Fibred permutation sets and the idempotents and units of monomial Burnside rings

Laurence Barker

Department of Mathematics, Bilkent University, 06533 Bilkent, Ankara, Turkey
Received 3 March 2003
Available online 11 September 2004
Communicated by Michel Broué

Abstract

We study the units of monomial Burnside rings and the idempotents of monomial Burnside algebras. Introducing a tenduction map, we realise the unit group and the torsion unit group as a Mackey functor.

Keywords: Monomial Burnside rings; Units of Burnside rings; Möbius inversion; Tenduction; Tensor induction

1. Introduction

Our main purpose is to realise unit groups of monomial Burnside rings as a Mackey functors over \( \mathbb{Z} \). The two given ingredients are a finite group \( G \) and a cyclic (or, more generally, supercyclic) group \( C \). Before studying the unit group of the monomial Burnside ring \( B(C, G) \) (defined below), it will be necessary to examine the primitive idempotents of monomial Burnside algebras \( RB(C, G) \) with coefficients in a suitable ring \( R \). In the case where \( C \) is cyclic of odd prime order \( p \), and \( G \) is of odd order, the unique subgroup of index 2 in the torsion-unit group of \( B(C, G) \) will be realized as a Mackey functor over the field \( \mathbb{F}_p \) of order \( p \).

E-mail address: barker@fen.bilkent.edu.tr.

0021-8693/5 – see front matter © 2004 Elsevier Inc. All rights reserved.
More important than proving any theorems about them, we must indicate why there is good reason to believe that monomial Burnside rings are worthy of attention. We shall argue that there are motives for seeking to extend the ordinary Burnside ring $B(G)$ and its unit group $B(G)^*$. That requires, to begin with, some comments on $B(G)$ and $B(G)^*$, themselves. First, let us note that they are much more than isomorphism invariants of the finite group $G$. They are Mackey functors, and so they are susceptible to local methods. The use of the ring $B(G)$ in representation theory, in connection with induction theorems, is very well known—see, for instance, Benson [1, Chapter 5]—but let us say a few words on the rather more enigmatic role of $B(G)^*$ in representation theory.

As an isomorphism invariant, $B(G)^*$ imposes very little constraint on the group $G$, since it is an elementary abelian 2-group, hence its only parameter is its rank. As a Mackey functor over the Galois field $\mathbb{F}_2$, there is much richer scope for study. The unit group $B(G)^*$ plays a role in the theory of $G$-spheres. Matsuda [20, Corollary 2.6] showed that the group of homotopy equivalences on the unit sphere $S(M)$ of an $\mathbb{R}G$-module $M$ maps injectively to $B(G)^*$, and bijectively when $M$ is, in a suitable sense, sufficiently large. Writing square brackets to denote isomorphism classes, and writing $\tilde{\Lambda}$ to indicate a reduced Lefschetz invariant, tom Dieck [12, Proposition 5.5.9] observed that the assignment $[M] \mapsto \tilde{\Lambda}(S(M))$ determines a linear map $\exp_G$ from the real representation ring to the unit group $B(G)^*$. Yoshida [27, Theorem A] extended $\exp_G$ to a map from the real valued virtual characters to $B(G)^*$. Further extension to complex virtual characters would surely be desirable, but calculation with concrete examples (say, with the cyclic group of order 4) will readily convince those who try it that any such extension of the domain also requires an extension of the codomain $B(G)^*$.

At various levels of generality, monomial Burnside rings, explicitly or implicitly, have been studied in contexts involving or closely related to induction theorems. See, for instance, Boltje [2–6], Boltje–Külshammer [7,8], Conlon [11], Dress [14]. In such contexts, applications rely on the fact that, if the supercyclic group $C$ is suitable and, in particular, sufficiently large, then $B(C,G)$ maps surjectively to the representation ring of $FG$-modules, where $F$ is a splitting field for $G$. In effect, $C$ is taken to be the group of torsion-units of $F$ (we shall explain this substitution in Section 2). One of the motives for the level of generality we have selected is that (for reasons which will transpire) the case where $|C|$ is prime seems to be of special interest.

Following Dress [14], let us introduce the notion of a monomial Burnside ring in an abstract manner that is entirely detached from its applications in linear representation theory. The theory of permutation sets for a finite group extends quite easily to a theory of fibred permutation sets where the role previously played by the points of the permutation set is now played by fibres, which are copies of a fixed abelian group $A$ called the fibre group. We define an $A$-fibred $G$-set to be an $A$-free $A \times G$-set with only finitely many orbits. The category of $A$-fibred $G$-sets admits a product and a coproduct given by, respectively, tensor product over $A$ and set-wise coproduct (disjoint union). See Section 2 for details. Hence, we obtain a Grothendieck ring, denoted by $B(A,G)$, called the monomial Burnside ring for $G$ with fibre group $A$. In the special case where $A$ is trivial, we recover the ordinary Burnside ring $B(1,G) = B(G)$. For a commutative ring $\Theta$, the algebra $\Theta B(A,G) = \Theta \otimes_{\mathbb{Z}} B(A,G)$ is called a monomial Burnside algebra with coefficient ring $\Theta$. 
In view of (all) the applications (that we know of), our interest is in the case where the fibre group is isomorphic to a subgroup of the torsion-unit group of an integral domain. Such groups are said to be supercyclic (an equivalent and abstract definition of the term is given in Section 3). Much of the material in this paper is a study of the idempotents of the monomial Burnside ring $KB(C, G)$, where $K$ is a field of characteristic zero. In the special case where $K$ has enough roots of unity and $C$ is the unit group of an algebraically closed field, this has already been done by Boltje [6]. We shall present this as an application of a monomial version of Möbius inversion. Regarded as a preliminary to a study of units, our treatment of idempotents is not maximally efficient, because we shall prove some results on idempotents that will not be needed when we turn to units. Furthermore, the approach through Möbius inversion is not the most direct way of deriving the idempotent formula. However, we consider idempotents and monomial Möbius inversion to be of interest in their own right (and the latter provides a meaningful rationale for the idempotent formula in Section 5).

Still, the main section in this paper is the final one, where we introduce a tenduction (tensor induction) functor on $A$-fibred $G$-sets (as always, $A$ arbitrary abelian) and a tenduction map on the Burnside ring $B(C, -)$ (as always, $C$ arbitrary supercyclic). The tenduction map serves as the transfer map for the unit group $B(C, -)^*$ as a Mackey functor.

Let us mention another possible avenue of investigation. Out of a need for terminology, let us say that a ring $\Lambda$ is a split extension of a subring $\Gamma$ provided $\Lambda = \Gamma \oplus I$, where $I$ is a two-sided ideal. Thus, $\Gamma$ is a quotient ring of $\Lambda$, as well as a subring. In Section 2, we shall observe that $B(C, G)$ is a split extension of $B(G)$; hardly remarkable, since we were already regarding $B(C, G)$ as a generalization of $B(G)$. But, in Section 7, we shall observe that, if the exponent of $G$ divides the order of $C$, then $B(C, G)$ is also a split extension of the group algebra of the abelianization of (the dual group of) $G$. If we now also assume that $G$ is abelian, we conclude that the unit group $B(C, G)^*$ is a split extension (in the usual sense) of the unit group $\left(ZG\right)^*$. In Section 8, we shall be making use of (a fairly trivial result in) the theory of unit groups of group rings. These observations seem to suggest the possibility of a Mackey functor approach to the study of unit groups of commutative group rings.

Some conventions: Any ring is understood to have a unity element, and any subring is understood to have the same unity element. The unit group of a ring $\Theta$ is denoted by $\Theta^*$ and the torsion-unit group, by $\Theta_\omega^*$. We write $\exp(G)$ to denote the exponent of $G$. To avoid conflicts of terminology, we shall use the term specialize to refer to restriction of functions.

2. Monomial Burnside rings in general

We make some preliminary comments on monomial Burnside rings and monomial Burnside algebras in the general case where the fibre group is the arbitrary abelian group $A$. The material was introduced long ago by Dress [14], and he considered the even more general case where $G$ acts on $A$. Our purpose is just to set up our preferred notation and perspective.

We write $A \times G = AG = \{ag: a \in A, g \in G\}$. An $A$-free $AG$-set with finitely many $A$-orbits is called an $A$-fibred $G$-set. An $A$-orbit of an $A$-fibred $G$-set is called a fibre.
Whenever an expression of the form $AX$ denotes an $A$-fibred $G$-set, it is to be understood that $X$ is a set of representatives of the fibres. Note that $X$ is finite. Of course, $G$ need not stabilize $X$. Any element of $AX$ can be written uniquely in the form $ax$ where $a \in A$ and $x \in X$.

Let $AX$ and $AY$ be $A$-fibred $G$-sets. The coproduct of $AX$ and $AY$ is defined to be their coproduct as sets (their disjoint union)

$$AX \sqcup AY = A(X \sqcup Y)$$

regarded in an evident way as an $A$-fibred $G$-set. With respect to the action of $A$ on $AX \times AY$ given by $a(\xi, \eta) = (a\xi, a^{-1}\eta)$, we let $\xi \otimes \eta$ denote the $A$-orbit of an element $(\xi, \eta)$ of $AX \times AY$, and we let $AX \otimes AY$ denote the set of $A$-orbits. Loosely, we call $AX \otimes AY$ the tensor product of $AX$ and $AY$ over $A$. We let $AG$ act on $AX \otimes AY$ by

$$ag(\xi \otimes \eta) = ag\xi \otimes g\eta.$$

We write $x \otimes y = xy$ and $XY = \{xy: x \in X, y \in Y\}$. The product of $AX$ and $AY$ is defined to be the tensor product

$$AX \otimes AY = AXY$$

regarded as an $A$-fibred $G$-set.

