EXTENSIONS OF STRONGLY Π-REGULAR RINGS

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ABSTRACT

An ideal I of a ring R is strongly π-regular if for any $x \in I$ there exist $n \in \mathbb{N}$ and $y \in I$ such that $x^n = x^{n+1}y$. We prove that every strongly π-regular ideal of a ring is a B-ideal. An ideal I is periodic provided that for any $x \in I$ there exist two distinct $m, n \in \mathbb{N}$ such that $x^m = x^n$. Furthermore, we prove that an ideal I of a ring R is periodic if and only if I is strongly π-regular and for any $u \in U(I)$, $u^{-1} \in \mathbb{Z}[u]$.

Key Words: strongly π-regular ideal; B-ideal; periodic ideal.

2010 Mathematics Subject Classification: 16E50, 16U50, 16E20.

1. INTRODUCTION

A ring R is strongly π-regular if for any $x \in R$ there exist $n \in \mathbb{N}, y \in R$ such that $x^n = x^{n+1}y$. For instance, all artinian rings and all algebraic algebra over a filed. Such rings are extensively studied by many authors from very different view points (cf. [1], [3-4], [7], [9-12] and [14]). We say that an ideal I of a ring R is strongly π-regular provided that for any $x \in I$ there exist $n \in \mathbb{N}, y \in I$ such that $x^n = x^{n+1}y$. Many properties of strongly π-regular rings were extended to strongly π-regular ideals in [6].

Recall that a ring R has stable range one provided that $aR + bR = R$ with $a, b \in R$ implies that there exists a $y \in R$ such that $a + by \in R$ is invertible. The stable range one condition is especially interesting because of Evans' Theorem, which states that a module cancels from direct sums whenever has stable range one. For general theory of stable range conditions, we refer the reader to [6]. An ideal I of a ring R is a B-ideal provided that $aR + bR = R$ with $a \in 1 + I, b \in R$ implies that there exists a $y \in R$ such that $a + by \in R$ is invertible. An ideal I is a ring R is stable provided that $aR + bR = R$ with $a \in I, b \in R$ implies that there exists a $y \in R$ such that $a + by \in R$ is invertible. As is well known, every B-ideal of a ring is stable, but the converse is not true.

In [1, Theorem 4], Ara proved that every strongly π-regular ring has stable range one. This was extended to ideals, i.e., every strongly π-regular ideal of a ring is stable (cf. [5]). The main purpose of this note is to extend these results, and show that every strongly π-regular ideal of a ring is a B-ideal. An ideal I of a ring R is periodic provided that for any

$x \in I$ there exists two distinct $m, n \in \mathbb{N}$ such that $x^m = x^n$. Furthermore, we show that an ideal I of a ring R is periodic if and only if I is strongly π-regular and for any $u \in U(I)$, $u^{-1} \in \mathbb{Z}[u]$. Several new properties of such ideals are also obtained.

Throughout, all rings are associative with an identity and all modules are unitary modules. $U(R)$ denotes the set of all invertible elements in the ring R and $U(I) = \big(1+I\big) \bigcap U(R)$.

2. STRONGLY Π-REGULAR IDEALS

The aim of this section is to investigate more elementary properties of strongly π-regular ideals and construct more related examples. For any $x \in R$, we define $\sigma_x : R \to R$ given by $\sigma_x(r) = xr$ for all $r \in R$.

Theorem 2.1. *Let I be an ideal of a ring R. Then the following are equivalent:*

(1) *I is strongly π-regular.*

(2) *For any $x \in I$, there exists $n \geq 1$ such that $R = ker(\sigma_x^n) \oplus im(\sigma_x^n)$.*

Proof. (1) $\Rightarrow$ (2) Let $x \in I$. In view of [6, Proposition 13.1.15], there exist $n \in \mathbb{N}, y \in I$ such that $x^n = x^{n+1}y$ and $xy = yx$. It is easy to check that $\sigma_x^n = \sigma_x^{n+1}\sigma_y$. If $a \in ker(\sigma_x^n) \bigcap im(\sigma_x^n)$, then $a = \sigma_x^n(r)$ and $\sigma_x^n(a) = 0$. This implies that $x^{2n}r = \sigma_x^{2n}(r) = 0$, and so $a = x^n r = x^{n+1}yr = yx^{n+1}r = y^n x^{2n} r = 0$. Hence, $ker(\sigma_x^n) \bigcap im(\sigma_x^n) = 0$. For any $r \in R$, we see that $r = \big(r - \sigma_x^n(y^n r)\big) + \sigma_x^n(y^n r)$, and then $R = ker(\sigma_x^n) + im(\sigma_x^n)$, as required.

