# TRIPLETS OF CLOSELY EMBEDDED HILBERT SPACES

PETRU COJUHARI AND AURELIAN GHEONDEA

Abstract. We obtain a general concept of triplet of Hilbert spaces with closed (unbounded) embeddings instead of continuous (bounded) ones. We provide a model and an abstract theorem as well for a triplet of closely embedded Hilbert spaces associated to positive selfadjoint operator $H$, that is called the Hamiltonian of the system, which is supposed to be one-to-one but may not have a bounded inverse. Existence and uniqueness results, as well as left-right symmetry, for these triplets of closely embedded Hilbert spaces are obtained. We motivate this abstract theory by a diversity of problems coming from homogeneous or weighted Sobolev spaces, Hilbert spaces of holomorphic functions, and weighted $L^2$ spaces. An application to weak solutions for a Dirichlet problem associated to a class of degenerate elliptic partial differential equations is presented. In this way, we propose a general method of proving the existence of weak solutions that avoids coercivity conditions and Poincaré-Sobolev type inequalities.

## 1. Introduction

The concept of rigged Hilbert space was introduced and investigated by I.M. Gelfand and A.G. Kostyuchenko [18], see [19] for further developments, in connection to the general problem of reconciliating the two basic paradigms of Quantum Mechanics, that of P.A.M. Dirac based on bras and kets and used mainly by physicists, with that of J. von Neumann based on positive selfadjoint operators in Hilbert spaces and used mainly by mathematicians. This reconciliation was essentially facilitated by the L. Schwartz's theory of distributions [29]. Briefly, a rigged Hilbert space is a triplet $(\mathcal{S};\mathcal{H};\mathcal{S}^*)$, in which $\mathcal{H}$ is a complex Hilbert space, $\mathcal{S}$ is a topological vector space that is continuously and densely embedded in $\mathcal{H}$, while $\mathcal{S}^*$ is the "dual space of $\mathcal{S}$ with respect to $\mathcal{H}$" and such that $\mathcal{H}$ is continuously and densely embedded in $\mathcal{S}^*$. The rigged Hilbert space formalism was later recognized and used by physicists as a powerful and rigorously mathematical tool for problems in quantum mechanics, e.g. see A. Bohm and M. Gadella [11], R. de la Madrid [22], and the rich bibliography cited there. In particular, a theory, consistent both mathematically and physically, of Gamow states and of quantum resonances was made possible, e.g. see the survey article of O. Civitarese and M. Gadella [13].

One of the built-in deficiency of the theory of rigged Hilbert spaces consists on the vague formalization of the meaning of "dual space of $\mathcal{S}$ with respect to $\mathcal{H}$". In this respect, an

2010 *Mathematics Subject Classification.* 47A70, 47B25, 47B34, 46E22, 46E35, 35H99, 35D30.

*Key words and phrases.* Closed embedding, triplet of Hilbert spaces, rigged Hilbert spaces, kernel operator, Hamiltonian, degenerate elliptic operators, Dirichlet problem, weak solutions.

The first named author acknowledges financial support from the Polish Ministry of Science and Higher Education: 11.11.420.04 and Grant NN201 546438 (2010-2013).

The second named author acknowledges financial support from the grant of the Romanian National Authority for Scientific Research, CNCS UEFISCDI, project number PN-II-ID-PCE-2011-3-0119.

important contribution to the theory of rigged Hilbert spaces is due to Yu.M. Berezansky [6], [7], and his school [8], [9], in which rigged Hilbert spaces are generated by scales of continuously embedded Hilbert spaces with certain properties. The basic concept in this approach is that of a *triplet of Hilbert spaces*. More precisely, this is denoted by $\mathcal{H}_+ \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_-$, where: $\mathcal{H}_+$, $\mathcal{H}_0$, and $\mathcal{H}_-$ are Hilbert spaces, the embeddings are continuous (bounded linear operators), the space $\mathcal{H}_+$ is dense in $\mathcal{H}_0$, the space $\mathcal{H}_0$ is dense in $\mathcal{H}_-$, and the space $\mathcal{H}_-$ is the conjugate dual of $\mathcal{H}_+$ with respect to $\mathcal{H}_0$, that is, $\|\varphi\|_- = \sup\{|\langle h, \varphi\rangle_{\mathcal{H}} \mid \|h\|_+ \leq 1\}$, for all $\varphi \in \mathcal{H}_0$. Extending these triplets on both sides, one gets a *scale of Hilbert spaces* that yields, by an inductive limit and, respectively, a projective limit, a *rigged Hilbert space* $\mathcal{S} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{S}^*$. In this respect, the rather vague notion of "duality through a Hilbert space" is made precise, as well.

In order to produce a triplet of Hilbert spaces, this method requires that the positive selfadjoint operator which generates it, and that we call the *Hamiltonian* of the system, should have a bounded inverse. In the following we briefly describe this construction, following [7] and [9], but with different notation and making explicit a technique of operator ranges, e.g. see [17] and the rich bibliography cited there. Let $H$ be a positive selfadjoint operator in a Hilbert space $\mathcal{H}$ such that $A = H^{-1}$ is a bounded operator. Then there exists $S \in \mathcal{B}(\mathcal{H})$ such that $A = S^*S$, e.g. $S = A^{1/2}$ does the job. Note that, necessarily, $S$ has trivial kernel and dense range, but may not be boundedly invertible. Let $\mathcal{R}(S)$ denote the range space $\operatorname{Ran}(S)$, hence a dense linear manifold in $\mathcal{H}$, organized as a Hilbert space with respect to the norm

$$\|f\|_S = \|u\|_{\mathcal{H}}, \quad f = Su,\ u \in \mathcal{H}. \tag{1.1}$$

Then $\mathcal{H}_+ = \mathcal{R}(S)$ is continuously embedded in $\mathcal{H}$, let $j_+$ denote this embedding, and note that $j_+ j_+^* = A$, the *kernel operator* of this embedding.

On $\mathcal{H}$ one can define a new norm $\|\cdot\|_-$ by the variational formula

$$\|f\|_- = \sup\{\frac{|\langle f, u\rangle_{\mathcal{H}}|}{\|u\|_+} \mid u \in \mathcal{H}_+ \setminus \{0\}\}, \tag{1.2}$$

and let $\mathcal{H}_-$ denote the completion of $\mathcal{H}$ under the norm $\|\cdot\|_-$. Then $\mathcal{H}$ is continuously embedded and dense in $\mathcal{H}_-$; let $j_-$ denote the bounded operator of embedding $\mathcal{H}$ into $\mathcal{H}_-$. Thus, $(\mathcal{H}_+; \mathcal{H}; \mathcal{H}_-)$ is a triplet of Hilbert spaces. The following theorem gathers a few remarkable facts, cf. [7] and [9].

**Theorem 1.1.** *Let $H$ be a positive selfadjoint operator in a Hilbert space $\mathcal{H}$ such that $A = H^{-1}$ is a bounded operator, and let $S \in \mathcal{B}(\mathcal{H})$ be such that $A = S^*S$. With notation as before $(\mathcal{H}_+; \mathcal{H}; \mathcal{H}_-)$ is a triplet of Hilber spaces. In addition:*

(a) *The operator $j_+^* \colon \mathcal{H} \to \mathcal{H}_+$, when viewed as an operator densely defined in $\mathcal{H}_-$ and valued in $\mathcal{H}_+$, can be uniquely extended to a unitary operator $\widetilde{V} \colon \mathcal{H}_- \to \mathcal{H}_+$.*

(b) *The kernel operator $A$ can be viewed as a linear operator densely defined in $\mathcal{H}_-$, with dense range in $\mathcal{H}_+$, and it is a restriction of the unitary operator $\widetilde{V}$, as in item* (a).

(c) *The Hamiltonian operator $H$ can be viewed as an operator densely defined in $\mathcal{H}_+$ and valued in $\mathcal{H}_-$, and then it has a unique unitary extension $\widetilde{H} \colon \mathcal{H}_+ \to \mathcal{H}_-$ such that $\widetilde{H} = \widetilde{V}^{-1}$.*

(d) *The operator* $\Theta\colon \mathcal{H}_- \to \mathcal{H}_+^*$ *(here* $\mathcal{H}_+^*$ *denotes the conjugate dual space of* $\mathcal{H}_+$*), defined by* $(\Theta y)(x) = \langle \widetilde{V} y, x\rangle_+$*, for* $y \in \mathcal{H}_-$ *and* $x \in \mathcal{H}_+$*, provides the canonical identification of* $\mathcal{H}_-$ *with* $\mathcal{H}_+^*$*.*

One of the most important applications of Theorem 1.1 is to the method of weak solutions for boundary value problems associated to certain partial differential equations. The assumption in Theorem 1.1 that the operator $H$ has a bounded inverse requires, in terms of the corresponding boundary value problem, the Lax-Milgram Theorem referring to a bilinear form that is bounded away from zero, the so-called coercivity condition, that is usually proven by means of subtle Poincaré-Sobolev type inequalities, which can be rather technical and restricting very much the range of applications, e.g. see L.C. Evans [16], E. Sanchez-Palencia [30], or R.E. Showalter [33]. Our point of view, as illustrated by the main results Theorem 4.1 and Theorem 5.1, is that this technical condition can be weakened by means of the more general concept of triplets of closely embedded Hilbert spaces that we propose herewith. In order to substantiate this, we provide in Section 6 an application of our main results to provide existence of weak solutions for Dirichlet problems associated to degenerate elliptic operators.

In Section 2 we show that there are strong motivations, coming from problems related to homogeneous Sobolev spaces, weighted Sobolev spaces, Hilbert spaces of holomorphic functions, weighted $L^2$ spaces, and others, that require dropping the assumption that the Hamiltonian operator $H$ admits a bounded inverse. In Section 5 we show that a sufficiently rich and consistent theory for triplets of Hilbert spaces can be obtained by replacing the notion of continuous embedding by that of a closed embedding, cf. [14], within a more general concept of triplet of closely embedded Hilbert spaces. More precisely, by employing this new concept of triplets of closely embedded Hilbert spaces, in Theorem 5.1 we essentially recover all of the properties (a)–(d) from Theorem 1.1 in the more general case when the Hamiltonian is free of any coercivity assumption and, in this way, providing an approach to the motivating problems listed before.

In order to single out the concept of a triplet of closely embedded Hilbert spaces we make use of our previous investigations on closed embeddings in [14]. The correct axioms of a triplet of closely embedded Hilbert spaces became clearer to us first as a consequence of a "test of validity" of this model on Dirichlet type spaces on the unit polydisc as in [15] and, secondly, as an abstract model generated by an arbitrary factorization $H = T^*T$ of the Hamiltonian operator, that we obtain in Section 4.

## 2. Some Motivations

In this section we record a few of the problems that lead us to considering generalizations of triplets of Hilbert spaces.

2.1. **Bessel Potential versus Riesz Potential.** We first point out a triplet of Hilbert spaces associated to continuous embeddings of some Sobolev Hilbert spaces in $L_2(\mathbb{R}^n)$, following the Remark 4.3 in [14]. We assume the reader to be familiar with the basic terminology and facts on various Sobolev spaces as presented, e.g. in the monographs of R.A. Adams [1], V.M. Maz'ja [23], S.L. Sobolev [34], or R.A. Adams and J. Fournier [2]. A few notation is recalled in Section 6.

Let $\mathcal{H} = L_2(\mathbb{R}^n)$, $\quad n \geq 3$, and let $H_1$ denote the operator $H_1 = (-\Delta + I)^l$, where $\Delta \equiv \sum_{k=1}^n \partial^2/\partial x_k^2$ is the Laplacian and $l$ is a positive number. For the case when $l$ is integer, see Section 6 for notation. As the domain of $H_1$, the Sobolev space $W_2^\alpha(\mathbb{R}^n)$, $\alpha = 2l$ is considered. $H_1$ represents on this domain a positive definite selfadjoint operator. In particular, $H$ is an invertible operator, and its inverse is bounded on $\mathcal{H}$. Next, we denote

$$S = (-\Delta + I)^{-l/2}.$$

The operator $S$ can be represented, e.g. see E.M. Stein [35], §V.3.1, as a convolution integral operator with kernel

$$G(x) = cK_{(n-l)/2}\ (|x|)|x|^{(l-n)/2},$$

where $K_\nu$ is the modified Bessel function of the third kind, $c$ is a positive constant, see e.g. N. Aronszajn and K.T. Smith [4], § II.3. Thus

$$(Su)(x) = \int_{\mathbb{R}^n} G_l(x-y)u(y)\,\mathrm{d}\,y, \quad u \in L_2(\mathbb{R}^n).$$

This integral operator is known as *the Bessel potential of order* $l$, e.g. see [35].

Note that $S$ can be also regarded as a pseudodifferential operator corresponding to the symbol $(1 + |\xi|^2)^{-l/2}$, i.e.

$$(Su)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (1 + |\xi|^2)^{-l/2} \widehat{u}(\xi) e^{-\mathrm{i}\langle x,\xi\rangle}\,\mathrm{d}\,\xi, \quad x \in \mathbb{R}^n,$$

where $\widehat{u} = \mathcal{F}u$ is the Fourier transform of the function $u \in L_2(\mathbb{R}^n)$ and $\langle x, \xi\rangle$ denotes the scalar product of the elements $x, \xi \in \mathbb{R}^n$. Obviously, $S$ maps $L_2(\mathbb{R}^n)$ onto $W_2^l(\mathbb{R}^n)$.

Following (1.1) we define an inner product on $\mathrm{Ran}(S)$ $(= W_2^l(\mathbb{R}^n))$ by setting

$$\langle Sf, Sg\rangle_S := \langle f, g\rangle_{\mathcal{H}}, \quad f, g \in \mathcal{H}.$$

We have

$$\langle u, v\rangle_S = \langle (-\Delta + I)^{l/2} u, (-\Delta + I)^{l/2} v\rangle_{\mathcal{H}}, \quad u, v \in \mathrm{Ran}(S),$$

and, respectively, for the corresponding norm

$$\|u\|_S = \|(-\Delta + I)^{l/2} u\|_{\mathcal{H}}, \quad u \in \mathrm{Ran}(S).$$

This norm is equivalent with the standard norm

$$\|u\|_{W_2^l(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} |u(x)|^2\,\mathrm{d}\,x + \int_{\mathbb{R}^n} |(\nabla_l u)(x)|^2\,\mathrm{d}\,x \right)^{1/2}$$

of the Sobolev space $W_2^l(\mathbb{R}^n)$. Consequently, $\mathrm{Ran}(S)$ endowed with the norm $\|\cdot\|_S$ coincides with the Sobolev space $W_2^l(\mathbb{R}^n)$. Thus $\mathcal{R}(S) = W_2^l(\mathbb{R}^n)$ algebraically and topologically. Moreover, $\mathcal{R}(S)$ is continuously embedded in $\mathcal{H}$ and the kernel operator of the canonical embedding is the Bessel potential $J_\alpha = (-\Delta + I)^{-\alpha/2}$ of order $\alpha = 2\ell$. Note that

$$\langle u, v\rangle_S = \langle Hu, v\rangle_{\mathcal{H}}, \quad u \in \mathrm{Dom}(H),\ v \in \mathcal{R}(S).$$

We can now apply Theorem 1.1 and get a triplet of Hilbert spaces $(W_2^l(\mathbb{R}^n), L_2(\mathbb{R}^n), W_2^{-l}(\mathbb{R}^n))$, where $W_2^{-l}(\mathbb{R}^n)$ denotes the conjugate dual space of $W_2^l(\mathbb{R}^n)$, and the Hamiltonian operator is $H_1 = (-\Delta + I)^l$.

Let now $H_2^l(\mathbb{R}^n)$ denote the homogeneous Sobolev space of all functions $u \in W_{2,\mathrm{loc}}^l(\mathbb{R}^n)$ for which $\|u\|_{2,l}^2 < \infty$, where

$$\|u\|_{2,l}^2 := \int_{\mathbb{R}^n} (|(\nabla_l u)(x)|^2 + |x|^{-2l}|u(x)|^2)\,\mathrm{d}\,x, \quad u \in C_0^\infty(\mathbb{R}^n). \tag{2.1}$$

The operator $H_0 = (-\Delta)^l$ is defined on its maximal domain, i.e. on the Sobolev space $W_2^\alpha(\mathbb{R}^n)$, $\alpha = 2l$, and it represents a selfadjoint operator in $\mathcal{H}$. When trying to perform a similar treatment as in the case corresponding to the operator $H_1 = (\Delta + I)^l$ and described before, it turns out that the Hamiltonian operator $H_0$ is one-to-one but it does not have a bounded inverse. Instead of the Bessel potential that yields a bounded integral operator, we get the Riesz potential that yields an unbounded integral operator. In Subsection 4.2 in [14] we described a way of treating this case by means of "closely embedding" the homogeneous Sobolev space $H_2^l(\mathbb{R}^n)$ into $L_2(\mathbb{R}^n)$, which is actually associated to the Hamiltonian $H_0$ and cannot be continuously embedded in $L_2(\mathbb{R}^n)$, and which, once again, makes a motivation for changing the definition of the triplet of Hilbert spaces with a more general one.

