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Decision Support

Style goods pricing with demand learning

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ABSTRACT

For many industries (e.g., apparel retailing) managing demand through price adjustments is often the only tool left to companies once the replenishment decisions are made. A significant amount of uncertainty about the magnitude and price sensitivity of demand can be resolved using the early sales information. In this study, a Bayesian model is developed to summarize sales information and pricing history in an efficient way. This model is incorporated into a periodic pricing model to optimize revenues for a given stock of items over a finite horizon. A computational study is carried out in order to find out the circumstances under which learning is most beneficial. The model is extended to allow for replenishments within the season, in order to understand global sourcing decisions made by apparel retailers. Some of the findings are empirically validated using data from U.S. apparel industry.

1. Introduction

Fashion goods such as ski-apparel or sunglasses are characterized by high degrees of demand uncertainty. Most of the merchandise in this category are new designs. Although some of the demand uncertainty may be resolved using sales history of similar merchandise offered in previous years, most of the uncertainty still remains due to the changing consumer tastes and economic conditions every year. Retailers of these items face long lead times and relatively short selling seasons that force them to order well in advance of the sales season with limited replenishment opportunities during the season. Demand and supply mismatches due to this inflexible and highly uncertain environment result in forced mark-downs or shortages. Frazier [22] estimates that the forced mark-downs average 8% of net retail sales in apparel industry, which he states is also an indication of as much as 20% in lost sales from stock-outs. He estimates that the overall resulting revenue losses of the industry may be as much as $25 billion.

In 1985, U.S. textile and apparel industry initiated a series of business practices and technological innovations, called Quick Response, to cut down these costs and to be able to compete with foreign industry enjoying lower wages. Quick Response aims to shorten lead times through improvements in production and information technology. As a result, production and ordering decision can be shifted closer to the selling season, which will help to resolve some uncertainty. Moreover, additional replenishment opportunities during the season may be created. See Hammond and Kelly [25] for a review of Quick Response and Şen [38,39] for reviews of operations and current business practices and trends in the U.S. apparel industry.

Despite the efforts of domestic manufacturers to remain competitive in this industry, retailers are using more and more imports to source their apparel, preferring cost advantage over responsiveness. For most imported apparel and some domestic apparel, managing demand through price adjustments is often the only tool left to retailers once the buying decisions take place. These adjustments are usually in the form of mark-downs in the apparel industry. Fisher et al. [20] note that 25% of all merchandise sold in department stores in 1990 was sold with mark-downs. Systems that can intelligently decide the timing and magnitude of such mark-downs may help balance the supply and demand and improve the profits of these companies operating with thin margins. Despite enormous amount of data made available to decision makers, such intelligent systems have found limited use in the apparel industry. Recent academic research such as Gallego and van Ryzin [23] and Bitran and Mondschein [6] successfully model dynamic pricing of a given stock of items when the demand is probabilistic and price sensitive. These studies assume that the retailer's estimate of the demand does not change over the course of the season. However, substantial amount of uncertainty about the demand process can be resolved using the early sales information.

The purpose of this paper is to develop a dynamic pricing model that incorporates demand learning. By demand learning, we mean learning by using the early sales information during the selling season as opposed to improving forecasts over time before the start of

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the season. Observing sales can facilitate learning about the magnitude of the demand or functional form of the demand–price relationship, or both. Demand learning can be used to eliminate a considerable portion of demand uncertainty in the apparel industry. A consultant at Dayton Hudson Corp. states "a week after an item hits the floor, a merchant knows whether it's going to be a dog or a best-seller" (Chain Store Age [12]). For our pricing only model, we assume that the ordering decision has already been made with the best use of pre-season information and no further replenishment opportunities are available to the retailer. Basically, the model uses a Bayesian approach to update retailer's estimate of a demand parameter. Our model enables us to summarize sales and price history in a direct way to set the problem as a computationally feasible dynamic program. We also conduct a numerical study to analyze the impact of different factors on pricing decisions. First, we study how the accuracy and degree of uncertainty of the initial demand magnitude estimates, starting stock levels and price sensitivity of customers impact optimal price paths and expected revenues. We are also interested in finding the conditions under which earlier sales information has the most impact on revenues and whether it is always optimal to use this information. We also study the impact of demand function uncertainty on expected revenues obtained through demand learning. Finally, we extend the model to account for the possibility of re-ordering during the selling season. This helps us to understand the possible trade-offs for using quicker but more costly domestic manufacturing to achieve such flexibility.

Next, we review literature on Bayesian learning in inventory control and dynamic pricing of fashion goods. We present our basic model in Section 3. Our computational analysis is in Section 4. Section 5 studies the effects of inventory flexibility during the horizon. Section 6 states our conclusions and avenues for future research.

2. Literature survey

Inventory models that incorporate the updating of demand forecasts have been studied extensively. Most of these models utilize a Bayesian approach to update demand parameters of a periodic inventory model. Demand in one period is assumed to be random with a known distribution but with an unknown parameter (or unknown parameters). This unknown parameter has a prior probability distribution, which reflects the initial estimates of the decision maker. Observed sales are then used to find a posterior distribution of the unknown parameter using Bayes' rule. As more observations become available, uncertainty is resolved and the distribution of the demand approaches its true distribution. The prior distribution of the unknown parameter should be such that the posterior distribution is similar to the prior, which could be calculated easily. In addition, the demand distribution and the distribution of the unknown parameter should enable the decision maker to summarize information such that a dynamic program to solve the problem is computationally feasible. See DeGroot [15, Chapter 9] for such distributions.

Demand learning in inventory theory using a Bayesian approach is first studied by Scarf [33]. He studies a simple periodic inventory problem in which at the beginning of each period the problem is how much to order with the assumption of linear inventory holding, shortage and ordering costs and an exponential family of demand distributions with an unknown parameter. The distribution of the unknown parameter is updated after each period using Bayes' rule. He formulates the problem as a stochastic dynamic program and among other results, shows that the optimal policy is to order up to a critical level and the critical level for each period is an increasing function of the past cumulative demand. Iglehart [26] extends the results of Scarf [33] to account for a range family of distributions and convex inventory holding and shortage costs. Azoury and Miller [3] show that in most cases non-Bayesian order quantities are greater than Bayesian order quantities, but also state that this may not always be true. The dynamic programs used in these studies have two-dimensional state spaces, one for the starting inventory level and one for the cumulative sales. Scarf [34] and Azoury [4] show that the two-dimensional dynamic program can be reduced to one-dimensional for some specific demand distributions.

A particular form of Bayesian approach to demand learning is assuming Poisson demand with an unknown rate in each period. The unknown demand rate's prior distribution is assumed to be Gamma, resulting in an unconditional prior distribution of demand, which can be shown to be Negative Binomial. Posterior distributions are also Gamma and Negative Binomial whose parameters can be calculated by using only cumulative demand. These specific distributions are used to model inventory decisions of aircraft spare parts by Brown and Rogers [10]. Popovic [32] extends the model to account for non-constant demand rates.

Demand learning models are most valuable to inventory problems of style goods that are characterized with moderate to extreme degrees of demand uncertainty that is resolvable significantly by observing early sales. Murray and Silver [30] use a Bayesian model in which the purchase probability of homogeneous customers is unknown but distributed priorly with a Beta distribution. This distribution is updated after each period to optimize inventory levels in succeeding periods. Chang and Fyfee [13] present an alternative approach to demand learning. Their model defines the demand in each period as a noise term plus a fraction of total demand, which is a random variable whose distribution is revised once the sales information becomes available each period. Bradford and Sugrue [9] use the Negative Binomial demand model described earlier to derive optimal inventory stocking policies in a two-period style-goods context.

Fisher and Raman [21] propose a production planning model for fashion goods that uses early sales information to improve forecasts. Their model, which is called Accurate Response, also considers the constraints in the production systems such as production capacity and minimum production quantities. Iyer and Bergen [27] study the Quick Response systems, where the retailers have more information about upcoming demand due to the decreased lead times. They use Bayesian learning to address whether the retailer or the manufacturer wins under such systems. Eppen and Iyer [16] develop a different methodology for Bayesian learning of demand. The demand process is assumed to be one of a set of pure demand processes with discrete prior distribution. This distribution is updated periodically based on Bayes' rule. This demand model is used in a dynamic programming formulation to derive the initial inventory levels and how much to divert periodically to a secondary outlet for a catalog merchandiser. Eppen and Iyer [17] use the same demand model to study the impact of backup agreements on expected profits and inventory levels for fashion goods. Gurnani and Tang [24] study the effect of forecast updating on ordering of seasonal products. Their model allows the retailer to order at two instants before the selling season. The forecast quality may be improved in the second instance, but the cost may either decrease or increase probabilistically.

All of the studies above ignore one crucial aspect of the problem: pricing. In economics literature, Lazear [28] studies clearance sales where he uses Bayesian learning to update the reservation price distribution after observing early sales in the season. However, his model considers the initial and the mark-down prices of a single item and thus lacks the dynamics of price adjustments for a stock of items. Balvers and Casimano [5] incorporate Bayesian learning in pricing models, but they assume a completely flexible supply and ignore inventories that link the pricing decisions. Style goods, on the other hand, face supply inflexibility as a result of short seasons, long lead times and limited production capacities. This characteristic of the problem gave rise to models such as those in Gallego and van Ryzin [23] and Bitran and Mondschein [6] that dynamically price the perishable good over the selling season. Both of these models assume that there is no replenishment opportunity and the only decisions to be made are the timing and magnitude of price changes over the course of the season. Gallego and van Ryzin [23] use a Poisson process for demand where the demand rate depends on the price of the product. Monotonicity results as a function of the remaining stock level and remaining time in the selling season are derived via a dynamic continuous-time model. Among other results, they show that the optimal profit of the deterministic problem, in which demand rates are assumed to be constant, gives an upper bound for the optimal expected profit. For the continuous price case, fixed-price heuristics are shown to be asymptotically optimal. For the discrete price case, a deterministic solution can be used to develop again asymptotically optimal heuristics. Feng and Gallego [18] derive the optimal policy for the two price case. In Bitran and Mondschein's [6] model, the purchase process for a given price is determined by a Poisson process for the store arrival and a reservation price distribution. They show that the model is equivalent to the model in Gallego and van Ryzin [23]. They also show that the loss associated with preferring a discrete-time rather than a continuous-time model is small. Smith and Achabal [35] study clearance pricing in retailing. Their model is deterministic, but incorporates impact of reduced assortment and seasonal changes on demand rates. Petruzzi and Dada [31] consider a periodic review model where the retailer is allowed to order new inventory as well as change the price at each period. However, the stochastic component of their demand model is very specific. If the retailer can fully satisfy the demand in any period, the uncertainty is completely resolved and the remaining problem is a deterministic one. Otherwise, the retailer updates the lower bound for the uncertain component, the remaining problem remains to be a stochastic one, with a new estimate for the uncertain component.

Recently, three closely related papers discuss Bayesian learning in pricing of style goods. Subrahmanyan and Shoemaker [37] develop a general periodic demand learning model to optimize pricing and stocking decisions. As in Eppen and Iyer [16,17], they use a set of possible demand distribution functions for each period and a discrete prior distribution that tabulates the probability of these possible demand distributions being the true demand distribution. This discrete distribution is updated after each period using the Bayes' rule. The information requirements are extremely large in a general model as updating requires the history of sales, inventory levels and prices in each period. They present computational results on specific demand and price parameters. Bitran and Wadhwa [7] consider only the pricing decisions utilizing the two-phased demand model and discrete-time dynamic programming formulation in Bitran and Mondschein [6]. A Poisson process for store arrival and a reservation price distribution are used to define the purchase process. They assume that uncertainty is involved in a parameter of this reservation price distribution. An updating procedure on this parameter is proposed such that the rate of the purchase process has Gamma priors and posteriors. The methodology allows them to summarize all sales and price information in two variables. They present computational results to show the impact of demand learning on prices and expected profits. Aviv and Pazgal [2] study a problem where the arrival process is Poisson, the arrival rate has a Gamma distribution and the retailer controls the price continuously. The resulting model is a continuous-time optimal control problem. Among other results, it is shown that initial high variance leads to higher prices and the expected revenues of the optimal pricing policy are compared with expected revenues from several other policies including a fixed price scheme. Our model differs from previous work in the literature, as we utilize demand learning to resolve uncertainty about the demand function as well as the magnitude of the demand in a periodic setting.