(2) $\Rightarrow$ (1) Write $1 = a + b$ with $a \in ker(\sigma_x^n)$ and $b \in im(\sigma_x^n)$. For any $x \in I$. $\sigma_x^n(1) = \sigma_x^n(b)$, and so $x^n \in x^{2n}R$. Thus, I is strongly π-regular. □

Corollary 2.2. *Let I be a strongly π-regular ideal of a ring R, and let $x \in I$. Then the following are equivalent:*

(1) *σ_x is a monomorphism.*

(2) *σ_x is an epimorphism.*

(3) *σ_x is an isomorphism.*

Proof. (1) $\Rightarrow$ (2) In view of Theorem 2.1, there exists $n \geq 1$ such that $R = ker(\sigma_x^n) \oplus im(\sigma_x^n)$. Since σ_x is a monomorphism, so is σ_x^n. Hence, $ker(\sigma_x^n) = 0$, and then $R = im(\sigma_x^n)$. This implies that σ_x is an epimorphism.

(2) $\Rightarrow$ (3) Since $R = ker(\sigma_x^n) \oplus im(\sigma_x^n)$, it follows from $R = im(\sigma_x^n)$ that $ker(\sigma_x^n) = 0$. Hence, σ_x is a monomorphism. Therefore σ_x is an isomorphism.

(3) $\Rightarrow$ (1) is trivial. □

Proposition 2.3. *Let I be an ideal of a ring R. Then the following are equivalent:*

(1) *I is strongly π-regular.*

(2) *For any $x \in I$, RxR is strongly π-regular.*

Proof. (1) ⇒ (2) Let $x \in I$. For any $a \in RxR$, there exists an element $b \in I$ such that $a^n = a^{n+1}b$ for some $n \in \mathbb{N}$. Hence, $a^n = a^{n+1}(ab^2)$. As $ab^2 \in RxR$, we see that RxR is strongly π-regular.

(2) ⇒ (1) For any $x \in I$, RxR is strongly π-regular, and so there exists a $y \in RxR$ such that $x^n = x^{n+1}y$. Clearly, $y \in I$, and therefore I is strongly π-regular. □

The index of a nilpotent element in a ring is the least positive integer n such that $x^n = 0$. The index $i(I)$ of an ideal I of a ring R is the supremum of the indices of all nilpotent elements of I. An ideal I of a ring R is of bounded index if $i(I) < \infty$. It is well known that $i(I) \leq n$ if and only if I contains no direct sums of $n+1$ nonzero pairwise isomorphic right ideals (cf. [9, Theorem 7.2]).

Theorem 2.4. *Let R be a ring, and let*

$$I = \{a \in R \mid i(RaR) < \infty\}.$$

Then I is a strongly π-regular ideal of R.

Proof. Let $x, y \in I$ and $z \in R$. Then $RxzR, RzxR \subseteq RxR$. This implies that $RxzR$ and $RzxR$ are strongly π-regular of bounded index. Hence, $xz, zx \in I$.

Obviously, $R(x-y)R \subseteq RxR+RyR$. For any $a \in R(x-y)R$, $a = c+d$ where $c \in RxR$ and $d \in RyR$. Since RxR is strongly π-regular, there exists some $n \in \mathbb{N}$ such that $c^n = c^{n+1}r$ for a $r \in R$. Let RyR is of bounded index m. Then $c^n = c^{nm+1}s$ for a $s \in R$. Hence, $a^{nm+1}s - a^n \in RyR$. As RyR is strongly π-regular, we can find $k \in \mathbb{N}$ and $d \in RyR$ such that

$$\begin{gathered}\big(a^{nm+1}s - a^n\big)^k = \big(a^{nm+1}s - a^n\big)^{k+1}d,\\ d = d\big(a^{nm+1}s - a^n\big)d,\\ d\big(a^{nm+1}s - a^n\big) = \big(a^{nm+1}s - a^n\big)d.\end{gathered}$$

Hence,

$$\begin{aligned} & \big((a^{nm+1}s - a^n) - (a^{nm+1}s - a^n)^2 d\big)^k \\ = \;& \big(a^{nm+1}s - a^n\big)^k \big(1 - (a^{nm+1}s - a^n)d\big)^k \\ = \;& \big(a^{nm+1}s - a^n\big)^k \big(1 - (a^{nm+1}s - a^n)d\big) \\ = \;& 0. \end{aligned}$$

Therefore $\big(a^{nm+1}s - a^n\big)^m = \big(a^{nm+1}s - a^n\big)^{m+1}t$. As a result, $a^{nm} \in a^{nm+1}R$. Hence, we can find a $r \in R$ such that $a^{nm} = a^{nm+1}(ar)$. Therefore I is a strongly π-regular ideal of R. □

Corollary 2.5. *Let R be a ring of bounded index. Then*

$$I = \{a \in R \mid RaR \text{ is strongly } \pi\text{-regular}\}$$

is the maximal strongly π-regular ideal of R.

