]/A_5 \to C_0'' := C_0'/(t_1 - 1) = \Lambda_2.$$

Since A_6/A_5 is (*a priori* a quotient of) the cyclic $\bar{\Lambda}_2$-module $\bar{\Lambda}_2 a_2$, the restriction of ∂_1 maps it isomorphically onto $(t_2 - 1)C_0'' = \bar{\Lambda}_2$, proving Statement (6).

For the other statements, it suffices to estimate the number of generators. With possible future applications in mind, we describe the structure of the intermediate quotients in the form (known module) $\twoheadrightarrow A_k/A_{k-1}$. In fact, all these epimorphisms are isomorphisms, see Remark 4.14 below.

In $\bar{A}[m]/A_4$, one has

$$t_3 = 1, \qquad a_1 = a_3 = 0, \qquad a_2' = -c_3';$$

the last relation follows from (3.14). Thus, A_5/A_4 is generated by c_3', and $\bar{A}[m]/A_6$ is generated by c_1, c_2, c_3; by (3.20) and Lemma 4.2,

$$\bar{\Lambda}_2 c_3' \twoheadrightarrow A_5/A_4, \qquad \Lambda_1 c_1 \oplus \Lambda_2 c_2 \oplus \mathbb{Z} c_3 \twoheadrightarrow \bar{A}[m]/A_6. \tag{4.5}$$

For the last summand $\mathbb{Z}c_3$, we use the fact that

$$(t_1 - 1)c_3 = -(t_1 - 1)a_2 = 0 \bmod A_6.$$

Thus, $\ell(\bar{A}[m]/A_6) \leqslant 2m + 1$, and Statements (5) and (7) are proved.

The module A_3/A_1 is generated by $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{c}_1$, $\tilde{c}_2$, $\tilde{c}_3$, and relations (3.10), (3.12), (3.14) imply

$$\tilde{a}_2 = -\tilde{c}_1, \qquad \tilde{a}_1 = -\tilde{c}_2, \qquad (t_1 - 1)(\tilde{c}_3 + \tilde{a}_2) = (t_2 - 1)\tilde{a}_1.$$

We can retain three generators $\tilde{c}_1$, $\tilde{c}_2$, $\tilde{c}_3$ only, rewriting the last relation in the form

$$(t_1 - 1)(\tilde{c}_3 - \tilde{c}_1) + (t_2 - 1)\tilde{c}_2 = 0. \tag{4.6}$$

Note also that $\varphi_m(t_3)A_3 = 0$, see (4.3).

In A_3/A_2, we have $(t_1 - 1)\tilde{c}_3 = (t_1 - 1)\tilde{a}_2 = 0$, hence also $(t_1 - 1)\tilde{c}_1 = 0$. Then (4.6) implies $(t_2 - 1)\tilde{c}_2 = 0$, and

$$\bar{\Lambda}_3 \tilde{c}_1 \oplus \bar{\Lambda}_3 \tilde{c}_2 \oplus \bar{\Lambda}_3 \tilde{c}_3 \twoheadrightarrow A_3/A_2, \tag{4.7}$$

see Lemma 4.2. This gives us Statement (3).

The module A_2/A_1 is generated by $\tilde{c}_1'$ and $\tilde{c}_3'$. By (4.3) and (4.1), we have

$$\varphi_m(t_i)(A_2/A_1) = 0 \quad \text{for all } i = 1, 2, 3. \tag{4.8}$$

Relations (3.11) and (4.6) imply $(t_1 t_3 - 1)(\tilde{c}_3' - \tilde{c}_1') = 0$; using (4.1), this can be rewritten as $(t_3 - t_2)\tilde{c}_3' = (t_1 - t_2)\tilde{c}_1'$. Let

$$u := (t_3 - t_2)\tilde{c}_3' = (t_1 - t_2)\tilde{c}_1'$$

and consider the cyclic submodule $A_2' \subset A_2/A_1$ generated by u. By Lemma 4.2,

$$\bar{\Lambda}_2 \tilde{c}_1' \oplus \bar{\Lambda}_2 \tilde{c}_3' \twoheadrightarrow (A_2/A_1)/A_2'. \tag{4.9}$$

On the other hand, $A_2' \subset \Lambda \tilde{c}_1' \cap \Lambda \tilde{c}_3'$; hence, $t_3^{-1} = t_2 = t_1^{-1}$ on this module and, by Lemma 4.2 again,

$$\bar{\Lambda}_2 u \twoheadrightarrow A_2' \quad \text{if } m \text{ is odd}. \tag{4.10}$$

This fact proves Statement (2) in the case of m odd.