More precisely, we consider the operator $T$ defined in the space $L_2(\mathbb{R}^n)$ by

$$(Tu)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |\xi|^{-l/2}\widehat{u}(\xi)e^{-\mathrm{i}\langle x,\xi\rangle}\,\mathrm{d}\,\xi, \quad x \in \mathbb{R}^n,$$

on the domain

$$\mathrm{Dom}(T) := \{u \in L_2(\mathbb{R}^n) \mid |\xi|^{-l/2}\widehat{u}(\xi) \in L_2(\mathbb{R}^n)\}.$$

The operator $T$ can be written formally as

$$T = (-\Delta)^{-l/2},$$

and it can be also considered as *the M. Riesz potential of order* $l$, e.g. see E.M. Stein [35], § V.1.1, that means that $T$ is the convolution integral operator with the kernel $|x|^{l-n}$, up to a constant,

$$(Tu)(x) = c\int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^{n-l}}\,\mathrm{d}\,y, \quad u \in \mathrm{Dom}(T).$$

$T$ represents a closed unbounded operator in $\mathcal{H}$ $(= L_2(\mathbb{R}^n))$, and, obviously, $\mathrm{Null}(T) = \{0\}$. The domain of $T$ is $\mathrm{Ran}(H_0^{1/2})$ and its range is $\mathrm{Dom}(H_0^{1/2})$, i.e. the Sobolev space $W_2^l(\mathbb{R}^n)$. In Theorem 4.4 in [14] it is proven that, by employing the more general notion of "closed embedding" and providing the necessary generalization of the "operator range" space $\mathcal{R}(T)$, see Subsection 3.2, one can prove that the homogeneous Sobolev space $H_2^l(\mathbb{R}^n) = \mathcal{R}(T)$.

2.2. **Weighted Sobolev Spaces.** Let $\Omega$ be a domain (nonempty open set) in $\mathbb{R}^N$. A *weight* $w$ on $\Omega$ is a measurable function $\omega\colon \Omega \to (0, +\infty)$. In this case, the weighted Hilbert space $L_w^2(\Omega)$ consists of all measurable functions $f\colon \Omega \to \mathbb{C}$ such that

$$\|f\|_{2,w}^2 = \int_\Omega |f(x)|^2 w(x)\,\mathrm{d}\,x < +\infty. \tag{2.2}$$

Following A. Kufner and B. Opic [27], a weight $w$ on $\Omega$ satisfies *condition* $B_2(\Omega)$ if $w^{-1} \in L_{\mathrm{loc}}^1(\Omega)$. An application of Schwarz Inequality shows that, if the weight $w$ satisfies condition $B_2(\Omega)$, then $L_w^2(\Omega)$ is continuously embedded in $L_{\mathrm{loc}}^1(\Omega)$, in particular $L_w^2(\Omega) \subset \mathcal{D}'(\Omega)$, the space of distributions on $\Omega$ and hence, for every function $u \in L_w^2(\Omega)$ and multi-index $\alpha \in \mathbb{N}_0^N$, the distributional derivatives $D^\alpha u$ make sense.

Letting $\mathcal{W} = \{w_j\}_{j=0}^N$ be a family of weights on $\Omega$, for any $u \in L^2_{w_0}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$ such that for $j = 1, \dots, N$ the distributional derivatives $\partial u/\partial x_j$ are regular distributions associated to functions in $L^2_{w_0}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$, one can define the norm

$$\|u\|_{2,\mathcal{W}} = \Big(\sum_{j=0}^N \|\partial u/\partial x_j\|^2_{2,w_j}\Big)^{1/2}. \tag{2.3}$$

If $W^1_2(\Omega;\mathcal{W})$ defines the weighted Sobolev space of all functions $u$ as before, endowed with the norm (2.3), and assuming that all weights $w_j$, for $j = 1, \dots, N$ belong to the class $B_2(\Omega)$, then $W^1_2(\Omega;\mathcal{W})$ is a Banach space, cf. Theorem 2.1 in [27]. However, as proven in Example 1.12 in [27], if $\Omega = (-1,1)$, $w_0(x) = x^2$, and $w_1(x) = x^4$, then $W^1_2(\Omega;\mathcal{W})$, with $\mathcal{W} = \{w_0, w_1\}$, is not complete with respect to the norm (2.3).

Because of the anomaly in the definition of the weighted Sobolev spaces $W^1_2(\Omega;\mathcal{W})$ described before, A. Kufner and B. Opic proposed in [27] to remove the "exceptional sets" $M_2(w_j)$ for all $j = 1, \dots, N$, where, for a given weight $w$ on $\Omega$, they defined

$$M_2(w) = \{x \in \Omega \mid \int_{\Omega\cap U(x)} w^{-1}(y)\,\mathrm{d}\,y = \infty \text{ for all neighbourhoods } U(x) \text{ of } x\}. \tag{2.4}$$

As proven in Theorem 3.3 in [27], if a weight $w$ is continuous a.e. on $\Omega$, then the exceptional set $M_2(w)$ has Lebesgue measure zero. However, there are situations when this set can be rather large, or even the whole $\Omega$.

**Example 2.1.** This example was obtained by Ö.F. Tekin as a Senior Project under the supervision of the second named author, during the Fall semester of 2011, [37]. Let $\Omega = (0,1)$ for $N = 1$ and define

$$w^{-1}(x) = \sum_{(m,n)\colon \frac{m}{2^n} > x} \frac{1}{(\frac{m}{2^n} - x)2^{3n}}, \quad x \in (0,1),$$

more precisely, for each $x \in (0,1)$, the terms are summed for all pairs of natural numbers $(m,n)$ such that $x < m/2^n$. Then $\omega$ is a weight on $(0,1)$ and the exceptional set $M_2(\Omega) = (0,1) = \Omega$.

These anomalies suggest that, as an alternative, one can define the weighted Sobolev space $W^1_2(\Omega;\mathcal{W})$ as the completion, under the norm (2.3), of the space of all functions $u$ for which the norm $\|\cdot\|_{2,\mathcal{W}}$ was originally defined. As noted in Remark 3.6 in [27], if this new definition is adopted, then the space $W^1_2(\Omega;\mathcal{W})$ may contain nonregular distributions and also functions whose distributional derivatives are not regular distributions, and hence they considered this definition to be unnatural. Our point of view is that, by considering the more general concepts of closed embeddings and triplets of closely embedded Hilbert spaces, and developing a sufficiently rich theory for them, this latter definition of weighted Sobolev spaces may be reconsidered, at least in view of some usual problems in the theory of Sobolev spaces.

2.3. **Dirichlet Type Spaces on the Polydisc.** For a fixed natural number $N$ consider the unit polydisc $\mathbb{D}^N = \mathbb{D} \times \cdots \times \mathbb{D}$, the direct product of $N$ copies of the unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. We consider $H(\mathbb{D}^N)$ the algebra of functions holomorphic in the polydisc, that is, the collection of all functions $f\colon \mathbb{D}^N \to \mathbb{C}$ that are holomorphic in each variable, equivalently, there exists $(a_k)_{k\in\mathbb{Z}_+^N}$ with the property that

$$f(z) = \sum_{k\in\mathbb{Z}_+^N} a_k z^k, \quad z \in \mathbb{D}^N, \tag{2.5}$$

where the series converges absolutely and uniformly on any compact subset in $\mathbb{D}^N$. Here and in the sequel, for any multi-index $k = (k_1, \ldots, k_N) \in \mathbb{Z}_+^N$ and any $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$ we let $z^k = z_1^{k_1} \cdots z_N^{k_N}$.

Let $\alpha \in \mathbb{R}^N$ be fixed. Following G.D. Taylor [36], for the one dimensional case, and D. Jupiter and D. Redett [21], for the multidimensional case, the *Dirichlet type space* $\mathcal{D}_\alpha$ is defined as the space of all functions $f \in H(\mathbb{D}^N)$ with representation (2.5) subject to the condition

$$\sum_{k\in\mathbb{Z}_+^N} (k+1)^\alpha |a_k|^2 < \infty, \tag{2.6}$$

where, $(k+1)^\alpha = (k_1+1)^{\alpha_1} \cdots (k_N+1)^{\alpha_N}$. By Proposition 2.5 in [21], the condition (2.6) implies that the function $f$ defined as in (2.5) is holomorphic in $\mathbb{D}^N$, so $\mathcal{D}_\alpha$ is a subspace of $H(\mathbb{D}^N)$ no matter whether we stipulate it in advance or not. The linear space $\mathcal{D}_\alpha$ is naturally organized as a Hilbert space with inner product $\langle\cdot,\cdot\rangle_\alpha$

$$\langle f, g\rangle_\alpha = \sum_{k\in\mathbb{Z}_+^N} (k+1)^\alpha a_k \overline{b_k}, \tag{2.7}$$

where $f$ has representation (2.5) and similarly $g(z) = \sum_{k\in\mathbb{Z}_+^N} b_k z^k$, for all $z \in \mathbb{D}^N$, and norm $\|\cdot\|_\alpha$ defined by

$$\|f\|_\alpha^2 = \sum_{k\in\mathbb{Z}_+^N} (k+1)^\alpha |a_k|^2. \tag{2.8}$$

For any $\alpha \in \mathbb{R}^N$, on the polydisc $\mathbb{D}^N$ the following kernel is defined

$$K^\alpha(w, z) = \sum_{k\in\mathbb{Z}_+^N} (k+1)^{-\alpha} \overline{w}^k z^k, \quad z, w \in \mathbb{D}^N, \tag{2.9}$$

where, for $w = (w_1, \ldots, w_N) \in \mathbb{D}^N$ one denotes $\overline{w} = (\overline{w}_1, \ldots, \overline{w}_N)$, the entry-wise complex conjugate. We let $K_w^\alpha = K^\alpha(w, \cdot)$. It turns out, as follows from Lemma 2.8 and Lemma 2.9 in [21], that $K^\alpha$ is the reproducing kernel for the space $\mathcal{D}_\alpha$ in the sense that the following two properties hold:

(rk1) $K_w^\alpha \in \mathcal{D}_\alpha$ for all $w \in \mathbb{D}^N$.
(rk2) $f(w) = \langle f, K_w^\alpha\rangle_\alpha$ for all $f \in \mathcal{D}_\alpha$ and all $w \in \mathbb{D}^N$.

A more general argument shows, e.g. see N. Aronszajn [3], that the set $\{K_w^\alpha \mid w \in \mathbb{D}^N\}$ is total in $\mathcal{D}_\alpha$ and that the kernel $K^\alpha$ is positive semidefinite,.

A partial order relation $\geq$ on $\mathbb{R}^N$ can be defined by $\alpha \geq \beta$ if and only if $\alpha_j \geq \beta_j$ for all $j = 1, \ldots, N$. In addition, $\alpha > \beta$ means $\alpha_j > \beta_j$ for all $j = 1, \ldots, N$.

The Dirichlet type space $\mathcal{D}_0$ coincides with the Hardy space $H^2(\mathbb{D})$. More precisely, following W. Rudin [28], let $\mathbb{T} = \partial\mathbb{D}$ denote the one-dimensional torus (the unit circle centered at 0 in the complex plane) and then let $\mathbb{T}^N = \mathbb{T}\times\cdot\times\mathbb{T}$ be the $N$-dimensional torus, also called *the distinguished boundary* of the unit polydisc $\mathbb{D}^N$, which is only a subset of $\partial\mathbb{D}^N$. We consider the product measure $\mathrm{d}\, m_N = \mathrm{d}\, m_1\times\cdots\times\mathrm{d}\, m_1$ on $\mathbb{D}^N$, where $\mathrm{d}\, m_1$ denotes the normalized Lebesgue measure on $\mathbb{T}$, and for any function $f\in H(\mathbb{D}^N)$ and $0\leq r<1$ let $f_r(z)=f(rz)$ for $z\in\mathbb{D}^N$. By definition, $f\in H(\mathbb{D}^N)$ belongs to $H^2(\mathbb{D}^N)$ if and only if

$$\sup_{0\leq r<1}\int_{\mathbb{T}^N}|f_r|^2\,\mathrm{d}\, m_N<\infty,$$

and the norm $\|\cdot\|_0$ and inner product $\langle\cdot,\cdot\rangle_0$ on the Hardy space $H^2(\mathbb{D}^N)$ are defined by

$$\|f\|_0^2=\sup_{0\leq r<1}\int_{\mathbb{T}^N}|f_r|^2\,\mathrm{d}\, m_N=\lim_{r\to 1-}\int_{\mathbb{T}^N}|f_r|^2\,\mathrm{d}\, m_N,\quad f\in H^2(\mathbb{D}^N),$$

$$\langle f,g\rangle_0=\lim_{r\to 1-}\int_{\mathbb{T}^N}f_r\overline{g_r}\,\mathrm{d}\, m_N,\quad f,g\in H^2(\mathbb{D}^N),$$

where, we can use the lower index 0 because it can be easily proven that this norm coincides with the norm $\|\cdot\|_0$ with definition as in (2.8) (here 0 is the multi-index with all entries null). Thus, $\mathcal{D}_0$ coincides as a Hilbert space with $H^2(\mathbb{D}^N)$. In addition, the reproducing kernel $K^0$ has a simple representation in this case, namely in the compact form

$$K^0(w,z)=\frac{1}{1-\overline{w}_1 z_1}\cdots\frac{1}{1-\overline{w}_N z_N}.$$

In the following proposition we point out that a natural triplet of Hilbert spaces can be made by rigging $\mathcal{D}_0=H^2(\mathbb{D}^N)$ when we consider multi-indices $\alpha\geq 0$. In order to describe precisely the operators associated to the triplet, like kernel operators, Hamiltonian, and so on, we need a class of linear operators that are in the family of *radial derivative operators*, cf. F. Beatrous and J. Burbea [5].

Let $\mathcal{P}_N$ denote the complex vector space of polynomial functions in $N$ complex variables, that is, those functions $f$ that admit a representation (2.5) for which $\{a_k\}_{k\in\mathbb{Z}_+^N}$ has finite support. We consider now the additive group $\mathbb{R}^N$ and a representation $T.\colon\mathbb{R}^N\to\mathcal{L}(\mathcal{P}_N)$, where $\mathcal{L}(\mathcal{P}_N)$ denotes the algebra of linear maps on the vector space $\mathcal{P}_N$, defined by

$$(T_\alpha f)(z)=\sum_{k\in\mathbb{Z}_+^N}(k+1)^\alpha a_k z^k,\quad \alpha\in\mathbb{R}^N\ z\in\mathbb{D}^N, \tag{2.10}$$

where the polynomial $f$ has representation (2.5) and $\{a_k\}_{k\in\mathbb{Z}_+^N}$ has finite support.

Theorem 1.1 provides the abstract framework to precisely describe a triplet of Hilbert spaces $(\mathcal{D}_\alpha;H^2(\mathbb{D}^N);\mathcal{D}_{-\alpha})$, when $\alpha\geq 0$. We record this in the following proposition, where the underlying spaces and operators are precisely described, for details see [15].