Proof. Since R is of bounded index, so is RaR for any $a \in R$. In view of Theorem 2.4, $I = \{a \in R \mid RaR$ is strongly π-regular$\}$ is a strongly π-regular ideal of R. Thus we complete the proof by Proposition 2.3. □

Example 2.6. *Let V be an infinite-dimensional vector space over a field F, let $R = End_F(V)$, and let $I = \{\sigma \in R \mid dim_F \sigma(V) < \infty\}$. Then I is strongly π-regular, while R is not strongly π-regular.*

Proof. Clearly, I is an ideal of the ring R. We have the descending chain $\sigma(V) \supseteq \sigma^2(V) \supseteq \cdots$. As $dim_F\sigma(V) < \infty$, we can find some $n \in \mathbb{N}$ such that $\sigma^n(V) = \sigma^{n+1}(V)$. Since V is a projective right F-module, we can find some $\tau \in R$ such that the following diagram

$$\begin{array}{ccc} & & V \\ & \tau \swarrow & \downarrow \sigma^n \\ V & \overset{\sigma^{n+1}}{\twoheadrightarrow} & \sigma^{n+1}(V) \end{array}$$

commutates, i.e., $\sigma^{n+1}\tau = \sigma^n$. Hence, $\sigma^n = \sigma^{n+1}(\sigma\tau^2)$. Therefore I is a strongly π-regular ideal of R. Let ϵ be an element of R such that $\varepsilon(x_i) = x_{i+1}$ where $\{x_1, x_2, \cdots\}$ is the basis of V. If R is strongly π-regular, there exists some $m \in \mathbb{N}$ such that $\varepsilon^m R = \varepsilon^{m+1}R$, and so $\varepsilon^m(V) = \varepsilon^{m+1}(V)$. As $\varepsilon^m(x_i) = x_{i+m}$ for all i, we see that $\varepsilon^m(V) = \sum\limits_{i>m} x_iF \neq \sum\limits_{i>m+1} x_iF = \varepsilon^{m+1}(V)$. This gives a contradiction. Therefore R is not a strongly π-regular ring. $\square$

Example 2.7. *Let V be an infinite-dimensional vector space over a field F, let $R = End_F(V)$, and let $S = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$. Then $I = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}$ is a strongly π-regular ideal of R, while S is not a strongly π-regular ring.*

Proof. By the discussion in Example 2.6, R is not strongly π-regular. Hence, S is not strongly π-regular. As $I^2 = 0$, one easily checks that I is a strongly π-regular ideal of the ring S. $\square$

An ideal I of a ring R is called a gsr-ideal if for any $a \in I$ there exists some integer $n \geq 2$ such that $aRa = a^nRa^n$. For instance, every ideal of strongly regular rings is a gsr-ideal.

Example 2.8. *Every gsr-ideal of a ring is strongly π-regular.*

Proof. Let I be a gsr-ideal of a ring R. Given $\overline{x}^2 = \overline{0}$ in $I/\big(I \bigcap J(R)\big)$, then $x^2 \in I \bigcap J(R)$. As I is a gsr-ideal, we see that $xRx = x^2Rx^2 \subseteq J(R)$, i.e., $(RxR)^2 \subseteq J(R)$. As $J(R)$ is semiprime, it follows that $RxR \subseteq J(R)$, and so $x \in J(R)$. That is, $\overline{x} = \overline{0}$. This implies that $I/\big(I \bigcap J(R)\big)$ is reduced. For any idempotent $e \in I/\big(I \bigcap J(R)\big)$ and any $a \in R/J(R)$, it follows from $\big(ea(\overline{1} - e)\big)^2 = 0$ that $ea(\overline{1} - e) = 0$, thus $ea = eae$. Likewise, $ae = eae$. This implies that $ea = ae$. As a result, every idempotent in $I/\big(I \bigcap J(R)\big)$ is central. For any $x \in I \bigcap J(R)$, there exists some $y \in R$ such that $x^2 = x^2yx^2$, and then $x^2(1 - yx^2) = 0$. This implies that $x^2 = 0$. Assume that $x^2 = 0$. As I is a gsr-ideal, we see that $xRx = x^2Rx^2 = 0$. That is, $(RxR)^2 \subseteq J(R)$, and so $x \in J(R)$. Therefore $I \bigcap J(R) = \{x \in I \mid x^2 = 0\}$. Let $x \in I$. Then there exists some $n \geq 2$ such that $xRx = x^nRx^n$. Hence, $x^2 = x^2yx^2$. As $x^2y \in I$ is an idempotent, we see that $x^2 - x^6y^2 \in I \bigcap J(R)$. By the preceding discussion, we get $\big(x^2 - x^6y^2\big)^2 = 0$. This implies that $x^4 = x^5r$ for some $r \in I$. Thus I is strongly π-regular. $\square$

3. STABLE RANGE CONDITION

For any $x, y \in R$, write $x \circ y = x + y + xy$. We use $x^{[n]}$ to stand for $\underbrace{x \circ \cdots \circ x}_{n}$ $(n \geq 1)$

and $x^{[0]} = 0$. The following result was firstly observed in [1, Lemma 1], we include a simple proof to make the paper is self-contained.