3. Model

3.1. Demand model

Assume that there are N points in time that the pricing decisions can be made. Without loss of generality, assume that each period in consideration is of unit length. The demand in each period has a Poisson distribution. The demand rate is separable and consists of two components: a base demand rate Λ, and a multiplier $\Psi(p)$ which is a function of the price p. The Poisson rate is equal to

$$\Lambda(p) = \Psi(p)\Lambda.$$

We assume that the functional form of the demand function is not known with certainty but is known to be from a family of K functions. In particular we assume

$$\Psi(p) = \psi_j(p) \quad \text{with probability } \theta_{j,0}, \quad \text{for } j = 1, 2, \ldots, K.$$

For each j, define $\bar{p}_j$ such that $\psi_j(\bar{p}_j) = 1$. Although our model does not depend on a particular demand function, in our computational study, we assume exponential price sensitivity $\Lambda(p) = a\mathrm{e}^{-\gamma_j p}$ and use

$$\psi_j(p) = \mathrm{e}^{-\gamma_j(p-\bar{p}_j)}. \tag{1}$$

Exponential price sensitivity and multiplicative demand functions are widely used in practice and research (see [35,36] for examples).

We assume that Λ is distributed as Gamma with parameters α and β. The distribution for Gamma is given by,

$$f(\lambda) = \frac{\beta^\alpha \lambda^{\alpha-1} \mathrm{e}^{-\beta\lambda}}{\Gamma(\alpha)}, \quad \lambda > 0.$$

The distribution of demand for a given price p, conditional on the demand function and base rate is given by,

$$f(x|p, \Lambda = \lambda, \Psi = \psi_j) = \frac{\mathrm{e}^{-\psi_j(p)\lambda}[\psi_j(p)\lambda]^x}{x!}, \quad \text{for } x = 0, 1, 2, \ldots$$

Then, the *prior* distribution (unconditional of Λ and Ψ) of demand is the following:

$$f(x|p) = \int_0^\infty \sum_{j=1}^K f(x|p, \Lambda = \lambda, \Psi = \psi_j)\theta_{j,0} f(\lambda)\,\mathrm{d}\lambda = \sum_{j=1}^K \theta_{j,0} \binom{\alpha + x - 1}{x} \left(\frac{\beta}{\beta + \psi_j(p)}\right)^\alpha \left(\frac{\psi_j(p)}{\beta + \psi_j(p)}\right)^x, \quad \text{for } x = 0, 1, 2, \ldots \tag{2}$$

Observing sales will facilitate learning on both the magnitude of demand (Λ) and the demand function (Ψ).

If the retailer charged a price of p_1 in the first period and the realized demand in period 1 was x_1, the *posterior* distribution of Λ and Ψ can be found using the Bayes' rule as follows:

$$f(\lambda,\psi_j|x_1,p_1)=\frac{f(x_1|p_1,\Lambda=\lambda,\Psi=\psi_j)\theta_{j,0}f(\lambda)}{\int_0^\infty\sum_{k=1}^K f(x_1|p_1,\Lambda=\lambda,\Psi=\psi_k)\theta_{k,0}f(\lambda)\,\mathrm{d}\lambda}=\frac{(1/x_1!)[\lambda\psi_j(p_1)]^{x_1}\mathrm{e}^{-\lambda\psi_j(p_1)}\theta_{j,0}[1/\Gamma(\alpha)]\beta^\alpha\lambda^{\alpha-1}\mathrm{e}^{-\beta\lambda}}{\int_0^\infty\sum_{k=1}^K(1/x_1!)[\lambda\psi_k(p_1)]^{x_1}\mathrm{e}^{-\lambda\psi_k(p_1)}\theta_{k,0}[1/\Gamma(\alpha)]\beta^\alpha\lambda^{\alpha-1}\mathrm{e}^{-\beta\lambda}\,\mathrm{d}\lambda}.$$

Integrating and simplifying, we get

$$f(\lambda,\psi_j|x_1,p_1)=\frac{\theta_{j,0}\lambda^{\alpha-1+x_1}\mathrm{e}^{-\lambda[\beta+\psi_j(p_1)]}[\psi_j(p_1)]^{x_1}}{\Gamma(\alpha+x_1)\sum_{k=1}^K\theta_{k,0}[\psi_k(p_1)]^{x_1}/[\beta+\psi_k(p_1)]^{\alpha+x_1-1}}. \tag{3}$$

Similarly, after observing x_1, x_2

$$f(\lambda,\psi_j|x_1,x_2,p_1,p_2)=\frac{\theta_{j,0}\lambda^{\alpha-1+x_1+x_2}\mathrm{e}^{-\lambda[\beta+\psi_j(p_1)+\psi_j(p_2)]}[\psi_j(p_1)]^{x_1}[\psi_j(p_2)]^{x_2}}{\Gamma(\alpha+x_1+x_2)\sum_{k=1}^K\theta_{k,0}[\psi_k(p_1)]^{x_1}[\psi_k(p_2)]^{x_2}/[\beta+\psi_k(p_1)+\psi_k(p_2)]^{\alpha+x_1+x_2-1}}. \tag{4}$$

In general, after observing $x_1, x_2, \ldots, x_{n-1}$

$$f(\lambda,\psi_j|x_1,\ldots,x_{n-1},p_1,\ldots,p_{n-1})=\frac{\theta_{j,0}\lambda^{\alpha-1+\sum_{\ell=1}^{n-1}x_\ell}\mathrm{e}^{-\left[\lambda+\sum_{\ell=1}^{n-1}\psi_j(p_\ell)\right]}\prod_{\ell=1}^{n-1}[\psi_j(p_\ell)]^{x_\ell}}{\Gamma\left(\alpha+\sum_{\ell=1}^{n-1}x_\ell\right)\sum_{k=1}^K\theta_{k,0}\prod_{\ell=1}^{n-1}[\psi_k(p_\ell)]^{x_\ell}\Big/\left[\beta+\sum_{\ell=1}^{n-1}\psi_k(p_\ell)\right]^{\alpha+\sum_{\ell=1}^{n-1}x_\ell-1}}. \tag{5}$$

Let $D_n(p)$ denote the demand in period n for a given price p. The distribution of $D_n(p)$ given the demand history $x_1, x_2, \ldots, x_n$ and price history $p_1, p_2, \ldots, p_n$ (unconditional of Λ and Ψ) can be derived as

$$f(x|x_1,\ldots,x_{n-1},p_1,\ldots,p_{n-1},p)=\frac{\sum_{k=1}^K\binom{\alpha+X_{n-1}+x-1}{x}\left(\frac{\beta+M_{k,n-1}}{\beta+M_{k,n-1}+\psi_k(p)}\right)^{\alpha+X_{n-1}}\left(\frac{\psi_k(p)}{\beta+M_{k,n-1}+\psi_k(p)}\right)^x\widetilde{M}_{k,n-1}\theta_{k,0}/(\beta+M_{k,n-1})^{\alpha+X_{n-1}}}{\sum_{k=1}^K\widetilde{M}_{k,n-1}\theta_{k,0}/(\beta+M_{k,n-1})^{\alpha+X_{n-1}}}, \tag{6}$$

where $X_{n-1}=\sum_{\ell=1}^{n-1}x_\ell$, $M_{k,n-1}=\sum_{\ell=1}^{n-1}\psi_k(p_\ell)$ and $\widetilde{M}_{k,n-1}=\sum_{\ell=1}^{n-1}[\psi_k(p_\ell)]^{x_\ell}$. X_{n-1}, $\mathbf{M_{n-1}}=[M_{1,n-1}\ldots M_{K,n-1}]$ and $\widetilde{\mathbf{M}}_{\mathbf{n-1}}=[\widetilde{M}_{1,n-1}\ldots\widetilde{M}_{K,n-1}]$ summarize all the information in periods $1,\ldots,n-1$ and are called the *sufficient statistics* for estimating demand in period n.

The distribution in (6) can be written as a mixture of K Negative Binomial distributions:

$$f(x|x_1,\ldots,x_{n-1},p_1,\ldots,p_{n-1},p)=\sum_{k=1}^K\theta_{j,n-1}f_k(x|x_1,\ldots,x_{n-1},p_1,\ldots,p_{n-1},p), \tag{7}$$

where

$$\theta_{j,n-1}=\frac{\theta_{j,0}\widetilde{M}_{j,n-1}/(\beta+M_{j,n-1})^{\alpha+X_{n-1}}}{\sum_{k=1}^K\theta_{k,0}\widetilde{M}_{k,n-1}/(\beta+M_{k,n-1})^{\alpha+X_{n-1}}},$$

and $f_k(x|x_1,\ldots,x_{n-1},p_1,\ldots,p_{n-1},p)$ is a Negative Binomial distribution with parameters $\alpha+X_{n-1}$ and $(\beta+M_{k,n-1})/[\beta+M_{k,n-1}+\psi_k(p)]$.

Using (7), we can find the mean and variance of $D_n(p)$ conditional on $x_1,\ldots,x_{n-1}$ and $p_1,\ldots,p_{n-1}$ as follows:

$$\begin{aligned}&E[D_n(p)|x_1,\ldots,x_{n-1},p_1,\ldots,p_{n-1}]\\&\quad=\sum_{k=1}^K\theta_{k,n-1}(\alpha+X_{n-1})\psi_k(p)/(\beta+M_{k,n-1}),\\&\mathrm{Var}[D_n(p)|x_1,\ldots,x_{n-1},p_1,\ldots,p_{n-1}]\\&\quad=\sum_{k=1}^K(\theta_{k,n-1})^2(\alpha+X_{n-1})\psi_k(p)\times[\beta+M_{k,n-1}+\psi_k(p)]/(\beta+M_{k,n-1})^2.\end{aligned}$$

It is also worthwhile to see how the mean and variance of the unconditional distribution of demand behaves as n increases. For simplicity of the exposition, assume that $\bar{p}_j=\bar{p}$ and price is equal to $\bar{p}$ throughout the season so that $\psi_k(p_\ell)=1$ for all ℓ and k. The expected value and variance of the unconditional demand are given by,

$$\begin{aligned}E[D_n(\bar{p})|x_1,\ldots,x_{n-1},p_1,\ldots,p_{n-1}]&=\frac{\alpha+\sum_{\ell=1}^{n-1}x_\ell}{\beta+n-1},\\\mathrm{Var}[D_n(\bar{p})|x_1,\ldots,x_{n-1},p_1,\ldots,p_{n-1}]&=\frac{(\alpha+\sum_{\ell=1}^{n-1}x_\ell)(\beta+n)}{(\beta+n-1)^2}.\end{aligned}$$

It is easy to see that as n approaches infinity, both the mean and variance approach $\bar{x}$, average of x_i, which is the true rate of the Poisson process. We note that the convergence is faster if β is smaller. This corresponds to higher degrees of uncertainty in the decision maker's initial estimate of demand rate, and thus more reliance on actual sales information in estimating future demand.