If $m = 2k$ is even, (4.10) still holds, but we need a stronger statement. Note that $\varphi_m(t)$ is divisible by $\varphi_k(t^2)$. Furthermore, one has a polynomial identity

$$t^{m-2} \sum_{r=0}^{m-1} t^{1-r} \varphi_r(t^2) = t\varphi_{k-1}(t^2)\varphi_m(t) + \varphi_k(t^2), \tag{4.11}$$

which is easily established by multiplying both sides by t^2-1. On the submodule A_2' we have $t_2 = t_1^{-1}$, see (4.1); hence, $s := t_2t_1^{-1} = t_2^2$. Then, representing u in the form $u = t_1(1-s)\tilde{c}_1'$, we have

$$t_2^{1-r}\varphi_r(t_2^2)u = t_2^{1-r}t_1\varphi_r(s)(1-s)\tilde{c}_1' = t_1^r(1-s^r)\tilde{c}_1' = (t_1^r - t_2^r)\tilde{c}_1', \quad r \in \mathbb{Z}. \tag{4.12}$$

Summing up over $r = 0, \ldots, m-1$ and using (4.8) and (4.11) at $t = t_2$, we conclude that $\varphi_k(t_2^2)u = 0$, *i.e.*,

$$\Lambda_2 u/\varphi_k(t_2^2) \twoheadrightarrow A_2' \quad \text{if } m = 2k \text{ is even,} \tag{4.13}$$

obtaining a stronger inequality $\ell(A_2') \leqslant \deg \varphi_k(t^2) = m-2$.

The final inequality in the statement of the lemma is the sum of items 1–7. □

4.2. **Proof of Theorem 1.3.** We assume that $m \geqslant 3$. By (2.4), it suffices to show that $\mathbf{T}\langle \bar{L}[m]\rangle = 0$, where $\bar{L}[m] := L[m] + R_0[m]$ is the divisor introduced in §3.4. Since $\Phi[m]$ is simply connected, we can use Theorem 2.2, reducing the problem to proving the inequality $\ell(H_1(\Phi[m] \smallsetminus \bar{L}[m])) \leqslant \operatorname{rk} \mathbf{K}\langle \bar{L}[m]\rangle$.

According to [1, 13], $\operatorname{rk} \mathbf{S}_m = 3(m-1)(m-2) + 1 + \delta_m$. On the other hand, $H_2(\bar{L}[m])$ is the free abelian group generated by the classes of the $3m^2$ lines and the additional class $[R_0[m]]$. Hence, $\operatorname{rk} \mathbf{K}\langle \bar{L}[m]\rangle = 9m - 6 - \delta_m$, and the statement follows from (3.21) and Lemma 4.4. □

Remark 4.14. It follows from the proof that all inequalities in the statement of Lemma 4.4 are, in fact, equalities, *i.e.*, no relation has been lost, even though some relations were multiplied by non-units. Furthermore, all epimorphisms (4.5), (4.7), (4.9), (4.10), (4.13) are isomorphisms.

Remark 4.15. We only use the inequality $\operatorname{rk} \mathbf{S}_m \leqslant 3(m-1)(m-2) + 1 + \delta_m$, *i.e.*, the fact that there is *at least* a certain number of relations between the components. In general, it would suffice to prove the inequality $\ell(A[\alpha]) \leqslant \operatorname{rk} \mathbf{K}\langle V[\alpha]\rangle + |G| - 1$, see Lemma 3.18.

Remark 4.16. The rank $\operatorname{rk} \mathbf{S}_m$ can easily be computed directly, by tensoring the module by $\mathbb{C}$ and counting the irreducible summands, which are all of dimension 1 (multi-eigenspaces of the three commuting finite order operators t_1, t_2, t_3).