Lemma 3.1. *Let $x_i, y_j \in R$, and let $p_i, q_j \in \mathbb{Z}(1 \leq i \leq m, 1 \leq j \leq n)$. If $\sum\limits_i p_i = \sum\limits_j q_j = 1$, then $\big(\sum\limits_i p_i x_i\big) \circ \big(\sum\limits_j q_j y_j\big) = \sum\limits_{i,j} (p_i q_j)(x_i \circ y_j)$; If $\sum\limits_i p_i = \sum\limits_j q_j = 0$, then $\big(\sum\limits_i p_i x_i\big)\big(\sum\limits_j q_j y_j\big) = \sum\limits_{i,j} (p_i q_j)(x_i \circ y_j)$.*

Proof. For any $p_i, q_j \in \mathbb{Z}$, one easily checks that $\sum\limits_{i,j}(p_i q_j)(x_i \circ y_j) = \big(\sum\limits_i p_i x_i\big)\big(\sum\limits_j q_j y_j\big) + \big(\sum\limits_j q_j\big)\big(\sum\limits_i p_i x_i\big) + \big(\sum\limits_i p_i\big)\big(\sum\limits_j q_j y_j\big)$. Therefore the result follows. □

Lemma 3.2. *Let I be a strongly π-regular ideal of a ring R. Then for any $x \in I$, there exists some $n \in \mathbb{N}$ such that $x^{[n]} = x^{[n+1]} \circ y = z \circ x^{[n+1]}$ for $y, z \in I$.*

Proof. Let $x \in I$. Then $-x - x^2 \in I$. Since I is a strongly π-regular ideal, there exists some $n \in \mathbb{N}$ such that $(-x - x^2)^n = (-x - x^2)^{n+1}s = s(-x - x^2)^{n+1}$. Clearly, $x - x^{[2]} = -x - x^2$. Thus,

$$\big(x - x^{[2]}\big)^n = \big(x - x^{[2]}\big)^{n+1}s = \big(x - x^{[2]}\big)^{2n}t,$$

where $t = s^n$. Since $\sum\limits_{i=0}^{n}(-1)^i \begin{pmatrix} n \\ i \end{pmatrix} = 0$, it follows from Lemma 3.1 that

$$\sum_{i=0}^{n}(-1)^i \begin{pmatrix} n \\ i \end{pmatrix} \big(x^{[n-i]} \circ (x^{[2]})^{[i]}\big) = \big(x - x^{[2]}\big)^n.$$

Thus,

$$\begin{aligned} \big(x - x^{[2]}\big)^n &= \sum_{i=0}^{n}(-1)^i \begin{pmatrix} n \\ i \end{pmatrix} x^{[n+i]} \\ &= x^{[n]} + \sum_{i=1}^{n}(-1)^i \begin{pmatrix} n \\ i \end{pmatrix} x^{[n+i]}. \end{aligned}$$

Let $u = \sum\limits_{i=1}^{n}(-1)^{i+1} \begin{pmatrix} n \\ i \end{pmatrix} x^{[i]}$. Then $u \circ x^{[n]} = x^{[n]} \circ u$. Since $\sum\limits_{i=1}^{n}(-1)^{i+1} \begin{pmatrix} n \\ i \end{pmatrix} = 1$, by using Lemma 2.1 again, $\big(x - x^{[2]}\big)^n = x^{[n]} - x^{[n]} \circ u$. Thus, we get

$$\begin{aligned} x^{[n]} - x^{[n]} \circ u &= \big(x^{[n]} - x^{[n]} \circ u\big)^2 t \\ &= \big(x^{[n]} - x^{[n]} \circ u\big)\big(x^{[n]} - x^{[n]} \circ u\big)(t - 0) \\ &= \big(x^{[2n]} - x^{[2n]} \circ u - x^{[2n]} \circ u + x^{[2n]} \circ u^{[2]}\big)(t - 0) \\ &= x^{[2n]} \circ \big(t - u \circ t - u \circ t + u^{[2]} \circ y + u + u - u^{[2]}\big) - x^{[2n]} \\ &= x^{[2n]} \circ (u^2 t) - x^{[2n]}. \end{aligned}$$