**Proposition 2.2.** *For any $\alpha\in\mathbb{R}^N$ with $\alpha\geq 0$, $(\mathcal{D}_\alpha;H^2(\mathbb{D}^N);\mathcal{D}_{-\alpha})$ is a triplet of Hilbert spaces with the following properties:*

(a) *The embeddings $j_\pm$ of $\mathcal{D}_\alpha$ in $H^2(\mathbb{D}^N)$ and, respectively, of $H^2(\mathbb{D}^N)$ in $\mathcal{D}_{-\alpha}$, are bounded and have dense ranges.*

(b) *The adjoint $j_+^*$ is defined by $j_+^*f=T_{-\alpha}f$ for all $f\in\mathrm{Dom}(j_+^*)=H^2(\mathbb{D}^N)\cap\mathcal{D}_{-\alpha}$.*

(c) *The kernel operator $A = j_+ j_+^*$ is a nonnegative bounded operator in the Hilbert space $H^2(\mathbb{D}^N)$, defined by $Af = T_{-\alpha} f$ for all $f \in H^2(\mathbb{D}^N)$ and is an integral operator with kernel $K^\alpha$, in the sense that, for all $f \in H^2(\mathbb{D}^N)$, we have*

$$(Af)(z) = \langle f, K_z^\alpha \rangle_0 = \lim_{r \to 1-} \int_{\mathbb{T}^N} f_r(w) K^\alpha(rw, z) \,\mathrm{d}\, m_N(w), \quad z \in \mathbb{D}^N. \tag{2.11}$$

(d) *The Hamiltonian operator $H = A^{-1}$ is a positive selfadjoint operator in $H^2(\mathbb{D}^N)$ defined by $Hf = T_\alpha f$ for all $f \in \mathrm{Dom}(H) = H^2(\mathbb{D}^N) \cap \mathcal{D}_{2\alpha}$.*

(e) *The canonical unitary identification of $\mathcal{D}_{-\alpha}$ with $\mathcal{D}_\alpha^*$ is defined by*

$$(\Theta g) f = \langle T_{-\alpha} f, g \rangle_\alpha, \quad f \in \mathcal{D}_{-\alpha}, \ g \in \mathcal{D}_\alpha.$$

*In addition, $\sigma(A) \setminus \{0\} = \{(k+1)^{-\alpha} \mid k \in \mathbb{Z}_+^N\}$ and $\sigma(H) \setminus \{0\} = \{(k+1)^\alpha \mid k \in \mathbb{Z}_+^N\}$. Moreover, if $\alpha_j > 0$ for all $j = 1, \ldots, N$, the kernel operator $A$ is Hilbert-Schmidt.*

This proposition can be used to describe a rigging $(\mathcal{S}(\mathbb{D}^n), H^2(\mathbb{D}^N), \mathcal{S}^*(\mathbb{D}^N))$, by Dirichlet type spaces and Bergman type spaces, see [15].

Because, in this special case of the unit polydisc, the coefficients on different directions are independent, a natural question that can be raised is what can be said when considering a multi-index $\alpha \in \mathbb{R}^N$ that contains positive as well as negative components, from the point of view of the triplet $(\mathcal{D}_\alpha; \mathcal{D}_0; \mathcal{D}_{-\alpha})$ as in Proposition 2.2. It is clear that, in this case, there is no continuous embedding of $\mathcal{D}_\alpha$ in $\mathcal{D}_0$. However, as proven directly in [15], the statements of Proposition 2.2 have natural generalizations, with very similar transcription, in terms of unbounded operators. This transcription, with appropriate definitions of closed embeddings and triplets of closed embeddings of Hilbert spaces, has been obtained directly in [15] because of the relative tractability of the problem, but an abstract model and questions on existence and uniqueness properties have not been considered there.

## 2.4. **Weighted $L^2$ Spaces.**

In connection with the Dirichlet type spaces as presented in Subsection 2.3, but also from a more general perspective, it is natural to consider triplets associated to weighted $L^2$ spaces. Let $(X; \mathfrak{A})$ be a measurable space on which we consider a $\sigma$-finite measure $\mu$. A function $\omega$ defined on $X$ is called a *weight* with respect to the measure space $(X; \mathfrak{A}; \mu)$ if it is measurable and $0 < \omega(x) < \infty$, for $\mu$-almost all $x \in X$. Note that $\mathcal{W}(X; \mu)$, the collection of weights with respect to $(X; \mathfrak{A}; \mu)$, is a multiplicative unital group. For an arbitrary $\omega \in \mathcal{W}(X; \mu)$, consider the measure $\nu$ whose Radon-Nikodym derivative with respect to $\mu$ is $\omega$, denoted $\mathrm{d}\,\nu = \omega \,\mathrm{d}\,\mu$, that is, for any $E \in \mathfrak{A}$ we have $\nu(E) = \int_E \omega \,\mathrm{d}\,\mu$. It is easy to seee, e.g. see [15], that $\nu$ is always $\sigma$-finite.

**Proposition 2.3.** *Let $\omega$ be a weight on the $\sigma$-finite measure space $(X; \mathfrak{A}; \mu)$ such that $\operatorname{ess\,inf}_X \omega > 0$. Let $\mathcal{H}_0 = L^2(X; \mu)$, $\mathcal{H}_+ = L^2_\omega(X; \mu)$ and $\mathcal{H}_- = L^2_{\omega^{-1}}(X; \mu)$. Then $(\mathcal{H}_+; \mathcal{H}_0; \mathcal{H}_-)$ is a triplet of Hilbert spaces for which:*

(a) *The embeddings $j_\pm$ of $\mathcal{H}_+$ in $\mathcal{H}_0$ and of $\mathcal{H}_0$ in $\mathcal{H}_-$ are bounded and have dense ranges.*

(b) *The adjoint $j_+^*$ is defined by $j_+^* h = \omega^{-1} h$ for all $h \in L^2(X; \mu)$.*

(c) *The kernel operator $A = j_+ j_+^*$ is a nonnegative bounded operator defined by $Ah = \omega^{-1} h$, for all $h \in L^2(X; \mu)$. Moreover, when viewed as an operator defined in $\mathcal{H}_-$ and valued in $\mathcal{H}_+$, $A$ admits a unique unitary extension $\widetilde{A} \colon \mathcal{H}_- \to \mathcal{H}_+$.*

(d) *The Hamiltonian* $H = A^{-1}$ *is defined by* $Hh = \omega h$ *for all* $h \in \mathrm{Dom}(H) = L^2_{\omega^2}(X;\mu)$. *Moreover, when viewed as an operator defined in* $\mathcal{H}_+$ *and valued in* $\mathcal{H}_-$, $H$ *can be uniquely extended to a unitary operator* $\widetilde{H} = \widetilde{A}^{-1}$.

(e) *The canonical unitary identification of* $\mathcal{H}_+^*$ *with* $\mathcal{H}_-$ *is the operator* $\Theta$ *is defined by*

$$(\Theta g)(f) := \langle \widetilde{A} f, g\rangle_+ = \int_X f\overline{g}\,\mathrm{d}\,\mu, \quad f \in \mathcal{H}_+,\ g \in \mathcal{H}_-, \tag{2.12}$$

*Consequently,* $\sigma(A) = \operatorname{ess\,ran}(\omega^{-1})$ *and* $\sigma(H) = \operatorname{ess\,ran}(\omega)$*, where* ess ran *denotes the* $\mu$*-essential range.*

A natural question that can be raised in connection with the preceding proposition is whether anything might be said when dropping the assumption $\operatorname{ess\,inf}\omega > 0$. Again, the embeddings cannot be continuous anymore, and hence we have to allow unbounded operators to show up. Once the notions of closed embeddings and triplets of closely embedded Hilbert spaces have been singled out as in [15], Proposition 2.3 can be naturally extended to cover the general case and we used this extension in order to provide a solution to the construction of triplets of closely embedded Hilbert spaces associated to any pair of Dirichlet type spaces, but questions on abstract models, existence and uniqueness properties, have not been considered yet.

## 3. Notation and Preliminary Results

A Hilbert space $\mathcal{H}_+$ is called *closely embedded* in the Hilbert space $\mathcal{H}$ if:

(ceh1) There exists a linear manifold $\mathcal{D} \subseteq \mathcal{H}_+ \cap \mathcal{H}$ that is dense in $\mathcal{H}_+$.
(ceh2) The embedding operator $j_+$ with domain $\mathcal{D}$ is closed, as an operator $\mathcal{H}_+ \to \mathcal{H}$.

The meaning of the axiom (ceh1) is that on $\mathcal{D}$ the algebraic structures of $\mathcal{H}_+$ and $\mathcal{H}$ agree, while the meaning of the axiom (ceh2) is that the embedding $j_+$ is explicitly defined by $j_+x = x$ for all $x \in \mathcal{D} \subseteq \mathcal{H}_+$ and, considered as an operator from $\mathcal{H}_+$ to $\mathcal{H}$, it is closed. Also, recall that in case $\mathcal{H}_+ \subseteq \mathcal{H}$ and the embedding operator $j_+ \colon \mathcal{H}_+ \to \mathcal{H}$ is continuous, one says that $\mathcal{H}_+$ is *continuously embedded* in $\mathcal{H}$, e.g. see P.A. Fillmore and J.P. Williams [17] and the bibliography cited there.

Following L. Schwartz [31], we call $A = j_+ j_+^*$ the *kernel operator* of the closely embedded Hilbert space $\mathcal{H}_+$ with respect to $\mathcal{H}$.

The abstract notion of closed embedding of Hilbert spaces was singled out in [14] following a generalized operator range model. In this section we point out two models, which are dual in a certain way, and that will be used in this article as the main technical ingredient of the triplets of closely embedded Hilbert spaces. Constructions similar to those of the spaces $\mathcal{D}(T)$ and $\mathcal{R}(T)$ have been recently considered in the theory of interpolation of Banach spaces, e.g. see M. Haase [20] and the rich bibliography cited there.

3.1. **The Space $\mathcal{D}(T)$.** In this subsection we introduce a model of closely embedded Hilbert space generated by a closed densely defined operator. For the beginning, we consider a linear operator $T$ defined on a linear submanifold of $\mathcal{H}$ and valued in $\mathcal{G}$, for two Hilbert spaces $\mathcal{H}$ and $\mathcal{G}$, and assume that its null space $\mathrm{Null}(T)$ is a closed subspace of $\mathcal{H}$. On the linear manifold $\mathrm{Dom}(T) \ominus \mathrm{Null}(T)$ we consider the norm

$$|x|_T := \|Tx\|_{\mathcal{G}}, \quad x \in \mathrm{Dom}(T) \ominus \mathrm{Null}(T), \tag{3.1}$$

and let $\mathcal{D}(T)$ be the Hilbert space completion of the pre-Hilbert space $\mathrm{Dom}(T) \ominus \mathrm{Null}(T)$ with respect to the norm $|\cdot|_T$ associated the inner product $(\cdot,\cdot)_T$

$$(x,y)_T = \langle Tx, Ty\rangle_{\mathcal{G}}, \quad x,y \in \mathrm{Dom}(T) \ominus \mathrm{Null}(T). \tag{3.2}$$

We consider the operator $i_T$ defined, as an operator in $\mathcal{D}(T)$ and valued in $\mathcal{H}$, as follows

$$i_T x := x, \quad x \in \mathrm{Dom}(i_T) = \mathrm{Dom}(T) \ominus \mathrm{Null}(T). \tag{3.3}$$

**Lemma 3.1.** *The operator $i_T$ is closed if and only if $T$ is a closed operator.*

*Proof.* Let us assume that $T$ is a closed operator. Then $\mathrm{Null}(T)$ is a closed subspace of $\mathcal{H}$, hence the definition of the operator $i_T$ makes sense. In order to prove that $i_T$ is closed, let $(x_n)$ be a sequence in $\mathrm{Dom}(i_T)$ such that $|x_n - x|_T \to 0$ and $\|i_T x_n - y\|_{\mathcal{H}} \to 0$, as $n \to \infty$, for some $x \in \mathcal{D}_T$ and $y \in \mathcal{H}$. By (3.1) it follows that the sequence $(Tx_n)$ is Cauchy in $\mathcal{G}$. Since $(x_n)$ is also Cauchy in $\mathcal{H}$, it follows that the sequence of pairs $((x_n, Tx_n))$ is Cauchy in the graph norm of $T$ and then, since $T$ is a closed operator, it follows that there exists $z \in \mathrm{Dom}(T)$ such that

$$\|x_n - z\|_{\mathcal{H}} + \|Tx_n - Tz\|_{\mathcal{G}} \to 0, \quad \text{as } n \to \infty.$$

Taking into account that $\|Tx_n - Tz\|_{\mathcal{G}} = |x_n - z|_T$ for all $n \geq 1$, we get $z = x$ modulo $\mathrm{Null}(T)$, hence $x \in \mathrm{Dom}(i_T)$. In addition, $x = y$, hence $i_T$ is a closed operator.

The proof of the converse implication follows a similar reasoning as before. □

The next proposition emphasizes the fact that the construction of $\mathcal{D}(T)$ is actually a renorming process.

**Proposition 3.2.** *The operator $Ti_T$ admits a unique isometric extension $\widehat{T}\colon \mathcal{D}(T) \to \mathcal{G}$.*

*Proof.* Since $\mathrm{Dom}(i_T) = \mathrm{Dom}(T) \ominus \mathrm{Null}(T)$ and $i_T$ acts like identity, it follows that $\mathrm{Dom}(Ti_T) = \mathrm{Dom}(i_T)$ which is dense in $\mathcal{D}(T)$. Also, for all $x \in \mathrm{Dom}(i_T)$ we have $\|Ti_T x\|_{\mathcal{G}} = \|Tx\|_{\mathcal{G}} = |x|_T$, hence $Ti_T$ is isometric. Therefore, $Ti_T$ extends uniquely to an isometric operator $\mathcal{D}(T) \to \mathcal{G}$. □

The most interesting case is when the operator $T$ is a closed and densely defined operator in a Hilbert space $\mathcal{H}$. The next proposition explores this case from the point of view of the closed embedding of $\mathcal{D}(T)$ in $\mathcal{H}$ and that of the kernel operator $A = i_T i_T^*$.

**Proposition 3.3.** *Let $T$ be a closed and densely defined operator on $\mathcal{H}$ and valued in $\mathcal{G}$, for two Hilbert spaces $\mathcal{H}$ and $\mathcal{G}$.*

(a) $\mathcal{D}(T)$ *is closely embedded in $\mathcal{H}$ and $i_T$ is the underlying closed embedding.*

(b) $\mathrm{Ran}(T^*) \subseteq \mathrm{Dom}(i_T^*)$ *and equality holds provided that* $\mathrm{Null}(T) = 0$.

(c) $\mathrm{Ran}(T^*T) \subseteq \mathrm{Dom}(i_T i_T^*)$ *and equality holds provided that* $\mathrm{Null}(T) = 0$. *In addition,*

$$(i_T i_T^*)(T^*T)x = x, \text{ for all } x \in \mathrm{Dom}(T^*T) \ominus \mathrm{Null}(T) \tag{3.4}$$

(d) $(i_T i_T^*)\,\mathrm{Ran}(T^*T) \subseteq \mathrm{Dom}(T^*T)$ *and equality holds provided that* $\mathrm{Null}(T) = 0$. *In addition,*

$$(T^*T)(i_T i_T^*)u = u, \text{ for all } u \in \mathrm{Ran}(T^*T). \tag{3.5}$$

*Proof.* (a) First note that, since $T$ is closed, its null space is closed, hence the construction of the Hilbert space $\mathcal{D}(T)$ and $i_T$ make sense. The operator $i_T$ is densely defined, by construction. By Lemma 3.1, $i_T$ is closed as well. Hence, the axioms (ceh1) and (ceh2) are fulfilled.

(b) Let $y \in \operatorname{Ran}(T^*)$ be arbitrary, hence $y = T^*x$ for some $x \in \operatorname{Dom}(T^*) \subseteq \mathcal{G}$. Then, for all $u \in \operatorname{Dom}(i_T) = \operatorname{Dom}(T) \ominus \operatorname{Null}(T)$ we have

$$\langle y, i_T u\rangle_{\mathcal{H}} = \langle T^*x, y\rangle_{\mathcal{H}} = \langle x, Tu\rangle \mathcal{G},$$

hence

$$|\langle y, i_T u\rangle_{\mathcal{H}}| \leq \|x\|_{\mathcal{G}}\, \|Tu\|_{\mathcal{G}} = \|x\|_{\mathcal{G}}\, |u|_T, \quad .$$

which implies that $y \in \operatorname{Dom}(i_T^*)$.