Having imposed a product and a coproduct on the category of $A$-fibred $G$-sets, we can now construct the Grothendieck ring $B(A,G)$. Let $[AX]$ denote the isomorphism class of a $A$-fibred $G$-set $AX$. As an abelian group, $B(A,G)$ is generated by the isomorphism classes of $A$-fibred $G$-sets. The relations are given by the condition that

$$[AX] + [AY] = [AX \sqcup AY] = [A(X \sqcup Y)].$$

The multiplication is such that

$$[AX][AY] = [AX \otimes AY] = [AXY].$$

Evidently, $B(A,G)$ is a commutative ring.

To analyse the ring $B(A,G)$, we need some more notation and terminology. Let $A \backslash AX$ denote the set of fibres of $AX$. Obviously, $AX$ is transitive as an $AG$-set if and only if $A \backslash AX$ is transitive as a $G$-set. In that case, $AX$ is said to be transitive as an $A$-fibred $G$-set. As an abelian group, $B(A,G)$ is freely generated by the isomorphism classes of transitive $A$-fibred $G$-sets.

We define an $A$-character of $G$ to be a group homomorphism $G \to A$. We define an $A$-subcharacter of $G$ to be a pair $(V, \nu)$ where $V \leq G$ and $\nu$ is an $A$-character of $V$. The $A$-subcharacters of $G$ admit an action of $G$ by conjugation: $\delta(V, \nu) = (\delta V, \delta \nu)$. Let $AG/V$ denote a transitive $A$-fibred $G$-set such that $V$ is the stabilizer of a fibre $Ax$, and $v x = v(\nu)x$ for all $v \in V$. When $\nu$ is the trivial $A$-character of $V$, we write $AG/V = AG/V$.

Proofs of the following three remarks are left as easy exercises.
Remark 2.1. Given $A$-subcharacters $(V, \nu)$ and $(W, \omega)$ of $G$, then $A\nu G/V$ is isomorphic to $A\omega G/W$ if and only if $(V, \nu)$ is $G$-conjugate to $(W, \omega)$. Every transitive $A$-fibred $G$-set is isomorphic to an $A$-fibred $G$-set of the form $A\nu G/V$.

Remark 2.2. As an abelian group

$$B(A, G) = \bigoplus_{(V, \nu)} \mathbb{Z}[A\nu G/V],$$

where $(V, \nu)$ runs over a set of representatives of the $G$-classes of $A$-subcharacters of $G$.

In particular, the abelian group $B(A, G)$ has finite rank if and only if the set of group homomorphisms $\text{Hom}(V, A)$ is finite for all $V \subseteq G$.

For subgroups $V \leq G \geq W$, an $A$-character $\nu$ of $V$ and an $A$-character $\omega$ of $W$, let us write $\nu.\omega$ for the $A$-character of $V \cap W$ such that $u \mapsto \nu(u)\omega(u)$ for $u \in V \cap W$. The multiplication on $B(A, G)$ is given by the following Mackey formula.

Remark 2.3. Given $A$-subcharacters $(V, \nu)$ and $(W, \omega)$ of $G$, we have

$$[A\nu G/V][A\omega G/W] = \sum_{VgW \subseteq G} [A_{\nu.\omega} G/V \cap gW],$$

where the notation indicates that $g$ runs over a set of representatives of the cosets of $V$ and $W$ in $G$.

Any map $\alpha : A_1 \to A_2$ of abelian groups induces a ring homomorphism $B(A_1, G) \to B(A_2, G)$ such that $[A_1 X] \mapsto [A_2 \otimes A_1 X]$, the tensor product being over $A_1$. In particular, extending $A$ to an overgroup $A'$, then the embedding $A \hookrightarrow A'$ induces an embedding $B(A, G) \hookrightarrow B(A', G)$. If $A$ has a complementary subgroup in $A'$, then (in the terminology of Section 1), $B(A', G)$ is a split extension of $B(A, G)$. In particular, $B(A, G)$ is a split extension of $B(G)$. Another consequence of these observations (equally trite but, again, tremendous) is as follows. Consider a commutative ring $\Theta$, its group of units $\Theta^*$ and its group of torsion-units $\Theta^\omega$. The embedding $\Theta^\omega \hookrightarrow \Theta^*$ induces an embedding $B(\Theta^\omega, G) \hookrightarrow B(\Theta^*, G)$. But $G$ is finite, so we can make the identification $B(\Theta^\omega, G) = B(\Theta^*, G)$.

The category of $A$-fibred $G$-sets, denoted $A$-$G$-$\text{SET}$, is full subcategory of the category of $AG$-sets. For a subgroup $F \leq G$, induction and restriction of $AF$-sets and $AG$-sets specialize to an induction functor

$$A \text{Ind}_F^G : A$F$-$\text{SET} \to A$G$-$\text{SET}$$

and a restriction functor

$$A \text{Res}_F^G : A$G$-$\text{SET} \to A$F$-$\text{SET}. $$
For \( g \in G \), there is also an evident conjugation functor

\[ A \text{Con}_F^g : A-F\text{-SET} \to A^{gF}\text{-SET}. \]

The functors \( A \text{Ind}_F^G, A \text{Res}_F^G, A \text{Con}_F^g \) induce the following linear maps on monomial Burnside rings:

- an induction map
  \[ A \text{Ind}_F^G : B(A, F) \to B(A, G), \]
- a restriction map
  \[ A \text{Res}_F^G : B(A, G) \to B(A, F), \]
- a conjugation map
  \[ A \text{Con}_F^g : B(A, F) \to B(A, ^gF). \]

Thus, for instance, \( A \text{Res}_F^G[AX] = [A \text{Res}_F^G(AX)] \). Sometimes, we omit the subscript \( A \).

Given a commutative ring \( \Lambda \), we call \( \Lambda B(A, G) = \Lambda \otimes \mathbb{Z} B(A, G) \) a monomial Burnside algebra over \( A \). The induction, restriction and conjugation maps on monomial Burnside rings \( B(A, -) \) extend uniquely to \( \Lambda \)-linear induction, restriction and conjugation maps on monomial Burnside algebras \( \Lambda B(A, -) \).

Recall that a Green functor over \( \Lambda \) is a Mackey functor \( M \) such that, for each group \( G \) in the domain, \( M(G) \) is a \( \Lambda \)-algebra, and a couple of extra axioms hold in addition to the axioms of a Mackey functor. See Thévenaz [23] for details. It is easy to see that \( B(A, -) \) is a Green functor over \( \mathbb{Z} \) and, more generally, \( \Lambda B(A, -) \) is a Green functor over \( \Lambda \).

Let us make a few comments on linearization. For the rest of this section, we assume that \( A \) is a subgroup of the unit group \( \Theta^* \) of a commutative ring \( \Theta \). Let \( R(\Theta G) \) denote the representation ring (also called the Green ring) associated with the category \( \Theta G\text{-MOD} \) of (finitely generated \( \Theta \)-free) \( \Theta G \)-modules. For a \( \Theta G \)-module \( M \), we write \( [M] \) for the isomorphism class of \( M \), regarded as an element of \( R(\Theta G) \). By linear extension, as above, the induction, restriction, and conjugation functors for \( \Theta G \)-modules give rise to induction, restriction, and conjugation maps between representation rings. Thus \( R(\Theta -) \) becomes a Green functor. The linearization functor

\[ \text{Lin}_G = \Theta \otimes_A - : A-G\text{-SET} \to \Theta G\text{-MOD} \]

gives rise to a ring homomorphism, the linearization map

\[ \text{lin}_G : B(A, G) \to R(\Theta G). \]
Thus \( \text{lin}_G[AX] = [\text{Lin}_G(AX)] \). Evidently, induction, restriction, and conjugation commute with linearization; \( \text{lin}_G \) is a morphism of Green functors.

Our reason for setting down all these obvious functorial formalities is that, in the case of a supercyclic fibre group \( C \), we shall be introducing a tenduction map to realize the unit group \( B(C, -)^* \) as a Mackey functor (not as Green functor). It will still be true that \( \text{lin}_G \) and extension of the fibre group are morphisms of Mackey functors. In fact, it will still be obvious. But the point is worth making because there is another apparently reasonable definition of tenduction that is not compatible with linearization or with extension of the fibre group.

3. Subcharacters and subelements

We collect together some definitions and observations concerning subcharacters and subelements. The purpose of the material will become apparent in later sections.

Given an element \( s \) of a \( G \)-set \( S \), we write \( [s]_G \) for the \( G \)-orbit of \( s \) in \( G \). Whenever we call the \( G \)-action conjugation, we also call \( [s]_G \) the \( G \)-class of \( s \), and we call \( N_G(s) \) the normalizer of \( s \) in \( G \). If elements \( s, t \in S \) belong to the same \( G \)-orbit, then we write \( s =_G t \). We let \( G \setminus S \) denote the set of \( G \)-orbits in \( S \).

A supernatural number, recall, is a formal product \( \prod p^{\alpha(p)} \), where \( p \) runs over the rational primes and each \( \alpha(p) \in \mathbb{N} \cup \{ \infty \} \). The arithmetic of the supernatural numbers is entirely multiplicative. With respect to the divisibility relation, the supernatural numbers comprise a lattice. In fact, every non-empty set of supernatural numbers has a lowest common multiple and a highest common factor.

A group is said to be supercyclic provided every finitely generated subgroup is finite and cyclic. Given a supercyclic group \( C \), then the order of \( C \), denoted \( |C| \), is defined to be the lowest common multiple of the orders of the elements of \( C \). In other words, the order of \( C \) is the exponent of \( C \), understood to be a supernatural number. For each supernatural number \( n \), there is a unique subgroup \( C_n \) of \( Q/Z \) having order \( |C_n| = n \). In the case where \( n \) is finite, \( C_n \) is generated by the coset of \( Z \) owning \( 1/n \). Thus, a supercyclic group is determined up to isomorphism by its order, and the supernatural numbers are precisely the orders of the supercyclic groups.