While our analysis so far assumes that the periods are identical except for the prices charged, our model allows us to permit seasonality and any other extensions as long as the multiplicative nature of the demand is preserved. That is, as long as we can state the demand rate in period ℓ as

$$\Lambda_\ell(p_\ell,\tau_\ell)=\Psi(p_\ell,\tau_\ell)\Lambda,$$

(where Ψ now is a more general random function of price p_ℓ and seasonality factor τ_ℓ) our model is applicable. Uneven period lengths are also easily accountable by considering the length as a seasonality factor.

3.1.1. Deterministic demand function

Our specification of the demand function Ψ from a set of functional forms ψ_j allows one to interpret the demand function as a non-deterministic one. If the function is given as $\Psi = \psi$ with certainty, then the distribution function of Λ in (5) reduces to Gamma distribution with parameters $\alpha + X_{n-1}$ and $\beta + M_{n-1}$

$$f(\lambda|x_1,\ldots,x_{n-1},p_1,\ldots,p_{n-1}) = \frac{[\beta + M_{n-1}]^{\alpha+X_{n-1}}\lambda^{\alpha+X_{n-1}-1}e^{-(\beta+M_{n-1})\lambda}}{\Gamma(\alpha+X_{n-1})}, \quad \lambda > 0,$$

where $M_{n-1} = \sum_{\ell=1}^{n-1}\psi(p_\ell)$.

In this case, the unconditional distribution of demand in period n that is given in (6) reduces to Negative Binomial distribution with parameters $\alpha + X_{n-1}$ and $[\beta + M_{n-1}]/[\beta + M_{n-1} + \psi(p)]$

$$f(x|x_1,\ldots,x_{n-1},p_1,\ldots,p_{n-1},p) = \binom{\alpha+X_{n-1}+x-1}{x}\left(\frac{\beta+M_{n-1}}{\beta+M_{n-1}+\psi(p)}\right)^{\alpha+X_{n-1}}\left(\frac{\psi(p)}{\beta+M_{n-1}+\psi(p)}\right)^{x}, \quad \text{for } x = 0,1,\ldots$$

Note that here the sufficient statistics are $X_{n-1} = \sum_{\ell=1}^{n-1}x_\ell$ and $M_{n-1} = \sum_{\ell=1}^{n-1}\psi(p_\ell)$.

The demand in the nth period given the demand and price history will have a mean of

$$E[D_n(p)|x_1,\ldots,x_{n-1},p_1,\ldots,p_{n-1}] = \frac{(\alpha+\sum_{\ell=1}^{n-1}x_\ell)\psi(p)}{\beta+\sum_{\ell=1}^{n-1}\psi(p_\ell)},$$

which basically means that the sales rate in the nth period is a linear function of sales rate in the earlier $n-1$ periods. This is in fact not surprising. Carlson [11] studied sales data of apparel merchandise from a major department store to see whether the sales rate after a mark-down is predictable. Given an initial price and a mark-down percentage, he has shown that past mark-down sales rate is in fact a linear function of pre mark-down sales rate. Our model completely agrees with this empirical result.

3.1.2. Deterministic demand rate

If the demand rate is given as $\Lambda = \lambda$ with certainty, the probability that the demand function is $\psi_j(p)$ in period n given a sales history of $x_1, x_2, \ldots, x_{n-1}$ and a price history of $p_1, p_2, \ldots, p_n$ can be written as

$$\theta_{j,n-1} = Pr\{\Psi = \psi_j|x_1,\ldots,x_{n-1},p_1,\ldots,p_{n-1}\} = \frac{e^{-\lambda\sum_{\ell=1}^{n-1}\psi_j(p_\ell)}\prod_{\ell=1}^{n-1}[\psi_j(p_\ell)]^{x_\ell}\theta_{j,0}}{\sum_{k=1}^{K}e^{-\lambda\sum_{\ell=1}^{n-1}\psi_k(p_\ell)}\prod_{\ell=1}^{n-1}[\psi_k(p_\ell)]^{x_\ell}\theta_{k,0}}.$$

Then, the unconditional (of Λ) distribution of demand in period n is given by

$$f(x|x_1,\ldots,x_{n-1},p_1,\ldots,p_{n-1},p) = \sum_{j=1}^{K}\theta_{j,n-1}\frac{e^{-\lambda\psi_j(p)}[\lambda\psi_j(p)]^x}{x!}$$

with mean equal to

$$E[D_n(p)|x_1,\ldots,x_{n-1},p_1,\ldots,p_{n-1}] = \sum_{j=1}^{K}\theta_{j,n-1}\lambda\psi_j(p).$$

3.2. Pricing model

The problem is to determine prices in periods $1,\ldots,N$ so that a fixed stock of I_0 items is sold with maximum expected revenue. Risk neutrality of the retailer (and thus expected revenue maximization) is a fairly standard assumption in the revenue management theory and practice (see [40]) and may be considered reasonable given that the revenue management and dynamic pricing decisions are implemented over many problem instances (flight departures, hotel nights, seasonal items, etc.). In other cases, there may be a need for incorporating the risk preferences of the retailer and this has only been recently studied (see, for example, [19,29]). Since our primary objective in this initial paper is to understand the impact of demand learning on pricing decisions, we follow the traditional literature on revenue management and assumed risk neutrality of the retailer. For simplicity of the presentation, we also assume that the inventory holding costs within the selling season are negligible. We note that it is very easy to relax this assumption in the context of our model.

We use a discrete-time dynamic programming model. Let $V_n(I_{n-1}, X_{n-1}, \mathbf{M_{n-1}}, \widetilde{\mathbf{M}}_{\mathbf{n-1}})$ be the maximum expected revenue from period n through N when the initial inventory is I_{n-1} and the cumulative sales is X_{n-1}, vector of cumulative price multipliers are $\mathbf{M}_{n-1}$ and $\widetilde{\mathbf{M}}_{\mathbf{n-1}}$. Note that

$$I_{n-1} = \max\{0, I_0 - X_{n-1}\}$$

and can be dropped from the formulation. But we keep I_{n-1} in our formulation for ease of exposition. Also let p_s be the salvage value for any inventory left unsold beyond period N.

The backward recursion formulation can be written as

$$\begin{aligned} V_n(I_{n-1},X_{n-1},\mathbf{M_{n-1}},\widetilde{\mathbf{M}}_{\mathbf{n-1}}) &= \max_{p_n \geqslant p_s} E[p_n\min\{D_n(p_n),I_{n-1}\} + V_{n+1}((I_{n-1}-D_n(p_n))^+, \\ &X_{n-1}+D_n(p_n),\mathbf{M_{n-1}}+\psi(p_n)),\widetilde{\mathbf{M}}_{\mathbf{n-1}}\widehat{\mathbf{M}}(p_n)|X_{n-1},\mathbf{M_{n-1}},\widetilde{\mathbf{M}}_{\mathbf{n-1}}]. \end{aligned} \tag{8}$$

where $\boldsymbol{\psi}(p_n) = [\psi_1(p_n)\ldots\psi_K(p_n)]$ and $\widehat{\mathbf{M}}(p_n)$ is a $K \times K$ diagonal matrix with entries $[\psi_1(p_n)]^{D_n(p_n)}, \ldots, [\psi_K(p_n)]^{D_n(p_n)}$ in the diagonal.

Boundary conditions are

$$V_{N+1}(I_N, X_N, \mathbf{M_{n-1}}, \widetilde{\mathbf{M}}_{\mathbf{n-1}}) = p_s I_N, \quad \text{for all } I_N, X_N, \mathbf{M_{n-1}}, \widetilde{\mathbf{M}}_{\mathbf{n-1}}, \tag{9}$$

$$V_n(0, X_{n-1}, \mathbf{M_{n-1}}, \widetilde{\mathbf{M}}_{\mathbf{n-1}}) = 0, \quad \text{for all } n, X_{n-1}, \mathbf{M_{n-1}}, \widetilde{\mathbf{M}}_{\mathbf{n-1}}. \tag{10}$$

The first condition states that any leftover merchandise has only salvage value when the season ends at the end of period N. We assume that this salvage value is deterministic and is known at time 0. The second condition states that the future expected profits are zero, when there is no merchandise left in stock since re-ordering is not allowed. This property also allows us to avoid the problem of censored demand information due to unsatisfied demand. In case of excess demand (when the inventory is exhausted), there are no further decisions to be made and no further information about demand is required. The dynamic program can be solved by starting with the Nth period and proceeding backwards.

We solved many problems with different sets of parameters to investigate the structural properties of the optimal policy. In all these problems, we observed that higher sales in earlier periods always translate into higher prices in future periods. The intuition behind this behavior is the following. First, higher sales in earlier periods mean (stochastically) higher demand in future periods because of the Bayesian nature of the demand distributions. Second, higher sales in earlier periods also mean lower left-over inventory for future periods since there are no further replenishment opportunities. Thus, higher sales in earlier periods inflate the expected demand while decreasing the available supply in future periods. This allows the seller to charge higher prices to balance the demand and supply. The second part of the argument (lower inventory calls for higher prices), is formally proved by Chun [14] for the Negative Binomial demand. The first part of the argument (stochastically larger demand calls for higher prices), however, is not true in general. See Bitran and Wadhwa [8] for counter examples and certain conditions that are required.

In order to show how the model works, we provide the following example.

Example: The retailer has 12 units to sell in a season with two periods of unit length. When the price is set to 1.00, the demand in each period is Poisson with a rate distributed with Gamma with parameters $\alpha = 2$ and $\beta = 0.5$. The retailer can charge different prices in these periods from a discrete set $\mathscr{P} = \{0.50, 0.55, \ldots, 0.95, 1.00\}$. The price affects the demand in an exponential manner with two possible elasticity parameters

$$\Psi(p) = \begin{cases} \psi_1(p) = e^{-2(p-1)} & \text{with probability } 0.5, \\ \psi_2(p) = e^{-4(p-1)} & \text{with probability } 0.5. \end{cases}$$

The mean total demand is given as follows for each price in $\mathscr{P}$.

Price	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
Mean demand	20.2	17.0	14.4	12.1	10.3	8.7	7.4	6.3	5.4	4.8	4.0

The problem is to find the price in the first period and the pricing policy in the second period so as to maximize the total revenues. We solve the problem with the dynamic program given in Eqs. (8)–(10). The optimal policy is to charge 0.95 in the first period and then charge the prices in the second row of the following table in the second period based on the demand realization in the first period.

x_1	0	1	2	3	4	5	6	7	8	9	10	11	12
p_2^*	0.50	0.55	0.60	0.65	0.75	0.80	0.85	0.95	1.00	1.00	1.00	1.00	1.00
$100 \times \theta_{11}$	53.5	52.7	52.0	51.2	50.5	49.7	49.0	48.2	47.5	46.7	46.0	45.2	44.5
$E[D_2(1)]^*$	1.21	1.81	2.41	3.01	3.61	4.21	4.81	5.41	6.01	6.61	7.20	7.80	8.39

The third row in the table shows the posterior probability that the demand elasticity parameter is 2. The third row is the expected demand in the second period if the retailer charges a price of 1. The resulting optimal expected revenue is 7.81, about 0.65 per unit. The table above also shows how the posterior probability that $\Psi = \psi_1$ and expected demand in period 2 (if a price of 1 is charged) changes based on the observed demand in period 1.

4. Computational study

We first note that although pricing through a demand learning model is the best the retailer can do, it is not necessarily optimal. The optimal policy depends on the true value of underlying base demand rate and the true demand function. The optimal prices can be computed by using a dynamic programming formulation, which uses the Poisson demand distribution with the true value of the demand rate and the true demand function. The performance of the demand learning model depends on how accurate the retailer's initial demand estimates are and how fast the retailer can learn about the true demand rate and demand function. Note that prior to the start of the season, the retailer assumes that the base demand rate is distributed Gamma with parameters α and β. The expected value and variance of this random variable are given by,

$$E[\lambda] = \frac{\alpha}{\beta} \quad \text{and} \quad \mathrm{Var}[\lambda] = \frac{\alpha}{\beta^2}.$$

Hence, α/β defines the initial point estimate. Coefficient of variation can be derived as $1/\sqrt{\alpha}$. Thus, given a fixed ratio α/β, the magnitude of α (or β) defines the variance of the initial estimate, and hence the decision maker's reliance on her prior beliefs about demand rate. For a fixed ratio α/β, when α (or β) is large, the retailer is confident about her initial estimate, and she hardly updates her demand rate estimate based on observed sales. As α (or β) gets smaller, more weight is given to the observed sales in estimating future demand.