Let us assume now that $\operatorname{Null}(T) = 0$ and consider an arbitrary vector $y \in \operatorname{Dom}(i_T^*)$. For any $x \in \operatorname{Ran}(T)$ there exists a unique vector $u_x \in \operatorname{Dom}(T) = \operatorname{Dom}(i_T)$ such that $x = Tu_x$ and $\|x\|_{\mathcal{G}} = |u_x|_T$. In this way, we can define a linear functional $\operatorname{Ran}(T) \ni x \mapsto \varphi_y(x) = \langle i_T u_x, y\rangle_{\mathcal{H}} = \langle u_x, i_T^* y\rangle_T$ and note that

$$|u_x|_T\, |i_T^* y|_T = \|x\|_{\mathcal{G}}\, |i_T^* y|_T, \quad x \in \operatorname{Ran}(T).$$

This shows that $\varphi_y$ has a continuous extension $\widetilde{\varphi}_y \colon \mathcal{G} \to \mathbb{C}$ and hence, there exists $g \in \mathcal{G}$ such that $\widetilde{\varphi}_y(x) = \langle x, g\rangle_{\mathcal{G}}$ for all $x \in \mathcal{G}$. Specializing this for arbitrary $x \in \operatorname{Ran}(T)$, it follows that, on the one hand,

$$\widetilde{\varphi}_y(x) = \langle x, g\rangle_G = \langle Tu_x, g\rangle_{\mathcal{G}},$$

while, on the other hand,

$$\widetilde{\varphi}_y(x) = \langle i_T u_x, y\rangle_{\mathcal{H}} = \langle u_x, y\rangle_{\mathcal{H}}.$$

Since $\operatorname{Dom}(T)$ is dense in $\mathcal{H}$ it follows that $y = T^*g$, that is, $y \in \operatorname{Ran}(T^*)$.

(c) Let $y \in \operatorname{Ran}(T^*T)$ be arbitrary, hence $y = T^*Tx$ for some $x \in \operatorname{Dom}(T^*T)$, that is, $x \in \operatorname{Dom}(T)$ and $Tx \in \operatorname{Dom}(T^*)$. Without loss of generality we can assume that $x \in \operatorname{Dom}(T) \ominus \operatorname{Null}(T) = \operatorname{Dom}(i_T)$. Then, for any $u \in \operatorname{Dom}(i_T)$ we have

$$\langle y, i_T u\rangle_{\mathcal{H}} = \langle T^*Tx, y\rangle_{\mathcal{H}} = \langle Tx, Tu\rangle_{\mathcal{G}} = (x, u)_T,$$

hence, the linear functional $\mathcal{D}(T) \supseteq \operatorname{Dom}(i_T) \ni u \mapsto \langle i_T u, y\rangle_{\mathcal{H}}$ is bounded. Therefore, $y \in \operatorname{Dom}(i_T^*)$ and $i_T^* y = x \in \operatorname{Dom}(i_T)$, in particular, $y \in \operatorname{Dom}(i_T i_T^*)$. Thus, we showed that $\operatorname{Ran}(T^*T) \subseteq \operatorname{Dom}(i_T i_T^*)$ and that $(i_T i_T^*)(T^*T)x = x$ for all $x \in \operatorname{Dom}(T^*T) \ominus \operatorname{Null}(T)$ (recall that $\operatorname{Null}(T) = \operatorname{Null}(T^*T)$).

If, in addition, $\operatorname{Null}(T) = 0$, then $\operatorname{Null}(T^*T) = \operatorname{Null}(T) = 0$ and then the representation $y = T^*Tx$ for $y \in \operatorname{Ran}(T^*T)$ and $x \in \operatorname{Dom}(T^*T)$ is unique and the reasoning from above can be reversed, hence $\operatorname{Ran}(T^*T) = \operatorname{Dom}(i_T i_T^*)$.

(d) As a consequence of the proof of (e), we also get that $(i_T i_T^*)$ maps $\operatorname{Ran}(T^*T)$ in $\operatorname{Dom}(T^*T)$ and that, for all $u \in \operatorname{Ran}(T^*T)$, we have $(T^*T)(i_T i_T^*)u = u$. In case $\operatorname{Null}(T) = 0$ then $(i_T i_T^*) \operatorname{Ran}(T^*T) = \operatorname{Dom}(T^*T)$ $\square$

**Remark 3.4.** We can view the Hilbert space $\mathcal{D}(T)$ and its closed embedding $i_T$ as a model for the abstract definition of a closed embedding. More precisely, let $(\mathcal{H}_+; \|\cdot\|_+)$ be a Hilbert space closely embedded in the Hilbert space $(\mathcal{H}; \|\cdot\|_{\mathcal{H}})$ and let $j_+$ denote the underlying closed embedding. Since $j_+$ is one-to-one, we can define a linear operator $T$ with $\operatorname{Dom}(T) = \operatorname{Ran}(j_+) \oplus \operatorname{Null}(j_+^*)$, viewed as a dense linear manifold in $\mathcal{H}$, and valued in $\mathcal{H}_+$, defined by

$T(x \oplus x_0) = j_+^{-1}x$, for all $x \in \mathrm{Ran}(j_+)$ and $x_0 \in \mathrm{Null}(j_+^*)$. Then $\mathrm{Null}(T) = \mathrm{Null}(j_+^*)$ and, for all $x \in \mathrm{Ran}(j_+)$ we have $x = j_+u$ for a unique $u = x \in \mathrm{Dom}(j_+)$, hence

$$\|x\|_+ = \|Tx\|_+ = |x|_T.$$

Thus, modulo a completion of $\mathrm{Dom}(j_+)$ which may be different, the Hilbert space $(\mathcal{D}(T); |\cdot|_T)$ coincides with the Hilbert space $(\mathcal{H}_+; \|\cdot\|_+)$.

3.2. **The Hilbert Space $\mathcal{R}(T)$.** In this subsection we recall a construction and its basic properties of Hilbert spaces associated to ranges of general linear operators that was used in [14] as the model that provided the abstract definition of a closed embedding of Hilbert spaces.

Let $T$ be a linear operator acting from a Hilbert space $\mathcal{G}$ to another Hilbert space $\mathcal{H}$ and such that its null space $\mathrm{Null}(T)$ is closed. Introduce a pre-Hilbert space structure on $\mathrm{Ran}(T)$ by the positive definite inner product $\langle\cdot,\cdot\rangle_T$ defined by

$$\langle u, v\rangle_T = \langle x, y\rangle_{\mathcal{G}} \tag{3.6}$$

for all $u = Tx$, $v = Ty$, $x, y \in \mathrm{Dom}(T)$ such that $x, y \perp \mathrm{Null}(T)$. Let $\mathcal{R}(T)$ be the completion of the pre-Hilbert space $\mathrm{Ran}(T)$ with respect to the corresponding norm $\|\cdot\|_T$, where $\|u\|_T^2 = \langle u, u\rangle_T$, for $u \in \mathrm{Ran}(T)$. The inner product and the norm on $\mathcal{R}(T)$ are denoted by $\langle\cdot,\cdot\rangle_T$ and, respectively, $\|\cdot\|_T$ throughout.

Further, consider the *embedding operator* $j_T \colon \mathrm{Dom}(j_T)(\subseteq \mathcal{R}(T)) \to \mathcal{H}$ with domain $\mathrm{Dom}(j_T) = \mathrm{Ran}(T)$ defined by

$$j_T u = u, \quad u \in \mathrm{Dom}(j_T) = \mathrm{Ran}(T). \tag{3.7}$$

Another way of viewing the definition of the Hilbert space $\mathcal{R}(T)$ is by means of a certain factorization of $T$.

**Lemma 3.5.** *Let $T$ be a linear operator with domain dense in the Hilbert space $\mathcal{G}$, valued in the Hilbert space $\mathcal{H}$, and with closed null space. We consider the Hilbert space $\mathcal{R}(T)$ and the embedding $j_T$ defined as in* (3.6) *and, respectively,* (3.7). *Then, there exists a unique coisometry $U_T \in \mathcal{B}(\mathcal{G}, \mathcal{R}(T))$, such that* $\mathrm{Null}(U_T) = \mathrm{Null}(T)$ *and* $T = j_T U_T$.

**Remark 3.6.** The assumption in Lemma 3.5 that $T$ is densely defined is not so important; if this is not the case then $U_T$ must have a larger null space only, in order to keep it unique. More precisely, $\mathrm{Null}(U_T) = \mathrm{Null}(T) \oplus (\mathcal{G} \ominus \mathrm{Dom}(T))$ and, consequently, $TP_{\overline{\mathrm{Dom}(T)}} \subseteq j_T U_T$, which turns out to be an equality since $\mathrm{Null}(T)$ is supposed to be a closed subspace in $\mathcal{G}$.

The most interesting situation, from our point of view, is when the embedding operator has some closability properties.

**Lemma 3.7.** *Let $T$ be an operator densely defined in $\mathcal{G}$, with range in $\mathcal{H}$, and with closed null space. With the notation as before, the operator $T$ is closed if and only if the embedding operator $j_T$ is closed.*

We denote by $\mathcal{C}(\mathcal{H}, \mathcal{G})$ the collection of all operators $T$ that are closed and densely defined from $\mathcal{H}$ and valued in $\mathcal{G}$. The following lemma is a direct consequence of Lemma 3.5 and Lemma 3.7.

**Lemma 3.8.** *Let $T \in \mathcal{C}(\mathcal{H}, \mathcal{G})$. Then* $\operatorname{Dom}(j_T^*) \supseteq \operatorname{Dom}(T^*)$. *If, in addition, $T$ is one-to-one, then* $\operatorname{Dom}(j_T^*) = \operatorname{Dom}(T^*)$

We also recall an extension of a characterization of operator ranges due to Yu.L. Shmulyan [32] and similar results of L. de Branges and J. Rovnyak [12], to the case of closed densely defined operators between Hilbert spaces, cf. [14].

**Theorem 3.9.** *Let $T \in \mathcal{C}(\mathcal{G}, \mathcal{H})$ be nonzero and $u \in \mathcal{H}$. Then $u \in \operatorname{Ran}(T)$ if and only if there exists $\mu_u \geq 0$ such that $|\langle u, v\rangle_{\mathcal{H}}| \leq \mu_u \|T^* v\|_{\mathcal{G}}$ for all $v \in \operatorname{Dom}(T^*)$. Moreover, if $u \in \operatorname{Ran}(T)$ then*

$$\|u\|_T = \sup\Big\{\frac{|\langle u, v\rangle_{\mathcal{H}}|}{\|T^* v\|_{\mathcal{G}}} \mid v \in \operatorname{Dom}(T^*),\ T^* v \neq 0\Big\},$$

*where $\|\cdot\|_T$ is the norm associated to the inner product defined as in* (3.6).

Let us observe that the definition of closely embedded Hilbert spaces is consistent with the model $\mathcal{R}(T)$, for $T \in \mathcal{C}(\mathcal{G}, \mathcal{H})$, more precisely, if $\mathcal{H}_+$ is closely embedded in $\mathcal{H}$ then $\mathcal{R}(j_+) = \mathcal{H}_+$ and $\|x\|_+ = \|x\|_{j_+}$.

The model for the abstract definition of closely embedded Hilbert spaces follows the results on the Hilbert space $\mathcal{R}(T)$. Thus, if $T \in \mathcal{C}(\mathcal{G}, \mathcal{H})$ then the Hilbert space $\mathcal{R}(T)$, with its canonical embedding $j_T$ as defined in (3.6) and (3.7), is a Hilbert space closely embedded in $\mathcal{H}$, e.g. by Lemma 3.7. Conversely, if $\mathcal{H}_+$ is a Hilbert space closely embedded in $\mathcal{H}$, and $j_+$ denotes its canonical closed embedding, then $\mathcal{H}_+$ can be naturally viewed as the Hilbert space of type $\mathcal{R}(j_+)$. This fact is actually more general.

**Proposition 3.10.** *Let $T \in \mathcal{C}(\mathcal{G}, \mathcal{H})$ and consider the Hilbert space $\mathcal{R}(T)$ closely embedded in $\mathcal{H}$, with its canonical closed embedding $j_T$. Then $TT^* = j_T j_T^*$.*

As in the case of continuous embeddings, one can prove that Hilbert spaces that are closely embedded in a given Hilbert space are uniquely determined by their kernel operators, but the uniqueness takes a slightly weaker form. This is illustrated by the following theorem.

**Theorem 3.11.** *Let $\mathcal{H}_+$ be a Hilbert space closely embedded in $\mathcal{H}$, with $j_+ : \mathcal{H}_+ \to \mathcal{H}$ its densely defined and closed embedding operator, and let $A = j_+ j_+^*$ be the kernel operator of $\mathcal{H}_+$. Then*

(a) $\operatorname{Ran}(A^{1/2}) = \operatorname{Dom}(j_+)$ *is dense in both $\mathcal{R}(A^{1/2})$ and $\mathcal{H}_+$.*

(b) *For all $x \in \operatorname{Ran}(A^{1/2})$ and all $y \in \operatorname{Dom}(A)$ we have $\langle x, y\rangle_{\mathcal{H}} = \langle x, Ay\rangle_+ = \langle x, Ay\rangle_{A^{1/2}}$.*

(c) $\operatorname{Ran}(A)$ *is dense in both $\mathcal{R}(A^{1/2})$ and $\mathcal{H}_+$.*

(d) *For any $x \in \operatorname{Dom}(j_+)$ we have*

$$\|x\|_+ = \sup\Big\{\frac{|\langle x, y\rangle_{\mathcal{H}}|}{\|A^{1/2} y\|_{\mathcal{H}}} \mid y \in \operatorname{Dom}(A^{1/2}),\ A^{1/2} y \neq 0\Big\}.$$

(e) *The identity operator* : $\operatorname{Ran}(A))(\subseteq \mathcal{R}(A^{1/2})) \to \mathcal{H}_+$ *uniquely extends to a unitary operator $V : \mathcal{R}(A^{1/2}) \to \mathcal{H}_+$ such that $VAx = j_+^* x$, for all $x \in \operatorname{Dom}(A)$.*

## 4. A Model of a Triplet of Closely Embedded Hilbert Spaces

In this section we develop a construction of a chain of two closed embeddings with certain duality properties related to a given positive selfadjoint operator with trivial null space, as a generalization of the classical notion of a triplet of Hilbert spaces. This construction will lead us to the axiomatization of triplets of closely embedded Hilbert spaces and will be essential in applications. Let $\mathcal{H}$ be a Hilbert space and $H$ a positive selfadjoint operator in $\mathcal{H}$, that we call the Hamiltonian. We assume that $H$ has trivial null space. Let $\mathcal{G}$ be another Hilbert space and let $T \in \mathcal{C}(\mathcal{H}, \mathcal{G})$ be such that it provides a factorization of the Hamiltonian

$$H = T^*T. \tag{4.1}$$

Then $T$ has trivial null space as well, and let $T^{-1}$ denote the algebraic inverse operator of $T$, that is, $\mathrm{Dom}(T^{-1}) = \mathrm{Ran}(T)$. We consider the Hilbert space $\mathcal{D}(T)$ as described in Subsection 3.1, more precisely, in our special case $\mathcal{D}(T)$ is the Hilbert space completion of $\mathrm{Dom}(T)$ with respect to the quadratic norm $|\cdot|_T$ defined as in (3.1), and the associated inner product $(\cdot,\cdot)_T$. The closed embedding $i_T$, defined as in (3.3), has domain $\mathrm{Dom}(T)$ dense in $\mathcal{D}(T)$ and range in $\mathcal{H}$. Observe that, without loss of generality, we can assume that $T$ has dense range (otherwise, replace $\mathcal{G}$ by the closure of $\mathrm{Ran}(T)$). For example, all these assumptions are met when $T = H^{1/2}$, and uniqueness modulo unitary equivalence holds as well, but having in mind future applications we want to keep this level of generality.

Throughout this section we keep the following two assumptions on $T$: $\mathrm{Null}(T) = \{0\}$ *and* $\mathrm{Ran}(T)$ *is dense in* $\mathcal{G}$. As mentioned in Subsection 3.1, the kernel operator $A$ of the closed embedding $i_T$ is a positive selfadjoint operator in $\mathcal{H}$

$$A = i_T i_T^* = j_{T^{-1}} j_{T^{-1}}^* = T^{-1} T^{-1*} = (T^*T)^{-1} \tag{4.2}$$

hence, in accordance with (4.1), $H = T^*T = A^{-1}$; the kernel operator is the inverse of the Hamiltonian, in the sense of one-to-one unbounded operators.

In the following we use Lemma 3.5. Thus, we have the coisometry $V_T \in \mathcal{B}(\mathcal{G}, \mathcal{D}(T))$, uniquely determined such that $T^{-1} = i_T V_T$ and $\mathrm{Null}(V_T) = \mathcal{G} \ominus \mathrm{Ran}(T)$. Due to our assumption that $\mathrm{Ran}(T)$ is dense in $\mathcal{G}$, the operator $V_T$ is actually unitary. Similarly, there exists a coisometry $U_{T^*} \in \mathcal{B}(\mathcal{G}, \mathcal{R}(T^*)$ such that $T^* = j_{T^*} U_{T^*}$, uniquely determined by the property $\mathrm{Null}(U_{T^*}) = \mathrm{Null}(T^*)$. Again, since $\mathrm{Ran}(T)$ is supposed to be dense in $\mathcal{G}$, it follows that $U_{T^*}$ is actually unitary.