Let \( C \) be a supercyclic group. Let \( O(G) \) denote the intersection of the kernels of the \( C \)-characters of \( G \). Thus \( O(G) \) is the minimal normal subgroup of \( G \) such that \( G/O(G) \) is abelian with exponent dividing \( |C| \). The group of \( C \)-characters of \( G \), denoted \( \hat{G} = \text{Hom}(G, C) \)

may be regarded as the dual of the group \( \overline{G} = G/O(G) \).

Recall that the \( C \)-subcharacters of \( G \) were defined (in a more general context) in Section 2. The \( G \)-set of \( C \)-subcharacters of \( G \) is written as

\[
\text{ch}(C, G) = \{(V, v) : V \lhd G, v \in \hat{V}\}.
\]
We define a \( C \)-subelement of \( G \) to be a pair \((H, hO(H))\) where \( h \in H \leq G \). Sometimes, we abbreviate \((H, hO(H))\) as \((H, h)\). Thus, two \( C \)-subelements \((H, h)\) and \((I, i)\) of \( G \) are equal if and only if \( H = I \) and \( hO(H) = iO(H) \). We let \( G \) act on the \( C \)-subelements of \( G \) by conjugation: \( \xi(h, h) = (\xi H, \xi h) \). The \( G \)-set of \( C \)-subelements of \( G \) is denoted by

\[
el(C, G) = \{(H, hO(H)): H \leq G, hO(H) \in \mathcal{P}\}.
\]

**Lemma 3.1.** We have \(|\el(C, G)| = |\ch(C, G)|\).

**Proof.** For each subgroup \( F \leq G \), the number of \( C \)-subelements with first coordinate \( F \) and the number of \( C \)-subcharacters with first coordinate \( F \) are both equal to \(|F : O(F)|\). \(\square\)

The next lemma is well-known.

**Lemma 3.2.** Let \( B \) be a finite group acting as automorphisms on a finite abelian group \( A \) and on the dual group \( \hat{A} \), the actions preserving the duality. Then \(|B \backslash A| = |B \backslash \hat{A}|\).

**Proof.** Embedding \( \mathbb{Q}/\mathbb{Z} \) in the unit group of \( \mathbb{C} \), we can regard the elements of the group \( \hat{A} = \text{Hom}(A, \mathbb{Q}/\mathbb{Z}) \) as functions \( A \rightarrow \mathbb{C} \). For \( \alpha \in \hat{A} \), let \( \alpha^+ : A \rightarrow \mathbb{C} \) be the sum of the \( B \)-conjugates of \( \alpha \). The set \( \{\alpha^+ : \alpha \in \hat{A}\} \) is linearly independent and has size \(|B \backslash \hat{A}|\). But each \( \alpha^+ \) is \( B \)-invariant, and can be regarded as a function \( B \backslash A \rightarrow \mathbb{C} \). Therefore \(|B \backslash A| \leq |B \backslash \hat{A}|\). The reverse inequality follows by duality. \(\square\)

**Lemma 3.3.** We have \(|G \backslash \el(C, G)| = |G \backslash \ch(C, G)|\).

**Proof.** Let \( F \leq G \). By Lemma 3.2, the number of \( G \)-classes of \( C \)-subelements with first coordinate conjugate to \( F \) is equal to the number of \( G \)-classes of \( C \)-subcharacters with first coordinate conjugate to \( F \). \(\square\)

### 4. Monomial Möbius inversion

The incidence function and the Möbius function for a finite poset are, essentially, mutually inverse matrices with rows and columns indexed by the elements of the poset. An important particular case is that where the poset is \( \text{sub}(G) \), the poset of subgroups of \( G \), partially ordered by inclusion. We shall replace \( \text{sub}(G) \) with the two sets \( \el(C, G) \) and \( \ch(C, G) \). These two sets are, in some sense, dual to each other: \( \el(C, G) \) indexes the rows of the monomial incidence function and the columns of the monomial Möbius function; \( \ch(C, G) \) indexes the columns of the monomial incidence function and the rows of the monomial Möbius function. As a banal but convenient manoeuvre, we shall also consider \( G \)-invariant versions of the various incidence and Möbius functions.
Let us introduce a little device of notation that sometimes facilitates symbolic manipulations. The Kronecker value of a proposition $S$ is defined to be the rational integer

$$[S] = \begin{cases} 1, & \text{if } S \text{ holds,} \\ 0, & \text{if } S \text{ fails.} \end{cases}$$

Consider a finite poset $P$. The incidence function of $P$ is defined to be the function $\xi : P \times P \rightarrow \mathbb{Z}$ such that $\xi(x, y) = [x \leq y]$ for $x, y \in P$. For an integer $n \geq -2$, we define a function $c_n : P \times P \rightarrow \mathbb{Z}$ such that $c_{-2}(y, x) = [y = x]$ and $c_{-1}(y, x) = [y < x]$ and, if $n \geq 0$, then $c_n(y, x)$ is the number of chains in $P$ having the form $y < z_0 < \cdots < z_n < x$.

The Möbius function of $P$ is defined to be the function $\mu : P \times P \rightarrow \mathbb{Z}$ given by

$$\mu(y, x) = \sum_{n=-2}^{\infty} (-1)^n c_n(y, x).$$

Since $P$ is finite, only finitely many of the terms of the sum are non-zero. (Our numbering convention reflects the fact that, if $y < x$, then $\mu(y, x)$ is the reduced Euler characteristic of the simplicial complex associated with the open interval bounded by $y$ and $x$.)

Let $A$ be an abelian group, and let $\theta$ and $\phi$ be functions $P \rightarrow A$. The equation

$$\theta(y) = \sum_{x \in P} \phi(x) \xi(x, y)$$

is called the totient equation. The equation

$$\phi(x) = \sum_{y \in P} \theta(y) \mu(y, x)$$

is called the inversion equation. The principle of Möbius inversion asserts that the totient equation holds for all $y \in P$ if and only if the inversion equation holds for all $x \in P$. A proof can be found in Kerber [19, Section 2.2]. The principle can also be expressed as the matrix equation

$$\sum_{y \in P} \xi(x, y) \mu(y, z) = [x = z] = \sum_{y \in P} \mu(x, y) \xi(y, z),$$

which holds for all $x, z \in P$. In particular, if $x < z$, then

$$\sum_{y \in P : x \leq y \leq z} \mu(y, z) = 0 = \sum_{y \in P : x \leq y \leq z} \mu(x, y).$$

The Möbius function $\mu$ is determined by this identity, together with the conditions that $\mu(x, x) = 1$ and $\mu(y, x) = 0$ when $y \not< x$. 
Suppose now that the finite poset $P$ is a $G$-poset. We define the $G$-invariant incidence function $\zeta_G$ and the $G$-invariant Möbius function $\mu_G$ to be the functions $P \times P \to \mathbb{Z}$ such that

$$
\zeta_G(x, y) = \sum_{x' \in [x]_G} \zeta(x', y), \quad \mu_G(y, x) = \sum_{y' \in [y]_G} \mu(y', x).
$$

Note that, if $x =_G x'$ and $y =_G y'$, then $\zeta(x, y) = \zeta(x', y')$ and $\mu(y, x) = \mu(y', x')$. So $\mu_G$ and $\zeta_G$ may be regarded as functions $G \setminus P \times G \setminus P \to \mathbb{Z}$. If the functions $\theta$ and $\phi$ are $G$-invariant (with $G$ acting trivially on $A$) then the totient equation can be rewritten as

$$
\theta(y) = \sum_{x \in G} \phi(x) \zeta_G(x, y)
$$

and the inversion equation can be rewritten as

$$
\phi(x) = \sum_{y \in G} \theta(y) \mu_G(y, x).
$$

Here, the notation indicates that the indices of the sums run over representatives of the $G$-orbits in $P$. The equivalence of the totient equation and the inversion equation can be expressed as the matrix equation

$$
\sum_{y \in G} \zeta_G(x, y) \mu_G(y, z) = [x =_G z] = \sum_{y \in G} \mu_G(x, y) \zeta_G(y, z).
$$

Let $K$ be a field of characteristic zero, and suppose that an embedding $C \hookrightarrow K^\omega$ is given. For a $C$-subcharacter $\omega$ of $G$ and a subset $T \subseteq G$, we write $\omega(T) = \sum_{t \in T} \omega(t)$ as an element of $K$. We write $\omega^{-1}$ to denote the inverse of $\omega$ in the group $\hat{G}$.

Whenever we write an expression of the form $\zeta(H, V)$ or $\mu(V, H)$, where $H$ and $V$ are subgroups of $G$, it is to be understood that $\zeta$ and $\mu$ are the incidence and Möbius functions of the $G$-poset sub$(G)$. We now generalize those two functions. We define a monomial incidence function

$$
\zeta : \text{el}(C, G) \times \text{ch}(C, G) \to K, \quad \zeta(H, h; V, v) = v(h) \zeta(H, V)
$$

and a monomial Möbius function

$$
\mu : \text{ch}(C, G) \times \text{el}(C, G) \to K, \quad \mu(V, v; H, h) = v^{-1}(V \cap hO(H)) \mu(V, H) / |V|
$$

where $(H, h) \in \text{el}(C, G)$ and $(V, v) \in \text{ch}(C, G)$. 
Let \((V, \nu)\) and \((W, \omega)\) be \(C\)-subcharacters of \(G\). Using orthonormality properties of characters,

\[
\sum_{(H, h) \in \text{el}(C, G)} \mu(V, \nu; H, h) \xi(H, h; W, \omega) = \sum_{H: V \leq H \leq W} \frac{\mu(V, H)}{|V: O(V)|} \sum_{v \in O(V) \subseteq V} v^{-1}(\nu) \omega(v) \\
= \sum_{H: V \leq H \leq W} \mu(V, W) [v = \text{Res}_V^W(\omega)] \\
= [(V, \nu) = (W, \omega)].
\]

By Remark 3.1, we can interpret the equality as an assertion that two square matrices are mutual inverses. Therefore, given \(C\)-subelements \((H, h)\) and \((I, i)\) of \(G\), we have

\[
\sum_{(V, \nu) \in \text{ch}(C, G)} \xi(H, h; V, \nu) \mu(V, \nu; I, i) = [(H, h) = (I, i)].
\]

Now consider functions \(\theta : \text{ch}(C, G) \to A\) and \(\phi : \text{el}(C, A) \to A\). By the comments above, the totient equation

\[
\theta(V, v) = \sum_{(H, h) \in \text{el}(C, G)} \phi(C, G) \xi(H, h; V, v)
\]

holds for all \((V, v) \in \text{ch}(C, G)\) if and only if the inversion equation

\[
\phi(H, h) = \sum_{(V, \nu) \in \text{ch}(C, G)} \theta(V, v) \mu(V, \nu; H, h)
\]

holds for all \((H, h) \in \text{el}(C, G)\).

