European Journal of Operational Research 237 (2014) 957–965

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Decision Support

Equilibrium in an ambiguity-averse mean–variance investors market ☆

Mustafa Ç. Pınar *

Department of Industrial Engineering, Bilkent University, 06800 Bilkent, Ankara, Turkey

ARTICLE INFO

Article history:
Received 2 July 2013
Accepted 5 February 2014
Available online 12 February 2014

Keywords:
Robust optimization
Mean–variance portfolio theory
Ellipsoidal uncertainty
Equilibrium price system

ABSTRACT

In a financial market composed of n risky assets and a riskless asset, where short sales are allowed and mean–variance investors can be ambiguity averse, i.e., diffident about mean return estimates where confidence is represented using ellipsoidal uncertainty sets, we derive a closed form portfolio rule based on a worst case max–min criterion. Then, in a market where all investors are ambiguity-averse mean–variance investors with access to given mean return and variance–covariance estimates, we investigate conditions regarding the existence of an equilibrium price system and give an explicit formula for the equilibrium prices. In addition to the usual equilibrium properties that continue to hold in our case, we show that the diffidence of investors in a homogeneously diffident (with bounded diffidence) mean–variance investors' market has a deflationary effect on equilibrium prices with respect to a pure mean–variance investors' market in equilibrium. Deflationary pressure on prices may also occur if one of the investors (in an ambiguity-neutral market) with no initial short position decides to adopt an ambiguity-averse attitude. We also establish a CAPM-like property that reduces to the classical CAPM in case all investors are ambiguity-neutral.

1. Introduction and background

A major theme in mathematical finance is the study of investors' portfolio decisions using the well-established theory of mean–variance that began with the seminal work of Markowitz (1987). The mean–variance portfolio theory then formed the basis of the celebrated Capital Asset Pricing Model (CAPM) (Sharpe, 1964), the most commonly used equilibrium and pricing model in the financial literature. However, it is a well-known fact that the investors' portfolio holdings in the mean–variance portfolio theory are very sensitive to the estimated mean returns of the risky assets; see e.g., Best and Grauer (1991a), Best and Grauer (1991b), Black and Litterman (1992). The purpose of the present paper is to investigate equilibrium relations in a financial market composed of n risky assets and a riskless asset using an approach that takes into account the imprecision in the mean return estimates. In our model, investors act as mean–variance investors with a degree of diffidence (or confidence) towards the mean return estimates of risky assets. We refer to this attitude of diffidence as *ambiguity aversion* to distinguish it from risk aversion quantified by a mean–variance objective function. Decision making under ambiguity aversion is an active research area in decision theory and economics; see e.g., Klibanoff, Marinacci, and Mukerji (2005, 2009). Our study follows the earlier work of Konno and Shirakawa (1994, 1995), and is in particular inspired by the previous work of Deng, Li, and Wang (2005) where the authors study a similar problem allowing the mean returns of risky assets to vary over a hyper-rectangle, i.e., an interval is specified for each mean return estimate and a max–min approach is used in the portfolio choice as in the present paper. We adopt an ellipsoidal uncertainty set for the mean-return vector instead of a hyper-rectangle, and obtain a closed-form portfolio rule using a worst-case max–min approach as in the *robust optimization* framework of Ben-Tal and Nemirovski (1999, 1998). In contrast, in Deng et al. (2005) a closed-form portfolio rule is not possible due to the polyhedral nature of their ambiguity representation. The ellipsoidal model controls the diffidence of investors using a single positive parameter ϵ while the interval model of Deng et al. (2005) requires the specification of an interval for each risky asset, and has to resort to numerical solution of a linear programming problem to find a worst-case rate of return vector in the hyper-rectangle. The linear programming nature of the procedure may cause several components of the rate of return vector in question to assume the lower or upper end values of the interval as a by-product of the simplex method (i.e., an extreme point of the hyper-rectangle will be found). Since the worst case return occurs at an extreme point of the hyper-rectangle, it corresponds to an extreme scenario where most (or all) risky assets assume their worst possible return values, which may translate into an unnecessarily

☆ Revision: December 2013.

* Tel.: +90 3122902603.
E-mail address: mustafap@bilkent.edu.tr

http://dx.doi.org/10.1016/j.ejor.2014.02.016
0377-2217/

conservative portfolio. Such extreme behavior does not occur with an ellipsoidal representation of the uncertainty set due to the nonlinear geometry of the ellipsoid. Besides, the ellipsoidal representation is also motivated by statistical considerations alluded to in Section 2. As in Deng et al. (2005), in the contributions of Konno and Shirakawa (1994, 1995) where short sales are not allowed, the formula for the equilibrium price vector requires the solution of an optimization problem as input to the formula whereas we have an explicit formula for the equilibrium price.

To the best of our knowledge, the present paper is one the few studies next to Deng et al. (2005), Wu, Song, Xu, and Liu (2009) to incorporate ambiguity aversion in asset returns in an equilibrium framework. However, unlike the present paper, in neither Deng et al. (2005) nor Wu et al. (2009) there is a truly closed-form result, and furthermore they do not study the impact of ambiguity aversion on equilibrium prices.

The seminal results on equilibrium in capital markets were established in the early works of Lintner (1965), Mossin (1966) and Sharpe (1964), which resulted in the celebrated CAPM; see Elton and Gruber (1991), Markowitz (1987) for textbook treatments of the subject. The theory of equilibrium in capital asset markets were later extended in several directions in e.g., Black (1972), Nielsen (1987, 1989, 1990, 1989). In a recent study, Rockafellar, Uryasev, and Zabarankin (2007) use the so-called *diversion measures* (an example is Conditional Value at Risk, CVaR) to investigate equilibrium in capital markets. Balbás, Balbás, and Balbás (2010) use coherent risk measures, expectation bounded risk measures and general deviations in optimal portfolio problems, and study CAPM-like relations. Grechuk and Zabarankin (2012) consider an optimal risk sharing problem among agents with utility functionals depending only on the expected value and a deviation measure of an uncertain payoff. They characterize Pareto optimal solutions and study the existence of an equilibrium. Kalinchenko, Uryasev, and Rockafellar (2012) use the generalized CAPM based on mixed Conditional Value at Risk deviation for calibrating the risk preferences of investors. Hasuike (2010) use fuzzy numbers to represent investors' preferences in an extension of the CAPM. Zabarankin, Pavlikov, and Uryasev (2014) uses the Conditional Drawdown-at-Risk (CDaR) measure to study optimal portfolio selection and CAPM-like equilibrium models. Won and Yannelis (2011) examine equilibrium with an application to financial markets without a riskless asset where uncertainty makes preferences incomplete. They assume a normal distribution for the mean return with an uncertain mean, and adopt a min–max approach using an ellipsoidal representation as in the present paper.