The kernel operator $B$ of the closed embedding of $\mathcal{H}$ in $\mathcal{R}(T^*)$ is

$$B = j_{T^*}^{-1} j_{T^*}^{-1*} = (j_{T^*}^* j_{T^*})^{-1}. \tag{4.3}$$

On the other hand, since $T^* = j_{T^*} U_{T^*}$, where $U_{T^*} \colon \mathcal{G} \to \mathcal{R}(T^*)$ is unitary, it follows that

$$TT^* = U_{T^*}^* j_{T^*}^* j_{T^*} U_{T^*},$$

which, when combined with (4.3), shows that

$$(TT^*)^{-1} = U_{T^*}^* (j_{T^*}^* j_{T^*})^{-1} U_{T^*} = U_{T^*}^* B U_{T^*}. \tag{4.4}$$

Since, via the polar decomposition for the closed densely defined operator $T$, the operators $TT^*$ and $T^*T$ are unitary equivalent, from (4.2) and (4.4) it follows that the two kernel operators $A$ and $B$ are unitary equivalent.

Further on, consider the unitary operator $U_{T^*}V_T^{-1}$, acting between $\mathcal{D}(T)$ and $\mathcal{R}(T^*)$, and denote this operator by $\widetilde{H}$. Then, $\widetilde{H}$ is an extension of the Hamiltonian operator $H$ and its inverse, that we denote by $\widetilde{A}$, is an extension of the kernel operator $A$. Indeed, this follows from the fact that $T^*T = j_{T^*}U_{T^*}V_T^{-1}i_T^{-1}$, and then taking into account of (4.1), and the fact that both $j_{T^*}$ and $i_T$ are closed embeddings.

Let us observe now that the kernel operator can be viewed as an operator acting from $\mathcal{R}(T^*)$ and valued in $\mathcal{D}(T)$. Indeed, taking into account (4.2), $\mathrm{Dom}(A) = \mathrm{Dom}(i_T i_T^*) \subseteq \mathrm{Ran}(T^*) \subseteq \mathcal{R}(T^*)$ and $\mathrm{Ran}(A) \subseteq \mathrm{Dom}(T) \subseteq \mathcal{D}(T)$. Since $H = A^{-1}$, it follows that the Hamiltonian operator $H$ can be viewed as acting from $\mathcal{D}(T)$ and valued in $\mathcal{R}(T^*)$.

In the following we show that the operator $H$, when viewed as an operator acting from $\mathcal{D}(T)$ and valued in $\mathcal{R}(T^*)$, is densely defined and has dense range. Indeed, in order to prove that the domain of $H$ is dense in $\mathcal{D}(T)$ it is sufficient (actually, equivalent) to proving that $\mathrm{Ran}(A)$ is dense in $\mathcal{D}(T)$. To see this, let $x \in \mathcal{D}(T)$ be such that $(x, Ay)_T = 0$ for all $y \in \mathrm{Dom}(A)$. We first prove that $(x, i_T^*)_T = 0$ for all $y \in \mathrm{Dom}(i_T^*)$. Indeed, since $A = i_T i_T^*$, it follows that $\mathrm{Dom}(A)$ is a core for $i_T^*$, hence, for any $y \in \mathrm{Dom}(i_T^*)$ there exists a sequence $(y_n)$ of vectors in $\mathrm{Dom}(A)$ such that $\|y_n - y\|_{\mathcal{H}} \to 0$ and $|i_T^* y - i_T^* y_n|_T \to 0$ as $n \to \infty$. Consequently, $0 = (x, Ay_n)_T = (x, i_T^* y_n)_T \to (x, i_T^* y)_T$ as $n \to \infty$, hence $(x, i_T^* y)_T = 0$. Since $y$ is arbitrary in $\mathrm{Dom}(i_T^*)$ and $\mathrm{Ran}(i_T^*)$ is dense in $\mathcal{D}_T$, it follows that $x = 0$. Thus, $\mathrm{Ran}(A) = \mathrm{Dom}(H)$ is dense in $\mathcal{D}(T)$. In a completely similar fashion, by using $j_{T^*}$ instead of $i_T$ and taking into account that $H = T^*T$, we prove that $\mathrm{Ran}(H)$ is dense in $\mathcal{R}(T^*)$.

The construction we got so far can be visualized by the compound diagram in Figure 1, where all the triangular diagrams are commutative, by definition, while the rectangular diagram is commutative in the weaker sense $j_{T^*}\widetilde{H} \supseteq H i_T$.

$$
\begin{tikzcd}
\mathcal{G} \arrow[d, "V_T"'] & & \mathcal{G} \arrow[dl, "T^*"'] \arrow[d, "U_{T^*}"] & \\
\mathcal{D}(T) \arrow[r, "i_T"] & \mathcal{H} \arrow[ul, "T"'] \arrow[r, shift left, "j_{T^*}^{-1}"] \arrow[dl, "i_T^*"'] \arrow[d, shift right, "A"'] & \mathcal{R}(T^*) \arrow[l, shift left, "j_{T^*}"] \arrow[d, shift right, "\widetilde{A}"'] & \\
\mathcal{D}(T) \arrow[r, "i_T"'] & \mathcal{H} \arrow[u, shift right, "H=A^{-1}"'] \arrow[r, shift left, "i_T^{-1}"] & \mathcal{D}(T) \arrow[l, shift left, "i_T"] \arrow[u, shift right, "\widetilde{H}"'] & \mathcal{G} \arrow[l, "V_T"'] \arrow[ul, "U_{T^*}"']
\end{tikzcd}
$$

Figure 1.

Let us observe now that, as a consequence of Theorem 3.9 when applied to $T^*$ instead of $T$, for all $y \in \mathrm{Dom}(T^*)$ we have the following variational formula

$$
\|y\|_{T^*} = \sup\Big\{ \frac{|\langle y, x\rangle_{\mathcal{H}}|}{|x|_T} \mid x \in \mathrm{Dom}(T) \setminus \{0\}\Big\}. \tag{4.5}
$$

Finally, we show that there is a canonical identification of $\mathcal{R}(T^*)$ with the conjugate dual space $\mathcal{D}(T)^*$. To see this, we define a linear operator

$$
\Theta \colon \mathcal{R}(T^*) \to \mathcal{D}(T)^*, \quad (\Theta\alpha)(x) := (\widetilde{A}\alpha, x)_T, \quad \alpha \in \mathcal{R}(T^*),\ x \in \mathcal{D}(T), \tag{4.6}
$$

and, taking into account that $\widetilde{A}$ is unitary it follows that $\Theta$ is unitary as well.

We summarize all the previous constructions and facts in the following

**Theorem 4.1.** *Let $H$ be a positive selfadjoint operator in the Hilbert space $\mathcal{H}$, with trivial null space. Let $T \in \mathcal{C}(\mathcal{H}, \mathcal{G})$ be such that* $\operatorname{Ran}(T)$ *is dense in $\mathcal{G}$ and $H = T^*T$. Then:*

(i) *The Hilbert space $\mathcal{D}(T)$ is closely embedded in $\mathcal{H}$ with its closed embedding $i_T$ having range dense in $\mathcal{H}$, and its kernel operator $A = i_T i_T^*$ coincides with $H^{-1}$.*
(ii) *$\mathcal{H}$ is closely embedded in the Hilbert space $\mathcal{R}(T^*)$ with its closed embedding $j_{T^*}^{-1}$ having range dense in $\mathcal{R}(T^*)$. The kernel operator $B = j_{T^*}^{-1} j_{T^*}^{-1*}$ of this closed embedding is unitary equivalent with $A = H^{-1}$.*
(iii) *The operator $i_T^* | \operatorname{Ran}(T^*)$ extends uniquely to a unitary operator $\widetilde{A}$ between the Hilbert spaces $\mathcal{R}(T^*)$ and $\mathcal{D}(T)$. In addition, $\widetilde{A}$ is the unique unitary extension of the kernel operator $A$, when viewed as an operator acting from $\mathcal{R}(T^*)$ and valued in $\mathcal{D}(T)$, as well.*
(iv) *The operator $H$ can be viewed as a linear operator with domain dense in $\mathcal{D}(T)$ and dense range in $\mathcal{R}(T^*)$, is isometric, extends uniquely to a unitary operator $\widetilde{H} \colon \mathcal{D}(T) \to \mathcal{R}(T^*)$, and $\widetilde{H} = \widetilde{A}^{-1}$.*
(v) *Letting $V_T \in \mathcal{B}(\mathcal{G}, \mathcal{D}_T)$ denote the unitary operator such that $T^{-1} = i_T V_T$ and $U_{T^*} \in \mathcal{B}(\mathcal{G}, \mathcal{R}(T^*))$ denote the unitary operator such that $T^* = U_{T^*} j_{T^*}$, we have $\widetilde{H} = U_{T^*} V_T^{-1}$.*
(vi) *The operator $\Theta$ defined by (4.6) provides a canonical identification of the Hilbert space $\mathcal{R}(T^*)$ with the conjugate dual space $\mathcal{D}(T)^*$ and, for all $y \in \operatorname{Dom}(T^*)$*

$$\|y\|_{T^*} = \sup\Big\{ \frac{|\langle y, x\rangle_{\mathcal{H}}|}{|x|_T} \mid x \in \operatorname{Dom}(T) \setminus \{0\} \Big\}.$$

## 5. Triplets of Closely Embedded Hilbert Spaces

In this section, we use the model obtained in Theorem 4.1 in order to derive an abstract definition for a triplet of closely embedded Hilbert spaces and then we approach existence, uniqueness, and other basic properties, as a left-right symmetry.

### 5.1. **Definition and Basic Properties.**

By definition, $(\mathcal{H}_+; \mathcal{H}_0; \mathcal{H}_-)$ is called a *triplet of closely embedded Hilbert spaces* if:

(th1) $\mathcal{H}_+$ is a Hilbert space closely embedded in the Hilbert space $\mathcal{H}_0$, with the closed embedding denoted by $j_+$, and such that $\operatorname{Ran}(j_+)$ is dense in $\mathcal{H}_0$.
(th2) $\mathcal{H}_0$ is closely embedded in the Hilbert space $\mathcal{H}_-$, with the closed embedding denoted by $j_-$, and such that $\operatorname{Ran}(j_-)$ is dense in $\mathcal{H}_-$.
(th3) $\operatorname{Dom}(j_+^*) \subseteq \operatorname{Dom}(j_-)$ and for every vector $y \in \operatorname{Dom}(j_-) \subseteq \mathcal{H}_0$ we have

$$\|y\|_- = \sup\Big\{ \frac{|\langle x, y\rangle_{\mathcal{H}_0}|}{\|x\|_+} \mid x \in \operatorname{Dom}(j_+),\ x \neq 0 \Big\}. \tag{5.1}$$

Let us first observe that, by (5.1) in axiom (th3), for all $y \in \operatorname{Dom}(j_-)$ and $x \in \operatorname{Dom}(j_+)$ we have $|\langle j_+ x, y\rangle_{\mathcal{H}_0}| = |\langle x, y\rangle_{\mathcal{H}_0}| \leq \|x\|_+ \|y\|_-$. By the definition of $\operatorname{Dom}(j_+^*)$ this means that $\operatorname{Dom}(j_-) \subseteq \operatorname{Dom}(j_+^*)$ hence, taking into account of $\operatorname{Dom}(j_+^*) \subseteq \operatorname{Dom}(j_-)$, the first condition in axiom (th3), it follows that actually

$$\operatorname{Dom}(j_+^*) = \operatorname{Dom}(j_-). \tag{5.2}$$

In the following we show that the axioms (th1)–(th3) are sufficient in order to obtain essentially all the properties that we get in Theorem 4.1. Given $(\mathcal{H}_+;\mathcal{H}_0;\mathcal{H}_-)$ a triplet of closely embedded Hilbert spaces and letting $j_\pm$ denote the closed embedding of $\mathcal{H}_+$ in $\mathcal{H}_0$ and, respectively, the closed embedding of $\mathcal{H}_0$ in $\mathcal{H}_-$, the operator $A = j_+j_+^*$ is positive selfadjoint in $\mathcal{H}_0$ and it is called the *kernel operator*. Also, since $\operatorname{Ran}(j_+)$ is dense in $\mathcal{H}_0$, it follows that $\operatorname{Ran}(A)$ is dense in $\mathcal{H}_0$ as well, equivalently $\operatorname{Null}(A) = \{0\}$. In particular, $H := A^{-1}$ is a positive selfadjoint operator in $\mathcal{H}_0$ and it is called the *Hamiltonian* of the triplet $(\mathcal{H}_+;\mathcal{H}_0;\mathcal{H}_-)$. Clearly, 0 is not an eigenvalue of $H$. In addition, let us observe that $\operatorname{Dom}(H) \subseteq \operatorname{Ran}(j_+) = \operatorname{Dom}(j_+) \subseteq \mathcal{H}_+$

Further on, for any $y \in \operatorname{Ran}(j_-)$, the linear functional $\mathcal{H}_+ \supseteq \operatorname{Ran}(j_+) \ni x \mapsto \langle x, y\rangle_{\mathcal{H}_0} \in \mathbb{C}$ is bounded and hence, via the Riesz Representation Theorem, there exists uniquely $z_y \in \mathcal{H}_+$ such that $\langle x, y\rangle_{\mathcal{H}_0} = \langle x, z_y\rangle_{\mathcal{H}_+}$ for all $x \in \operatorname{Ran}(j_+) = \operatorname{Dom}(j_+)$, and $\|z_y\|_+ = \|y\|_-$. Thus, a linear operator $V\colon \operatorname{Dom}(j_-)(\subseteq \mathcal{H}_-) \to \mathcal{H}_+$ is uniquely defined by $Vy = z_y$, and it is isometric, in particular it is extended uniquely to an isometry $\widetilde{V}\colon \mathcal{H}_- \to \mathcal{H}_+$. In addition, for all $x \in \operatorname{Dom}(j_+) = \operatorname{Ran}(j_+)$ and all $y \in \operatorname{Dom}(j_-) = \operatorname{Ran}(j_-)$ we have

$$\langle j_+x, y\rangle_{\mathcal{H}_0} = \langle x, y\rangle_{\mathcal{H}_0} = \langle x, z_y\rangle_+ = \langle x, Vy\rangle_+,$$

that is, $V$ is $j_+^*$ when viewed as a linear operator from $\mathcal{H}_-$ and valued in $\mathcal{H}_+$. Consequently, $\operatorname{Ran}(V) \supseteq \operatorname{Ran}(j_+^*)$, which is dense in $\mathcal{H}_+$. Thus, we have shown that the isometric operator $\widetilde{V}$ is actually unitary $\mathcal{H}_- \to \mathcal{H}_+$.

We observe that the kernel operator $A$ can be viewed also as acting from $\mathcal{H}_-$ and valued in $\mathcal{H}_+$. Indeed, $A = j_+j_+^*$, hence $\operatorname{Dom}(A) \subseteq \operatorname{Dom}(j_+^*) = \operatorname{Dom}(j_-) \subseteq \mathcal{H}_-$ and, clearly, $\operatorname{Ran}(A) \subseteq \operatorname{Ran}(j_+) \subseteq \mathcal{H}_+$. On the other hand, for any $y \in \operatorname{Dom}(A) \subseteq \mathcal{H}_-$ and any $x \in \operatorname{Dom}(j_+) \subseteq \mathcal{H}_+$ we have $\langle Ay, x\rangle_+ = \langle j_+j_+^*y, x\rangle_+ = \langle j_+^*y, x\rangle_+$, hence $A$ is a restriction of the operator $V$ defined before.

In the following we prove that $\operatorname{Ran}(A)$ is dense in $\mathcal{H}_+$. To see this, let $x \in \mathcal{H}_+$ be such that $\langle x, Ay\rangle_+ = 0$ for all $y \in \operatorname{Dom}(A)$. We claim that $\langle x, j_+^*y\rangle_+ = 0$ for all $y \in \operatorname{Dom}(j_+^*)$. Indeed, since $\operatorname{Dom}(j_+^*)$ is a core for $A$, it follows that for any $y \in \operatorname{Dom}(j_+^*)$ there exists a sequence $(y_n)$ of vectors in $\operatorname{Dom}(A)$ such that $\|y_n - y\|_{\mathcal{H}_0} \to 0$ and $\|j_+^*y_n - j_+^*y\|_+ \to 0$ as $n \to \infty$, hence $0 = \langle x, Ay\rangle_+ = \langle x, j_+^*y_n\rangle_+ \to \langle x, j_+^*y\rangle_+$ as $n \to \infty$. Taking into account that the range of $V = j_+^*$, considered as an operator from $\mathcal{H}_-$ to $\mathcal{H}_+$, is dense in $\mathcal{H}_+$, it follows that $x = 0$. Thus, we conclude that $\operatorname{Ran}(A)$ is dense in $\mathcal{H}_+$.