We analyze three different models in our computational study. Under *Perfect Information* model, the true value of the underlying base rate and true demand function are known, and an optimal policy is derived using Poisson distributed demand with rate $\psi(p)\lambda$. Under *No Learning* model, the decision maker only knows $\alpha, \beta, \theta_{1,0}, \theta_{2,0}, \ldots, \theta_{K,0}$ and $\psi_1, \psi_2, \ldots, \psi_K$ and an optimal policy is derived using the initial mixture of Negative Binomial distributions whose distribution is given in (2). This distribution is *not* updated as the sales are observed. Under *Learning* model, the decision maker also only knows $\alpha, \beta, \theta_{1,0}, \theta_{2,0}, \ldots, \theta_{K,0}$ and $\psi_1, \psi_2, \ldots, \psi_K$ at the beginning of the season, however the demand distribution is updated using observed sales following the learning model as given in (7).

In order to understand the impact and value of learning, the performance of the policies that are derived under Learning and No Learning models are evaluated using Poisson distributed demand with the true value of the base rate. We should note again however that this rate is not revealed to the decision maker before the season (for otherwise, the decision maker would simply use Perfect Information model to maximize its revenues) and thus evaluation of Learning and No Learning models based on the true Poisson rate cannot appropriately guide the decision maker *before* the season.

Our primary objective in the computational study is to discover the conditions under which the early sales information has the most impact on revenues by comparing the revenues of Learning model with that of No Learning model. While doing this we also generate the optimal revenues for Perfect Information model. We specifically study the impacts of accuracy of the initial estimate, the variance of the initial estimate, price elasticity of demand on the profit from all three models.

For the purposes of computational study, we assume that there is only one chance to change the price during the season. The resulting model is a special two-period case of the model described earlier. We assume a season of length 1 and assume two equal periods of length 0.5. We allow the first and second prices to be in the set $\{0.50, 0.55, 0.60, \ldots, 0.95, 1.00\}$. We do not put any restrictions on the direction of the price change in the second period, i.e., the second period price can be higher or lower than the first period price. We assume that the salvage value is zero. We use exponential price sensitivity, i.e., demand functions of the form $\psi(p) = e^{-\gamma(p-1)}$. In Sections 4.1–4.3, we assume that the retailer has perfect knowledge about the demand function, (i.e., $\Psi = \psi$ with probability 1), and investigate the impact of learning about the demand rate only. Therefore the demand model used in Sections 4.1–4.3 is one that is explained in Section 3.1.1. In Section 4.4, we investigate the impact of demand function uncertainty and demand rate uncertainty simultaneously and use the general demand model given in Section 3.1.

4.1. The impact of the accuracy of the initial point estimate of demand rate

In this part of the study, we assess the impact of the initial estimate on profits of Learning and No Learning models in a variety of settings. For the price sensitivity of demand, we use a moderate value, e.g., $\gamma = 3$.

The analysis is done in two steps; first we keep the initial point estimate constant and vary the true rate of the Poisson distribution and later we keep the true rate of the Poisson distribution constant and vary the initial point estimate. Note that the value of the initial estimate is α/β. In the first part of the analysis, we set $\alpha/\beta = 20$. However in order to study also the impact of decision maker's reliance on the initial estimate, we use two scenarios. In high variance case, $\alpha = 10$ and $\beta = 0.5$, resulting in a variance of 40 for the gamma distribution (or a coefficient of variation of $1/\sqrt{10}$). In low variance case, $\alpha = 40$ and $\beta = 2$ resulting in a variance of 10 for the gamma distribution (or a coefficient of variation of $1/\sqrt{40}$). We also use different values for the starting inventory level, in order to incorporate the impact of imbalance between supply and demand in pricing decisions. This first step of the analysis is summarized in Table 1. The revenues of Learning and No Learning models are provided in percent of the optimal revenues that are generated by Perfect Information model. The row titled L/N % shows the performance of Learning model against No Learning model ($100 \times$ expected revenue with Learning model/expected revenue with No Learning model).

Note that λ is the true Poisson rate when the price is set at the maximum price 1.00. The true Poisson rate takes on values 10, 15, 20, 25 and 30, while the decision maker's initial point estimate is fixed at 20. Note also that optimal policies for No Learning and Learning models

Table 1
The impact of initial estimate, revenues as a function of λ

I_0				λ				
				10	15	20	25	30
10	α	β	Perfect Information	9.0361	9.8697	9.9918	9.9997	10.0000
	10	0.5	Learning	99.47	99.87	99.90	99.98	100.00
			No Learning	97.02	99.95	100.00	100.00	100.00
			L/N (%)	102.53	99.92	99.90	99.98	100.00
	40	2	Learning	97.57	99.98	100.00	100.00	100.00
			No Learning	96.86	99.94	100.00	100.00	100.00
			L/N (%)	100.74	100.05	100.00	100.00	100.00
20	α	β	Perfect Information	14.2552	16.8405	18.6529	19.6623	19.9511
	10	0.5	Learning	89.40	98.03	99.54	99.19	99.49
			No Learning	83.08	96.85	99.98	99.33	99.45
			L/N (%)	107.61	101.22	99.56	99.86	100.04
	40	2	Learning	84.96	97.24	99.96	99.45	99.57
			No Learning	83.05	96.74	99.99	99.48	99.58
			L/N (%)	102.29	100.52	99.97	99.97	99.99
30	α	β	Perfect Information	17.8773	21.7092	24.4823	26.6369	28.3606
	10	0.5	Learning	87.64	96.64	99.28	98.57	95.88
			No Learning	83.21	96.40	99.99	96.81	91.89
			L/N (%)	105.33	100.24	99.29	101.81	104.34
	40	2	Learning	86.70	97.67	99.88	97.05	92.15
			No Learning	83.21	96.40	99.99	96.90	92.00
			L/N (%)	104.19	101.32	99.88	100.15	100.17

are evaluated using Poisson distribution with the true rate. When we compare the revenues obtained from No Learning and Learning models, we conclude that learning from observed sales is most beneficial when the initial point estimate is inaccurate and when the variance is high (the decision maker relies less on the initial estimate and is more willing to update her estimate based on observed sales). This gives an opportunity to Learning model to quickly identify the inaccuracy of the initial estimate and correct the estimate for the second period. The benefits are more pronounced when the true Poisson rate is lower (e.g., $\lambda = 10$) than the initial estimate and the initial inventory levels are high (e.g., $I_0 = 20$ and $I_0 = 30$). Since the maximum allowed price is 1.00, pricing is more instrumental when the demand rate is significantly lower than the initial inventory.

Notice that in 13 cases, No Learning model is performing better than Learning model. These are the cases where the initial estimate is fairly accurate and updating the demand distribution using a random sample can therefore reduce the revenues. The reductions are minimal when the variance is low (the decision maker relies more on the initial estimate and is less willing to update its estimate based on observed sales). It should be noted, however, that the savings due to Learning model when the initial estimate is inaccurate is much higher than the losses due to Learning model when the initial estimate is accurate.

Finally we should note that when the initial inventory is low (i.e., $I_0 = 10$), pricing is not very useful as the maximum price is set at 1.00. Therefore, the difference between Learning and No Learning models are minimal, and both models can perform very close to Perfect Information model.

The second step of the analysis is summarized in Table 2. In the second step of the analysis we fixed the true Poisson rate (λ) at 20 and let the initial point estimate (α/β) take on values 10, 15, 20, 25 and 30. In order to eliminate the impact of the variance in the analysis, we fixed the coefficient of variation of the gamma distribution (which is equal to $\sqrt{(\alpha/\beta^2)/(\alpha/\beta)} = 1/\sqrt{\alpha}$ to $1/\sqrt{10}$ for high variance case, and to $1/\sqrt{40}$ for low variance case).

In addition to results that are similar to those that are obtained in the first step, the second step provides an additional interesting observation. While the maximum revenue is achieved when the estimate is accurate in No Learning model, the same is not necessarily true for Learning model. When the initial inventory is 10 for both high and low variance, and when the initial inventory is 20 for high variance, the maximum revenue is achieved when the decision maker is in fact overestimating the demand. By overestimating the demand, the decision maker is less likely to charge lower than the maximum price in the second based on a random sample.

4.2. The impact of the variance of the initial estimate of demand rate

In this part of the study, we investigate the impact of the variance of the initial estimate on the performance of Learning and No Learning models. Note again that the variance of the initial estimate reflects the decision maker's reliance on its initial estimate and how much she is willing to update her estimate based on observed sales for Learning model.

The analysis is summarized in Table 3 for an initial inventory level of 20, and Table 4 for an initial inventory level of 30. For both tables, parameter α of the Gamma distribution takes on values 5, 10, 15, 25, 40 and 80 while the parameter β of the Gamma distribution takes on values 0.25, 0.5, 0.75, 1.25, 2 and 4, respectively. This keeps the mean of the Gamma distribution constant at 20, while the variance of the Gamma distribution takes on values 80, 40, 26.67, 16, 10, and 5. The tables show the optimal first period price, expected optimal second period price and optimal expected revenue for Perfect Information model to form a benchmark. As mentioned earlier, No Learning model uses the same the Negative Binomial distribution when deciding the first period price and deriving a policy for the second period price, while Learning model uses an updated Negative Binomial distribution for the second period. However, the expected revenues and expected second period prices reported in Table 3 and Table 4 use the true Poisson distribution when taking the expectations. In the less likely case that the initial inventory is totally depleted in the first period, we take the second period price to be 1.00 when calculating expected second period price.