Remark 4.17. By (3.26), when computing the torsion, one can replace $A[\alpha]$ with the smaller module $A^\circ[\alpha]$. *A posteriori*, $A^\circ[m]$ is the $\Lambda[m]$-module spanned by the three generators c_1, c_2, c_3 subject to a single relation

$$(t_1-1)(t_3-1)c_1 = (t_2-1)(t_3-1)c_2 + (t_1-1)(t_3-1)c_3,$$

see [5]. In this form, some of the results of this paper generalize to Fermat varieties of higher dimension, see [6]. Note, though, that this one-relator presentation of $A^\circ[\alpha]$ does not extend to more general Delsarte surfaces; see [5] for further details.

4.3. **A toy example.** In conclusion, we consider a very simple example, answering the generalized question, see Problem 2.6, in the special case of a covering ramified over at most three lines.

Theorem 4.18. *If the covering $\Phi[\alpha] \to \Phi$ is unramified over at least one of the lines R_j, $j = 0, 1, 2, 3$, then $\mathbf{T}\langle V[\alpha]\rangle = 0$.*

Proof. We can assume that the covering is unramified over R_3, *i.e.*, the epimorphism $\alpha\colon \mathbb{G} \to G$ sends g_3 to 0. Then, obviously, $\operatorname{Ker}\alpha = \mathbb{Z}g_3 \oplus (\mathbb{G}_{12} \cap \operatorname{Ker}\alpha)$ and, by Proposition 3.22, we have $H_1(\Phi[\alpha]) = 0$, *i.e.*, Theorem 2.2 is applicable.

By (3.16), we have $t_3 = 1$ on $A[\alpha]$, and relations (3.10), (3.12), (3.14) become

$$(t_2 - 1)a_3 = (t_1 - 1)a_3 = 0, \qquad (t_1 - 1)(c_3 + a_2) = (t_2 - 1)a_1.$$

Introducing the generator $a_2' := c_3 + a_2$ instead of a_2, we see that the submodule $A^\circ[\alpha] \subset A[\alpha]$ introduced in §3.5 is a direct summand (as a Λ-module), and all relations in $A^\circ[\alpha]$ are $t_3 = 1$ and (3.9), (3.11), (3.13). The three latter translate into independent relations $(t_2 - 1)c_1 = (t_1 - 1)c_2 = (t_1t_2 - 1)c_3 = 0$, and $A^\circ[\alpha]$ is a direct sum of three group rings:

$$A^\circ[\alpha] = \mathbb{Z}[G/\alpha(g_2)]c_1 \oplus \mathbb{Z}[G/\alpha(g_1)]c_2 \oplus \mathbb{Z}[G/\alpha(g_1 + g_2)]c_3.$$

By (3.26), one has $\mathbf{T}\langle V[\alpha]\rangle \cong \operatorname{Tors} A^\circ[\alpha] = 0$. □

Corollary 4.19 (of (2.5) and Theorem 4.18)**.** *If a covering* $\operatorname{pr}\colon \Phi[\alpha] \to \Phi$ *as in Theorem 4.18 has degree* m *prime to* 6*, then* $\operatorname{Pic}\Phi[\alpha] = \mathbf{S}\langle V[\alpha]\rangle$*.* ◁

4.4. **Proof of Theorem 1.4 and Corollary 1.5.** Corollary 1.5 is an immediate consequence of Theorem 1.4 and the fact that $\operatorname{Pic}\Sigma_{p,q}$ is rationally generated by the classes of the lines, see [3]. In view of Theorem 2.2, the statement of Theorem 1.4 is purely homological, and we can deform $\Sigma_{p,q}$ to the Fermat surface $\Phi[m]$; then, the m^2 lines in question deform to the components of $L_1[m]$, and $\mathbf{S}_{p,q} = \mathbf{S}\langle L_1[m]\rangle$. Similar to (2.4), the latter group equals $\mathbf{S}\langle \bar{L}_1[m]\rangle$, where $\bar{L}_1[m] := L_1[m] + R_0[m]$.