In a similar fashion we can prove that $\operatorname{Dom}(A)$ is dense in $\mathcal{H}_-$. Since $A$, when viewed as a linear operator from $\mathcal{H}_-$ to $\mathcal{H}_+$, is a restriction of the operator $V$ (formally the same with $j_+^*$) which is isometric, it follows that the linear operator $A$, when viewed as a linear operator from $\mathcal{H}_-$ to $\mathcal{H}_+$, is isometric and that it has a unique unitary extension $\widetilde{A}\colon \mathcal{H}_- \to \mathcal{H}_+$, which is exactly $\widetilde{V}$.

Similarly, the Hamiltonian operator can be viewed as a linear operator densely defined in $\mathcal{H}_+$ and with range in $\mathcal{H}_-$: recall that $\operatorname{Dom}(j_+^*) = \operatorname{Dom}(j_-)$ and hence that it is a subspace of $\mathcal{H}_-$. Since $H = A^{-1}$, it follows that $H$ is a restriction of $V^{-1}$, it is isometric, with domain dense in $\mathcal{H}_+$ and range dense in $\mathcal{H}_-$, hence it has a unique unitary extension $\widetilde{H} = \widetilde{A}^{-1} = \widetilde{V}^{-1}\colon \mathcal{H}_+ \to \mathcal{H}_-$.

For a better understanding of all these proven facts we depict these constructions by the following diagram:

$$\begin{tikzcd}
& \mathrm{Dom}(j_+^*) \arrow[dl, "V"'] \arrow[d, hook] \arrow[dr, hook] & \\
\mathcal{H}_+ \arrow[r, "j_+"] & \mathcal{H}_0 \arrow[r, shift left, "j_-"] \arrow[d, shift right, "A"'] \arrow[dl, "j_+^{-1}"'] & \mathcal{H}_- \arrow[l, shift left, "j_-^{-1}"] \arrow[d, shift right, "\widetilde{V}=\widetilde{A}"'] \\
\mathcal{H}_+ \arrow[r, "j_+"'] & \mathcal{H}_0 \arrow[u, shift right, "H=A^{-1}"'] \arrow[r, shift left, "j_+^{-1}"] & \mathcal{H}_+ \arrow[l, shift left, "j_+"] \arrow[u, shift right, "\widetilde{H}=\widetilde{A}^{-1}"']
\end{tikzcd}$$

Figure 2.

In Figure 2, all the triangular diagrams are commutative, by definition. The lower right rectangular diagram is commutative in a weaker sense, namely $j_-H \subseteq \widetilde{H} j_+^{-1}$.

Finally, we show that there exists a natural identification of $\mathcal{H}_-$ with the conjugate dual space of $\mathcal{H}_+$, more precisely, we consider the operator $\Theta\colon \mathcal{H}_- \to \mathcal{H}_+^*$ defined by

$$(\Theta y)(x) := \langle \widetilde{V} y, x\rangle_+, \quad y \in \mathcal{H}_-,\ x \in \mathcal{H}_+.$$

To see this, note that for any $l \in \mathcal{H}_+^*$ there exists uniquely $z \in \mathcal{H}_+$ such that $l(x) = \langle z, x\rangle_+$, for all $x \in \mathcal{H}_+$. Letting $y = \widetilde{V}^{-1} z \in \mathcal{H}_-$ it follows

$$l(x) = \langle z, x\rangle_+ = \langle \widetilde{V} y, x\rangle_+ = (\Theta y)(x), \quad x \in \mathcal{H}_+.$$

Thus, $\Theta$ is surjective. In addition, with the notation as before, we have

$$\|\Theta y\| = \|\widetilde{V} y\|_+ = \|y\|_-, \quad y \in \mathcal{H}_-,$$

hence $\Theta$ is unitary, as claimed.

We gather all these proven facts in the following

**Theorem 5.1.** *Let $(\mathcal{H}_+; \mathcal{H}_0; \mathcal{H}_-)$ be a triplet of closely embedded Hilbert spaces, and let $j_\pm$ denote the corresponding closed embeddings of $\mathcal{H}_+$ in $\mathcal{H}_0$ and, respectively, of $\mathcal{H}_0$ in $\mathcal{H}_-$. Then:*

(a) *The kernel operator $A = j_+ j_+^*$ is positive selfadjoint in $\mathcal{H}_0$ and $0$ is not an eigenvalue for $A$. Also, the Hamiltonian operator $H = A^{-1}$ is a positive selfadjoint operator in $\mathcal{H}_0$ for which $0$ is not an eigenvalue.*

(b) $\mathrm{Dom}(j_+^*) = \mathrm{Dom}(j_-)$*, the closed embeddings $j_+$ and $j_-$ are simultaneously continuous or not, and the operator $V = j_+^*\colon \mathrm{Dom}(j_+^*)(\subseteq \mathcal{H}_-) \to \mathcal{H}_+$ extends uniquely to a unitary operator $\widetilde{V}\colon \mathcal{H}_- \to \mathcal{H}_+$.*

(c) *The kernel operator $A$ can be viewed as an operator densely defined in $\mathcal{H}_-$ with dense range in $\mathcal{H}_+$, and it is a restriction of the unitary operator $\widetilde{V}$.*

(d) *The Hamiltonian operator $H$ can be viewed as an operator densely defined in $\mathcal{H}_+$ with range dense in $\mathcal{H}_-$, and it is uniquely extended to a unitary operator $\widetilde{H}\colon \mathcal{H}_+ \to \mathcal{H}_-$, and $\widetilde{H} = \widetilde{V}^{-1}$.*

(e) *The operator $\Theta$ defined by $(\Theta y)(x) = \langle \widetilde{V} y, x\rangle_+$, for all $y \in \mathcal{H}_-$ and all $x \in \mathcal{H}_+$ provides a unitary identification of $\mathcal{H}_-$ with the conjugate dual space $\mathcal{H}_+^*$.*

5.2. **Existence and Uniqueness.** We can now approach questions related to existence and uniqueness of triplets of closely embedded Hilbert spaces, similar to results known for the classical triplets of Hilbert spaces, cf. [6]. First we show that, in a triplet of closely embedded Hilbert spaces $(\mathcal{H}_+;\mathcal{H}_0;\mathcal{H}_-)$, the essential part, in a weaker sense, is the left-hand one, that is, the closed embedding of $\mathcal{H}_+$ into $\mathcal{H}_0$.

**Theorem 5.2.** *Assume that $\mathcal{H}_0$ and $\mathcal{H}_+$ are two Hilbert spaces such that $\mathcal{H}_+$ is closely embedded in $\mathcal{H}_0$, with $j_+$ denoting this closed embedding, and such that* $\operatorname{Ran}(j_+)$ *is dense in $\mathcal{H}_0$.*

(1) *One can always extend this closed embedding to the triplet $(\mathcal{H}_+;\mathcal{H}_0;\mathcal{R}(j_+^{-1*}))$ of closely embedded Hilbert spaces.*

(2) *Let $(\mathcal{H}_+;\mathcal{H}_0;\mathcal{H}_-)$ be any other extension of the closed embedding $j_+$ to a triplet of closely embedded Hilbert spaces, let $A = j_+j_+^*$ be its kernel operator, and let $j_-$ denote the closed embedding of $\mathcal{H}_0$ in $\mathcal{H}_-$. Then, there exists a unique unitary operator $\Phi_-\colon \mathcal{H}_- \to \mathcal{R}(j_+^*)$ such that when restricted to* $\operatorname{Dom}(j_-)$ *acts as the identity operator.*

*Proof.* (1) Indeed, the kernel operator $A = j_+j_+^*$ of $\mathcal{H}_+$ is a positive selfadjoint operator in $\mathcal{H}_0$ and it is one-to-one, since $\operatorname{Ran}(j_+)$ is supposed to be dense in $\mathcal{H}_0$. Then $H = A^{-1}$ is a one-to-one positive selfadjoint operator in $\mathcal{H}_0$ and letting $T = j_+^{-1}$ we have $H = T^*T$, with $T$ closed, densely defined, and one-to-one, as an operator from $\mathcal{H}_0$ into $\mathcal{H}_+$. Then we apply Theorem 4.1, more precisely, we define $\mathcal{H}_- = \mathcal{R}(T^*) = \mathcal{R}(j_+^{-1*})$.

(2) Since $\operatorname{Dom}(j_+^*) = \operatorname{Dom}(j_-)$ we can use the operators $\widetilde{V}$ in Theorem 4.1 and Theorem 5.1 to prove that the identity operator on $\operatorname{Dom}(j_-)$ when viewed as a linear operator from $\mathcal{H}_-$ and with range in $\mathcal{R}(j_+^*)$ extends uniquely to a unitary operator. $\square$

As a consequence of the previous theorem we can prove that the concept of triplet of Hilbert spaces with closed embeddings has a certain "left-right" symmetry, which, in general, the classical triplets of Hilbert spaces do not share.

**Proposition 5.3.** *Let $(\mathcal{H}_+;\mathcal{H}_0;\mathcal{H}_-)$ be a triplet of closely embedded Hilbert spaces. Then $(\mathcal{H}_-;\mathcal{H}_0;\mathcal{H}_+)$ is also a triplet of closely embedded Hilbert spaces, more precisely:*

(1) *If $j_+$ and $j_-$ denote the closed embeddings of $\mathcal{H}_+$ in $\mathcal{H}_0$ and, respectively, of $\mathcal{H}_0$ in $\mathcal{H}_-$, then $j_-^{-1}$ and $j_+^{-1}$ are the closed embeddings of $\mathcal{H}_-$ in $\mathcal{H}_0$ and, respectively, of $\mathcal{H}_0$ in $\mathcal{H}_+$.*

(2) *If $H$ and $A$ denote the Hamiltonian, respectively, the kernel operator of the triplet $(\mathcal{H}_+;\mathcal{H}_0;\mathcal{H}_-)$, then $A$ and $H$ are the Hamiltonian and, respectively, the kernel operator of the triplet $(\mathcal{H}_-;\mathcal{H}_0;\mathcal{H}_+)$.*

*Proof.* We first prove the statement assuming that the given triplet is in the model form, that is, for some Hilbert space $\mathcal{G}$ and some operator $T \in \mathcal{C}(\mathcal{H}_0,\mathcal{G})$ that is one-to-one and has dense range, we have $\mathcal{H}_+ = \mathcal{D}(T)$ and $\mathcal{H}_- = \mathcal{R}(T^*)$, with the closed embeddings $j_+ = i_T$ and, respectively, $j_- = j_{T^*}^{-1}$, as in Section 4. Then, observe that $S = T^{*-1} \in \mathcal{C}(\mathcal{H}_0,\mathcal{G})$ is one-to-one and has dense range and that, inspecting the corresponding constructions in subsections 3.1 and 3.2, we have $\mathcal{D}(S) = \mathcal{R}(T^*) = \mathcal{H}_-$ and $\mathcal{R}(S^*) = \mathcal{D}(T) = \mathcal{H}_+$. By Theorem 4.1 it follows that $(\mathcal{H}_-;\mathcal{H}_0;\mathcal{H}_+)$ is now a triplet of closely embedded Hilbert spaces as well, with closed embeddings $j_{T^*}$ and, respectively, $i_T^{-1}$. Thus, assertion (1) is proven, in this special case.

In order to prove assertion (2), note that, by Proposition 3.10, we have $j_{T^*} j_{T^*}^* = T^* T = H$, hence $H$ is the kernel operator of the triplet $(\mathcal{H}_-; \mathcal{H}_0; \mathcal{H}_+)$, and then $A$ becomes its Hamiltonian operator.

The general case now follows from assertion (2) in Theorem 5.2 that shows that, without loss of generality, we can assume that $\mathcal{H}_+ = \mathcal{D}(T)$ and $\mathcal{H}_- = \mathcal{R}(T^*)$ for some $T \in \mathcal{C}(\mathcal{H}_0, \mathcal{G})$ which is one-to-one and has dense range, more precisely, we can take $\mathcal{G} = \mathcal{H}_+$ and $T = j_+$, the closed embedding of $\mathcal{H}_+$ in $\mathcal{H}_0$. $\square$

We are now in a position to approach existence and uniqueness of triplets of Hilbert spaces in terms of a given Hamiltonian operator.

**Theorem 5.4.** *Let $H$ be an arbitrary positive selfadjoint operator in a Hilbert space $\mathcal{H}_0$ for which $0$ is not an eigenvalue.*

(1) *With notation as in subsections 3.1 and 3.2, $(\mathcal{D}(H^{1/2}); \mathcal{H}_0; \mathcal{R}(H^{1/2}))$ is a triplet of closely embedded Hilbert spaces such that $H$ is its Hamiltonian.*

(2) *Let $(\mathcal{H}_+; \mathcal{H}_0; \mathcal{H}_-)$ be any other triplet of closely embedded Hilbert spaces with the same Hamiltonian $H$. Then:*

(a) $\mathrm{Dom}(H^{1/2})$ *is dense in both $\mathcal{D}(H^{1/2})$ and $\mathcal{H}_+$.*
(b) *For any $x \in \mathrm{Dom}(H^{1/2})$ and any $y \in \mathrm{Dom}(H)$ we have $\langle x, Hy\rangle_{\mathcal{H}_0} = \langle x, y\rangle_+ = (x, y)_{H^{1/2}}$.*
(c) $\mathrm{Dom}(H)$ *is dense in both $\mathcal{D}(H^{1/2})$ and $\mathcal{H}_+$.*
(d) *For any $x \in \mathrm{Dom}(j_+) = \mathrm{Dom}(H^{1/2})$ we have*

$$\|x\|_+ = \sup\left\{ \frac{|\langle x, H^{1/2} z\rangle_{\mathcal{H}_0}|}{\|z\|_{\mathcal{H}_0}} \mid z \in \mathrm{Dom}(H^{1/2}) \right\}.$$

(e) *The identity operator* $: \mathrm{Dom}(H)(\subseteq \mathcal{D}(H^{1/2}) \to \mathcal{H}_+$ *extends uniquely to a unitary operator $\Phi_+ \colon \mathcal{D}(H^{1/2}) \to \mathcal{H}_+$ such that $\Phi_+ y = j_+^* H y$ for all $y \in \mathrm{Dom}(H)$.*

*Proof.* (1) Indeed, we can apply Theorem 4.1 to $T = H^{1/2}$, since $T$ is one-to-one as well.