In the present paper, we investigate the equilibrium implications of ambiguity aversion defined as diffidence vis à vis estimated mean returns. In particular, in a capital market in equilibrium where all investors fully trust estimated mean rates of return (they are ambiguity-neutral), if one investor decides to adopt an ambiguity-averse position, this shift may create a downward pressure on equilibrium prices. In uniform markets where all investors are ambiguity averse, the effect of ambiguity aversion is also deflationary with respect to a fully confident (ambiguity-neutral) investors market.

In summary, the contributions of the present are as follows:

- we use an ellipsoidal representation of the ambiguity in mean returns which avoids extreme scenarios, and thus alleviates the overly conservative nature of the resulting portfolios,
- our ellipsoidal ambiguity model allows for a truly closed-form portfolio rule,
- we establish a sufficient condition for a unique equilibrium price vector in financial markets with ambiguity averse investors, as well as a necessary and sufficient condition for existence of non-negative equilibrium prices,
- we have an explicit formula for the equilibrium price system in a market of mean–variance and ambiguity-averse investors, which reduces to a formula for the equilibrium prices of a market of mean–variance investors,
- we show the deflationary effect of the ambiguity aversion on risky asset prices,
- we establish a generalization of the CAPM which reverts to the original CAPM when all investors are ambiguity-neutral.

The paper is organized as follows. In Section 2 we examine the problem of portfolio choice of an ambiguity-averse investor using an ellipsoidal ambiguity set and worst case max–min criterion. We derive an explicit optimal portfolio rule. In Section 3, we study conditions under which an equilibrium system of prices exist in different capital markets characterized by the presence of ambiguity-averse or neutral investors, and give an explicit formula for equilibrium prices. We illustrate the results with a numerical example. Section 4 gives some properties of equilibrium. In particular, separation and proportion properties are shown, as well as a CAPM-like result which reduces to the classical CAPM when investors have full confidence in the estimates of mean rate of return. We conclude in Section 5 with a summary and future research directions.

2. Ambiguity-averse mean–variance investor's portfolio rule

Let the price per share of asset j be denoted $p_j, j = 1, 2, \ldots, n$ for the first n risky assets in the market, we assume the price of the $n+1$th riskless asset to be equal to one. We denote by x_j^0 the number of shares of asset j held initially by the investor while we use x_j to denote the number of shares of asset j held by the investor after the transaction, for all $j = 1, \ldots, n+1$. Unlimited short positions are allowed, i.e., there is no sign restriction on x_j.

The n risky assets have random rate of return vector $r = (r_1, r_2, \ldots, r_n)$ and estimate of mean rate of return vector $\hat{r} = (\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_n)$ (that we shall also refer to as the *nominal rate of return*) with variance–covariance matrix estimate Γ which is assumed positive definite. The $(n+1)$th position is reserved for the riskless asset with deterministic rate of return equal to R. The investor has a risk aversion coefficient $\omega \in (0, 1)$ and an initial endowment W_0 assumed positive such that

$$W_0 = \sum_{j=1}^{n} p_j x_j^0 + x_{n+1}^0.$$

Since there are no withdrawals from and injections to the portfolio, we still have, after the transaction,

$$W_0 = \sum_{j=1}^{n} p_j x_j + x_{n+1}.$$

Dividing the last equation by W_0 and defining the proportions $y_j \equiv \frac{p_j x_j}{W_0}$ for $j = 1, \ldots, n+1$ we have that

$$\sum_{j=1}^{n+1} y_j = 1.$$

If we denote the true (unknown) mean rates of return by $\mathbf{r}_j$ for $j = 1, \ldots, n$ the mean rate of return of portfolio x (with proportions y_j) is equal to

$$\sum_{j=1}^{n} \mathbf{r}_j y_j + R y_{n+1}$$

with variance equal to $\sum_{j=1}^{n}\sum_{k=1}^{n} \Gamma_{jk} y_j y_k = \mathbf{y}^T \Gamma \mathbf{y}$ where $\mathbf{y}$ denotes the vector with components $(y_1, \ldots, y_n)$. Note that the random end-of-period wealth W_1 is given as

$$W_1 = W_0\left[\sum_{i=1}^{n} r_i y_i + R y_{n+1}\right].$$

The investor is also ambiguity averse with ambiguity aversion coefficient ϵ such that his/her confidence in the mean rate of return vector estimate is expressed as a belief that the true mean rate of return lies in the ellipsoidal set

$$U_r = \{\mathbf{r} \| \Gamma^{-1/2}(\mathbf{r} - \hat{r})\|_2 \leqslant \epsilon\},$$

that is, an n-dimensional ellipsoid centered at $\hat{r}$ (the estimated mean return vector) with radius ϵ. The idea is that the decisions of an ambiguity averse investor are made by considering the worst case occurrences of the true mean rate of return $\mathbf{r}$ within the set U_r. Therefore, more conservative portfolio choices are made when the volume of the ellipsoid is larger, i.e. for greater values of ϵ, while an ambiguity-neutral investor with no doubt about errors in the estimated values sets ϵ equal to zero. The differences between the true mean rate of return $\mathbf{r}$ and its forecast $\hat{r}$ depend on the variance of the returns, hence they are scaled by the inverse of the covariance matrix. To quote Fabozzi, Kolm, Pachamanova, and Focardi (2007): "The parameter ϵ corresponds to the overall amount of scaled deviations of the realized returns from the forecasts against which the investor would like to be protected". Garlappi, Uppal, and Wang (2007) show that the ellipsoidal representation of the ambiguity of estimates may also lead to more stable portfolio strategies, delivering a higher out-of-sample Sharpe ratio compared to the classical Markowitz portfolios. It is also well-known (see Johnson & Wichern (1997, p. 212)), that the random variable

$$(\mathbf{r} - \hat{r})^T \Gamma^{-1}(\mathbf{r} - \hat{r}),$$

has a known distribution (F-distribution under standard assumptions on the time series of returns), and this fact can be exploited using a quantile framework to set meaningful values for ϵ in practical computation with return data, c.f. Garlappi et al. (2007).

The ambiguity-averse mean–variance investor is interested in choosing his/her optimal portfolio according to the solution of the following problem

$$\max_{\mathbf{y}} \min_{\mathbf{r}\in U_r} (1-\omega)(\mathbf{r}^T\mathbf{y} + (1 - e^T\mathbf{y})R) - \omega \mathbf{y}^T \Gamma \mathbf{y}$$

where e represents an n-vector of ones and the scalar $\omega \in (0,1)$ represents the degree of risk aversion of the investor. The larger the value of ω, the more risk averse (in the sense of aversion to variance of portfolio return) the investor. Processing the inner min we obtain as usual the problem:

$$\max_{\mathbf{y}} (1-\omega)(\hat{r}^T\mathbf{y} + (1 - e^T\mathbf{y})R - \epsilon\|\Gamma^{1/2}\mathbf{y}\|_2) - \omega \mathbf{y}^T \Gamma \mathbf{y}$$

that is referred to as AAMVP (abbreviation of Ambiguity Averse Mean–Variance Portfolio). Let $\hat{\mu} = \hat{r} - Re$. Hence we can re-write AAMVP as

$$\max_{\mathbf{y}} (1-\omega)(\hat{\mu}^T\mathbf{y} + R - \epsilon\|\Gamma^{1/2}\mathbf{y}\|_2) - \omega \mathbf{y}^T \Gamma \mathbf{y}.$$

Let us define the market optimal Sharpe ratio as $H^2 = \hat{\mu}^T \Gamma^{-1} \hat{\mu}$.