Table 2
The impact of initial estimate, revenues as a function of α/β

I_0				α/β				
				10	15	20	25	30
10	α	β	Perfect Information	9.9918	9.9918	9.9918	9.9918	9.9918
	10	0.5	No Learning	99.64	99.97	100.00	100.00	100.00
			Learning	99.54	99.86	99.90	99.97	99.97
			L/N (%)	99.91	99.89	99.90	99.97	99.97
	40	2	No Learning	99.73	99.97	100.00	100.00	100.00
			Learning	99.73	99.97	100.00	100.00	100.00
			L/N (%)	100.00	100.00	100.00	100.00	100.00
20	α	β	Perfect Information	18.6529	18.6529	18.6529	18.6529	18.6529
	10	0.5	No Learning	87.04	96.27	99.98	99.29	98.31
			Learning	87.51	96.25	99.54	99.71	99.84
			L/N (%)	100.55	99.97	99.56	100.42	101.55
	40	2	No Learning	87.19	96.27	99.99	99.29	98.06
			Learning	87.18	96.45	99.96	99.77	99.08
			L/N (%)	99.99	100.19	99.97	100.48	101.03
30	α	β	Perfect Information	24.4823	24.4823	24.4823	24.4823	24.4823
	10	0.5	No Learning	82.03	97.21	99.99	97.33	92.44
			Learning	82.70	97.96	99.28	98.30	96.21
			L/N (%)	100.81	100.77	99.29	100.99	104.07
	40	2	No Learning	82.14	97.21	99.99	97.33	92.37
			Learning	82.46	97.73	99.88	98.07	94.16
			L/N (%)	100.38	100.53	99.88	100.76	101.94

Table 3
The impact of variance ($I_0 = 20$)

λ	Perfect Information					α					
	p_1^*	$E[p_2^*]$	Revenue			5	10	15	25	40	80
10	0.8	0.7383	14.2552	Learning	p_1^*	1	1	1	1	1	1
					$E[p_2^*]$	0.7128	0.7562	0.7670	0.7922	0.8129	0.8216
					%	91.92	89.40	88.65	86.68	84.96	84.19
				No Learning	p_1^*	1	1	1	1	1	1
					$E[p_2^*]$	0.8345	0.8345	0.8345	0.8345	0.8349	0.8401
					%	83.08	83.08	83.08	83.08	83.05	82.65
				L/N (%)		110.64	107.61	106.71	104.33	102.29	101.87
20	1	0.9400	18.6529	Learning	p_1^*	1	1	1	1	1	1
					$E[p_2^*]$	0.9088	0.9198	0.9208	0.9288	0.9330	0.9334
					%	98.95	99.54	99.61	99.83	99.96	99.97
				No Learning	p_1^*	1	1	1	1	1	1
					$E[p_2^*]$	0.9298	0.9298	0.9298	0.9298	0.9355	0.94
					%	99.98	99.98	99.98	99.98	99.99	100.00
				L/N (%)		98.97	99.56	99.63	99.86	99.97	99.97
30	1	0.9999	19.9511	Learning	p_1^*	1	1	1	1	1	1
					$E[p_2^*]$	0.9872	0.9881	0.9882	0.9893	0.9895	0.9895
					%	99.43	99.49	99.49	99.55	99.57	99.57
				No Learning	p_1^*	1	1	1	1	1	1
					$E[p_2^*]$	0.9863	0.9863	0.9863	0.9863	0.9896	0.9902
					%	99.45	99.45	99.45	99.45	99.58	99.61
				L/N (%)		99.98	100.04	100.04	100.11	99.99	99.96

Table 4
The impact of variance ($I_0 = 30$)

λ	Perfect Information					α					
	p_1^*	$E[p_2^*]$	Revenue			5	10	15	25	40	80
10	0.65	0.6327	17.8773	Learning	p_1^*	0.90	0.90	0.90	0.90	0.85	0.85
					$E[p_2^*]$	0.5967	0.6215	0.6314	0.6582	0.6828	0.6995
					%	89.12	87.64	87.08	85.22	86.70	85.47
				No Learning	p_1^*	0.85	0.85	0.85	0.85	0.85	0.85
					$E[p_2^*]$	0.8345	0.8345	0.8345	0.8345	0.8349	0.8401
					%	83.09	83.21	83.21	83.21	83.21	83.20
				L/N (%)		107.25	105.33	104.65	102.42	104.19	102.72
20	0.85	0.8565	24.4823	Learning	p_1^*	0.90	0.90	0.90	0.90	0.85	0.85
					$E[p_2^*]$	0.8035	0.8081	0.8067	0.8091	0.8518	0.8541
					%	98.97	99.28	99.35	99.65	99.88	99.94
				No Learning	p_1^*	0.85	0.85	0.85	0.85	0.85	0.85
					$E[p_2^*]$	0.85	0.85	0.85	0.85	0.852	0.8565
					%	99.99	99.99	99.99	99.99	99.99	100.00
				L/N (%)		98.98	99.29	99.36	99.66	99.88	99.94
30	1	0.9496	28.3606	Learning	p_1^*	0.90	0.90	0.90	0.90	0.85	0.85
					$E[p_2^*]$	0.9536	0.9536	0.9495	0.9466	0.9738	0.9739
					%	95.87	95.88	95.77	95.66	92.15	92.16
				No Learning	p_1^*	0.85	0.85	0.85	0.85	0.85	0.85
					$E[p_2^*]$	0.9657	0.9657	0.9657	0.9657	0.9695	0.9712
					%	91.89	91.89	91.89	91.89	92.00	92.07
				L/N (%)		104.33	104.34	104.22	104.10	100.17	100.09

When the initial inventory (I_0) is 20, we note that No Learning and Learning models set the initial price to 1.00 for all variance levels (Table 3). When the initial inventory (I_0) is 20 and the true Poisson rate (λ) is 10, we observe that Perfect Information model sets the initial price to 0.80, significantly lower than No Learning and Learning models. However, as the variance gets higher, Learning model is better able to correct its estimate and thus charges lower prices in the second period. This is in contrast to No Learning model where the second period price and the revenue is insensitive to the variance.

When the initial inventory (I_0) is 20 and the true Poisson rate (λ) is 20, we observe that Perfect Information model sets the initial price to 1.00. The revenues of No Learning and Learning models are also quite close to the optimal revenue obtained in Perfect Information model. However, we note that when the variance is high for Learning model, the decision maker runs the risk of charging a less than optimal price as she may interpret a randomly low demand in the first period as a sign for low demand overall.

When the initial inventory (I_0) is 20 and the true Poisson rate (λ) is 30, we again observe that Perfect Information model sets the initial price to 1.00. Since the demand rate is quite high as compared to the supply, expected optimal second price also needs to be close to 1.00. Similar to the case when the true Poisson rate (λ) is 20, the decision maker still has the risk of charging a less than optimal second period price, based on a randomly low demand in the first period when he uses Learning model. This is especially true for high variance case.

When the initial inventory (I_0) is 30, we note that No Learning model sets the initial price to 0.85 for all variance levels, while Learning model sets the initial price to 0.90 when the variance is high and to 0.85 when the variance is low (Table 4). The difference between the initial prices of Learning and No Learning models shows that the fact that the decision maker will learn from observed sales may lead the decision maker to different decisions, even before she observes sales.

Table 4 shows results similar to those in Table 3, except that now, Learning model provides significant benefits also when the true Poisson rate (λ) is 30. The value of learning is more pronounced, when the starting inventory level is high, i.e., correcting an underestimation pays more.

4.3. The impact of price elasticity

In Table 5, the impact of price elasticity of demand is analyzed for Perfect Information, Learning and No Learning models. In this specific analysis, the parameter α of the Gamma distribution is taken as 10 and the parameter β of the Gamma distribution is taken as 0.5 leading to an initial estimate with mean 20. The γ value, which controls the price elasticity of demand, takes on 7 values between 1.0 and 4.0, where $\gamma = 1.0$ models inelastic demand (for this case, the optimal price is 1.00 since reducing the price will not modify demand). We first note that the optimal revenue is an increasing function of the price elasticity of demand for Perfect Information model as the decision maker is better able to manipulate the demand.

For $\lambda = 10$, the decision maker is initially overestimating the demand for both Learning and No Learning models and charges an initial price higher than the optimal price in Perfect Information model. However, Learning model can partially correct its estimate based on observed sales and improve its revenue by reducing the price in the second period. The revenue of Learning model increases as the price sensitivity increases since the price reductions are more effective with high price sensitivity. As the price sensitivity increases, we observe that the difference between Perfect Information and Learning and the difference between Learning and No Learning also increase as the information is more useful with a more elastic demand.

For $\lambda = 20$, Learning model performs worse than No Learning model for all demand elasticities. This is because the initial estimate is accurate, and the decision maker is better off if she does not change her estimate based on observed sales. We also see that performance of Learning and No Learning models improves as γ increases, which shows that when the demand is accurately estimated, an elastic demand will always help.

For $\lambda = 30$, the decision maker is initially underestimating the demand for both Learning and No Learning models. Note that with Perfect Information model, the initial price needs to be 1.00, and we hardly need a reduction in price in the second period. However, with inaccurate information, both Learning and No Learning models can ask for price reductions in the second period especially when the demand is highly elastic. However, we should see that the relationship between γ and the performances of Learning and No Learning models is not clear to have any further conclusions.

Table 5
The impact of price elasticity ($I_0 = 20$)

λ			γ						
			1.0	1.5	2.0	2.5	3.0	3.5	4.0
10	Perfect Information	p_1^*	1.00	0.75	0.70	0.75	0.80	0.80	0.80
		$E[p_2^*]$	1.0000	0.7338	0.7141	0.7204	0.7383	0.7764	0.8211
		Revenue	9.9972	10.8237	12.2038	13.3753	14.2552	14.9601	15.4778
	Learning	p_1^*	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		$E[p_2^*]$	1.0000	0.8129	0.7509	0.7397	0.7562	0.7665	0.7799
		%	100.00	95.62	91.10	89.77	89.40	90.04	90.90
	No Learning	p_1^*	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		$E[p_2^*]$	1.0000	0.9106	0.8622	0.8419	0.8345	0.8362	0.8451
		%	100.00	94.20	87.42	84.05	83.08	82.88	82.82
	L/N (%)		100.00	101.51	104.21	106.80	107.61	108.65	109.76
20	Perfect Information	p_1^*	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		$E[p_2^*]$	1.0000	0.9730	0.9524	0.9461	0.9400	0.9352	0.9356
		Revenue	18.2233	18.2669	18.3928	18.5295	18.6529	18.762	18.8537
	Learning	p_1^*	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		$E[p_2^*]$	1.0000	0.9609	0.9375	0.9148	0.9198	0.9151	0.9171
		%	100.00	99.95	99.81	99.52	99.54	99.41	99.34
	No Learning	p_1^*	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		$E[p_2^*]$	1.0000	0.9851	0.9622	0.9462	0.9298	0.9298	0.926
		%	100.00	99.96	99.95	100.00	99.98	99.96	99.88
	L/N (%)		100.00	99.99	99.86	99.52	99.56	99.45	99.46
30	Perfect Information	p_1^*	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		$E[p_2^*]$	1.0000	1.0000	1.0000	0.9999	0.9999	0.9996	0.9996
		Revenue	19.9505	19.9505	19.9506	19.9508	19.9511	19.9516	19.9521
	Learning	p_1^*	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		$E[p_2^*]$	1.0000	0.9967	0.9932	0.9876	0.9881	0.9849	0.985
		%	100.00	99.86	99.71	99.45	99.49	99.36	99.37
	No Learning	p_1^*	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		$E[p_2^*]$	1.00	0.9989	0.9955	0.9918	0.9863	0.9863	0.9822
		%	100.00	99.97	99.83	99.68	99.45	99.45	99.30
	L/N (%)		100.00	99.90	99.87	99.77	100.04	99.91	100.07

4.4. Demand function uncertainty

The analysis so far assumes that the demand function Ψ is known and the retailer has uncertainty only about the magnitude of the demand. Using the general model in Section 3, we now study the case where the retailer has imperfect information regarding the demand function as well. While our model allows us to study the case of misspecification of the demand function as well, we only consider the case of misspecification of the parameter of a specific demand function. In particular, we assume exponential price sensitivity, as in Sections 4.1–4.3, but assumed that the parameter γ is one of five values, i.e.,

$$\Psi(p) = \begin{cases} \psi_1(p) = \mathrm{e}^{-1(p-1)} & \text{with probability } \theta_1, \\ \psi_2(p) = \mathrm{e}^{-2(p-1)} & \text{with probability } \theta_2, \\ \psi_3(p) = \mathrm{e}^{-3(p-1)} & \text{with probability } \theta_3, \\ \psi_4(p) = \mathrm{e}^{-4(p-1)} & \text{with probability } \theta_4, \\ \psi_5(p) = \mathrm{e}^{-5(p-1)} & \text{with probability } \theta_5. \end{cases}$$

We based our analysis on a single problem where the initial inventory I_0 is 30, the true value of the arrival rate λ is 20 and the true value of the price sensitivity parameter γ is 3. The season is again composed of two periods of length 0.5. The retailer charges a price in the set $\mathscr{P} = \{0.50, 0.55, \ldots, 0.95, 1.00\}$. With Perfect Information, the optimal expected revenue is 24.4823. Optimal first period price for this problem is 0.85 and expected optimal second period price is 0.8565. We study the impact of prior imperfect information about the magnitude and function of the demand on the optimal expected revenues of Learning and No Learning models.