(2) The argument is essentially contained in Theorem 3.11, only that this is rephrased in terms of the Hamiltonian $H$ instead of its inverse, the kernel operator $A$. $\square$

## 6. Weak Solutions for a Class of Dirichlet Problems

In this section we apply the abstract results on triplets of closely embedded Hilbert spaces to weak solutions for a Dirichlet problem associated to a class of degenerate elliptic partial differential equations. We briefly fix the notation and recall some of the underlying facts related to Sobolev spaces. Let $\Omega$ be an open (nonempty) set of the $N$-dimensional euclidean space $\mathbb{R}^N$. We use the notation $D_j = \mathrm{i}\frac{\partial}{\partial x_j}$, $(j = 1, \ldots, N)$ for the operators of differentiation with respect to the coordinates of points $x = (x_1, \ldots, x_N)$ in $\mathbb{R}^N$, and, for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}_+^N$, let $x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$, $D^\alpha = D_1^{\alpha_1} \cdots D_N^{\alpha_N}$. $\nabla_l = (D^\alpha)_{|\alpha|=l}$ denotes the gradient of order $l$, where $l$ is a fixed nonnegative integer. Denoting $m = m(N, l)$ to be the number of all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_N)$ such that $|\alpha| = \alpha_1 + \cdots + \alpha_N = l$, $\nabla_l$ can be viewed as an operator acting from $L_2(\Omega)$ into $L_2(\Omega; \mathbb{C}^m)$ defined on its maximal domain, the Sobolev space $W_2^l(\Omega)$, by

$$\nabla_l u = (D^\alpha u)_{|\alpha|=l}, \qquad u \in W_2^l(\Omega).$$

Recall that the Sobolev space $W_2^l(\Omega)$ consists of those functions $u \in L_2(\Omega)$ whose distributional derivatives $D^\alpha u$ belong to $L_2(\Omega)$ for all $\alpha \in \mathbb{Z}_+^N$, $|\alpha| \leq l$. Equipped with the norm

$$\|u\|_{W_2^l(\Omega)} = \Bigl( \sum_{|\alpha|\leq m} \|D^\alpha u\|^2_{L_2(\Omega)} \Bigr)^{1/2}, \tag{6.1}$$

$W_2^l(\Omega)$ becomes a Hilbert space that is continuously embedded in $L_2(\Omega)$. Also, recall that $\overset{\circ}{W}{}_2^{\,l}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in the space $W_2^l(\Omega)$. Besides, we will use the spaces $\overset{\circ}{L}{}_p^{\,l}(\Omega)$ (for $p = 1, 2$). The space $\overset{\circ}{L}{}_p^{\,l}(\Omega)$, $(1 \leq p < \infty)$ is defined as the completion of $C_0^\infty(\Omega)$ under the metric corresponding to

$$\|u\|_{p,l} := \|\nabla_l u\|_{L_p(\Omega)} = \Bigl( \int_\Omega \Bigl( \sum_{|\alpha|=l} |D^\alpha u(x)|^2 \Bigr)^{p/2} \mathrm{d}\, x \Bigr)^{1/p}, \quad u \in C_0^\infty(\Omega).$$

The elements of $\overset{\circ}{L}{}_p^{\,l}(\Omega)$ can be realized as locally integrable functions on $\Omega$ vanishing at the boundary $\partial\Omega$ and having distributional derivatives of order $l$ in $L_p(\Omega)$. Moreover, these functions, after modification on a set of zero measure, are absolutely continuous on every line which is parallel to the coordinate axes, cf. O. Nikodym [24], S.M. Nikolski [25] (see also V.M. Maz'ja [23]).

Further, suppose that on $\Omega$ there is defined an $m \times m$ matrix valued measurable function $a$, more precisely, $a(x) = [a_{\alpha\beta}(x)]$, $|\alpha|, |\beta| = l$, $x \in \Omega$, where the scalar valued functions $a_{\alpha,\beta}$ are measurable on $\Omega$ for all multi-indices $|\alpha|, |\beta| = l$. We impose the following conditions.

(C1) *For almost all (with respect to the $n$-dimensional standard Lebesgue measure) $x \in \Omega$, the matrix $a(x)$ is nonnegative (positive semidefinite), that is,*

$$\sum_{|\alpha|,|\beta|=l} a_{\alpha\beta}(x)\overline{\eta}_\beta \eta_\alpha \geq 0, \ \textit{for all } \eta = (\eta_\alpha)_{|\alpha|=l} \in \mathbb{C}^m.$$

According to the condition (C1), there exists an $m \times m$ matrix valued measurable function $b$ on $\Omega$, such that

$$a(x) = b(x)^* b(x), \ \text{for almost all } x \in \Omega,$$

where $b(x)^*$ denotes the Hermitian conjugate matrix of the matrix $b(x)$. Here and hereafter, it is convenient to consider $m \times m$ matrices as linear transformations in $\mathbb{C}^m$. Also, $|\cdot|$ denotes the unitary norm (the $\ell_2$ norm) in $\mathbb{C}^m$.

(C2) *There is a nonnegative measurable function $c$ on $\Omega$ such that, for almost all $x \in \Omega$ and all $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{C}^N$,*

$$|b(x)\widetilde{\xi}| \geq c(x)|\tilde{\xi}|,$$

*where $\tilde{\xi} = (\xi^\alpha)_{|\alpha|=l}$ is the vector in $\mathbb{C}^m$ with $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_N^{\alpha_N}$.*

(C3) *All the entries $b_{\alpha\beta}$ of the $m \times m$ matrix valued function $b$ are functions in $L_{1,\mathrm{loc}}(\Omega)$.*

(C4) *The function $c$ in* (C2) *has the property that $1/c \in L_2(\Omega)$.*

Under the conditions (C1)–(C4), we consider the operator $T$ acting from $L_2(\Omega)$ to $L_2(\Omega; \mathbb{C}^m)$ and defined by

$$(Tu)(x) = b(x)\nabla_l u(x), \qquad \text{for almost all } x \in \Omega, \tag{6.2}$$

on its domain

$$\mathrm{Dom}(T) = \{u \in \overset{\circ}{W}{}_2^{\,l}(\Omega) \mid b\nabla_l u \in L_2(\Omega; \mathbb{C}^m)\}. \tag{6.3}$$

Our aim is to describe, in view of the abstract model proposed in Section 4, the triplet of closely embedded Hilbert spaces $(\mathcal{D}(T); L_2(\Omega); \mathcal{R}(T^*))$ associated with the operator $T$ defined at (6.2) and (6.3). In terms of these results, we obtain information about weak solutions for the corresponding operator equation involving the Hamiltonian operator $H = T^*T$ of the triplet, which in fact is a Dirichlet boundary value problem in $L_2(\Omega)$ with homogeneous boundary values. This problem is associated to the differential sesqui-linear form

$$\begin{aligned} a[u,v] &= \int_\Omega \langle a(x)\nabla_l(x), \nabla_l(x)\rangle \,\mathrm{d}\,x \\ &= \sum_{|\alpha|=|\beta|=l} \int_\Omega a_{\alpha\beta}(x) D^\beta u(x)\overline{D^\alpha v(x)} dx, \quad u, v \in C_0^\infty(\Omega), \end{aligned} \tag{6.4}$$

which, as will be seen, can be extended up to elements of $\mathcal{D}(T)$. The problem can be reformulated as follows : *given $f \in \mathcal{D}(T)^*$ (which is canonically identified withe $\mathcal{R}(T^*)$), find $v \in \mathcal{D}(T)$ such that*

$$a[u,v] = \langle u, f\rangle \text{ for all } u \in \mathcal{D}(T), \tag{6.5}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between $\mathcal{D}(T)$ and $\mathcal{D}(T)^*$. The problem in (6.5) can be considered only for $u \in \mathrm{Dom}(T)$, or, even more restrictively, only for $u \in C_0^\infty(\Omega)$.

We first prove a useful inequality.

**Lemma 6.1.** *Under the conditions* (C1) *through* (C4), *there holds the inequality*

$$\int_\Omega |\nabla_l u(x)| \,\mathrm{d}\,x \leq C\left(\int_\Omega |b(x)\nabla_l u(x)|^2 \,\mathrm{d}\,x\right)^{\frac{1}{2}}, \quad u \in C_0^\infty(\Omega), \tag{6.6}$$

*where*

$$C = \left(\int_\Omega c(x)^{-2} \,\mathrm{d}\,x\right)^{\frac{1}{2}}.$$

*Proof.* For any function $u \in C_0^\infty(\Omega)$, due to condition (C2), we have

$$|b(x)\nabla_l u(x)| \geq c(x)|\nabla_l u(x)|, \text{ for almost all } x \in \Omega.$$

Hence

$$|\nabla_l u(x)| \leq c(x)^{-1}|b(x)\nabla_l u(x)|, \text{ for almost all } x \in \Omega,$$

and then, integrating over $\Omega$ and then using Schwarz inequality, we obtain

$$\begin{aligned} \int_\Omega |\nabla_l u(x)| \,dx &\leq \int_\Omega c(x)^{-1}|b(x)\nabla_l u(x)| \,\mathrm{d}\,x \\ &\leq \left(\int_\Omega c(x)^{-2}dx\right)^{\frac{1}{2}}\left(\int_\Omega |b(x)\nabla_l u(x)|^2 \,\mathrm{d}\,x\right)^{\frac{1}{2}}, \end{aligned}$$

hence the inequality (6.6). ∎

Secondly we investigate the topological properties of the operator $T$.

**Lemma 6.2.** *Under the conditions* (C1)–(C4)*, the operator* $T$*, defined at* (6.2) *and* (6.3)*, is closed, densely defined, and injective.*

*Proof.* By (C3), all entries $b_{\alpha\beta}$ of $b$ are functions in $L_{1,\mathrm{loc}}(\Omega)$, therefore $C_0^\infty(\Omega) \subseteq \operatorname{Dom}(T)$, hence $T$ is densely defined. The injectivity of $T$ follows from the inequality (6.6) given in Lemma 6.1.

In order to prove that $T$ is closed, let $(u_n)$ be a sequence of elements $u_n \in \operatorname{Dom}(T)$, i.e. $u_n \in \overset{\circ}{W}{}_2^l(\Omega)$ for which $b\nabla_l u_n \in L_2(\Omega; \mathbb{C}^m)$, such that $Tu_n \to v$ in the norm of $L_2(\Omega; \mathbb{C}^m)$ and $u_n \to u$ in the norm of $L_2(\Omega)$. It follows that

$$\int_\Omega |b\nabla_l(u_n - u_k)|^2 \,\mathrm{d}\, x \to 0, \text{ as } n, k \to \infty$$

and, by Lemma 6.1,

$$\int_\Omega |\nabla_l(u_n - u_k)| \,\mathrm{d}\, x \to 0, \text{ as } n, k \to \infty$$

that is,

$$\|u_n - u_k\|_{\overset{\circ}{L}{}_1^l(\Omega)} \to 0 \text{ as } n, k \to \infty.$$

Since $\overset{\circ}{L}{}_1^l(\Omega)$ is a complete space and the gradient of functions in $\overset{\circ}{L}{}_1^l(\Omega)$, considered in the sense of distributions, coincides almost everywhere with the gradient considered in the sense of ordinary derivatives, see Theorem 1.1.3/1 in V.G. Maz'ja [23], it follows that there is an element $\widetilde{u} \in \overset{\circ}{L}{}_1^l(\Omega)$ such that

$$\int_\Omega |\nabla_l(u_n - \tilde{u})| \,\mathrm{d}\, x \to 0 \text{ as } n \to \infty.$$

Note also that, without loss of generality, we can assume that $u_n \to u$ pointwise almost everywhere on $\Omega$: otherwise, we may use a subsequence of $(u_n)$.

For any $\varphi \in C_0^\infty(\Omega)$ we have $u\overline{\nabla_l^*\varphi} \in L_1(\Omega)$, and then by the Dominated Convergence Theorem of Lebesgue, one gets

$$\begin{aligned}
\langle u, \nabla_l^*\varphi\rangle_{L_2(\Omega)} &= \int_\Omega u\overline{\nabla_l^*\varphi} \,\mathrm{d}\, x \\
&= \lim_{n\to\infty} \int_\Omega u_n \overline{\nabla_l^*\varphi} \,\mathrm{d}\, x = \lim_{n\to\infty} \int_\Omega \langle \nabla_l u_n, \varphi\rangle \,\mathrm{d}\, x \\
&= \int_\Omega \langle \nabla_l \widetilde{u}, \varphi\rangle \,\mathrm{d}\, x.
\end{aligned}$$

Therefore, $u \in \operatorname{Dom}(\nabla_l)$ and $\nabla_l u = \nabla_l \widetilde{u}$, hence

$$\int_\Omega \mid \nabla_l(u_n - u) \mid \mathrm{d}\, x \to 0 \text{ as } n \to \infty.$$

Moreover,

$$\|u - \widetilde{u}\|_{\overset{\circ}{L}{}_1^l(\Omega)} \leq \|u - u_n\|_{\overset{\circ}{L}{}_1^l(\Omega)} + \|u_n - \widetilde{u}\|_{\overset{\circ}{L}{}_1^l(\Omega)} \to 0,$$

so $u = \widetilde{u} \in \overset{\circ}{L}{}_1^l(\Omega)$.

Also, we have

$$\lim_{n\to\infty} \nabla_l u_n(x) = \nabla_l u(x), \qquad \text{for almost all } x \in \Omega$$

Then, by Fatou's Lemma,

$$\int_\Omega |b\nabla_l(u_n - u)|^2 \le \liminf_k \int_\Omega |b\nabla_l(u_n - u_k)|^2 dx \le \epsilon.$$

It follows that

$$b\nabla_l u = b\nabla_l(u - u_n) + b\nabla_l u_n \in L_2(\Omega; \mathbb{C}^m)$$

and

$$\int_\Omega |b\nabla_l(u_n - u)|^2 \to 0 \text{ as } n \to \infty.$$

Therefore $u \in \mathrm{Dom}(T)$, $v = b\nabla_l u$, i.e. $v = Tu$, and the closedness of $T$ is proven. □

As a consequence of Lemma 6.2, we can now apply Theorem 4.1 and the underlying constructions to the operator $T$. To this end, it will be convenient to consider $T$ as an operator acting from $L_2(\Omega)$ to the space obtained by the closure of $\mathrm{Ran}(T)$ in $L_2(\Omega; \mathbb{C}^m)$. Obviously, all properties in the previous lemma remain true for this restriction as well. We now follow the model space $\mathcal{D}(T)$ as in Subsection 3.1 and define

$$|u|_T := \left( \int_\Omega |b(x)\nabla_l u(x)|^2 \,\mathrm{d}\, x \right)^{\frac{1}{2}}, \; u \in \mathrm{Dom}(T). \tag{6.7}$$

Recall that $b$ is determined by $a(x) = b^*(x)b(x)$ for almost all $x \in \Omega$ and note that, due to the conditions (C2) through (C4), this is a pre-Hilbert norm on $\mathrm{Dom}(T)$. The corresponding inner product is given by,

$$(u, v)_T = \int_\Omega \langle b(x)\nabla_l u(x), b(x)\nabla_l v(x)\rangle \,\mathrm{d}\, x \tag{6.8}$$

for $u, v \in \mathrm{Dom}(T)$. Let $\mathcal{D}(T)$ denote the Hilbert space obtained by an abstract completion of $\mathrm{Dom}(T)$ with respect to the norm $|\cdot|_T$ defined at (6.7). In order to use efficiently this space, we have to choose a special representation of the space $\mathcal{D}(T)$ that can be realized inside the space $\overset{\circ}{L}{}_1^l(\Omega)$, with elements functions on $\Omega$.

**Proposition 6.3.** *The Hilbert space $\mathcal{D}(T)$ has a realization that is continuously embedded in $\overset{\circ}{L}{}_1^l(\Omega)$.*

*Proof.* Let $u$ be an arbitrary element of the space $\mathcal{D}(T)$. Then, there exists a sequence $(u_n)$, with all elements in $\mathrm{Dom}(T)$, such that

$$|u_n - u|_T \to 0 \text{ as } n \to \infty.$$

In particular,

$$|u_n - u_k|_T^2 = \int_\Omega |b(x)(\nabla_l(u_n - u_k)(x)|^2 \,\mathrm{d}\, x \to 0 \text{ as } n, k \to \infty.$$

In view of the inequality in Lemma 6.1, it follows that

$$\|u_n - u_k\|_{\overset{\circ}{L}{}_1^l(\Omega)} \to 0 \text{ as } n, k \to \infty.$$

Since $\overset{\circ}{L}{}_1^l(\Omega)$ is complete there exists a function $v \in \overset{\circ}{L}{}_1^l(\Omega)$ such that

$$\|u_n - v\|_{\overset{\circ}{L}{}_1^l(\Omega)} \to 0 \text{ as } n \to \infty.$$

The element $v$ depends only on $u$, more precisely, it is not depending on the chosen sequence $(u_n)$. Therefore, it can be defined an operator $J_a \colon \mathcal{D}(T) \to \overset{\circ}{L}{}_1^l(\Omega)$ by setting

$$J_a u = v, \quad u \in \mathcal{D}(T).$$

$J_a$ is an injective operator. To see this, if $J_a u = 0$, then for a suitable sequence $(u_n)$, $u_n \in \mathrm{Dom}(T)$, we have $| u_n - u |_T^2 \to 0$, and

$$\|u_n\|_{\overset{\circ}{L}{}_1^l(\Omega)} \sim \int_\Omega |\nabla_l u_n(x)| \,\mathrm{d}\, x \to 0 \text{ as } n \to \infty.$$

It can be assumed that $\nabla_l u_n \to 0$ almost everywhere, otherwise, we may pass to a subsequence of $(u_n)$. For any $\epsilon > 0$ and sufficiently large $n$ and $k$, there holds

$$|u_n - u_k|_T^2 = \int_\Omega |b\nabla_l(u_n - u_k)|^2 \,\mathrm{d}\, x < \epsilon,$$

and, by applying Fatou's Lemma,

$$\begin{aligned}\|u_n\|_T^2 &= \int_\Omega |b\nabla_l u_n|^2 \,\mathrm{d}\, x = \int_\Omega \lim_{k\to\infty} |b\nabla_l(u_n - u_k)|^2 \,\mathrm{d}\, x \\ &\leq \liminf_k \int_\Omega | \, b\nabla_l(u_n - u_k) \, |^2 \,\mathrm{d}\, x \leq \epsilon.\end{aligned}$$

Thus, $u_n \to 0$ in $\mathcal{D}(T)$ and hence $u = 0$. We conclude that the operator $J_a$ is injective, therefore the space $\mathcal{D}(T)$ can be realized by means of functions in $\overset{\circ}{L}{}_1^l(\Omega)$. Moreover, the embedding of $\mathcal{D}(T)$ into $\overset{\circ}{L}{}_1^l(\Omega)$ is continuous, that again is a consequence of the inequality in Lemma 6.1 which, obviously, can be extended for all $u \in \mathcal{D}(T)$. $\square$

As a consequence of Proposition 6.3, let $\overset{\circ}{\mathcal{H}}{}_a^l(\Omega)$ denote the concrete realization $\mathcal{D}(T)$ continuously embedded into $\overset{\circ}{L}{}_1^l(\Omega)$. Moreover, according to the assertions in items (i) and (ii) of Theorem 4.1, this space $\overset{\circ}{\mathcal{H}}{}_a^l(\Omega)$ is closely embedded in $L_2(\Omega)$ and, in turn, $L_2(\Omega)$ is closely embedded in the conjugate space $\big(\overset{\circ}{\mathcal{H}}{}_a^l(a)\big)^*$, that we denote by $\overset{\circ}{\mathcal{H}}{}_a^{-l}(\Omega)$. Moreover, $(\overset{\circ}{\mathcal{H}}{}_a^l(\Omega);\ L_2(\Omega);\ \overset{\circ}{\mathcal{H}}{}_a^{-l}(\Omega))$ is a triplet of closely embedded Hilbert spaces in the sense of the definition as in Subsection 5.1.