Proposition 1. *If $\epsilon < H$ then AAMVP admits the unique optimal solution*

$$y^* = \left(\frac{1-\omega}{2\omega}\right)\left(\frac{H-\epsilon}{H}\right)\Gamma^{-1}\hat{\mu}, y^*_{n+1} = 1 - \sum_{j=1}^{n}\left(\frac{1-\omega}{2\omega}\right)\left(\frac{H-\epsilon}{H}\right)(\Gamma^{-1}\hat{\mu})_j$$

i.e., an ambiguity-averse mean–variance investor with limited diffidence ($\epsilon < H$) makes the optimal portfolio choice in the risky assets

$$x_j^* = \left(\frac{W_0}{p_j}\right)\left(\frac{1-\omega}{2\omega}\right)\left(\frac{H-\epsilon}{H}\right)(\Gamma^{-1}\hat{\mu})_j, \quad j = 1, \ldots, n.$$

If $\epsilon \geqslant H$, then it is optimal for an AAMVP investor to keep all initial wealth in the riskless asset.

Proof. The function is strictly concave. The first-order necessary and sufficient conditions (assuming a solution $\mathbf{y} \neq 0$) yields the candidate solution:

$$\mathbf{y}^* = \left(\frac{(1-\omega)\sigma}{(1-\omega)\epsilon + 2\sigma\omega}\right)\Gamma^{-1}\hat{\mu},$$

where we defined $\sigma \equiv \sqrt{\mathbf{y}^T\Gamma\mathbf{y}}$. Using the definition of σ we obtain $(1-\omega)^2H^2 = ((1-\omega)\epsilon + 2\sigma\omega)^2$. Developing the parentheses on the right side we obtain a quadratic equation in σ

$$4\omega^2\sigma^2 + 4\omega(1-\omega)\epsilon\sigma + (1-\omega)^2(\epsilon^2 - H^2) = 0$$

with the positive root $\sigma_+ = \frac{(1-\omega)(H-\epsilon)}{2\omega}$ provided that $\epsilon < H$. Then the result follows by simple algebra. If $\epsilon \geqslant H$ then our supposition that a non-zero solution exists has been falsified, in which case we revert to the origin as the optimal solution. □

Notice that when the investor is not ambiguity averse, i.e., $\epsilon = 0$, one recovers the optimal portfolio rule of a mean–variance investor, namely,

$$\mathbf{y}^* = \left(\frac{1-\omega}{2\omega}\right)\Gamma^{-1}\hat{\mu}.$$

The factor $\frac{H-\epsilon}{H} < 1$ in the optimal portfolio of a diffident investor whose diffidence is bounded above by the slope of the Capital Market Line (we shall refer to such investors as *mildly diffident*, we shall also be using the terms *bounded diffidence* or *limited diffidence* in the same context), tends to curtail both long and short positions with respect to the portfolio of a fully confident (i.e., ambiguity-neutral) investor.

An alternative proof would proceed by exchanging the max and the min as in Deng et al. (2005). Solving the max problem first for fixed $\mathbf{r}$, one finds the point

$$\mathbf{y} = \frac{1-\omega}{2\omega}\Gamma^{-1}(\mathbf{r} - Re). \tag{1}$$

Then minimizing the resulting maximum

$$(1-\omega)R + \frac{(1-\omega)^2}{4\omega}(\mathbf{r} - Re)^T\Gamma^{-1}(\mathbf{r} - Re)$$

over the set U_r one finds the *worst case rate of return* r^* as the unique minimizer of the above function (this is missing in the analysis of Deng et al. (2005)):

$$r^* = \frac{H-\epsilon}{H}\hat{r} + \frac{\epsilon R}{H}e, \tag{2}$$

which when plugged into (1) for r results in the solution we have obtained in Proposition 1.

3. Existence of an equilibrium price system

In this section we shall analyze the existence of an equilibrium price system in capital markets where investors adopt or relinquish an ambiguity-averse attitude. First, we shall look at markets where all investors are either ambiguity-averse or ambiguity-neutral. We refer to such markets as *uniform markets*. Then, we investigate the effect on equilibrium prices of introducing an ambiguity-averse investor in a market of ambiguity-neutral investors. We shall refer to such markets as *mixed*.

We denote the price system by the vector $(p_1, p_2, \ldots, p_n)$ for the n risky assets. The price of the riskless asset is assumed to be equal to one. We make the following assumptions:

1. The total number of shares of asset j is $x_j^0, j = 1, 2, \ldots, n+1$.
2. Investors $i = 1, \ldots, m$ make their static portfolio choices according to the ambiguity-averse mean–variance portfolio model AAMVP of the previous section; they all agree on the nominal excess return vector $\hat{\mu}$ (i.e., they all agree on the same nominal rate of return vector $\hat{r}$ and the same riskless rate R) and positive-definite variance–covariance matrix Γ.
3. Investor i invests an initial wealth W_i^0 in an initial portfolio $(x_{i1}^0, x_{i2}^0, \ldots, x_{in+1}^0)$.
4. Investor i has risk aversion coefficient ω_i and ambiguity aversion coefficient (diffidence level) ϵ_i.

We have

$$\sum_{i=1}^{m} x_{ij}^0 = x_j^0, j = 1, 2, \ldots, n+1, \tag{3}$$

$$\sum_{j=1}^{n} p_j x_{ij}^0 + x_{in+1}^0 = W_i^0. \tag{4}$$

3.1. Uniform markets

Using the result from the previous section we have that each investor i holds the percentage portfolio

$$y_{ij}^* = \frac{1-\omega_i}{2\omega_i}\frac{H-\epsilon_i}{H}(\Gamma^{-1}\hat{\mu})_j, j = 1, 2, \ldots, n, \tag{5}$$

$$y_{in+1}^* = 1 - \sum_{j=1}^{n} y_{ij}^* = 1 - \frac{1-\omega_i}{2\omega_i}\frac{H-\epsilon_i}{H}\sum_{j=1}^{n}(\Gamma^{-1}\hat{\mu})_j \tag{6}$$

under the assumption that each investor i operates under limited diffidence, i.e., $\epsilon_i < H, i = 1, \ldots, m$. Passing to the corresponding asset portfolio holdings (shares) x_{ij}^* we have

$$x_{ij}^* = \frac{W_i^0 y_{ij}^*}{p_j} = \frac{W_i^0}{p_j}\frac{1-\omega_i}{2\omega_i}\frac{H-\epsilon_i}{H}(\Gamma^{-1}\hat{\mu})_j, \quad j = 1, 2, \ldots, n, \tag{7}$$

$$x_{in+1}^* = W_i^0 y_{in+1}^* = W_i^0\left(1 - \sum_{j=1}^{n} y_{ij}^*\right) = W_i^0\left(1 - \frac{1-\omega_i}{2\omega_i}\frac{H-\epsilon_i}{H}\sum_{j=1}^{n}(\Gamma^{-1}\hat{\mu})_j\right). \tag{8}$$