Patching $L_2[m]$ and $L_3[m]$, *cf.* §3.4, we conclude that

$$\mathbf{T}\langle \bar{L}_1[m]\rangle = \operatorname{Tors}\bar{A}'[m], \qquad \bar{A}'[m] := \bar{A}[m]/(\Lambda c_2 + \Lambda c_3).$$

Filtering this module as in Lemma 4.4 and analyzing the proof of the lemma, we see that Statements (1), (4), and (6) hold without change, whereas the other statements can be rewritten as follows:

(2) $\ell(A_2/A_1) = 0$ due to (4.6),
(3) $\ell(A_3/A_2) \leqslant m - 1$, see (4.7),
(5) $\ell(A_5/A_4) = 0$, see (4.5),
(7) $\ell(A_7/A_6) \leqslant m$, see (4.5).

Summing this up, we obtain $\ell(\bar{A}'[m]) \leqslant m^3 + 2m - 2$. On the other hand, one has $\operatorname{rk}\mathbf{S}\langle \bar{L}_1[m]\rangle = (m-1)^2 + 1$, see [3]; hence, $\operatorname{rk}\mathbf{K}\langle \bar{L}_1[m]\rangle = 2m - 1$ and, as in §4.2, we conclude that $\bar{A}'[m]$ is a free abelian group. □

References

1. Noboru Aoki and Tetsuji Shioda, *Generators of the Néron-Severi group of a Fermat surface*, Arithmetic and geometry, Vol. I, Progr. Math., vol. 35, Birkhäuser Boston, Boston, MA, 1983, pp. 1–12. MR 717587 (85m:14049)
2. E. Artin, *Theory of braids*, Ann. of Math. (2) **48** (1947), 101–126. MR 0019087 (8,367a)
3. Samuel Boissière and Alessandra Sarti, *On the Néron-Severi group of surfaces with many lines*, Proc. Amer. Math. Soc. **136** (2008), no. 11, 3861–3867. MR 2425725 (2009e:14058)
4. Alex Degtyarev, *Topology of algebraic curves: An approach via dessins d'enfants*, De Gruyter Studies in Mathematics, vol. 44, Walter de Gruyter & Co., Berlin, 2012. MR 2952675
5. ———, *On the Picard group of a Delsarte surface*, to appear, `arXiv:1307.0382`, 2013.
6. Alex Degtyarev and Ichiro Shimada, *On the topology of projective subspaces in complex Fermat varieties*, to appear, `arXiv:1405.4683`, 2014.

7. Albrecht Dold, *Lectures on algebraic topology*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 200, Springer-Verlag, Berlin, 1980. MR 606196 (82c:55001)
8. A. T. Fomenko and D. B. Fuks, *Kurs gomotopicheskoi topologii*, “Nauka”, Moscow, 1989, With an English summary. MR 1027592 (92a:55001)
9. Heisuke Hironaka, *Triangulations of algebraic sets*, Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974), Amer. Math. Soc., Providence, R.I., 1975, pp. 165–185. MR 0374131 (51 #10331)
10. M. Mizukami, *Birational mappings from quartic surfaces to Kummer surfaces*, Master’s thesis, University of Tokyo, 1975.
11. Matthias Schütt, Tetsuji Shioda, and Ronald van Luijk, *Lines on Fermat surfaces*, J. Number Theory **130** (2010), no. 9, 1939–1963. MR 2653207 (2011i:14064)
12. Ichiro Shimada and Nobuyoshi Takahashi, *Primitivity of sublattices generated by classes of curves on an algebraic surface*, Comment. Math. Univ. St. Pauli **59** (2010), no. 2, 77–95. MR 2789841 (2012c:14079)
13. Tetsuji Shioda, *On the Picard number of a Fermat surface*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), no. 3, 725–734 (1982). MR 656049 (83f:14029)
14. ———, *An explicit algorithm for computing the Picard number of certain algebraic surfaces*, Amer. J. Math. **108** (1986), no. 2, 415–432. MR 833362 (87g:14033)
15. E. R. van Kampen, *On the fundamental group of an algebraic curve*, Amer. J. Math. **55** (1933), 255–260.
16. Oscar Zariski, *On the Problem of Existence of Algebraic Functions of Two Variables Possessing a Given Branch Curve*, Amer. J. Math. **51** (1929), no. 2, 305–328. MR 1506719

Department of Mathematics, Bilkent University, 06800 Ankara, Turkey
E-mail address: degt@fen.bilkent.edu.tr