In Table 6, we use three different sets of prior probabilities to study the impact of demand function uncertainty: $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) \in \{(0,\ 0,\ 1,\ 0,\ 0), (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\}$. The first set corresponds to full and accurate information; the second set corresponds to a moderate level of uncertainty and the third set corresponds to a high level uncertainty regarding the demand function. The retailer's knowledge of the magnitude of demand is governed again by the parameters of the Gamma distribution. α/β reflects the accuracy of the initial point estimate and takes on five values: 10, 15, 20, 25, and 30. For this purpose only β is varied so that the coefficient of variation stays constant at $1/\sqrt{\alpha}$.

The results in Table 6 show that Learning model provide revenue increases over No Learning model in 17 instances out of 20 instances with demand function uncertainty. For the remaining three cases, Learning model performs very close to No Learning model (and to Perfect Information model). The impact of the demand function uncertainty on the revenues of Learning model depends heavily on the accuracy of

Table 6
The impact of demand function uncertainty: $\lambda = 20$, $\gamma = 3$, $I_0 = 30$

α	β	$(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$	Learning			No Learning			L/N (%)
			p_1^*	$E[p_2^*]$	%	p_1^*	$E[p_2^*]$	%	
10	1.000	(0, 0, 1, 0, 0)	0.65	0.9748	82.70	0.65	0.9494	82.03	100.81
		$(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	0.70	0.9209	89.30	0.65	0.9550	82.19	108.65
		$(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$	0.75	0.8591	93.91	0.70	0.9004	88.76	105.80
	0.667	(0, 0, 1, 0, 0)	0.80	0.8608	97.96	0.80	0.8250	97.21	100.77
		$(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	0.80	0.8569	97.94	0.80	0.8327	97.58	100.37
		$(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$	0.85	0.8274	99.48	0.80	0.8513	98.18	101.33
	0.500	(0, 0, 1, 0, 0)	0.90	0.8081	99.28	0.85	0.8500	99.99	99.29
		$(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	0.90	0.8119	99.32	0.85	0.8577	99.98	99.35
		$(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$	0.90	0.8238	99.64	0.85	0.8692	99.87	99.77
	0.400	(0, 0, 1, 0, 0)	0.95	0.8024	98.30	0.95	0.8472	97.33	100.99
		$(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	0.95	0.8100	98.30	0.95	0.8488	97.22	101.11
		$(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$	1.00	0.8131	96.28	0.95	0.8653	96.19	100.09
	0.333	(0, 0, 1, 0, 0)	1.00	0.8063	96.21	1.00	0.8723	92.44	104.07
		$(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	1.00	0.8157	96.04	1.00	0.8756	92.14	104.23
		$(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$	1.00	0.8366	95.02	1.00	0.8838	91.46	103.89
40	4.000	(0, 0, 1, 0, 0)	0.65	0.9658	82.46	0.65	0.9539	82.14	100.38
		$(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	0.70	0.9018	88.73	0.65	0.9550	82.19	107.95
		$(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$	0.75	0.8439	93.42	0.70	0.9006	88.78	105.23
	2.667	(0, 0, 1, 0, 0)	0.80	0.8454	97.73	0.80	0.8249	97.21	100.53
		$(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	0.80	0.8454	97.85	0.80	0.8357	97.64	100.22
		$(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$	0.80	0.8478	98.02	0.80	0.8551	98.29	99.73
	2.000	(0, 0, 1, 0, 0)	0.85	0.8518	99.87	0.85	0.8520	99.99	99.88
		$(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	0.90	0.8175	99.78	0.85	0.8677	99.96	99.82
		$(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$	0.90	0.8317	99.78	0.85	0.8804	99.73	100.05
	1.600	(0, 0, 1, 0, 0)	0.95	0.8307	98.07	0.95	0.8472	97.33	100.76
		$(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	0.95	0.8372	97.87	0.95	0.8543	96.89	101.01
		$(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$	0.95	0.8527	97.11	0.95	0.8678	96.08	101.07
	1.333	(0, 0, 1, 0, 0)	1.00	0.8507	94.16	1.00	0.8735	92.37	101.94
		$(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	1.00	0.8542	93.91	1.00	0.8830	91.57	102.55
		$(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$	1.00	0.8685	92.80	1.00	0.8957	90.50	102.55

retailer's initial estimate of the magnitude of the demand. There are many instances for which having more precise information about the demand function may in fact hurt the retailer if her demand magnitude estimate is inaccurate. For example, in instance with $\alpha = 10$ and $\beta = 1$, the retailer's expected demand rate is 10, when in fact the true value of the demand rate is *20*. In this case, precisely knowing the demand function ($\gamma = 3$ with probability 1) generates 82.70% of the optimal revenue, while prior probabilities $\left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)$ and $\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$ can generate 89.30% and 93.91% of the optimal revenue, respectively, under Learning Model. This can be explained as follows. With a demand rate estimate of 10 and initial inventory of 30 units, the retailer would like to charge a low price in both periods to maximize its expected revenue. However, the imprecise information about the demand function will guide the retailer to be more conservative in her pricing decision (i.e., she will charge a higher price) in the first period and this will lead to first period prices to be closer to the optimal price (0.85). Note that Learning model still outperforms No Learning model in these cases, since the retailer learns about the demand function and magnitude in Learning model and use this information for more efficient pricing in the second period. With a few exceptions, the difference between Learning model and No Learning model is more significant when there is uncertainty about the demand function. However, the relative performance of Learning model is not monotone in the level of uncertainty. One can also see from the results that the revenue gains by using Learning model as opposed to No Learning model are larger when the retailer's variance of the estimate of the demand rate is higher ($\alpha = 10$).

In Table 7, we introduce inaccuracies to the retailer's initial estimate of the demand function by using four sets of prior probabilities $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) \in \{(0,\ 0,\ 1,\ 0,\ 0), \left(0, 0, \frac{3}{4}, \frac{1}{4}, 0\right), \left(0, 0, \frac{1}{2}, \frac{1}{2}, 0\right), \left(0, 0, \frac{1}{4}, \frac{3}{4}, 0\right)\}$. In the first set, the retailer has full and accurate information ($\gamma = 3$ with probability 1) about the demand function. In the second, third and fourth sets, the retailer uses prior probabilities $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$ for the event $\gamma = 4$. In 26 out of 30 instances with demand function uncertainty, Learning model generates higher revenues than No Learn-

Table 7
The impact of demand function uncertainty: $\lambda = 20$, $\gamma = 3$, $I_0 = 30$

α	β	$(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$	Learning			No Learning			L/N (%)
			p_1^*	$E[p_2^*]$	%	p_1^*	$E[p_2^*]$	%	
10	1.000	$(0,0,1,0,0)$	0.65	0.9748	82.70	0.65	0.9494	82.03	100.81
		$(0,0,\frac{3}{4},\frac{1}{4},0)$	0.70	0.9201	89.25	0.70	0.8820	88.18	101.22
		$(0,0,\frac{1}{2},\frac{1}{2},0)$	0.70	0.9173	89.20	0.70	0.8781	88.09	101.26
		$(0,0,\frac{1}{4},\frac{3}{4},0)$	0.75	0.8657	94.04	0.70	0.8805	88.21	106.61
	0.667	$(0,0,1,0,0)$	0.80	0.8608	97.96	0.80	0.8250	97.21	100.77
		$(0,0,\frac{3}{4},\frac{1}{4},0)$	0.80	0.8523	97.81	0.80	0.8327	97.58	100.24
		$(0,0,\frac{1}{2},\frac{1}{2},0)$	0.80	0.8494	97.78	0.80	0.8312	97.55	100.23
		$(0,0,\frac{1}{4},\frac{3}{4},0)$	0.85	0.8100	99.02	0.80	0.8329	97.64	101.42
	0.500	$(0,0,1,0,0)$	0.90	0.8081	99.28	0.85	0.8500	99.99	99.29
		$(0,0,\frac{3}{4},\frac{1}{4},0)$	0.90	0.8067	99.35	0.90	0.8135	99.85	99.49
		$(0,0,\frac{1}{2},\frac{1}{2},0)$	0.90	0.8119	99.32	0.90	0.8187	99.81	99.51
		$(0,0,\frac{1}{4},\frac{3}{4},0)$	0.90	0.8235	99.62	0.90	0.8190	99.80	99.82
	0.400	$(0,0,1,0,0)$	0.95	0.8024	98.30	0.95	0.8472	97.33	100.99
		$(0,0,\frac{3}{4},\frac{1}{4},0)$	0.95	0.8058	98.47	0.95	0.8466	97.35	101.16
		$(0,0,\frac{1}{2},\frac{1}{2},0)$	0.95	0.8058	98.47	0.95	0.8481	97.24	101.27
		$(0,0,\frac{1}{4},\frac{3}{4},0)$	0.95	0.8269	98.08	0.95	0.8528	96.88	101.24
	0.333	$(0,0,1,0,0)$	1.00	0.8063	96.21	1.00	0.8723	92.44	104.07
		$(0,0,\frac{3}{4},\frac{1}{4},0)$	1.00	0.8131	96.20	1.00	0.8698	92.60	103.89
		$(0,0,\frac{1}{2},\frac{1}{2},0)$	1.00	0.8120	96.27	1.00	0.8724	92.34	104.26
		$(0,0,\frac{1}{4},\frac{3}{4},0)$	1.00	0.8219	95.92	1.00	0.8724	92.34	103.88
40	4.000	$(0,0,1,0,0)$	0.65	0.9658	82.46	0.65	0.9539	82.14	100.38
		$(0,0,\frac{3}{4},\frac{1}{4},0)$	0.70	0.9016	88.72	0.65	0.9482	82.07	108.10
		$(0,0,\frac{1}{2},\frac{1}{2},0)$	0.70	0.9024	88.76	0.70	0.8839	88.27	100.56
		$(0,0,\frac{1}{4},\frac{3}{4},0)$	0.70	0.9026	88.77	0.70	0.8845	88.31	100.52
	2.667	$(0,0,1,0,0)$	0.80	0.8454	97.73	0.80	0.8249	97.21	100.53
		$(0,0,\frac{3}{4},\frac{1}{4},0)$	0.80	0.8430	97.73	0.80	0.8327	97.58	100.16
		$(0,0,\frac{1}{2},\frac{1}{2},0)$	0.80	0.8456	97.87	0.80	0.8393	97.82	100.05
		$(0,0,\frac{1}{4},\frac{3}{4},0)$	0.80	0.8478	98.02	0.80	0.8443	98.00	100.02
	2.000	$(0,0,1,0,0)$	0.85	0.8518	99.87	0.85	0.8520	99.99	99.88
		$(0,0,\frac{3}{4},\frac{1}{4},0)$	0.90	0.8175	99.78	0.85	0.8577	99.98	99.80
		$(0,0,\frac{1}{2},\frac{1}{2},0)$	0.90	0.8210	99.84	0.90	0.8310	99.72	100.12
		$(0,0,\frac{1}{4},\frac{3}{4},0)$	0.90	0.8278	99.83	0.90	0.8312	99.71	100.12
	1.600	$(0,0,1,0,0)$	0.95	0.8307	98.07	0.95	0.8472	97.33	100.76
		$(0,0,\frac{3}{4},\frac{1}{4},0)$	0.95	0.8372	97.87	0.95	0.8543	96.89	101.01
		$(0,0,\frac{1}{2},\frac{1}{2},0)$	0.95	0.8420	97.63	0.95	0.8551	96.83	100.83
		$(0,0,\frac{1}{4},\frac{3}{4},0)$	0.95	0.8465	97.42	0.95	0.8647	96.21	101.26
	1.333	$(0,0,1,0,0)$	1.00	0.8507	94.16	1.00	0.8735	92.37	101.94
		$(0,0,\frac{3}{4},\frac{1}{4},0)$	1.00	0.8542	93.91	1.00	0.8819	91.64	102.48
		$(0,0,\frac{1}{2},\frac{1}{2},0)$	1.00	0.8533	93.97	1.00	0.8838	91.46	102.75
		$(0,0,\frac{1}{4},\frac{3}{4},0)$	1.00	0.8665	92.95	1.00	0.8838	91.46	101.63

ing model. In the remaining four instances, both models generate close to optimum revenues and the gap is insignificant. The impact of the accuracy of the retailer's initial estimate of the demand function uncertainty on the revenues of Learning model depends again heavily on the accuracy of retailer's initial estimate of the magnitude of the demand. In many instances, the retailer may in fact get hurt by the accuracy of his initial estimate of demand function if his estimate of the demand magnitude is not accurate. For example, when $\alpha = 10$ and $\beta = 1.00$, the retailer's initial estimate of the demand rate is 10, when in fact the demand rate λ is 20. In this case, precise information about the demand function ($\theta_3 = 1$) generates a revenue of 82.70% of the optimal revenue with Learning model, while increasing levels of inaccuracy provided by $\theta_3 = \frac{3}{4}$, $\theta_3 = \frac{1}{2}$ and $\theta_3 = \frac{1}{4}$ generate 89.25%, 89.20% and 94.04% of the optimal revenues, respectively. The explanation is similar to one provided for Table 6. Again, with a few exceptions, the performance gap between Learning and No Learning models is higher when there is more inaccuracy in retailer's initial estimate of the demand function. This is particularly true when the retailer's initial estimate of the demand magnitude is accurate.