The market clearing condition requires the following equation to hold:

$$\sum_{i=1}^{m} x_{ij}^* = x_j^0, j = 1, 2, \ldots, n+1, \tag{9}$$

i.e., we have

$$\sum_{i=1}^{m}\frac{W_i^0}{p_j}\frac{1-\omega_i}{2\omega_i}\frac{H-\epsilon_i}{H}(\Gamma^{-1}\hat{\mu})_j = x_j^0, j = 1, 2, \ldots, n+1, \tag{10}$$

Re-arranging this equation and recalling (4) we have the equation system with n equations and n unknowns:

$$p_j x_j^0 = (\Gamma^{-1}\hat{\mu})_j \sum_{i=1}^{m}\left(\frac{1-\omega_i}{2\omega_i}\right)\left(\frac{H-\epsilon_i}{H}\right)\left(\sum_{k=1}^{n} p_k x_{ik}^0 + x_{in+1}^0\right),$$

$$j = 1, 2, \ldots, n. \tag{11}$$

Now, define for convenience $\zeta_j = (\Gamma^{-1}\hat{\mu})_j$ for $j = 1, 2, \ldots, n$ and

$$\alpha = \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\frac{1-\omega_i}{2\omega_i}\right)\left(\frac{H-\epsilon_i}{H}\right)\frac{x_{ij}^0}{x_j^0}\zeta_j.$$

Proposition 2. *In an ambiguity-averse mean–variance investors' market where every investor has limited diffidence (i.e., $\epsilon_i < H$ for all $i = 1, \ldots, m$) if $\alpha \neq 1$, then there exists a unique solution p^* to the equilibrium system (11) given by*

$$p_j^* = \frac{1}{1-\alpha}\frac{\zeta_j}{x_j^0}\sum_{i=1}^{m}\left(\frac{1-\omega_i}{2\omega_i}\right)\left(\frac{H-\epsilon_i}{H}\right)x_{in+1}^0, \quad j = 1, \ldots, n. \tag{12}$$

If $\sum_{i=1}^{m}\left(\frac{1-\omega_i}{2\omega_i}\right)\left(\frac{H-\epsilon_i}{H}\right)x_{ij}^0 \geqslant 0, j = 1, 2, \ldots, n+1$, and no investor is short on risky assets, i.e., $\zeta_j \geqslant 0$ for all $j = 1, \ldots, n$, then the market admits a unique non-negative equilibrium price vector p^* if and only if $\alpha < 1$.

Proof. The proof is almost identical to the proof of Theorem 4.1 in Deng et al. (2005) with minor modifications. Let

$$c_j = \sum_{i=1}^{m}\left(\frac{1-\omega_i}{2\omega_i}\right)\left(\frac{H-\epsilon_i}{H}\right)x_{ij}^0, \; j = 1, 2, \ldots, n+1,$$

and

$$d_j = \zeta_j / x_j^0, \; j = 1, 2, \ldots, n.$$

Let c be the vector with components $(c_1, \ldots, c_n)$ and d the vector with components $(d_1, \ldots, d_n)$. Then we can express α as $\alpha = c^T d$. The system (11) can now be re-written as

$$\begin{aligned} p_j &= \frac{\zeta_j}{x_j^0}\sum_{i=1}^{m}\left(\frac{1-\omega_i}{2\omega_i}\right)\left(\frac{H-\epsilon_i}{H}\right)\sum_{k=1}^{n} p_k x_{ik}^0 + \frac{\zeta_j}{x_j^0}\sum_{i=1}^{m}\left(\frac{1-\omega_i}{2\omega_i}\right)\left(\frac{H-\epsilon_i}{H}\right)x_{in+1}^0 \\ &= d_j\sum_{k=1}^{n} p_k \sum_{i=1}^{m}\left(\frac{1-\omega_i}{2\omega_i}\right)\left(\frac{H-\epsilon_i}{H}\right)x_{ik}^0 + d_j c_{n+1} \\ &= \sum_{k=1}^{n} c_k p_k + d_j c_{n+1}, \quad j = 1, 2, \ldots, n. \end{aligned}$$

In vector form we have the equation

$$p = d(c^T p) + c_{n+1} d,$$

or, equivalently

$$(I - dc^T)p = c_{n+1} d.$$

Then, when $\alpha \neq 1$ the system has the unique solution

$$p = c_{n+1}(I - dc^T)^{-1} d = c_{n+1}\left(I + \frac{dc^T}{1-\alpha}\right)d = \frac{c_{n+1}}{1-\alpha}d, \tag{13}$$

where the second equality follows from the Sherman–Morrison–Woodbury formula.[1] The rest of the proof consists of applying Farkas Lemma (c.f. chapter 2 of Mangasarian (1994)) to the system

$$(I - dc^T)p = c_{n+1} d, \quad p \geqslant 0,$$

and its alternative

$$(I - cd^T)y \leqslant 0, \quad d^T y > 0,$$

under the conditions $c \geqslant 0, \zeta_j \geqslant 0$ for all $j = 1, \ldots, n$ and $\alpha < 1$. If $\alpha < 1$, then the unique solution in (13) is non-negative. If $\alpha \geqslant 1$, then $y = c$ satisfies the alternative system, hence no non-negative equilibrium prices exist. □

The scalar α plays an important role in the existence of equilibrium results (see also Deng et al. (2005), Konno & Shirakawa (1995) and the scalar γ in Corollary 1 below). However, a financial interpretation of the condition involving α is missing from the literature. Note that the double summation in α, considered without

[1] $(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1} u}$.

the ratio term $\frac{x_{ij}^0}{x_j^0}$ (which represents the investor i's initial fraction of shares of asset j) would give the total of fraction portfolio holdings (y_{ij}^*) in the market, summed over all investors and all risky assets. Thus, the scalar α gives a measure of the weighted total of fraction portfolio holdings where each y_{ij}^* is weighted by the corresponding ratio $\frac{x_{ij}^0}{x_j^0}$. If this weighted total is strictly less than 1, an equilibrium price exists as is shown in the proposition above. The condition is also necessary. The condition

$$\sum_{i=1}^{m}\left(\frac{1-\omega_i}{2\omega_i}\right)\left(\frac{H-\epsilon_i}{H}\right)x_{ij}^0 \geqslant 0, \quad j=1,2,\ldots,n+1$$

also represents a weighted total of initial portfolio holdings over all investors in the market. The weight $\left(\frac{1-\omega_i}{2\omega_i}\right)\left(\frac{H-\epsilon_i}{H}\right)$ encodes the risk aversion and ambiguity aversion attitudes of the investor.