Much as before, we define a \(G\)-invariant monomial incidence function \(\xi_G\) and a \(G\)-invariant monomial Möbius function \(\mu_G\) such that

\[
\xi_G(H, h; V, v) = \sum_{(H', h') \in [H, h]_G} \xi(H', h'; V, v),
\]

\[
\mu_G(V, v; H, h) = \sum_{(V', v') \in [V, v]_G} \mu(V', v'; H, h).
\]

**Theorem 4.1** (Monomial Möbius inversion). Given \(G\)-invariant functions \(\theta : \text{ch}(C, G) \to A\) and \(\phi : \text{el}(C, G) \to A\), then the totient equation

\[
\theta(V, v) = \sum_{(H, h) \in \text{el}(C, G)} \phi(C, G) \xi_G(H, h; V, v)
\]
holds for all \((V, v) \in \text{ch}(C, G)\) if and only if the inversion equation

\[
\phi(H, h) = \sum_{(V, v) \in G} \theta(V, v) \mu_G(V, v; H, h)
\]

holds for all \((H, h) \in \text{el}(C, G)\).

**Proof.** This is clear from the preceding discussion. \(\square\)

In our proof of Theorem 4.1, we somehow managed to avoid using Lemma 3.3. Oddly enough, the implication is the other way around: the theorem yields another proof of the lemma.

We mention that, by letting the matrices act on column vectors rather than on row vectors, all of the downwards sums can be replaced by upwards sums. For instance, the totient equation

\[
\theta(H, h) = \sum_{(V, v) \in G} \zeta_G(H, h; V, v) \phi(V, v)
\]

is equivalent to the inversion equation

\[
\phi(V, v) = \sum_{(H, h) \in G} \mu_G(V, v; H, h) \theta(H, h).
\]

### 5. An idempotent formula

In this section and the next one, we shall examine the primitive idempotents of the monomial Burnside algebra \(KB(C, G)\), where \(K\) is a field of characteristic zero. Throughout the present section, we shall assume that \(K\) has enough roots of unity for all our purposes. We shall remind ourselves of this standing hypothesis whenever we want to, because it will be dropped in the next section. As in Section 4, we embed \(C\) in the group of torsion-units \(K^\times\). Of course, the idempotents of \(KB(C, G)\) do not depend on the choice of the embedding \(C \hookrightarrow K^\times\), but we shall be making use of the embedding in our description of the idempotents.

A formula for the primitive idempotents of \(QB(G)\) was discovered independently by Gluck [16] and Yoshida [26]. Similar formulas for the primitive idempotents of \(CD(G)\) and \(CD^P(G)\) were given by Boltje [6, Section 3]. In this section, we unify and generalize those results. We give a formula the primitive idempotents of \(KB(C, G)\). We also characterize the induction and restriction maps in terms of the primitive idempotents.

Remark 2.2 tells us that, as a direct sum of 1-dimensional \(K\)-vector spaces,

\[
KB(C, G) = \bigoplus_{(V, v) \in \text{ch}(C, G)} K [C_v G / V].
\]
The notation, here, indicates that \((V, \nu)\) runs over a set of representatives of the \(G\)-classes of \(C\)-subcharacters of \(G\). The primitive idempotents of \(KB(C, G)\) will be indexed by the \(G\)-classes of \(C\)-subelements of \(G\). The primitive idempotent indexed by the \((G\text{-class of the})\) \(C\)-subelement \((H, h)\) will be denoted \(e_{H,h}^G\). We shall prove that \(KB(C, G)\) is semi-simple and

\[
KB(C, G) = \bigoplus_{(H, h) \in \text{el}(C, G)} Ke_{H,h}^G,
\]

as a direct sum of algebras isomorphic to \(K\). Let us call the elements of \(KB(C, G)\) having the form \([C, G/V]\) the \textit{transitive elements}. Equations (1) and (2) express the coordinate systems associated with, respectively, the basis of transitive elements and the basis of primitive idempotents. The aim of this section is to determine the transformation matrices between the two coordinate systems. Let us express the transformation matrices as

\[
[C, G/V] = \sum_{(H, h) \in \text{el}(C, G)} m_G(H, h; V, \nu)e_{H,h}^G, \tag{3}
\]

\[
e_{H,h}^G = \sum_{(V, \nu) \in \text{ch}(C, G)} m_{-1}^{-1}_G(V, \nu; H, h)[C, G/V]. \tag{4}
\]

After establishing the decomposition in Eq. (2), we shall give formulas for the matrix entries \(m_G(H, h; V, \nu)\) and \(m_{-1}^{-1}_G(V, \nu; H, h)\).

Our first step is to define species (algebra maps to the ground field)

\[
s_{H,h}^G : KB(C, G) \to K.
\]

Consider a \(C\)-fibred \(G\)-set \(CX\) and a \(C\)-subelement \((H, h)\) of \(G\). Given a fibre \(Cx\) in \(CX\) stabilized by \(H\), let us write \(\phi_x\) for the \(C\)-character of \(H\) such that \(hx = \phi_x(h)x\) for all \(h \in H\). Note that \(\phi_x\) is independent of the choice of the element \(x\) of the fibre \(Cx\). We define \(s_{H,h}^G\) to be the linear map such that

\[
s_{H,h}^G[CX] = \sum_{Cx} \phi_x(h),
\]

where \(Cx\) runs over the fibres in \(CX\) that are stabilized by \(H\). Let us show that \(s_{H,h}^G\) is a species. Consider another \(C\)-fibred \(G\)-set \(CY\). A fibre \(Cxy \subseteq CXY\) is stabilized by \(H\) if and only if the fibres \(Cx \subseteq CX\) and \(Cy \subseteq CY\) are stabilized by \(H\). In that case, \(\phi_{xy} = \phi_x\phi_y\). Therefore \(s_{H,h}^G([CX])s_{H,h}^G([CY]) = s_{H,h}^G([CX \times CY])\), as required.

The next result is immediate from Dress [14, Theorem 1′(c)]. We mention that, in our special case, his argument simplifies and the first and second halves of the conclusion follow easily from Lemmas 3.2 and 3.3, respectively.

\section*{Lemma 5.1 (Dress).} Recall that \(K\) is sufficiently large. Given \(C\)-subelements \((H, h)\) and \((I, i)\) of \(G\), then \(s_{H,h}^G = s_{I,i}^G\) if and only if \((H, h) = G(I, i)\). Every species of \(KB(C, G)\) is of the form \(s_{H,h}^G\), and the species span the dual space of \(KB(C, G)\).
By the lemma, there exists a unique element $e_{H,h}^G \in KB(C, G)$ such that

$$s_{I,i}^{G}(e_{H,h}^{G}) = \lfloor (I, i) = G(H, h) \rfloor.$$ 

In the proof of the lemma, we saw that the algebra $KB(C, G)$ is isomorphic to a direct sum of copies of $K$. So each $e_{H,h}^G$ is a primitive idempotent and $Ke_{H,h}^G \cong K$. The decomposition in Eq. (2) is now established.

We can now turn to the problem of evaluating the matrix entries in Eqs. (3) and (4). We have

$$m_G(H, h; V, v) = s_{H,h}^G[C_G/V] = \sum_{gV \subseteq G} s_{gV}(h) \lfloor H \leq gV \rfloor.$$ 

Comparing with the definition (in Section 4) of the appropriate incidence function, we obtain

$$m_G(H, h; V, v) = \frac{|N_G(H, h)|}{|V|} \zeta_G(H, h; V, v).$$  (5)

Theorem 4.1 says that the matrix with entries $\mu_G(V, v; H, h)$ is the inverse of the matrix with entries $\zeta_G(H, h; V, v)$. Therefore

$$m_{-1}^{-1}(V, v; H, h) = \frac{|V|}{|N_G(H, h)|} \mu_G(V, v; H, h).$$  (6)

We have proved the following result.

**Theorem 5.2 (Idempotent formula).** Recall that $K$ is sufficiently large. There is a bijective correspondence $e_{H,h}^G \leftrightarrow [H, h]_G$ between the primitive idempotents $e_{H,h}^G$ of $KB(C, G)$ and the $G$-conjugacy classes $[H, h]_G$ of $C$-subelements $(H, h)$ of $G$. We have

$$|N_G(H, h)|e_{H,h}^G = \sum_{(V, v) \in ch(C, G)} |V|\mu_G(V, v; H, h)[C_G/V].$$

The following remark is immediate from the definitions that we have made, but it is worth emphasising, because we use it very frequently (often without mentioning it).