Further on, by Theorem 4.1 (vi), the conjugate space of $\overset{\circ}{\mathcal{H}}{}_a^l(\Omega)$, that is, $\overset{\circ}{\mathcal{H}}{}_a^{-l}(\Omega)$, is canonically identified with $\mathcal{R}(T^*)$. In general, this is not a space of distributions on $\Omega$. On the other hand, for every $f \in (\overset{\circ}{\mathcal{H}}{}_a^l(\Omega))^*$ there exist elements $g \in L_2(\Omega; \mathbb{C}^m)$ such that

$$f(u) = \int_\Omega \langle g(x), b(x)\nabla_l u(x)\rangle \,\mathrm{d}\, x, \quad u \in \overset{\circ}{W}{}_2^l(\Omega). \tag{6.9}$$

Moreover,

$$\|f\|_{(\overset{\circ}{\mathcal{H}}{}^{l}_a(\Omega))^*} = \inf\{\|g\|_{L_2(\Omega;\mathbb{C}^m)} \mid g \in L_2(\Omega;\mathbb{C}^m) \text{ such that (6.9) holds }\}.$$

The Hamiltonian $H = T^*T$ of the triplet can be viewed as an operator associated with the differential sesqui-linear form $a$ defined as in (6.4). We recall that $a$, on $C_0^\infty(\Omega)$, coincides with the inner product $(\cdot,\cdot)_T$, and hence $a$ can be extended on $\mathcal{D}(T)$ by

$$a[u,v] = (u,v)_T, \quad u,v \in \mathcal{D}(T).$$

On the other hand, due to Theorem 4.1 (iv), $H$ admits an extension to a unitary operator $\widetilde{H}$ acting between $\overset{\circ}{\mathcal{H}}{}^{l}_a(\Omega)$ and $\overset{\circ}{\mathcal{H}}{}^{-l}_a(\Omega)$. Therefore, the form $a$ extended on $\overset{\circ}{\mathcal{H}}{}^{l}_a(\Omega)$, is associated with $\widetilde{H}$. Consequently, the problem defined by (6.5) is equivalent with the operator equation

$$\widetilde{H}v = f, \quad f \in \overset{\circ}{\mathcal{H}}{}^{-l}_a(\Omega). \tag{6.10}$$

Thus, a solution of (6.4) is treated as a weak solution for (6.10). It is sufficient to verify (6.10) for $u \in \overset{\circ}{W}{}^{l}_a(\Omega)$ or on another dense subspace in $\overset{\circ}{\mathcal{H}}{}^{l}_a(\Omega)$ as, for instance, $C_0^\infty(\Omega)$.

The preceeding considerations can be summarized in the following

**Theorem 6.4.** *For $\Omega$ a domain in $\mathbb{R}^N$ and $l \in \mathbb{N}$, let $a(x) = [a_{\alpha\beta}(x)] = b(x)^*b(x)$, $|\alpha|,|\beta| = l$, $x \in \Omega$, satisfy the conditions* (C1)–(C4), *and consider the differential sesqui-linear form*

$$a[u,v] = \int_\Omega \langle a(x)\nabla_l(x), \nabla_l(x)\rangle \,\mathrm{d}\,x = \sum_{|\alpha|=|\beta|=l} \int_\Omega a_{\alpha\beta}(x) D^\beta u(x)\overline{D^\alpha v(x)} dx, \quad u,v \in C_0^\infty(\Omega),$$

*Then:*

(1) *The operator $T$ acting from $L_2(\Omega)$ to $L_2(\Omega;\mathbb{C}^m)$ and defined by $(Tu)(x) = b(x)\nabla_l u(x)$ for $x \in \Omega$ and $u \in \operatorname{Dom}(T) = \{u \in \overset{\circ}{W}{}^{l}_2(\Omega) \mid b\nabla_l u \in L_2(\Omega;\mathbb{C}^m)\}$ is closed, densely defined, and injective.*

(2) *The pre-Hilbert space $\operatorname{Dom}(T)$ with norm $|u|_T = (\int_\Omega |b(x)\nabla_l u(x)|^2 \,\mathrm{d}\,x)^{\frac{1}{2}}$, has a unique Hilbert space completion, denoted by $\mathcal{H}^l_a(\Omega)$, that is continuously embedded into $\overset{\circ}{L}{}^{l}_1(\Omega)$.*

(3) *The conjugate space of $\overset{\circ}{\mathcal{H}}{}^{l}_a(\Omega)$, denoted by $\overset{\circ}{\mathcal{H}}{}^{-l}_a(\Omega)$, can be realized in such a way that, for any $f \in \overset{\circ}{\mathcal{H}}{}^{-l}_a(\Omega)$ there exist elements $g \in L_2(\Omega;\mathbb{C}^m)$ such that*

$$f(u) = \int_\Omega \langle g(x), b(x)\nabla_l u(x)\rangle \,\mathrm{d}\,x, \quad u \in \overset{\circ}{W}{}^{l}_2(\Omega), \tag{6.11}$$

*and*

$$\|f\|_{\overset{\circ}{\mathcal{H}}{}^{-l}_a(\Omega)} = \inf\{\|g\|_{L_2(\Omega;\mathbb{C}^m)} \mid g \in L_2(\Omega;\mathbb{C}^m) \text{ such that (6.11) holds }\}.$$

(4) *$(\overset{\circ}{\mathcal{H}}{}^{l}_a(\Omega); L_2(\Omega); \overset{\circ}{\mathcal{H}}{}^{-l}_a(\Omega))$ is a triplet of closely embedded Hilbert spaces.*

(5) *For every $f \in \overset{\circ}{\mathcal{H}}{}^{-l}_a(\Omega)$ there exists a unique $v \in \mathcal{H}^l_a(\Omega)$ that solves the Dirichlet problem associated to the sesquilinear form $a$, in the sense that*

$$a[u,v] = \langle u, f\rangle \text{ for all } u \in \mathcal{H}^l_a(\Omega).$$

*More precisely, $v = \widetilde{H}^{-1} f$, where $\widetilde{H}$ is the unitary operator acting between $\overset{\circ}{\mathcal{H}}{}_a^{\,l}(\Omega)$ and $\overset{\circ}{\mathcal{H}}{}_a^{\,-l}(\Omega)$ that uniquely extends the positive selfadjoint operator $H = T^*T$ in $L_2(\Omega)$.*

**Remark 6.5.** Since the Hamiltonian $H$ is associated to a differential form, it can be treated as a formal differential operator

$$Hu = \sum_{|\alpha|=l} \sum_{|\beta|=l} D^\alpha (a_{\alpha,\beta}(x) D^\beta u). \tag{6.12}$$

However, it should be emphasized that $H$ in (6.12) should be rather considered a symbol that may not be a differential operator at all, due to the fact that the coefficients $a_{\alpha,\beta}$ are not assumed to be differentiable. If we impose conditions of smoothness on the boundary of $\Omega$ and on $a_{\alpha,\beta}$ then $H$ in (6.12) may be a differential operator.

**Remark 6.6.** In case $1/c \in L_\infty(\Omega)$ the following inequality can be proved

$$\int_\Omega |\nabla_l u(x)|^2 dx \leq C \int_\Omega |b(x) \nabla_l u(x)|^2 \,\mathrm{d}\, x, \quad u \in C_0^\infty(\Omega),$$

where $C$ is a constant that is independent of $u$. In this case, with arguments similar to those used in the proof Proposition 6.3, $\overset{\circ}{\mathcal{H}}{}_a^{\,l}(\Omega)$ is a space of functions that admits a natural continuous embedding into $\overset{\circ}{L}{}_2^{\,l}(\Omega)$. For a bounded domain $\Omega$, due to the Poincaré Inequality

$$\|u\|_{L_2(\Omega)} \leq c \|u\|_{\alpha,\beta}, \quad u \in C_0^\infty(\Omega),$$

the norm $\|\cdot\|_{2,l}$ is equivalent to the Sobolev norm $\|\cdot\|_{W_2^l(\Omega)}$. It follows $\overset{\circ}{L}{}_2^{\,l}(\Omega) = \overset{\circ}{W}{}_2^{\,l}(\Omega)$, the space $\overset{\circ}{\mathcal{H}}{}_a^{\,l}(\Omega)$ is realized as a subspace of $\overset{\circ}{W}{}_2^{\,l}(\Omega)$, and, in this case, $\overset{\circ}{\mathcal{H}}{}_a^{\,l}(\Omega)$ is continuously embedded in $L_2(\Omega)$.

## References

[1] R.A. Adams, *Sobolev Spaces*, Academic Press, New York - San Francisco - London 1975.

[2] R.A. Adams, J. Fournier, *Sobolev Spaces*, Second edition, Pure and Applied Mathematics (Amsterdam), Vol. 140, Elsevier/Academic Press, Amsterdam 2003.

[3] N. Aronszajn, Theory of reproducing kernels, *Trans. Amer. Math. Soc.*, **68**(1950), 337–404.

[4] N. Aronszajn, K.T. Smith, Theory of Bessel potentials, *Ann. Inst. Fourier*, **11**(1961), 385–475.

[5] F. Beatrous, J. Burbea, Holomorphic Sobolev spaces on the ball, *Dissertationes Math. (Rozprawy Mat.)*, **276** (1989), 60 pages.

[6] Yu.M. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators*, Amer. Math. Soc., Providence RI 1968.

[7] Yu.M. Berezanskii, *Selfadjoint Operators in Spaces of Functions of Infinitely Many Variables*, Transl. Math. Mon. Vol. 63, American Mathematical Society, Providence RI 1986.

[8] Yu.M. Berezansky, Y.G. Kondratiev, *Spectral Methods in Infinite-Dimensional Analysis*, Mathematical Physics and Applied Mathematics, Kluwer Academic Publishers, Dordrecht, 1995

[9] Yu.M. Berezansky, Z.G. Sheftel, G.F. Us, *Functional Analysis*, Vol. I and II, Birkhäuser Verlag, Basel 1996.

[10] M.S. Birman, M.Z. Solomjak: *Spectral Theory of Self-Adjoint Operators in Hilbert Space*, D. Reidel Publishing Company, Dordrecht-Boston-Lancaster-Tokyo 1987.

[11] A. Bohm, M. Gadella, *Dirac Kets, Gamow Vectors and Gel'fand Triplets. The Rigged Hilbert Space Formulation of Quantum Mechanics*, Lecture Notes in Physics Vol. 348, Springer-Verlag, Berlin 1989.

[12] L. de Branges, J. Rovnyak: Canonical models in quantum scattering theory, in *Perturbation Theory and its Applications in Quantum Mechanics (Proc. Adv. Sem. Math. Res. Center, U.S. Army, Theoret. Chem. Inst., Univ. of Wisconsin, Madison, Wis., 1965)*, pp. 295–392, Wiley, New York 1966.

[13] O. Civitarese, M. Gadella, Physical and mathematical aspects of Gamow states, *Physics Reports*, **396**(2004), 41–113.

[14] P. Cojuhari, A. Gheondea, Closed embeddings of Hilbert spaces, *J. Math. Anal. Appl.* **369**(2010), 60–75.

[15] P. Cojuhari, A. Gheondea, Triplets of Dirichlet type spaces on the unit polydisc, *Complex Analysis and Operator Theory*, **7**(2013), 1525–1544.

[16] L.C. Evans, *Parial Differential Equations*, Amer. Math. Soc., Providence RI 1998.

[17] P.A. Fillmore, J.P. Williams: On operator ranges, *Adv. Math.* **7**(1971), 254–281.

[18] I.M. Gelfand, A.G. Kostychenko, On eigenfunction expansions of differenital an other operators [Russian], *Dokl. Akad. Nauk SSSR*, **103**(1955), 349-352.

[19] I.M. Gelfand, N.Y. Vilenkin, *Generalized Functions*, Vol. IV, Academic Press, New York 1964.

[20] M. Haase, *The Functional Calculus for Sectorial Operators*, Operator Theory: Advances and Applications, Vol. 169, Birkhäuser Verlag, Basel 2006.

[21] D. Jupiter, D. Redett, Multipliers on Dirichlet type spaces, *Acta. Sci. Math. (Szeged)*, **72**(2006), 179–203.

[22] R. de la Madrid, *Quantum Mechanics in Rigged Hilbert Space Language*, PhD Dissertation, University of Texas at Austin, 2002.

[23] V.G. Maz'ja, *Sobolev Spaces,* Springer-Verlag, Berlin, 1985.

[24] O. Nikodým, Sur une classe de fonctions considerées dans l'étude de probleme de Dirichlet, *Fund. Math.* **21**(1933), 129–150.

[25] S.M. Nikolski, On a class of function spaces, *Uspehi Mat. Nauk*, **9**(1956), 203–212.

[26] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin 1966.

[27] A. Kufner, B. Opic, How to define reasonably weighted Sobolev spaces, *Commentationes Mathematicae Universitatis Carolinae*, **25**(1984), 537–554.

[28] W. Rudin, *Function Theory in Polydiscs*, W.A. Benjamin, New York – Amsterdam 1969.

[29] L. Schwartz, *Théorie des distributions*, Hermann et Cie., Paris 1950.

[30] E. Sanchez-Palencia, *Nonhomogeneous Media and Vibration Theory*, Lecture Notes in Physics Vol. 127, Springer-Verlag, Berlin-New York 1980.

[31] L. Schwartz: Sous espace Hilbertiens d'espaces vectoriel topologiques et noyaux associés (noyaux reproduisants), *J. Analyse Math.*, **13**(1964), 115–256.

[32] Yu.L. Shmulyan, Two-sided division in a ring of operators [Russian], *Mat. Zametki*, **1**(1967), 605–610.

[33] R.E. Showalter, *Hilbert Space Methods for Partial Differential Equations*, Electronic Monographs in Differential Equations, San Marcos, TX 1994.

[34] S.L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, Translations of Mathematical Monographs, 90. Amer. Math. Soc., Providence, RI, 1991

[35] E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton 1970.

[36] G.D. Taylor, Multipliers on $D_\alpha$, *Trans. Amer. Math. Soc.* **123**(1966), 229–240.

[37] Ö.F. Tekin, Completeness of weighted Sobolev spaces, *Senior Project I*, Dept. of Math., Bilkent University, Bilkent 2011.



Department of Applied Mathematics, AGH University of Science and Technology, Al. Mickievicza 30, 30-059 Cracow, Poland

*E-mail address*: cojuhari@uci.agh.edu.pl

Department of Mathematics, Bilkent University, 06800 Bilkent, Ankara, Turkey, *and* Institutul de Matematică al Academiei Române, C.P. 1-764, 014700 Bucureşti, România

*E-mail address*: aurelian@fen.bilkent.edu.tr and A.Gheondea@imar.ro