The existence of strictly positive prices is a harder question that is rarely addressed (with the exception of Rockafellar et al. (2007)) although zero prices would hardly make economic sense in practice. Interestingly, we can also prove the following negative result on the existence of a strictly positive system of equilibrium prices. If the condition of Proposition 2 $\sum_{i=1}^{m}\left(\frac{1-\omega_i}{2\omega_i}\right)\left(\frac{H-\epsilon_i}{H}\right)x_{ij}^0 \geqslant 0, \quad j=1,2,\ldots,n$ partially holds (only for the risky assets), i.e., a weighted total of initial portfolio holdings of risky assets over all investors in the market is non-negative, while this total is negative for the riskless asset then it is not possible to have positive equilibrium prices in the market.

Proposition 3. *If $\sum_{i=1}^{m}\left(\frac{1-\omega_i}{2\omega_i}\right)\left(\frac{H-\epsilon_i}{H}\right)x_{ij}^0 \geqslant 0, j=1,2,\ldots,n, c_{n+1}<0$, no investor is short on risky assets, i.e., $\zeta_j \geqslant 0$ for all $j=1,\ldots,n$, and $\alpha \in (0,1)$ then a strictly positive equilibrium price system does not exist in an ambiguity-averse mean–variance investors' market where every investor has limited diffidence (i.e., $\epsilon_i < H$ for all $i=1,\ldots,m$).*

Proof. We shall invoke the non-homogeneous Stiemke theorem (Stiemke, 1915) for the system:

$$(I-dc^T)p = c_{n+1}d, \ p>0,$$

The alternative of the above system according to Stiemke's theorem[2] is the system

$$\begin{pmatrix} I-cd^T \\ -c_{n+1}d \end{pmatrix} x \geqslant 0, \begin{pmatrix} I-cd^T \\ -c_{n+1}d \end{pmatrix} x \neq 0,$$

If $x=c$ then

$$\begin{pmatrix} I-cd^T \\ -c_{n+1}d \end{pmatrix} x = \begin{pmatrix} c(1-\alpha) \\ -c_{n+1}\alpha \end{pmatrix}.$$

Since by assumption we have $\sum_{i=1}^{m}\left(\frac{1-\omega_i}{2\omega_i}\right)\left(\frac{H-\epsilon_i}{H}\right)x_{ij}^0 \geqslant 0, j=1,2,\ldots,n$, we have $c \geqslant 0$. Due to the hypotheses that $\alpha \in (0,1)$ and $c_{n+1}<0$ we have $x=c$ that satisfies the alternative system. □

If the market consists of fully confident (in the mean rate of return estimates) investors (i.e., ambiguity-neutral), we have the following equilibrium result in a mean–variance capital market. Let us define for convenience

$$\gamma = \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\frac{1-\omega_i}{2\omega_i}\right)\frac{x_{ij}^0}{x_j^0}\zeta_j.$$

[2] Stiemke's Theorem: Either $A^Ty=b, y>0$ has a solution or $Ax \geqslant 0, -b^Tx \geqslant 0$, $\begin{pmatrix} Ax \\ -b^Tx \end{pmatrix} \neq 0$ has a solution, but never both, c.f. Chapter 6 of Panik (1993).

Corollary 1. *In a mean–variance investors' market (with no ambiguity aversion) if $\gamma \neq 1$, then there exists a unique solution p^* to the equilibrium system (11) given by*

$$p_j^{\mathrm{mv}} = \frac{1}{1-\gamma}\frac{\zeta_j}{x_j^0}\sum_{i=1}^{m}\left(\frac{1-\omega_i}{2\omega_i}\right)x_{in+1}^0, \quad j=1,\ldots,n. \tag{14}$$

If $\sum_{i=1}^{m}\left(\frac{1-\omega_i}{2\omega_i}\right)x_{ij}^0 \geqslant 0, j=1,2,\ldots,n+1$, and $\zeta_j \geqslant 0$ for all $j=1,\ldots,n$, then the market admits a unique non-negative equilibrium price vector p^* if and only if $\gamma<1$.

As in Proposition 2 the scalar γ gives a measure of the weighted total of fraction portfolio holdings where each y_{ij}^* is weighted by the corresponding ratio $\frac{x_{ij}^0}{x_j^0}$.

An interesting case is when all ambiguity-averse investors agree on the same level of limited diffidence, i.e., $\epsilon_i = \epsilon < H$ for all $i=1,\ldots,m$. In that case, the equilibrium price vector p^* has a simplified expression:

$$p_j^H = \frac{H-\epsilon}{H(1-\alpha)}\frac{\zeta_j}{x_j^0}\sum_{i=1}^{m}\left(\frac{1-\omega_i}{2\omega_i}\right)x_{in+1}^0, j=1,\ldots,n. \tag{15}$$

Obviously, the above expression implies $p_j^H = \frac{(H-\epsilon)(1-\gamma)}{H-(H-\epsilon)\gamma}p_j^{\mathrm{mv}}$. Now, since we have

$$0 < \frac{(H-\epsilon)(1-\gamma)}{H-(H-\epsilon)\gamma} = \frac{H-H\gamma+\epsilon\gamma-\epsilon}{H-H\gamma+\epsilon\gamma} < 1$$

as $\gamma<1$ in equilibrium, and $H>\epsilon>0$. Therefore, in a homogeneously and mildly diffident ambiguity-averse mean–variance investors' market (where diffidence is bounded above by the slope of the Capital Market Line), equilibrium prices are under downward pressure with respect to a purely confident mean–variance investors' market. We summarize these observations below.

Proposition 4. *In a homogeneously and mildly diffident (where all investors have the same $\epsilon<H$) ambiguity-averse mean–variance investors' market in equilibrium prices are smaller than the equilibrium prices in a pure mean–variance investors' market.*

Another interesting observation is the following. Assume no investor has an initial liability, i.e., $x_{ij}^0>0$ for all $i=1,\ldots,m$ and $\zeta_j>0$ for all $j=1,\ldots,n$. Then we have the immediate consequence that $\alpha<\gamma$. This implies straightforwardly that $p_j^* < p_j^{\mathrm{mv}}$, for all $j=1,\ldots,n$. In other words, in an ambiguity-averse mean–variance investors market with bounded diffidence, if all investors have long initial positions, then equilibrium leads to smaller prices compared to the equilibrium prices of purely mean–variance investors' market, everything else being equal. Hence, the introduction of ambiguity aversion or diffidence in rate of return estimates into a market with all positive initial positions creates a deflationary pressure on equilibrium prices.