5. Inventory flexibility

The analysis so far assumes that there are no further replenishment opportunities available once the selling season starts. In the apparel industry, this corresponds to the case when the retailer orders from overseas and is not able to order during the season because of the long lead times relative to the selling seasons. Obviously, this limits the retailer's control during the selling season to pricing only, which sharply diminishes its responsiveness. As a result, some retailers are willing to use domestic suppliers and be able to order frequently, even though domestic suppliers are more costly. With domestic suppliers, the retailer is also able to make its initial order much closer to the season, when there is more information, hence less variance, about the demand process.

Some companies are using two (or sometimes even three) different suppliers for the very same product: an off-shore low-cost supplier for the initial large orders, and a domestic high-cost supplier for replenishments during the selling season (Apparel Industry Magazine [1]).

We study the value of this additional flexibility in the context of our pricing model. In a related study, Gurnani and Tang [24] study the impact of forecast improvements by having the flexibility to order at two instances, one of them being closer to the season. Their model differs from ours as they do not consider the possibility of ordering during the season by utilizing a structured learning from observed sales. Also they do not consider any pricing during the season. While their model allows the cost to go up or down as the merchandise is ordered closer to the season, we always assume that the ordering later is more costly reflecting the reality in the apparel industry.

In our model, the off-shore strategy will allow the company to order only once, but possibly with a low unit cost c^o. The domestic strategy will allow the company to order before and during the selling season, but possibly with a high unit cost c^d. The blended strategy, on the other hand, will allow the company to make its initial order at a unit cost c^o, but later replenishments at the unit cost c^d. We assume that there are no other costs involved, the pricing and inventory decisions are made simultaneously at the start of the each period; period lengths are equal for each strategy and the lead time is zero for all strategies.

To be able to compare these three strategies, we need to extend our pricing model to allow for inventory decisions. The problem is to determine prices and stock levels in periods $1, \ldots, N$ so that total expected profit is maximized. We use a discrete-time dynamic programming model. For this particular model, we assume that the retailer has no uncertainty regarding the demand function. The model is explained in the following.

Let $V_n(I_{n-1}, X_{n-1}, M_{n-1})$ be the maximum expected profit from period n through N where the starting inventory is I_{n-1} and the cumulative sales and cumulative price multipliers are X_{n-1} and M_{n-1}, respectively. Also let B_n be the starting inventory level for period n, after the retailer receives its orders. Thus, the retailer acquires $B_n - I_{n-1}$ new units in the beginning of period n. Let p_n be the price set in period n and let c_n be the acquisition cost per unit in period n.

Backward recursion can be written as

$$V_n(I_{n-1}, X_{n-1}, M_{n-1}) = \max_{p_n \geqslant p_s, B_n \geqslant I_{n-1}} E[-c_n(B_n - I_{n-1}) + p_n \min\{D_n, B_n\} + V_{n+1}((B_n - D_n)^+, X_{n-1} + D_n, M_{n-1} + m(p_n)) | X_{n-1}, M_{n-1}, p_n].$$

Boundary conditions are

$$V_{N+1}(I_N, X_N, M_N) = p_s I_N, \quad \text{for all } I_N, X_N, M_N,$$
$$X_0 = M_0 = I_0 = 0.$$

The first condition states that any left over merchandise has only salvage value (p_s) when the season ends at the end of period N. We also assume that it is independent of the cumulative sales up to period N. The dynamic program can be solved by starting with the Nth period and proceeding backwards.

For the off-shore strategy, the model can be used with the following acquisition costs.

$$c_1 = c^o,$$
$$c_n = \infty, \quad n = 2, \ldots, N.$$

For the domestic strategy, we simply have

$$c_n = c^d, \quad n = 1, \ldots, N.$$

For the blended strategy, we have,

$$c_1 = c^o,$$
$$c_n = c^d, \quad n = 2, \ldots, N.$$

Let $V^o(c^o, c^d)$, $V^d(c^o, c^d)$ and $V^b(c^o, c^d)$ be the optimal profits for the off-shore, domestic and blended strategies respectively. Without any analytical derivations, it is easy to see the following.

Observation 1. *When the off-shore cost is higher than or equal to the domestic cost (which is not likely), the domestic strategy outperforms the off-shore strategy. That is, for* $c^o \geqslant c^d$, $V^o(c^o, c^d) \leqslant V^d(c^o, c^d)$.

Intuition:. A domestic policy can simply imitate the optimal off-shore policy by ordering as much as the optimal off-shore policy does in the first period and ordering zero units in later periods. Since the acquisition costs are lower for the domestic orders, this policy generates more profit than the optimal off-shore policy.

Observation 2. *When the off-shore cost is lower than the domestic cost (which is typical), the blended strategy outperforms both strategies. That is, for* $c^o < c^d$, $V^b(c^o, c^d) \geqslant V^o(c^o, c^d)$ and $V^b(c^o, c^d) > V^d(c^o, c^d)$.

Intuition:. A blended policy can imitate the optimal off-shore policy by simply ordering as much as the off-shore policy does in the first period and ordering zero units in later periods. Since the acquisition costs are the same for the blended and off-shore strategies in the first period, this policy generates the same profit with the optimal off-shore profit. Likewise, another blended policy can imitate the optimal domestic policy by simply ordering as much as the optimal domestic policy does in each period. Since first period's acquisition costs are lower for the blended strategy, this policy generates more profit than the optimal domestic policy.

While these comparisons are trivial, a question of interest is under which other circumstances the retailers should favor domestic policies over off-shore policies and under which circumstances the gap between the blended and domestic and off-shore policies are minimal. This is important as acquisition costs may not be the only concern for a retailer. For example, using an additional supplier may involve additional fixed setup costs and complicate the coordination of the sourcing process, which disadvantage the blended strategies. Also, in our study we do not consider the inventory holding and other logistics costs that may be incurred within the selling season. Inclusion of inventory holding costs to the model may favor domestic and blended strategies against the off-shore strategy as domestic purchases may be used for frequent replenishments and may reduce inventory levels. However, if unit inventory holding costs are proportional to the unit cost and domestic cost is excessively higher than the import cost, inventory reduction effect will be less apparent.

We use the computational design in Section 4 to answer above questions. Again, the mean demand is $\alpha/\beta = 20$ and we have two periods of equal length. Different from the analysis in Section 4, the starting inventory level is optimized for all strategies. We assume that the maximum price to charge is 1.00. We set the off-shore acquisition cost to 0.5 and vary the domestic acquisition cost to study the effect of acquisition costs on different strategies. We note that in this analysis, we use Learning model as described in Section 3, and the expected profits are evaluated using the Negative Binomial distribution (with initial parameters in the first period and with updated parameters in the second period). We do not use the evaluations based on the true Poisson rate, as this is not available to the decision maker until after the season, and the decision maker makes her sourcing decisions based on her prior beliefs and how she updates her beliefs based on sales during the season. Fig. 1 shows the (expected) optimal profits of off-shore, domestic and blended strategies when $\gamma = 2$ and when variance equals 1.5μ or 3μ. The optimal profits are normalized with the profit of the optimal off-shore policy when variance equals to 3μ. We first note that the optimal off-shore profits do not vary with the domestic acquisition costs. Blended strategies, as shown above, outperform the domestic and off-shore strategies. Clearly, optimal blended and domestic profits decrease with acquisition costs. However, optimal blended profit curves are rather flat, as blended strategies prefers to order more from the off-shore supplier as the domestic supplier becomes more expensive. In fact, optimal blended profits approach optimal off-shore profits as domestic acquisition costs increase. The reduction in profits is more dramatic for domestic policies as they have to live with the expensive domestic suppliers. While domestic policies outperform off-shore policies for low domestic acquisition costs, off-shore policies are favorable as the domestic suppliers become more costly.

For this particular example, domestic and off-shore profit curves intersect when the normalized domestic cost is 1.1 for $\sigma^2 = 3\mu$. This means that the "break-even" point where off-shore profit equals domestic profit is when the unit domestic acquisition cost is 10% more

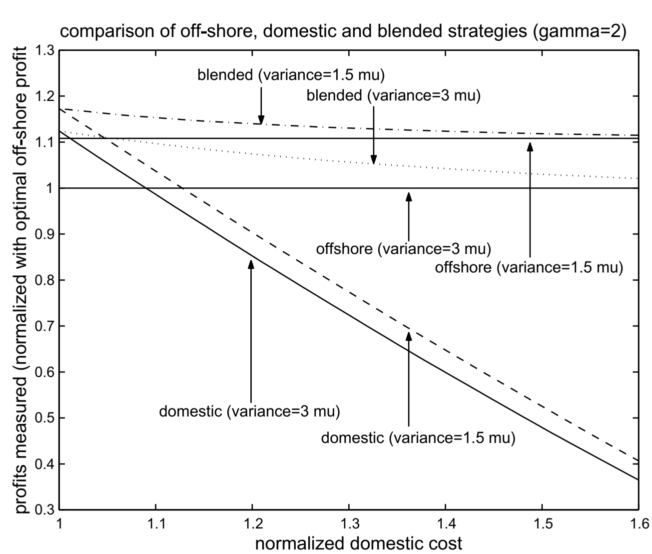

Fig. 1. Comparison of off-shore, domestic and blended strategies ($\gamma = 2$): Impact of domestic cost.

than the unit off-shore acquisition cost. Any unit domestic acquisition cost 10% more than the unit off-shore acquisition cost will lead the retailers to source their merchandise off-shore.

Another important factor for the efficiency of the off-shore, domestic and blended policies is the variance of the demand process. Typically, apparel retailers choose domestic suppliers for their high fashion content-high variance merchandise, while standard low fashion content-low variance merchandise can be sourced overseas. Also, the policy itself may help to reduce the variance as the domestic strategies can order closer to the season. Fig. 2 shows the optimal profits for domestic, off-shore and blended strategies for three different domestic acquisition costs (1.00, 1.06 or 1.12 times the off-shore acquisition cost). Again, the profits are normalized with the optimal off-shore profit for $\sigma^2 = 3\mu$. Note that blended and domestic policies are equivalent when the cost equals 1.00. These policies outperform any other policy. As the variance increases, all profits decrease. The off-shore optimal profit curve is steeper as off-shore policies are subject to more variance since they order only once. Optimal domestic policy when the cost equals 1.06 is inferior to the off-shore policy for low variance levels, but becomes favorable as the variance increases. Note again that the base line is the optimal off-shore profit when the variance is 3μ. If we can reduce the variance to 1.75μ by using a domestic policy, even the domestic acquisition costs of 1.06 can be desirable.