**Remark 5.3.** With respect to the basis of primitive idempotents, the coordinate decomposition of an element $\zeta \in KB(C, G)$ is

$$\zeta = \sum_{(H, h) \in Gel(C, G)} s_{(H, h)}^{G} (\zeta)e_{(H, h)}^{G}.$$ 

With respect to that coordinate decomposition, the induction and restriction maps are given by the matrix equations in the next two propositions.
Proposition 5.4. Given $F \leq G$ and a $C$-subelement $(J, j)$ of $F$, then
\[ \text{ind}_F^G(e_{J,j}^F) = |N_G(J, j) : N_F(J, j)| e_{J,j}^G. \]

Proof. Consider a $C$-subelement $(H, h)$ of $G$. The Mackey decomposition formula
\[ \text{res}_H^G \text{ind}_F^G = \sum_{H \subseteq G} \text{ind}_{H \cap F}^H \text{res}_{H \cap F}^F, \]

clearly holds for monomial Burnside algebras. Therefore
\[ s_{H,h}^G(\text{ind}_F^G(e_{J,j}^F)) = s_{H,h}^H(\text{res}_H^G(\text{ind}_F^G(e_{J,j}^F))) = \sum_{H \subseteq G} [(H, h) =_F \mathbb{S}(J, j)] \]
\[ = |\{gF \subseteq G: (H, h) =_F \mathbb{S}(J, j)\}| \]
\[ = \frac{|N_G(H, h)|}{|N_F(J, j)|} [(H, h) =_G (J, j)]. \]

Proposition 5.5. Given $F \leq G$ and a $C$-subelement $(H, h)$ of $G$, then
\[ \text{res}_F^G(e_{H,h}^F) = \sum_{(J,j)} e_{J,j}^F, \]

where $(J, j)$ runs over representatives of the $F$-classes of $C$-subelements of $F$ such that $(J, j)$ is $G$-conjugate to $(H, h)$.

Proof. For an arbitrary $C$-subelement $(J, j)$ of $F$, we have
\[ s_{J,j}^F(\text{res}_F^G(e_{H,h}^F)) = s_{J,j}^G(e_{H,h}^F). \]

Corollary 5.6. Recall that $K$ is sufficiently large. Given $F \leq G$ then, as a direct sum of ideals,
\[ KB(C, G) = \text{Im} \text{(ind}_F^G) \oplus \text{Ker} \text{(res}_F^G). \]

Proof. Let $(H, h) \in \text{ch}(C, G)$. By Proposition 5.4, $e_{H,h}^G \in \text{Im} \text{(ind}_F^G)$ if and only if $H \leq_G F$. By Proposition 5.5, $e_{H,h}^G \in \text{Ker} \text{(res}_F^G)$ if and only if $H \nleq_G F$. \[ \square \]

6. The primitive idempotents over a characteristic zero field

We now allow $K$ to be any field of characteristic zero. Our technique for determining the primitive idempotents of $KB(C, G)$ is based on Galois actions. It is a variant of the means by which Berman dealt with the analogous problem for group algebras. See, for instance, Karpilovsky [18, Section 8.9].
Throughout, we let $C_K$ be the maximum subgroup of $C$ such that $K^\omega$ has a subgroup isomorphic to $C_K$. Thus, $|C_K| = \gcd(|C|, |K^\omega|)$. We embed $C_K$ in $K$ arbitrarily. Let us begin by determining an upper bound (with respect to divisibility) on the number of roots of unity needed for the primitive idempotents to be as described in the previous section.

**Proposition 6.1.** Let $r = \gcd(|C|, \exp(G))$. Suppose that $K$ has primitive $r$th roots of unity. Then the primitive idempotents of $K B(C, G)$ are precisely the idempotents $e^G_{H, h}$. Furthermore

$$KB(C, G) = \bigoplus_{(H, h) \in Gel(C, G)} Ke^G_{H, h}$$

as a direct sum of algebras $K e^G_{H, h} \cong K$.

**Proof.** The values of the Möbius function in Eq. (6) all belong to $K$. So the assertion follows from Theorem 5.2. □

Before turning to the general case, let us recall some generalities concerning the primitive idempotents of an artinian subring $\Phi$ of a commutative artinian ring $\Theta$. The rings $\Theta$ and $\Phi$ each have only finitely many idempotents, and they sum to the (same) unity element. Each primitive idempotent of $\Phi$ is a sum of primitive idempotents of $\Theta$. Each primitive idempotent of $\Theta$ is a summand of a unique primitive idempotent of $\Phi$. When primitive idempotents $e$ and $f$ of $\Theta$ are summands of the same primitive idempotent of $\Phi$, we say that $e$ and $f$ are equivalent with respect to $\Phi$.

**Remark 6.2.** With $\Phi$ and $\Theta$ as above, the primitive idempotents $e$ of $\Phi$ are in a bijective correspondence with the equivalence classes $[e]$ of the primitive idempotents $e$ of $\Theta$. The correspondence is such that $e \leftrightarrow e$ provided $e$ is the sum of the primitive idempotents of $\Theta$ that are equivalent to $e$.

Thus, if the primitive idempotents of $\Theta$ have been determined, and if the above equivalence relation has been determined, then the primitive idempotents of $\Phi$ have been determined too. We shall use this principle to specify the primitive idempotents of $K B(C, G)$ in the absence of the hypothesis in Proposition 6.1. Our strategy is to extend $K$ to a field $K[C]$, defined below, and then to put $\Theta = K[C]B(C, G)$ and $\Phi = KB(C, G)$. We shall see that the primitive idempotents $e^G_{H, h}$ of $K[C]B(C, G)$ are permuted by the Galois group $\text{Gal}(K[C]/K)$. It will turn out that the equivalence classes $[e^G_{H, h}]$ are the orbits of $\text{Gal}(K[C]/K)$.

Let $K[C]$ be the minimal extension field of $K$ such that $K[C]$ has primitive $m$th roots of unity for every natural number $m$ dividing the supernatural number $|C|$. Then $K[C]^{\omega}$ has a subgroup isomorphic to $C$. Let us choose and fix an embedding $C \hookrightarrow K[C]^{\omega}$ that extends the embedding $C_K \hookrightarrow K^{\omega}$.

A field extension having the form $K[C]/K$ is called a supercyclotomic extension. Just as the theory of supercyclic groups is much the same as the theory of cyclic groups, the
theory of supercyclotomic extensions is much the same as the theory of cyclotomic extensions. Actually, for fixed \( G \), we could work with a cyclotomic extension but, once we have made some little observations on the Galois theory of supercyclotomic extensions, we shall actually find it simpler to work with a (profinite) Galois group that applies globally.

Given a subgroup \( D \subseteq C \), then we can regard \( K[D] \) as the subfield of \( K[C] \) generated over \( K \) by \( D \). If we let \( D \) run over the finite subgroups of \( C \), then \( K[C] = \bigcup_D K[D] \). In particular, \( K[C] \) is a union of (finite degree) Galois extension fields of \( K \). So \( K[C] \) is a normal separable extension field of \( K \). In particular, the (possibly infinite) Galois group \( \text{Gal}(K[C]/K) \) has fixed field \( K \).

Let \( \text{Aut}(C/CK) \) denote the subgroup of \( \text{Aut}(C) \) fixing \( CK \).

**Remark 6.3.** Let \( C' \) be such that \( CK \leq C' \leq C \). Then:

1. The action of \( \text{Gal}(K[C]/K) \) as automorphisms of \( K[C'] \) gives rise to a group epimorphism \( \text{Gal}(K[C]/K) \rightarrow \text{Gal}(K[C']/K) \).
2. The action of \( \text{Aut}(C/CK) \) as automorphisms of \( C' \) gives rise to a group epimorphism \( \text{Aut}(C/CK) \rightarrow \text{Aut}(C'/CK) \).

**Remark 6.4.** The action of \( \text{Gal}(K[C]/K) \) as automorphisms of the subgroup \( C \) of \( K[C]^* \) gives rise to a group isomorphism \( \text{Gal}(K[C]/K) \leftrightarrow \text{Aut}(C/CK) \).

Via the isomorphism in Remark 6.4, we identify \( \text{Gal}(K[C]/K) \) with \( \text{Aut}(C/CK) \). Then the two epimorphisms in Remark 6.3 coincide. Let us write

\[
\Gamma = \text{Gal}(K[C]/K) = \text{Aut}(C/CK).
\]

Given an element \( \gamma \in \Gamma \), and a \( C \)-subcharacter \((V, v)\) of \( G \), we write \( \gamma v = \gamma_v \) and \( \gamma(V, v) = (V, \gamma_v) \). Thus, \( \text{ch}(C, G) \) becomes a \( \Gamma \)-set. Consider a \( C \)-subelement \((H, h)\) of \( G \). The groups \( \hat{H} \) and \( \hat{H} \) are mutual duals, so the action of \( \Gamma \) on \( \hat{H} \) induces an action on \( \hat{H} \). Explicitly, \( \gamma \) sends each element \( hO(H) \in \hat{H} \) to the unique element \( \gamma hO(H) \in \hat{H} \) such that \( \gamma \phi(\gamma h) = \phi(h) \) for all \( \phi \in \hat{H} \). We write \( \gamma(H, h) = (H, \gamma h) \). Thus, \( \text{el}(C, G) \) becomes a \( \Gamma \)-set. The actions of \( \Gamma \) and \( G \) on \( \text{ch}(C, G) \) commute, likewise on \( \text{el}(C, G) \).

We have realized \( \text{ch}(C, G) \) and \( \text{el}(C, G) \) as permutation sets of the direct product \( \Gamma G = \Gamma \times G \).

By Proposition 6.1, the primitive idempotents of \( K[C]B(C, G) \) are the elements having the form \( e_{G,H,h}^G \). Let

\[
e_{H,h}^G = \sum_{(I,i)} e_{I,i}^G,
\]

where \((I, i)\) runs over representatives of the \( G \)-classes of \( C \)-subelements of \( G \) such that \((H, h)\) and \((I, i)\) are \( \Gamma G \)-conjugate.
Theorem 6.5. There is a bijective correspondence \( e_{G,K}^{H,h} \leftrightarrow [H, h]_{\Gamma_G} \) between the primitive idempotents \( e_{G,K}^{H,h} \) of \( KB(C,G) \) and the \( \Gamma_G \)-classes \( [H, h]_{\Gamma_G} \) of \( C \)-subelements \( (H, h) \) of \( G \).