A Numerical Example. For illustration we consider an example with three investors and three assets (two risky assets and one riskless asset). The relevant data for the risky assets are specified as follows

$$\hat{\mu} = (0.1287\ 0.1096)^T$$

$$\Gamma = \begin{bmatrix} 0.4218 & 0.0530 \\ 0.0530 & 0.2230 \end{bmatrix}.$$

We assume $x_j^0 = 10$ for all three assets $j=1,2,3$, and the initial portfolio holdings

$$[4\ 3\ 3]^T; [6\ 2\ 2]^T; [3\ 3\ 4]^T$$

for each asset respectively, e.g., investor 1 holds initially 4 shares of asset 1, 6 shares of asset 2 and 3 units of the riskless asset. We have $H=0.2822$ and $\zeta = \Gamma^{-1}\hat{\mu} = [0.2509\ 0.4319]^T$. In Fig. 1 we plot the

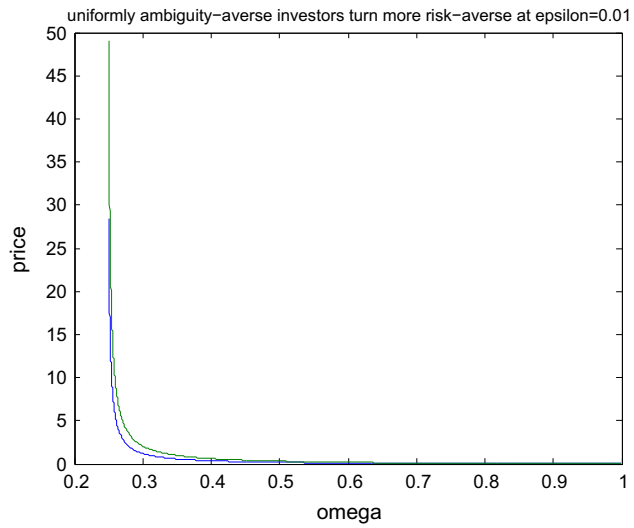

Fig. 1. Effect of increasing risk aversion coefficient ω when all investors are equally ambiguity averse with $\epsilon_i = 0.01$ for $i = 1, 2, 3$.

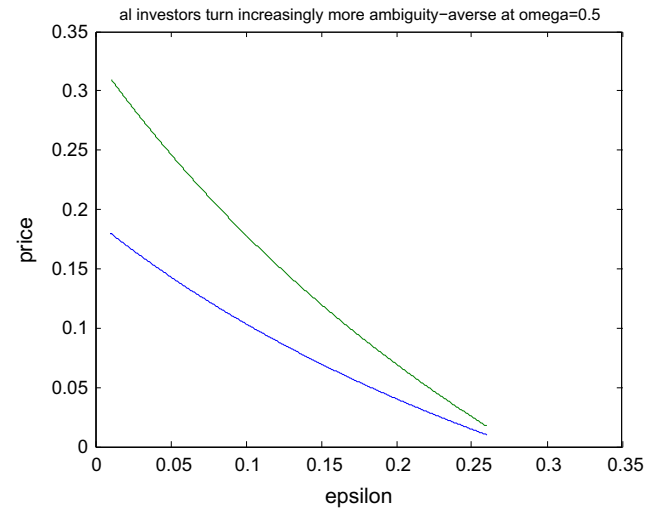

Fig. 3. Effect of increasing ambiguity aversion equally across the board with $\omega_i = 0.5$ for $i = 1, 2, 3$.

evolution of the prices of the two risky assets in a uniformly ambiguity-averse investors' market with $\epsilon_i = 0.01$ for $i = 1, 2, 3$. Increasing ω, i.e., increasing the risk aversion of investors (expressed as an increasing emphasis on a smaller variance of portfolio return) equally for all investors while ambiguity aversion remains fixed across the board has a sharp deflationary effect on asset prices. In Figs. 2 and 3 we show the impact of increasing ambiguity aversion equally for all investors at two different levels of risk aversion, $\omega = 0.25$ and $\omega = 0.5$, respectively. Both figures show clearly the deflationary effect on asset prices of increasing ambiguity aversion at both levels of risk aversion. The decrease in prices in response to an increase in ambiguity aversion is much more pronounced when the investors are less risk-averse at $\omega = 0.25$.

3.2. Mixed markets

Consider now a uniform market with ambiguity-neutral investors where an investor decides to adopt an ambiguity-averse position. For simplicity we shall examine the case where we have two investors. Investor indexed 1 is ambiguity-neutral with risk aversion coefficient ω_1, investor indexed 2 is ambiguity-averse with coefficient $\epsilon < H$ and risk aversion coefficient ω_2. All other assumptions about the assets traded in the market are still valid.

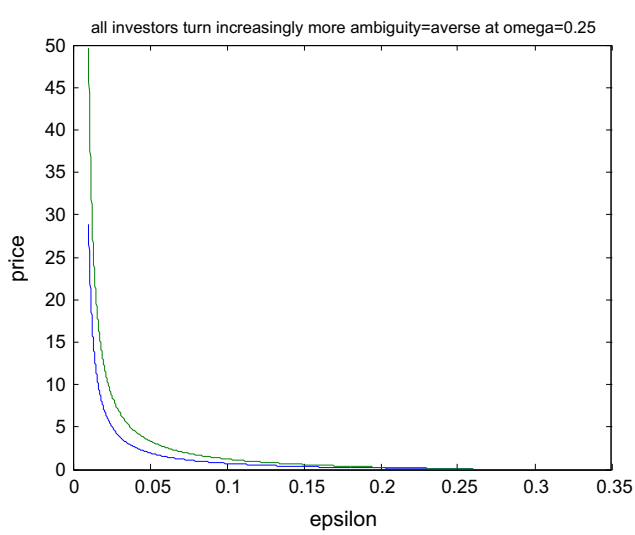

Fig. 2. Effect of increasing ambiguity aversion equally across the board with $\omega_i = 0.25$ for $i = 1, 2, 3$.

The ambiguity-neutral investor makes the portfolio choice

$$x_{1j} = \frac{W_1^0}{p_j}\frac{1-\omega_1}{2\omega_1}\zeta_j, j = 1, \ldots, n, x_{1n+1} = W_1^0\left(1 - \frac{1-\omega_1}{2\omega_1}\sum_{j=1}^{n}\zeta_j\right),$$

while the ambiguity-averse investor makes the choice

$$x_{2j} = \frac{W_2^0}{p_j}\frac{(1-\omega_2)(H-\epsilon)}{2H\omega_2}\zeta_j, j = 1, \ldots, n,$$

$$x_{2n+1} = W_2^0\left(1 - \frac{(1-\omega_2)(H-\epsilon)}{2H\omega_2}\sum_{j=1}^{n}\zeta_j\right).$$

As in the proof of Proposition 2 we define

$$c_j^1 = \frac{1-\omega_1}{2\omega_1}x_{1j}^0 \; j = 1, \ldots, n+1;$$

for investor 1, and

$$\tilde{c}_j^2 = \frac{1-\omega_2}{2\omega_2}\frac{H-\epsilon}{H}x_{2j}^0 \; j = 1, \ldots, n+1;$$

for investor 2, and $d_j = \zeta_j / x_j^0$ for $j = 1, \ldots, n$. Then we can express the equilibrium price system for the mixed market as

$$p^m = \frac{c_{n+1}^1 + \tilde{c}_{n+1}^2}{1-\alpha^m}d,$$

where $\alpha^m = d^T(c^1 + \tilde{c}^2)$, and we assume that the conditions guaranteeing the non-negativity of p^m as expressed in Proposition 2 hold.