Let $v = x^{[2n]} \circ (u^2 t) - x^{[2n]}$. Then

$$\begin{aligned} x^{[n]} &= x^{[n]} \circ u + v \\ &= \big(x^{[n]} \circ u + v - 0\big) \circ u + v \\ &= x^{[n]} \circ u^{[2]} + \big(v \circ u - u\big) + v \\ &\vdots \\ &= x^{[n]} \circ u^{[n+1]} + \sum_{i=0}^{n} \big(v \circ u^{[i]} - u^{[i]}\big) \end{aligned}$$

Further,

$$\begin{aligned} v\circ u^{[i]}-u^{[i]} &= \big(x^{[2n]}\circ(u^2t)-x^{[2n]}\big)\circ u^{[i]}-u^{[i]}\\ &= \big(x^{[2n]}\circ(u^2t)-x^{[2n]}+0\big)\circ u^{[i]}-u^{[i]}\\ &= x^{[2n]}\circ(u^2t)\circ u^{[i]}-x^{[2n]}\circ u^{[i]}\end{aligned}$$

Hence

$$x^{[n]} = x^{[n]}\circ u^{[n+1]}+\sum_{i=0}^{n}\big(x^{[2n]}\circ(u^2t)\circ u^{[i]}-x^{[2n]}\circ u^{[i]}\big)$$

Further, we see that

$$\sum_{i=0}^{n}\big(x^{[2n]}\circ(u^2t)\circ u^{[i]}-x^{[2n]}\circ u^{[i]}\big) = x^{[2n]}\circ\big(\sum_{i=0}^{n}((u^2t)\circ u^{[i]}-u^{[i]})+0\big)-x^{[2n]}.$$

As $\sum_{i=1}^{n}(-1)^{i+1}\left(\begin{array}{c} n\\ i\end{array}\right)=1$, we see that

$$\begin{aligned} u^{[n+1]} &= \big(\sum_{i=1}^{n}(-1)^{i+1}\left(\begin{array}{c} n\\ i\end{array}\right)x^{[i]}\big)^{[n+1]}\\ &= \sum_{i_1+\cdots+i_n=n+1} C_{i_1\cdots i_n}x^{[i_1+2i_2+\cdots+ni_n]}\\ &= \sum_{i_1+\cdots+i_n=n+1} C_{i_1\cdots i_n}x^{[n]}\circ x^{[1+i_2+\cdots+(n-1)i_n]}.\end{aligned}$$

It is easy to check that $\sum\limits_{i_1+\cdots+i_n=n+1} C_{i_1\cdots i_n}=\big(\sum\limits_{i=1}^{n}(-1)^{i+1}\left(\begin{array}{c} n\\ i\end{array}\right)\big)^{n+1}=1$, and so $u^{[n+1]}=x^{[n]}\circ v$, where $v=\sum\limits_{i_1+\cdots+i_n=n+1} C_{i_1\cdots i_n}x^{[1+i_2+\cdots+(n-1)i_n]}$. Therefore

$$\begin{aligned} x^{[n]} &= x^{[2n]}\circ v+x^{[2n]}\circ\big(\sum_{i=0}^{n}((u^2t)\circ u^{[i]}-u^{[i]})\big)-x^{[2n]}\\ &= x^{[2n]}\circ\big(v+(\sum_{i=0}^{n}((u^2t)\circ u^{[i]}-u^{[i]}))-0\big)\\ &= x^{[2n]}\circ\big(v+\sum_{i=0}^{n}((u^2t)\circ u^{[i]}-u^{[i]})\big)\end{aligned}$$

Let $y=x^{[n-1]}\circ\big(v+\sum\limits_{i=0}^{n}((u^2t)\circ u^{[i]}-u^{[i]})\big)$. Then $x^{[n]}=x^{[n+1]}\circ y$ with $y\in I$. Likewise, $x^{[n]}=z\circ x^{[n+1]}$ for a $z\in I$, as required. □

Theorem 3.3. *Every strongly π-regular ideal of a ring is a B-ideal.*

Proof. Let I be a strongly π-regular ideal of a ring R. Let $a\in 1+I$. Then $a-1\in I$. In view of Lemma 2.2, we can find some $n\in\mathbb{N}, b,c\in 1+I$ such that $(a-1)^{[n]}=(a-1)^{[n+1]}\circ(b-1)=(c-1)\circ(a-1)^{[n+1]}$. One easily checks that $(a-1)^{[n]}=a^n-1$ and $(a-1)^{[n+1]}=a^{n+1}-1$. Therefore $a^n=a^{n+1}b=ca^{n+1}$, and so $a^n\in a^{n+1}R\bigcap Ra^{n+1}$. According to [6, Proposition 13.1.2], $a\in 1+I$ is strongly π-regular. According to [6, Theorem 13.1.7], I is a B-ideal. □