Finally, the price sensitivity of the customers also affects the relative efficiency of these policies. We expect the off-shore strategy to be more sensitive to price sensitivity (γ) as pricing is the only control for such strategy once the season starts. Fig. 3 shows the optimal profits for varying levels of γ. Note that when $\gamma = 1$, the demand is inelastic and it is optimal to keep the price at its maximum. Thus, optimal profits at $\gamma = 1$ represent the optimal profits when the only control over the process is through inventory. All profits increase, as the price sensitivity increases. As expected, optimal off-shore profit increases faster with the price sensitivity. The off-shore strategy outperforms the domestic strategy with cost 1.06 when γ is close to 3.

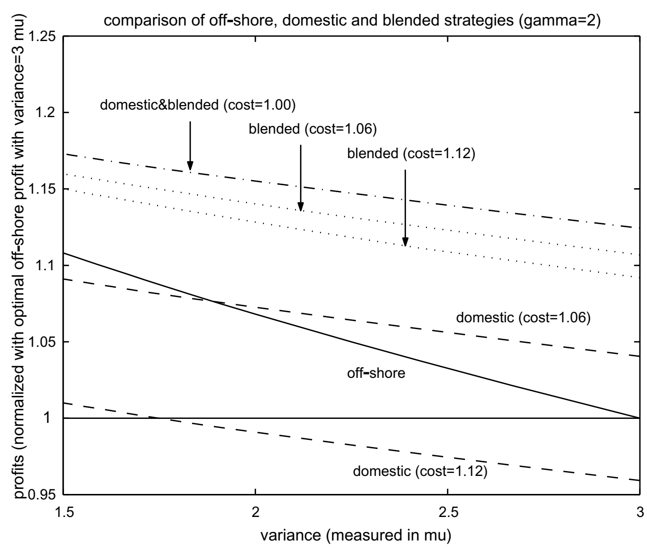

Fig. 2. Comparison of off-shore, domestic and blended strategies ($\gamma = 2$): Impact of demand variability.

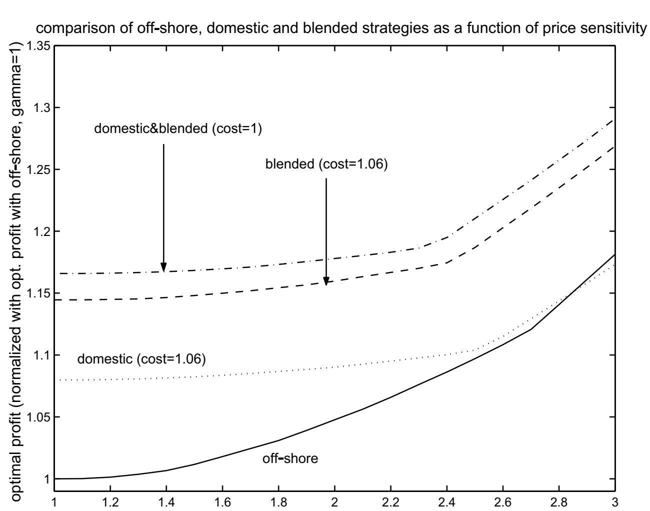

Fig. 3. Comparison of off-shore, domestic and blended strategies as a function of price sensitivity.

Combining these ideas, we generate the regions in which one strategy is favorable to the other. Fig. 4 shows the trade-off curves for off-shore and domestic strategies for $\gamma = 2$ and $\gamma = 3$. For variances and domestic costs on these lines, off-shore and domestic strategies generate the same profit. As domestic cost increases and/or variance decreases, off-shore strategy becomes more desirable and vice versa. Note that, the region for which off-shore strategy is more profitable is larger when $\gamma = 3$, reflecting the increased strength of off-shore strategies with price sensitive demand. In general, as γ increases, the trade-off curve moves to southeast. Although it is difficult to detect visually, we observe that the trade-off curves are concave in variance, possibly becoming flatter as variance gets larger. This means that if the variance gets excessively high, variance differences would have less impact and supplier selection decisions would depend more on cost differences and price sensitivity of demand.

The above analysis is based on the fact that the products are subject to same level of uncertainty under both strategies. However, in most cases, the choice of strategy itself may affect the level of uncertainty. As the retailers are able to order closer to the season with domestic strategies, they are able to know more about the consumer tastes that will shape the demand in the coming season and hence they face a more stable demand when they make their ordering decisions. To incorporate the possible reductions in variance, we choose a base case, which is an off-shore strategy with acquisition cost equals 0.5 and variance (σ^2) equals 3μ. The trade-off curves in Fig. 5 shows the increases in cost and reductions in variance with domestic strategy for which domestic and off-shore strategies generate equal profits. For example, for $\gamma = 3$, if the domestic cost is about 7% higher than off-shore cost, the domestic strategy will still result in higher profits, if the variance is reduced by more than 30% as a result. Alternatively, if the variance is reduced about 30%, the domestic strategy will generate higher profits only if the cost does not increase by more than 7%. Again, optimality region is larger for off-shore strategy for more price sensitive demand.

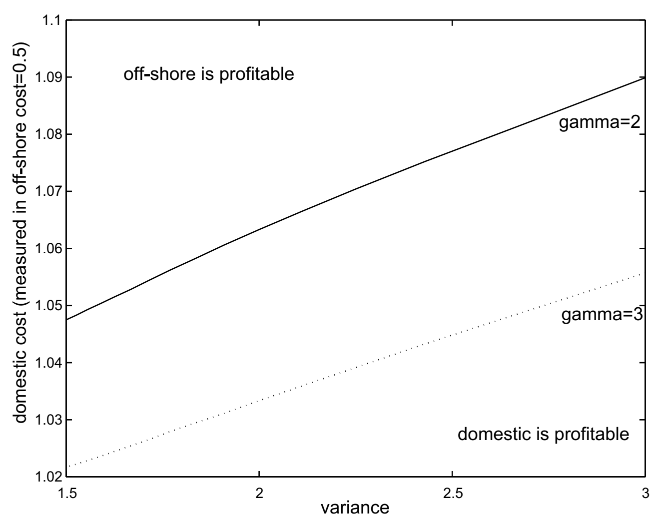

Fig. 4. Trade-off curves for off-shore and domestic production.

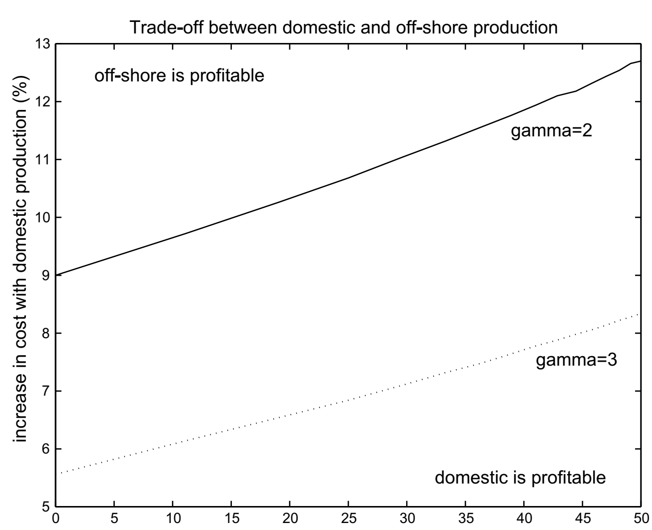

Fig. 5. Trade-off between domestic and off-shore production.

Table 8
Market share and cost of imports in apparel in 2002

	Average domestic price[a] (c^d)	Average import price[b] (c^o)	Ratio (c^d/c^o)	Market share of imports[c]
Men's				
Sweaters	15.27	10.42	1.47	98.3
Swimwear	12.93	4.52	2.86	100.0[d]
Suits	100.64	62.71	1.60	74.0
Women's				
Sweaters	12.24	9.69	1.26	86.1
Swimwear	13.61	6.10	2.23	70.8
Dresses	20.10	9.08	2.21	64.0

Data compiled from U.S. Census Bureau [41]. Prices are per dozen in U.S. dollars.
[a] Average cost ($) per unit for manufacturers' shipments.
[b] Average cost ($) (cost + insurance + freight) per unit from imports for consumption.
[c] Derived by dividing imports for consumption to apparent consumption in the U.S. market.
[d] Missing quantity data for 2002 imports is estimated using data from 2001.

Each retailer faces its own trade-off curve for each apparel item it offers and makes its decision to source it overseas or domestically. An aggregation of these individual decisions determines the market share of imports and domestic production in the domestic market. In Table 8, we provide the average unit import and domestic costs and the market share of imports for selected apparel in the U.S. market in 2002.

We observe that for all product categories, imports have a substantial cost advantage. This is particularly true for men's and women's swim-wear. While this translates into a perfect market domination of imports in men's swim-wear, domestic manufacturers still control 70.8% of the market for women's swim-wear. Similar arguments are valid for sweaters for men and women and suits for men and dresses for women. This shows that the cost advantage is not the only factor in supplier selection. Although it is very difficult to find an aggregate measure for the variance of demand in apparel, we are certainly aware of the importance of fashion in women's apparel. Popular styles and colors change every year, making it very difficult to forecast demand for a particular SKU. From both our computational analysis and industry data, we see that predictability of demand plays a considerable role in sourcing decisions. When the variance effect is less apparent (as in sweaters product category, or as in men's apparel in general as compared to women's apparel), we observe that cost difference is the main driver for such decisions.

6. Conclusion

In this paper, we study the pricing decisions of a perishable products retailer in the existence of demand learning. This is one of the first studies that incorporate "structured" Bayesian updating in the context of pricing for perishable products. The resulting model is computationally feasible and easy to understand and implement. We think that our model is most useful for apparel retailers, as this industry is identified with high levels of uncertainty, most of which can be resolved after observing sales during the earlier weeks of the selling season. Moreover, information required for the application of our model is readily available through point-of-sales scanners.

Through our computational study, we are able to understand the economics of pricing in this context. First, we observe that the optimal price in a given period is a non-decreasing function of sales in the earlier periods when demand learning takes place. Second, we pinpoint the circumstances under which this learning based on observed sales has the most value. We first study the impact of the accuracy and the variance of the initial estimate of demand magnitude and the price elasticity of demand when the retailer has Perfect Information about the demand function. Our major finding here is that demand learning is most beneficial when the initial estimate is inaccurate, the demand/supply mismatches necessitate price changes and demand is sensitive to price changes. We then study the case where the retailer learns about the demand function in addition to the demand magnitude. In this case, we observe that the accuracy and variance of the retailer's initial estimate of demand magnitude and demand function have a joint effect on the benefits of learning. Finally, we study the impact of an opportunity to procure merchandise during the season, in addition to the up-front procurement before the selling season. This helps us to see how supplier selection decisions are affected by the volatility and price-sensitivity of demand and the procurement costs. We support our conclusions with aggregate data from the apparel industry.

We note that our model with inventory flexibility can be extended to incorporate lead times, inventory holding costs and set-ups (cost and/or time) that may be attached to each purchase. We refrain from doing so, as their effects on costs are fairly trivial and their inclusion may complicate the presentation of the model.

Several avenues for future research are in order. First, the model could be extended to allow for the case where the time to switch price is also a decision variable. This way, one can study the impact of learning on optimal markdown times. An interesting question is whether Learning model will always delay the pricing decisions in an effort to learn more about the underlying demand. Second, the impact of more than two periods and mark-down only restrictions can be studied.

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