Now, we compare the equilibrium price system p^m to the equilibrium price system of a uniform ambiguity-neutral investors market. I.e., if investor 2 were to be ambiguity-neutral as well, we would have the following price system p^p:

$$p^p = \frac{c_{n+1}^1 + c_{n+1}^2}{1-\alpha^p}d,$$

where $\alpha^p = d^T(c^1 + c^2)$ with

$$c_j^2 = \frac{1-\omega_2}{2\omega_2}x_{2j}^0 \; j = 1, \ldots, n+1.$$

We assume again the conditions guaranteeing the non-negativity of p^p expressed in Corollary 1 hold. Now, it is a simple exercise to see that

$$\tilde{c}_j^2 = \frac{H-\epsilon}{H} c_j^2,\ j = 1, \ldots, n+1$$

and

$$\alpha^m = \alpha^p + \left(\frac{H-\epsilon}{H} - 1\right) d^T c^2.$$

These observations imply that

$$p^m = \frac{c_{n+1}^1 + \frac{H-\epsilon}{H} c_{n+1}^2}{1 - \alpha^p + \left(1 - \frac{H-\epsilon}{H}\right) d^T c^2} d.$$

Therefore, if $c_{n+1}^2 > 0$ and $d^T c^2 > 0$, we have $p^m < p^p$, i.e., if an investor with positive initial holdings moves from ambiguity-neutral position to (bounded) ambiguity-averse position, this change has a deflationary effect on equilibrium prices. We summarize this result below. We define

$$c_j^i = \frac{1-\omega_i}{2\omega_i} x_{ij}^0\ j = 1, \ldots, n+1, \quad i = 1, \ldots, m$$

for every investor i, and refer to the n-vector with components $(c_1^i, \ldots, c_n^i)$ as c^i.

Proposition 5. *In a uniform market of m ambiguity-neutral mean–variance investors in equilibrium, assume investor m adopts an ambiguity-averse attitude with coefficient $\epsilon < H$. Then the following statements hold:*

1. *A non-negative equilibrium price system*

$$p^m = \frac{\sum_{i=1}^{m-1} c_{n+1}^i + \tilde{c}_{n+1}^m}{1 - \alpha^m} d$$

exists, if and only if $\alpha^m < 1$ where α^m is defined as

$$\alpha^m = d^T \left(\sum_{i=1}^{m-1} c^i + \tilde{c}^m\right).$$

2. *If the initial holdings x_{mj}^0 for all $j = 1, \ldots, n+1$ of investor m are positive, the equilibrium prices of the mixed market are smaller than the equilibrium prices of the uniform market.*

The above result is not surprising if one bears in mind that an ambiguity-averse investor holds smaller long positions in risky assets compared to an ambiguity-neutral investor, which leads to a decreased demand for risky assets, and hence exerts a downward pressure on equilibrium prices.

A similar analysis can be made when the ambiguity aversion of one investor is not classified as mildly diffident, but rather, *significantly diffident*, i.e., with $\epsilon \geqslant H$, in which case this investor would put all his/her initial wealth into the riskless asset. It can be shown again that such behavior leads to a drop in equilibrium prices. This is left as an exercise.

4. Properties of the equilibrium price system

We devote this section to the study of some interesting properties of portfolios in equilibrium. More precisely, we follow the references Deng et al. (2005), Konno and Shirakawa (1994), Konno and Shirakawa (1995) to examine the properties of the portfolios in equilibrium in a market of mildly diffident mean–variance investors. Define the *master fund* $z_j^* = \zeta_j / e^T \zeta$ for all $j = 1, \ldots, n$. We begin with the following *two-fund separation property*. Let us define $A = e^T \Gamma^{-1} e$ and $B = e^T \Gamma^{-1} \hat{r}$.

Proposition 6. *Let the price system in the mildly diffident mean–variance investors' market be as defined in (12). Then, after the transaction, each investor i holds*

(i) *a portfolio composed of the riskless asset and a non-negative multiple λ_i of the initial total holdings $x^0 = (x_1^0, x_2^0, \ldots, x_n^0)$ of risky assets, where $\sum_{i=1}^m \lambda_i = 1$ and*

$$\lambda_i = \frac{(1-\alpha) W_i^0 (1-\omega_i)(H-\epsilon_i)}{\omega_i \sum_{k=1}^m \frac{1-\omega_k}{\omega_k} (H-\epsilon_k) x_{kn+1}^0}$$

for $i = 1, \ldots, m$;

(ii) *a percentage portfolio which is a linear $(\xi_i, 1-\xi_i)$ combination of the percentage riskless portfolio $(0, 0, \ldots, 0, 1)$ and the (augmented) master fund $(z_1^*, z_2^*, \ldots, z_n^*, 0)$ consisting only of risky assets where $\xi_i = \frac{1-\omega_i}{\omega_i} \frac{H-\epsilon}{H} (B - RA)$.*

Proof. Recall that in equilibrium each investor i holds the optimal portfolio

$$x_{ij}^* = \frac{W_i^0 (1-\omega_i)(H-\epsilon_i)\zeta_j}{2\omega_i H p_j^*} = \frac{W_i^0 (1-\omega_i)(H-\epsilon_i)\zeta_j}{2\omega_i \frac{1}{1-\alpha} \frac{\zeta_j}{x_j^0} \sum_{k=1}^m \frac{1-\omega_k}{2\omega_k (H-\epsilon_k)} x_{kn+1}^0}$$

$$= \frac{W_i^0 (1-\omega_i)(H-\epsilon_i)(1-\alpha)}{\omega_i \sum_{k=1}^m \frac{1-\omega_k}{\omega_k} (H-\epsilon_k) x_{kn+1}^0} x_j^0.$$

Since we have $\sum_{i=1}^m x_{ij}^* = x_0^j$, we infer immediately that $\sum_{i=1}^m \lambda_i = 1$. This proves part (i).

For part (ii), recall that $y_{ij}^* = \frac{(1-\omega_i)(H-\epsilon_i)}{2\omega_i H} \zeta_j$ and $y_{in+1}^* = 1 - \frac{(1-\omega_i)(H-\epsilon_i)}{2\omega_i H} e^T \zeta$. Since $e^T \zeta = B - RA$, we can re-write $y_{ij}^* = \frac{(1-\omega_i)(H-\epsilon_i)}{2\omega_i H} (B - RA) z_j^*$ and $y_{in+1}^* = 1 - \frac{(1-\omega_i)(H-\epsilon_i)}{2\omega_i H} (B - RA)$. Hence, the result follows. □

We note that the weight ξ_i in part (ii) of the previous result is smaller than the corresponding weight that would result if ϵ_i were taken equal to zero, i.e., the investor were ambiguity-neutral. This observation implies that ambiguity aversion leads to giving less weight to master fund z^*.