Corollary 3.4. *Let I be a strongly π-regular ideal of a ring R, and let A be a finitely generated projective right R-module. If $A=AI$, then for any right R-modules B and C, $A\oplus B\cong A\oplus C$ implies that $B\cong C$.*

Proof. For any $x\in I$, we have $n\in\mathbb{N}$ and $y\in R$ such that $x^n=x^{n+1}y$ and $xy=yx$. Hence $x^n=x^nzx^n$, where $z=y^n$. Let $g=zx^n$ and $e=g+(1-g)x^ng$. Then $e\in Rx$ is an idempotent. In addition, we have $1-e=(1-g)\big(1-x^ng\big)=(1-g)\big(1-x^n\big)\in Rx$. Set $f=1-e$. Then there exists an idempotent $f\in I$ such that $f\in Rx$ and $1-f\in Rx$. Therefore I is an exchange ideal of R. In view of Theorem 3.3, I is a B-ideal. Therefore we complete the proof by [6, Lemma 13.1.9]. □

Corollary 3.5. *Let I be a strongly π-regular ideal of a ring R, and let $a,b\in 1+I$. If $aR=bR$, then $a=bu$ for for some $u\in U(R)$.*

Proof. Write $ax=b$ and $a=by$. As $a,b\in 1+I$, we see that $x,y\in 1+I$. In view of Theorem 3.3, I is a B-ideal. Since $yx+(1-yx)=1$, there exists an element $z\in R$ such that $u:=y+(1-yx)z\in U(R)$. Therefore $bu=b\big(y+(1-yx)z\big)=by=a$, as required. □

Corollary 3.6. *Let I be a strongly π-regular ideal of a ring R, and let $A\in M_n(I)$ be regular. Then A is the product of an idempotent matrix and an invertible matrix.*

Proof. By virtue of Theorem 3.3, I is a B-ideal. As $A\in M_n(I)$ is regular, we have a $B\in M_n(I)$ such that $A=ABA$. Since $AB+(I_n-AB)=I_n$, we get $\big(A+(I_n-AB)\big)B+(I_n-AB)(I_n-B)=I_n$ where $A+(I_n-AB)\in I_n+M_n(I)$. Thus, we can find a $Y\in M_n(R)$ such that $U:=A+(I_n-AB)+(I_n-AB)(I_n-B)Y\in GL_n(R)$. Therefore $A=ABA=AB\big(A+(I_n-AB)+(I_n-AB)(I_n-B)Y\big)=ABU$, as required. □

Let A is an algebra over a field F. An element a of an algebra A over a field F is said to be algebraic over F if a is the root of some non-constant polynomial in $F[x]$. An ideal I of A is said to be an algebraic ideal of A if every element in I is algebraic over F.

Proposition 3.7. *Let A is an algebra over a field F, and let I be an algebraic ideal of A. Then I is a B-ideal.*

Proof. For any $a\in I$, a is the root of some non-constant polynomial in $F[x]$. So we can find $a_m,\cdots,a_n\in F$ such that $a_na^n+a_{n-1}a^{n-1}+\cdots+a_ma^m=0$, where $a_m\neq 0$. Hence, $a^m=-(a_na^n+\cdots+a_{m+1}a^{m+1})a_m^{-1}=-a^{m+1}(a_na^{n-m-1}+\cdots+a_{m+1})a_m^{-1}$. Set $b=-(a_na^{n-m-1}+\cdots+a_{m+1})a_m^{-1}$. Then $a^m=a^{m+1}b$. Therefore I is strongly π-regular, and so we complete the proof by Theorem 3.3. □

In the proof of Theorem 3.3, we show that for any $a\in 1+I$, there exists some $n\in\mathbb{N}$ such that $a^n\in a^{n+1}b$ for a $b\in 1+I$ if I is a strongly π-regular ideal. A natural problem asks that if the converse of the preceding assertion is true. The answer is negative from the following counterexample. Let $p\in\mathbb{Z}$ be a prime and set $\mathbb{Z}_{(p)}=\{a/b \mid b\notin \mathbb{Z}p\,(a/b \text{ in lowest terms})\}$. Then $\mathbb{Z}_{(p)}$ is a local ring with maximal $p\mathbb{Z}_{(p)}$. Thus, the Jacobson radical $p\mathbb{Z}_{(p)}$ satisfies the condition above. Choose $p/(p+1)\in p\mathbb{Z}_{(p)}$. Then $p/(p+1)\in J\big(\mathbb{Z}_{(p)}\big)$ is not nilpotent. This shows that $p\mathbb{Z}_{(p)}$ is not strongly π-regular.