Proof. Put \( \Theta = K[C]B(C,G) \) and \( \Phi = KB(C,G) \). By Remark 6.2, we are required to show that two primitive idempotents \( e_{G,K}^{H,h} \) and \( e_{G,K}^{I,i} \) of \( \Theta \) are equivalent with respect to \( \Phi \) if and only if the \( C \)-subelements \( (H, h) \) and \( (I, i) \) are \( \Gamma_G \)-conjugate. Since \( e_{G,K}^{H,h} = e_{G,K}^{I,i} \) if and only if \( (H, h) \) and \( (I, i) \) are \( G \)-conjugate, we need only consider actions of \( \Gamma \). We let \( \Gamma \) act as ring automorphisms of \( \Theta \) such that \( \gamma(\lambda[C_vG/V]) = (\gamma\lambda)[C_vG/V] \), where \( \gamma \in \Gamma \) and \( \lambda \in K[C] \). Since the \( \Gamma \)-fixed subfield of \( K[C] \) is \( K \), the \( \Gamma \)-fixed subring of \( \Theta \) is \( \Phi \). The primitive idempotents of \( \Theta \) are permuted by \( \Gamma \), and the orbit sums are the primitive idempotents of \( \Phi \). From the definition of the monomial Möbius function and the definition of the action of \( \Gamma \) on the \( C \)-subelements, we have

\[
\gamma(\mu(V, v; H, \gamma h)) = \mu(V, v; H, h).
\]

By Theorem 5.2, \( \gamma e_{H,\gamma h}^G = e_{H,h}^G \). So \( e_{H,h}^G \) is the \( \Gamma \)-orbit sum of \( e_{H,h}^G \). \( \Box \)

Corollary 6.6. The numbers \( |\Gamma_G \setminus \text{el}(C,G)| \) and \( |\Gamma_G \setminus \text{ch}(C,G)| \) are both equal to the number of primitive idempotents of \( KB(C,G) \).

Proof. By Theorem 6.5, there are precisely \( |\Gamma_G \setminus \text{el}(C,G)| \) primitive idempotents of \( KB(C,G) \). The equality \( |\Gamma_G \setminus \text{el}(C,G)| = |\Gamma_G \setminus \text{ch}(C,G)| \) holds by an argument similar to the proof of Lemma 3.3. \( \Box \)

It is worth working through what the theorem says in the case \( K = \mathbb{Q} \), which is likely to be the main case of interest. The analysis will involve cyclic \( C \)-sections (defined below), which have appeared before, in the case where \( C \) has prime order, in work of Bouc [10].

One motive for considering coefficients in \( \mathbb{Q} \) is that the tenduction map, defined in Section 9, is a polynomial function with coefficients in \( \mathbb{Q} \). Although the tenduction map exists for coefficients in \( \mathbb{Z} \), we shall see that the polynomial formula is not closed when the coefficient ring is an arbitrary ring of cyclotomic integers.

We define a cyclic \( C \)-section of \( G \) to be a pair \( (V, U) \) such that \( U \subseteq V \leq G \) and \( V/U \) is cyclic with order dividing \( |C| \). We define a \( C \)-subcycle of \( G \) to be a pair \( (H, Z) \) where \( H \leq G \) and \( Z \) is a cyclic subgroup of \( \hat{H} \). As an abuse of notation, given \( h \in H \), we abbreviate \( (H, \langle hO(H)\rangle) \) as \( (H, \langle h\rangle) \).

Lemma 6.7. Suppose that \( K = \mathbb{Q} \), whence \( \Gamma = \text{Aut}(C) \).

1. Two \( C \)-subcharacters \( (V, v) \) and \( (W, \omega) \) are \( \Gamma \)-conjugate if and only if \( (V, \text{Ker}(v)) = (W, \text{Ker}(\omega)) \); they are \( \Gamma_G \)-conjugate if and only if \( (V, \text{Ker}(v)) \equiv_G (W, \text{Ker}(\omega)) \).
Thus, the $\Gamma G$-classes of $C$-subcharacters of $G$ are in a bijective correspondence with the $G$-classes of cyclic $C$-sections of $G$.

(2) Two $C$-subelements $(H, h)$ and $(I, i)$ are $\Gamma$-conjugate if and only if $(H, \langle h \rangle) = (I, \langle i \rangle)$; they are $\Gamma G$-conjugate if and only if $(H, \langle h \rangle) \cong_G (I, \langle i \rangle)$. Thus, the $\Gamma G$-classes of $C$-subelements of $G$ are in a bijective correspondence with the $G$-classes of $C$-subcycles of $G$.

Proof. If the supernatural number $|C|$ is even, then $|C_Q| = 2$, otherwise $|C_Q| = 1$. Either way, every automorphism of $C$ fixes $C_Q$. So

$$\Gamma = \text{Gal}(\mathbb{Q}[C]/\mathbb{Q}) = \text{Aut}(C/C_Q) = \text{Aut}(C).$$

Let $r$ be as in Proposition 6.1, and let $\mathbb{Q}_r$ be the cyclotomic number field obtained from $\mathbb{Q}$ by adjoining primitive $r$th roots of unity. Let $\mathbb{Z}/r$ denote the ring of residue classes of rational integers modulo $r$. Making evident identifications, we write

$$\Gamma_r = \text{Gal}(\mathbb{Q}_r/\mathbb{Q}) = \text{Aut}(C_r) = (\mathbb{Z}/r)^*.$$

Let $\pi$ be the group epimorphism $\Gamma \to \Gamma_r$ specified in Remark 6.3. Given an element $\gamma \in \Gamma$, we interpret $\pi(\gamma)$ as a rational integer coprime to $r$ and well-defined up to congruence modulo $r$. Let $(V, \nu) \in \text{ch}(C, G)$ and $(H, h) \in \text{el}(C, G)$. The $\pi(\gamma)$th power $\nu^{\pi(\gamma)}$ is well-defined, because the order of the group element $\nu \in \hat{V}$ divides $r$. A similar comment holds for the element $hO(H) \in \overline{H}$. Since $\Gamma$ acts on $\text{ch}(C, G)$ and $\text{el}(C, G)$ via $\pi$, we have

$$\gamma(V, \nu) = (V, \nu^{\pi(\gamma)}), \quad \gamma(H, h) = (H, h^{\pi(\gamma)^{-1}}).$$

So the $\Gamma$-conjugates of $(V, \nu)$ are the $C$-subcharacters having the form $(V, \nu^m)$ where $m$ is coprime to $r$. Part (1) is now established. The $\Gamma$-conjugates of $(H, h)$ are the $C$-subelements having the form $(H, h^m)$ where, again, $m$ is coprime to $r$. Hence, part (2).

By Theorem 6.5 and part (2) of Lemma 6.7,

$$e^{G, \mathbb{Q}}_{H, h} = \sum_{(I, i)} e^{G}_{I, i},$$

where $(I, i)$ runs over representatives of the $G$-classes of $C$-subcharacters of $G$ such that $\langle I, \langle i \rangle \rangle =_G (H, \langle h \rangle)$. We have proved the following corollary.

Corollary 6.8. There is a bijective correspondence $e^{G, \mathbb{Q}}_{H, h} \leftrightarrow [H, \langle h \rangle]_G$ between the primitive idempotents $e^{G, \mathbb{Q}}_{H, h}$ of $\mathbb{Q}B(C, G)$ and the $G$-classes $[H, \langle h \rangle]$ of $C$-subcycles $(H, \langle h \rangle)$ of $G$. In particular, the number of primitive idempotents of $\mathbb{Q}B(C, G)$ is equal to the number of $G$-classes of $C$-subcycles of $G$, and it is also equal to the number of $G$-classes of cyclic $C$-sections in $G$. 


7. The primitive idempotents over a ring of integers

Let $R$ be an integral domain of characteristic zero such that no rational prime is invertible in $R$. The monomial Burnside ring $RB(C, G)$ is, of course, an extension of $B(C, G)$. As we observed in Section 2, $B(C, G)$ is a split extension of $B(G)$. Thus, in particular, we are regarding $B(G)$ as a subring of $RB(C, G)$. The main result in this section, Theorem 7.3, says that all the idempotents of $RB(C, G)$ belong to $B(G)$. The theorem was discovered independently by Boltje (private communication).

Throughout this section, we take $K$ to be a sufficiently large field containing $R$. As usual, we choose and fix an arbitrary embedding of $C$ in $K^\omega$. Since the idempotents of interest are (will eventually turn out to be) idempotents of the ordinary Burnside ring $B(G)$, let us explain how the notation simplifies in that case. In an evident way, the 1-subcharacters of $G$ can be identified with the subgroups of $G$, and similarly for the 1-subelements of $G$. When calculating with elements of $B(G)$, we judiciously delete the finer details of the notation in the equations of Section 5. Thus

$$RB(G) = \bigoplus_{V \leq G} R[G/V], \quad KB(G) = \bigoplus_{H \leq G} K e_H^G,$$

$$\frac{|V|}{N_G(H)[G/V]} = \sum_{H \leq G} \xi(H, V)e_H^G, \quad \frac{|N_G(H)|}{|V|} e_H^G = \sum_{V \leq G} \mu(V, H)[G/V].$$

These equations are due to Gluck [16] and Yoshida [26].