Let the vector y^M and z^M be defined with components

$$y_j^M = \frac{x_j^0 p_j}{\sum_{j=1}^{n+1} x_j^0 p_j}, \quad j = 1, 2, \ldots, n+1,$$

and

$$z_j^M = \frac{x_j^0 p_j}{\sum_{j=1}^{n+1} x_j^0 p_j}, \quad j = 1, 2, \ldots, n,$$

called, respectively, the market portfolio of all assets and the market portfolio of risky assets in Deng et al. (2005). We also have the following *proportion property*.

Proposition 7. *Let the capital market be in equilibrium. Then the following hold:*

(i) *the market portfolio y^M is proportional to the market portfolio z^M of risky assets;*

(ii) *the market portfolio z^M of risky assets is identical to z^*.*

Proof. Using the definition of y^M we have

$$y_j^M = \frac{\sum_{i=1}^m x_{ij}^* p_j}{\sum_{j=1}^n \sum_{i=1}^m x_{ij}^* p_j + \sum_{i=1}^m x_{in+1}^*}$$

$$= \frac{\left[\sum_{i=1}^m \frac{(1-\omega_i)(H-\epsilon_i)}{2\omega_i H} W_i^0\right] \zeta_j}{\sum_{j=1}^n \left[\sum_{i=1}^m \frac{(1-\omega_i)(H-\epsilon_i)}{2\omega_i H} W_i^0\right] \zeta_j + \sum_{i=1}^m W_i^0 \left(1 - \frac{(1-\omega_i)(H-\epsilon_i)}{2\omega_i H} e^t \zeta\right)}$$

$$= \frac{\left[\sum_{i=1}^m \frac{(1-\omega_i)(H-\epsilon_i)}{2\omega_i H} W_i^0\right] \zeta_j}{\sum_{i=1}^m W_i^0} = \frac{(B - RA)\left[\sum_{i=1}^m \frac{(1-\omega_i)(H-\epsilon_i)}{2\omega_i H} W_i^0\right]}{\sum_{i=1}^m W_i^0} z_j^*.$$

For the second part we have

$$z_j^M = \frac{\sum_{i=1}^m x_{ij}^* p_j}{\sum_{j=1}^n \sum_{i=1}^m x_{ij}^* p_j} = \frac{\left[\sum_{i=1}^m \frac{(1-\omega_i)(H-\epsilon_i)}{2\omega_i H} W_i^0\right]\zeta_j}{\sum_{j=1}^n \left[\sum_{i=1}^m \frac{(1-\omega_i)(H-\epsilon_i)}{2\omega_i H} W_i^0\right]\zeta_j} = z_j^*. \qquad \square$$

Let the random (uncertain) rate of return of the market portfolio be denoted by

$$r_M = \sum_{j=1}^n r_j z_j^M,$$

with the worst-case value

$$\bar{r}_M = \mathbb{E}[r_M] = \sum_{j=1}^n r_j^* z_j^M.$$

where r^* is as defined in (2). It is the rate of return where the maximum in the min – max portfolio selection model AAMVP of Section 2 is attained. Then, we have the following CAPM-like property which expresses the nominal excess rate of return of risky asset j as proportional to the worst-case excess rate of return of the market portfolio of risky assets. In addition to the terms that are encountered in the classical CAPM, the proportionality also depends on the square root of the market optimal Sharpe ratio H^2 and the ambiguity aversion coefficient ϵ.

Proposition 8. *Let a capital market of homogeneously diffident investors with common $\epsilon < H$ be in equilibrium. Then the excess nominal rate of return on each risky asset is proportional to the excess worst-case rate of return on the market portfolio of risky assets; i.e., the following holds*

$$\hat{r}_j - R = \frac{H\mathrm{cov}[r_j, r_M]}{(H-\epsilon)\mathrm{Var}[r_M]}(\bar{r}_M - R), \quad j = 1, 2, \ldots, n.$$

Proof. Let us re-write $z^M = z^* = \frac{H}{(H-\epsilon)(B-RA)}\Gamma^{-1}(r^* - Re)$ where r^* is defined in (2) of Section 2. Then, we have

$$\mathrm{Var}[r_M] = (z^M)^T \Gamma z^M = \frac{H(r_M - R)}{(H-\epsilon)(B-RA)}$$

and

$$\mathrm{cov}[r_j, r_M] = e_j^T \Gamma z^M = \frac{H\left(r_j^* - R\right)}{(H-\epsilon)(B-RA)}$$

where e_j is the n-vector with all components equal to zero except the jth component which is equal to one. Then, the result follows by taking the ratio $\frac{\mathrm{cov}[r_j, r_M]}{\mathrm{Var}[r_M]} = \frac{r_j^* - R}{r_M - R}$ and recalling the definition (2) of r^*. $\square$

Note that this result reduces to the classical CAPM when $\epsilon = 0$, i.e., there is no ambiguity aversion, r^* reduces to $\hat{r}$ (which we can take as the true mean rate of return when no ambiguity aversion is present), and the coefficient $\frac{H}{H-\epsilon}$ is equal to one. A possible interpretation of the previous result in terms of the classical CAPM is as follows. Recall that in classical CAPM, the factor of proportionality $\frac{\mathrm{cov}[r_j, r_M]}{\mathrm{Var}[r_M]}$ is called the *beta* of asset j (written β_j) and tells us how the nominal risk of this asset is correlated with the nominal risk of the whole market. If β_j is positive, then the risk of the asset is positively related to the market, and the investor holding that asset is partaking to the risk of the market and gets a premium for taking this position. If β_j is negative, the risk of the asset is inversely related with the risk of the market, i.e., if the market pays well, the asset pays poorly and vice versa. In our version of the CAPM like result, we have the *beta* that is scaled by the ratio $\frac{H}{H-\epsilon}$ which is a number larger than one when we have $0 < \epsilon < H$. Therefore, the constant of proportionality and hence the *new beta* which relates in our case the nominal excess return to the total worst case return of the market is larger than the beta of the classical CAPM.

5. Conclusions

In this paper, we analyzed existence of equilibrium in a financial market composed of risky assets and a riskless asset, where mean–variance investors can display aversion to ambiguity, i.e., aversion to imprecision in the estimated mean rates of return of risky assets. We first derived a closed-form optimal portfolio rule for a mean–variance investor with aversion to ambiguity modeled using an ellipsoidal uncertainty set, borrowing the concept from robust optimization. The optimal portfolio rule reduces to the portfolio choice of a mean–variance investor when the investor is ambiguity-neutral. We examined conditions under which an equilibrium exists in a market of ambiguity-averse investors as well as conditions that lead to deflationary pressure on equilibrium prices with respect to a pure mean–variance investors' (i.e., ambiguity-neutral) market. A CAPM-like result is derived, which reduces to the usual CAPM in the absence of ambiguity aversion.

Future research can address equilibrium in the absence of the riskless asset, limitations or exclusion of short sales, equilibrium with other risk measures such as robust CVaR or expected shortfall under mean return ambiguity, and equilibrium under transaction costs.

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