4. PERIODIC IDEALS

An ideal I of a ring R is periodic provided that for any $x\in I$ there exist distinct $m,n\in\mathbb{N}$ such that $x^m=x^n$. We note that an ideal I of a ring R is periodic if and only if for any $a\in I$, there exists a potent element $p\in I$ such that $a-p$ is nilpotent and $ap=pa$.

Lemma 4.1. *Let I be an ideal of a ring R. If I is periodic, then for any $x \in 1+I$ there exist $m \in \mathbb{N}, f(t) \in \mathbb{Z}[t]$ such that $x^m = x^{m+1}f(x)$.*

Proof. For any $a \in I$, there exists some $n \in \mathbb{N}$ such that $a^n = a^{n+1}\big(a^{m-n-1}\big)$ where $m \geq n+1$. For any $x \in 1+I$, we see that $x-1 \in I$. As in the proof in Lemma 3.2, we can find a $f(t) \in R[t]$ such that $(x-1)^{[n]} = (x-1)^{[n+1]} \circ (f(x)-1)$. One easily checks that $(x-1)^{[n]} = x^n - 1$ and $(x-1)^{[n+1]} = x^{n+1}-1$. Therefore $x^n = x^{n+1}f(x)$, as required. □

Lemma 4.2. *Let R be a ring, and let $c \in R$. If there exist a monic $f(t) \in \mathbb{Z}[t]$ and some $m \in \mathbb{N}$ such that $mc = 0$ and $f(c) = 0$, then there exist $s, t \in \mathbb{N}(s \neq t)$ such that $c^s = c^t$.*

Proof. Clearly, $\mathbb{Z}c \subseteq \{0, c, \cdots, (m-1)c\}$. Write $f(t) = t^k + b_1t^{k-1} + \cdots + b_{k-1}t + b_k \in \mathbb{Z}[t]$. Then $c^{k+1} = -b_1c^k - \cdots - b_{k-1}c^2 - b_kc$. This implies that $\{c, c^2, c^3, \cdots, c^l, \cdots\} \subseteq \{c, c^2, c^3, \cdots, c^k, 0, c, \cdots, (m-1)c, c^2, \cdots, (m-1)c^2, \cdots, c^k, \cdots, (m-1)c^k\}$. That is, $\{c, c^2, c^3, \cdots, c^k, \cdots\}$ is a finite set. Hence, we can find some $s, t \in \mathbb{N}, s \neq t$ such that $c^s = c^t$, as desired. □

As is well known, a ring R is periodic if and only if for any $x \in R$, there exists $n \in \mathbb{N}$ and $f(t) \in \mathbb{Z}[t]$ such that $x^n = x^{n+1}f(x)$. We extend this result to periodic ideals.

Lemma 4.3. *Let I be an ideal of a ring R. If for any $x \in I$, there exist $n \in \mathbb{N}$ and $f(t) \in \mathbb{Z}[t]$ such that $x^n = x^{n+1}f(x)$, then I is periodic.*

Proof. Let $x \in I$. If x is nilpotent, then we can find some $n \in \mathbb{N}$ such that $x^n = x^{n+1} = 0$. Thus, we may assume that $x \in I$ is not nilpotent. By hypothesis, there exists $n \in \mathbb{N}$ and $g(t) \in \mathbb{Z}[t]$ such that $x^n = x^{n+1}g(x)$. Thus, $x^n = x^{n+1}f(x)$, where $f(x) = x\big(g(x)\big)^2 \in \mathbb{Z}[t]$. In addition, $f(0) = 0$. Let $e = x^n\big(f(x)\big)^n$. Then $0 \neq e = e^2 \in R$ and $x^n = x^ne$. Set $S = eRe$ and $\alpha = ex = xe$. Then $f(\alpha) = ef(x)$. Further,

$$\alpha^n\big(f(\alpha)\big)^n = e, \alpha^n = x^n, \alpha^n = \alpha^{n+1}f(\alpha).$$

Thus, $e = \alpha^n\big(f(\alpha)\big)^n = \alpha^{n+1}\big(f(\alpha)\big)^{n+1} = \alpha^n\big(f(\alpha)\big)^n\alpha f(\alpha) = e\alpha f(\alpha) = \alpha f(\alpha)$ in S. Write $f(t) = a_1t + \cdots + a_nt^n$. Then $\alpha\big(a_1\alpha + \cdots + a_n\alpha^n\big) = e$. This implies that $(\alpha^{-1})^{n+1} - a_1(\alpha^{-1})^{n-1} - \cdots - a_ne = 0$. Let $g(t) = t^{n+1} - a_1t^{n-1} - \cdots - a_n \in \mathbb{Z}[t]$. Then $g(t)$ is a monic polynomial such that $g(\alpha^{-1}) = 0$.