Before proving the theorem, it is convenient to abstract some technicalities that will also be useful in the next section. Consider the mutually dual abelian groups $\hat{G}$ and $\overline{G}$. Given elements $\omega \in \hat{G}$ and $\overline{\omega} \in \overline{G}$, then $\omega(\overline{\omega})$ is an element of $C$. We have embedded $C$ in $K^\omega$, so we may regard $\omega(\overline{\omega})$ as an element of $K$. The group algebra $K \hat{G}$ decomposes as a direct sum of algebras

$$K \hat{G} = \bigoplus_{\overline{\omega} \in \overline{G}} Ke_{\overline{\omega}},$$

where each $e_{\overline{\omega}}$ is a primitive idempotent and

$$\omega = \sum_{\overline{\omega} \in \overline{G}} \omega(\overline{\omega}) e_{\overline{\omega}}$$

for all $\omega \in \hat{G}$. More generally, the group algebras $K \hat{G}$ and $K \overline{G}$ are mutually dual vector spaces over $K$. Thus, for each $\eta \in K \hat{G}$ we have an element $\eta(\overline{\omega}) \in K$, and we can write

$$\eta = \sum_{\overline{\omega} \in \overline{G}} \eta(\overline{\omega}) e_{\overline{\omega}}.$$
For a commutative ring \( \Theta \), let \( \phi_\Theta \) be the \( \Theta \)-algebra monomorphism \( \Theta \hat{\rightarrow} \Theta B(C, G) \) given by

\[
\phi_\Theta(\omega) = [C\omega G / G].
\]

Let \( \theta_\Theta \) be the \( \Theta \)-algebra epimorphism \( RB(C, G) \rightarrow \Theta \hat{\rightarrow} \) given by

\[
\theta_\Theta[C_G \nu G / V] = \begin{cases} 
\nu, & \text{if } V = G, \\
0, & \text{otherwise}.
\end{cases}
\]

We have decorated the symbols \( \phi_\Theta \) and \( \theta_\Theta \) with the subscript \( \Theta \) because, in this section and the next, we shall be varying the coefficient ring, and we shall sometimes need to be clear as to which coefficient ring is under consideration. However, \( \phi_\Theta \) and \( \theta_\Theta \) are just the \( \Theta \)-linear extensions of \( \phi_Z \) and \( \theta_Z \). We may drop the subscript when no ambiguity can arise.

Of course, \( \phi \) and \( \theta \) also depend on \( C \) and \( G \) but, in all our discussions involving \( \phi \) and \( \theta \), the groups \( C \) and \( G \) will be fixed. Since \( \theta_\Theta \) is a left-inverse of \( \phi_\Theta \), we have

\[
RB(C, G) = \phi_R(R \hat{\rightarrow} G) \oplus \ker(\theta_\Theta).
\]

Thus (in the terminology of Section 1), \( \phi_\Theta \) and \( \theta_\Theta \) realize \( RB(C, G) \) as a split extension of \( \theta \hat{\rightarrow} G \).

**Lemma 7.1.** Given \( \zeta \in KB(C, G) \) then \( \theta(\zeta) = \sum_{g \in G} s_{G,G}^G(\zeta) e_{\vec{g}} \).

**Proof.** In view of the coordinate decomposition in Remark 5.3, we need only evaluate \( \theta \) on the primitive idempotents of \( KB(C, G) \). Given \( \omega \in \hat{G} \) and \( g \in G \), then

\[
\mu_G(G, \omega; G, g) = \mu(G, \omega; G, g) = \omega(g) - 1/|\hat{G}|
\]

where \( \vec{g} \) denotes the image of \( g \) in \( \vec{G} \). Theorem 5.2 yields

\[
\theta(e_{G,G}^G) = \sum_{\omega \in \hat{G}} \omega(g)^{-1} e_{\vec{g}} = e_{\vec{g}}
\]

and \( \theta(e_{H,h}^G) = 0 \) for \( H < G \). \( \square \)

**Lemma 7.2.** An element \( \zeta \in KB(C, G) \) belongs to \( KB(G) \) if and only if \( s_{H,h}^G(\zeta) = s_{H,1}^G(\zeta) \) for all \( C \)-subelements \( (H, h) \) of \( G \). In that case, \( s_{H,1}^G(\zeta) = s_H^G(\zeta) \).

**Proof.** Given a \( G \)-set \( S \), then \( s_{H,h}^G[CS] = s_H^G[S] \), which is independent of \( h \). Therefore \( KB(G) \) is contained in the space of vectors satisfying the specified criterion. The reverse inclusion holds by considering dimensions. \( \square \)

As we noted above, the following theorem was discovered independently by Boltje.
Theorem 7.3. The idempotents of \( RB(C, G) \) are precisely the idempotents of \( B(G) \).

Proof. We must show that an idempotent \( \zeta \) of \( RB(C, G) \) satisfies the criterion in Lemma 7.2. By considering restrictions and arguing by induction on \( |G| \), we reduce to the task of showing that \( s^G_{G,g}(\zeta) = s^G_{G,1}(\zeta) \) for all \( g \in G \). But \( \theta_R(\zeta) \) is an idempotent of \( G \). Since no rational prime is invertible in \( R \), the only idempotents of \( G \) are 0 and 1. By Lemma 7.1, if \( \theta_R(\zeta) = 0 \), then \( s^G_{G,g}(\zeta) = 0 \) for all \( g \), while if \( \theta_R(\zeta) = 1 \) then \( s^G_{G,g}(\zeta) = 1 \) for all \( g \). \( \square \)

To finish the matter off, the primitive idempotents of \( RB(C, G) \) ought to be described explicitly in terms of the primitive idempotents of \( KB(C, G) \). For that, we shall need another lemma.

Lemma 7.4. For \( H \subseteq G \), the primitive idempotent \( e^G_H \in KB(G) \) decomposes as a sum of primitive idempotents of \( KB(C, G) \), thus

\[
e^G_H = \sum_{(I,i)} e^G_{I,i},
\]

where \( (I,i) \) runs over representatives of the \( G \)-classes of \( C \)-subelements of \( G \) such that \( I =_G H \).

Proof. Using Lemma 7.2, \( s^G_{I,i}(e^G_H) = \theta^G_{I,i}(e^G_H) = [I =_G H] \). \( \square \)

For a perfect subgroup \( Q \) of \( G \), let

\[
e^G_Q = \sum_{H} e^G_H,
\]

where \( H \) runs over representatives of the \( G \)-classes of subgroups of \( G \) such that the infinitely derived subgroup of \( H \) is \( G \)-conjugate to \( Q \). Dress [13] showed that there is a bijective correspondence \( e^G_Q \leftrightarrow [Q]_G \) between the primitive idempotents \( e^G_Q \) of \( RB(G) \) and the \( G \)-classes \( [Q]_G \) of perfect subgroups \( Q \) of \( G \). The result can also be found in the books by Benson [1, Corollary 5.4.8] and tom Dieck [12, Section 1.4]. Hence, via Lemma 7.4, we have the following result.

Proposition 7.5. The primitive idempotents of \( RB(C, G) \) coincide with the primitive idempotents of \( B(G) \). They are precisely the elements having the form

\[
e^G_Q = \sum_{(I,i)} e^G_{I,i},
\]

where \( Q \) is a perfect subgroup of \( G \), and \( (I,i) \) runs over representatives of the \( G \)-classes of \( C \)-subelements of \( G \) such that the infinitely derived subgroup of \( I \) is \( G \)-conjugate to \( Q \).
What remains open is the problem of determining the primitive idempotents of \( RB(C,G) \) when we drop the hypothesis that none of the rational primes are invertible in \( R \). In view of the 2-local decomposition of \( B(G)^* \) in Yoshida [27], it would be desirable to solve the problem in the case where all except one of the rational primes are invertible in the (characteristic zero) coefficient ring.

8. Units

We turn now to a study of the unit group \( B(C,G)^* \), the torsion-unit group \( B(C,G)^{\omega} \), and some subgroups of \( B(C,G)^{\omega} \). Let us recall some features of the unit group \( B(G)^* \) of the ordinary Burnside ring. It is well known that \( B(G)^* \) is an elementary abelian 2-group; to see this, observe that the set of species \( \mathbb{Q}B(G) \rightarrow \mathbb{Q} \) is a basis for the dual space of \( \mathbb{Q}B(G) \). So \( B(G)^* \) can be regarded as a vector space over the Galois field \( \mathbb{F}_2 \).

Long ago, tom Dieck [12, Proposition 1.5.1] observed that, supposing \(|G|\) is odd then, without the Odd Order Theorem, \( G \) is solvable if and only if \(|B(G)^*| = 2\). That led him [12, Problem 1.5.2] to propose that, in the study of \( B(G)^* \), the “2-primary structure of \( G \) is relevant” and he also signalled interest in the case where \( G \) is a 2-group. Yoshida [27] later vindicated the prediction as to the “2-primary structure.” Tornehave [24] and Yağan [25] have shown that the 2-group case has rich special features. But, by the Odd Order Theorem, \( B(G)^* \) captures nothing at all when \(|G|\) is odd. For arbitrary \( G \), and an odd prime \( p \), there is scant reason to expect much of a connection between \( B(G)^* \) and the “\( p \)-primary structure” of \( G \).

For monomial Burnside rings, it is quite a different story. By Proposition 8.1, the abelian group \( B(C,G)^{\omega} \) is finite. Let us write \( B(C,G)^{\text{even}} \) and \( B(C,G)^{\text{odd}} \) for the Sylow 2-subgroup and the Hall 2′-subgroup of \( B(C,G)^{\omega} \). Theorem 9.6 implies that the decomposition \( B(C,\sim)^{\omega} = B(C,\sim)^{\text{even}} \oplus B(C,\sim)^{\text{odd}} \) is a direct sum of Mackey functors over \( \mathbb{Z} \).

Proposition 8.1 implies that, for an odd prime \( p \), the group \( B(C_p,G)^{\text{odd}} \) is an elementary abelian \( p \)-group, in other words, \( B(C_p,G)^{\text{odd}} \) is a Mackey functor over \( \mathbb{F}_p \). Furthermore, Proposition 8.2 implies that, if \(|G|\) is odd, then \( B(C_p,G)^{\text{odd}} \) has index 2 in \( B(C_p,G)^{\omega} \). It seems reasonable to propose the Mackey functor \( B(C_p,\sim)^{\text{odd}} \) as an odd prime analogue of \( B(\sim)^* \).