Let $T = \{me \in S \mid m \in \mathbb{Z}\}$. Then T is a subring of S. For any $me \in I$, by hypothesis, there exists a $g(t) \in \mathbb{Z}[t]$ such that $(me)^p = (me)^{p+1}g(me) \in (me)^{p+1}T$. This implies that T is strongly π-regular. Construct a map $\varphi : \mathbb{Z} \to T, m \to me$. Then $\mathbb{Z}/Ker\varphi \cong T$. As $\mathbb{Z}$ is not strongly π-regular, we see that $Ker\varphi \neq 0$. Hence, $T \cong \mathbb{Z}_q$ for some $q \in \mathbb{N}$. Thus, $qe = 0$. As a result, $q\alpha^{-1} = 0$. In view of Lemma 4.2, we can find some $s, t \in \mathbb{N}(s \neq t)$ such that $(\alpha^{-1})^s = (\alpha^{-1})^t$. This implies that $\alpha^s = \alpha^t$. Hence, $x^{ns} = x^{st}$, as asserted. □

Theorem 4.4. *Let I be an ideal of a ring R. Then I is periodic if and only if*

(1) *I is strongly π-regular.*

(2) *For any $u \in U(I)$, $u^{-1} \in \mathbb{Z}[u]$.*

Proof. Suppose that I is periodic. Then I is strongly π-regular. For any $u \in U(I)$, it follows by Lemma 4.1 that there exist $m \in \mathbb{N}, f(t) \in \mathbb{Z}[t]$ such that $u^m = u^{m+1}f(u)$. Hence, $uf(u) = 1$, and so $u^{-1} \in \mathbb{Z}[u]$.

Suppose that (1) and (2) hold. For any $x \in I$, there exist $m \in \mathbb{N}$ and $y \in I$ such that $x^m = x^myx^m, y = yx^my$ and $xy = yx$ from [6, Proposition 13.1.15]. Set $u = 1 - x^my + x^m$.

Then $u^{-1} = 1 - x^m y + y$. Hence, $u \in U(I)$. By hypothesis, there exists an $g(t) \in \mathbb{Z}[t]$ such that $ug(u) = 1$. Further, $x^m = x^m y\big(1 - x^m y + x^m\big) = x^m yu$. Hence, $x^m u^{-1} = x^m y$, and so $x^m = x^m y x^m = x^{2m} g(u) = x^{2m} x^m \big(g(u)\big)^2$. Write $\big(g(u)\big)^2 = b_0 + b_1 u + \cdots + b_n u^n \in \mathbb{Z}[u]$. For any $i \geq 0$, it is easy to check that $x^m u^i = x^m\big(1 - x^m y + x^m\big)^i \in \mathbb{Z}[x]$. This implies that $x^m\big(g(u)\big)^2 \in \mathbb{Z}[x]$. According to Lemma 4.3, I is periodic. □

It follows by Theorem 4.4 and Theorem 3.3 that every periodic ideal of a ring is a B-ideal.

Corollary 4.5. *Let I be a strongly π-regular ideal of a ring R. If $U(I)$ is torsion, then I is periodic.*

Proof. For any $u \in U(I)$, there exists some $m \in \mathbb{N}$ such that $u^m = 1$. Hence, $u^{-1} = u^{m-1} \in \mathbb{Z}[u]$. According to Theorem 4.4, we complete the proof. □

Example 4.6. *Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ and $I = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$. Then I is a nilpotent ideal of R; hence, I is strongly π-regular. Clearly, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in U(I)$, but $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^m \neq 0$ for any $m \in \mathbb{N}$. Thus, $U(I)$ is torsion.* □

The example above shows that the converse of Corollary 4.6 is not true. But we can derive the following.

Proposition 4.7. *Let I be an ideal of a ring R. If $char(R) \neq 0$, then I is periodic if and only if*

(1) *I is strongly π-regular.*

(2) *$U(I)$ is torsion.*

Proof. Suppose that I is periodic. Then I is strongly π-regular. Let $x \in U(I)$. Then x is not nilpotent. By virtue of Lemma 4.1, there exist $m \in \mathbb{N}, f(t) \in \mathbb{Z}[t]$ such that $x^m = x^{m+1} f(x)$. As in the proof of Lemma 4.3, we have a monic polynomial $g(t) \in \mathbb{Z}[t]$ such that $g(\alpha^{-1}) = 0$. As $char(R) \neq 0$, we assume that $char(R) = q \neq 0$. Then $q\alpha^{-1} = 0$. According to Lemma 4.2, we can find two distinct $s, t \in \mathbb{N}$ such that $(\alpha^{-1})^s = (\alpha^{-1})^t$. Similarly to Lemma 4.3, $x^{ns} = x^{st}$, and so x is torsion. Therefore $U(I)$ is torsion. The converse is true by Corollary 4.5. □

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