All we shall do for the whole unit group \( B(C,G)^* \), in this section, is to explain how \( B(C,G)^* \) can be regarded as a split extension of \( (\mathbb{Z}G)^* \). As special cases of two ring homomorphisms defined in the previous section, consider the ring monomorphism

\[
\phi_Z : \mathbb{Z}G \rightarrow B(C,G)
\]

and the ring epimorphism

\[
\theta_Z : B(C,G) \rightarrow \mathbb{Z}G.
\]

Specializing to the unit groups, we have a group monomorphism

\[
\phi^* : (\mathbb{Z}G)^* \rightarrow B(C,G)^*
\]
and a group epimorphism
\[ \theta^*: B(C, G)^* \to (\hat{\mathbb{Z}G})^*. \]
Observing that \( \theta^* \) is a left-inverse for \( \phi^* \), we see that
\[ B(C, G)^* = \phi^*(\hat{\mathbb{Z}G})^* \oplus \ker(\theta^*). \]

**Proposition 8.1.** Let \( r = \gcd(|C|, \exp(G)) \) and \( \rho = \text{lcm}(2, r) \). The group \( B(C, G)^* \) has exponent dividing \( \rho \) and rank at most \( |G\ \epsilon(C, G)| = |G\ \chi(C, G)| \).

**Proof.** Let \( K \) be the cyclotomic field generated over \( \mathbb{Q} \) by the primitive \( r \)th roots of unity. The torsion-unit group \( K^\omega \) is cyclic with order \( \rho \). Let \( \zeta \in B(C, G)^\omega \). By Proposition 6.1, \( \zeta \) decomposes as a linear combination of primitive idempotents as in Remark 5.3 and each coordinate \( s_{G,H,h}(\zeta) \) is a root of unity in \( K \).

**Proposition 8.2.** If \( |G| \) is odd, then \( B(C, G)^\omega = \{ \pm 1 \} \times B(G)^{\text{odd}} \).

**Proof.** Given a torsion-unit \( \zeta \) in \( B(C, G) \), then \( s_{1,1}(\zeta) \) must be \( \pm 1 \) because it is a rational integer and a unit. Assume that \( s_{1,1}(\zeta) = 1 \). We are to prove that \( \zeta \) has odd order. Let \( K \) be the cyclotomic field as in the proof of the previous proposition. In view of the decomposition in Remark 5.3, we are to show that \( s_{G,H,h}(\pi(\zeta)) \) has odd order. An inductive argument on \( |G| \) deals with the case \( H < G \).

In Section 2, we noted that \( B(C, G) \) is a split extension of \( B(G) \). The projection \( \pi: B(C, G) \to B(G) \) is given by \( [CX] \mapsto [1 \otimes_C CX] = [C\backslash CX] \). We have \( s_{G,H}(\pi(\zeta)) = s_{G,H,1}(\zeta) \) for all \( H \leq G \). As noted by tom Dieck [12, Proposition 1.5.1], \( B(G)^* = \{ \pm 1 \} \); the result can be deduced quickly from Yoshida’s criterion [27, Proposition 6.5] together with the Odd Order Theorem. Therefore, \( s_{G}^G(\pi(\zeta)) = s_{1}^G(\pi(\zeta)) \). We have shown that \( s_{G,1}(\zeta) = 1 \).

Consider the ring epimorphism \( \theta_K: KB(C, G) \to K\hat{G} \) and its specialization
\[ \theta^\omega: B(C, G)^\omega \to (\hat{\mathbb{Z}G})^\omega. \]
Since \( \hat{G} \) is abelian, a weak version of Higman’s theorem says that
\[ (\hat{\mathbb{Z}G})^\omega = \{ \pm 1 \} \times \hat{G}. \]
See, for instance, Sehgal [21, Corollary 1.6] or Serre [22, Exercise 6.3]. From Section 7, recall that the elements \( \eta \) of \( K\hat{G} \) have the coordinate decomposition \( \eta = \sum_{g} \eta(g) e_g \). When \( \eta \in (\hat{\mathbb{Z}G})^\omega \), we have \( \eta(1) = \pm 1 \), with the positive value if and only if \( \eta \) belongs to the Hall 2'-subgroup \( \hat{G} \) of \( (\hat{\mathbb{Z}G})^\omega \). Since \( s_{G,1}^G(\zeta) = 1 \), we see from Lemma 7.1 that \( \theta^\omega(\zeta) \) belongs to the Hall 2' subgroup \( \hat{G} \) and, for the same reason, \( s_{G,g}^G(\zeta) \) has odd order for all \( g \in G \).

Let us mention some combinatorial bounds that can be proved using the same techniques. We only sketch the arguments. Using Proposition 7.5, it is easy to show that the
rank of $B(C, G)^{\text{even}}$ is at most the number of $G$-classes of cyclic $C$-sections in $G$. Using Remark 5.3 and Dirichlet’s unit theorem, it is not hard to show that the free-rank of $B(C, G)^*$ is zero or at most $|G\setminus \text{el}(C, G)|(\phi(r)/2 - 1)$, where $r$ is as above, and $\phi$ denotes the Euler totient function. In particular, if $\text{lcm}(2, r) \leq 6$, then every unit in $B(C, G)$ is a torsion unit. A very similar (but more refined) use of Dirichlet’s unit theorem appears in the proof of a general version of Higman’s theorem given in Karpilovsky [17, Section 8.9]. Evidently, the theory of $B(C, G)^*$ is related to the theory of commutative group rings.

9. Tenduction

The purpose of this section is to realize the unit group $B(C, -)^*$ as a Mackey functor. It will follow immediately that $B(C, G)^{\omega}$ and $B(C, G)^{\text{even}}$ and $B(C, G)^{\text{odd}}$ are Mackey subfunctors. For a subgroup $F$ of $G$, we shall define a product-preserving map

$$t_{C \text{ten}_F}^G : B(C, F) \rightarrow B(C, G)$$

called the tenduction map for coefficient ring $\mathbb{Z}$. Except where emphasis is needed, we shall tend to omit the left decorations. Since $t_{C \text{ten}_F}^G$ is product-preserving, it specializes to a group homomorphism

$$t_{C}^G : B(C, F)^* \rightarrow B(C, G)^*.$$

We shall find that $B(C, -)^*$, equipped with the tenduction, restriction and conjugation maps, is a Mackey functor.

The construction of the tenduction map is not at all straightforward. Let us summarize the steps we shall be taking. We shall introduce a functor

$$\text{A Ten}_F^G : \text{A-F-SET} \rightarrow \text{A-G-SET}$$

called the tenduction functor. Immediately from the definition, it will be clear that $\text{A Ten}_F^G$ preserves products: given $\text{A-fibred } F$-sets $AX$ and $AY$, then

$$\text{Ten}_F^G(AX \otimes AY) = \text{Ten}_F^G(AX) \otimes \text{Ten}_F^G(AY).$$

Then, we shall confine our attention to the case where the fibre group is a supercyclic group $C$. Let $K$ be a field with characteristic zero. Most of the work will be in finding a formula for the coordinates $s_{H, a}^G[\text{Ten}_F^G(CX)]$ in the case where $K$ has enough roots of unity. Extending that formula, and still assuming that $K$ has enough roots of unity, we shall be able to introduce a function

$$K_{C}^G : KB(C, F) \rightarrow KB(C, G)$$

called the tenduction map for coefficient ring $K$. The map $t_{C}^G$ will be related to the tenduction functor $\text{Ten}_F^G$ by the condition

$$c \text{ten}_F^G[CX] = [c \text{Ten}_F^G(CX)].$$
From the defining formula for $ten_G^F$, we shall show that the tenduction map specializes to a map $ten_G^F$ from $B(C, F)$ to $B(C, G)$. We shall then (and only then) be in a position to define $ten_G^F$ for arbitrary $K$.

This complicated zigzag manoeuvre—over sufficiently large $K$, over $\mathbb{Z}$, then over arbitrary $K$—is necessary. Let us issue a couple of warnings. First warning: Eq. (8) does not determine $ten_G^F$ as a product-preserving map. Even in the case $F = G = 1$, there are two distinct product-preserving maps satisfying Eq. (8). Second warning: if $K$ is the field of fractions of the integral domain $R$, then the tenduction map $Kten_G^F$ need not specialize to a function $RB(C, F) \to RB(C, G)$ and it need not specialize to a function $RB(C, F) \to RB(C, G)^*$. This is so even in the case where $C$ is trivial and $R$ is a ring of cyclotomic integers. We shall give a counter-example below.

It may be illuminating to make a comparison with the induction map $\text{Ind}_G^F$, which is, of course, a sum-preserving map on $B(A, F)$ satisfying

$$A\text{ind}_G^F[AX] = [A\text{Ind}_G^F(AX)].$$

The induction map $A\text{ind}_G^F$ on $KB(A, F)$ is not the unique sum-preserving map satisfying Eq. (9). It is the unique $K$-linear map satisfying Eq. (9). Linear extension can be used in that way because the induction functor $\text{Ind}_G^F$ preserves coproducts. The tenduction functor $\text{Ten}_G^F$, though, does not preserve coproducts. To construct a tenduction map, linear extension is not an option.

Tenduction for permutation sets was defined by tom Dieck [12, Section 5.13]. The difficulty discussed above was noticed by Yoshida [27, Section 3b], who consolidated the definition using a technique introduced by Dress [15]. We shall be following the same approach. Let us recall the key notion behind the technique. Let $\{p_1, \ldots, p_m\}$ and $\{q_1, \ldots, q_n\}$ be bases for abelian groups $P$ and $Q$, respectively. A function $\theta : P \to Q$ is said to be polynomial provided there exist polynomials

$$\theta_1, \ldots, \theta_n \in \mathbb{Q}[X_1, \ldots, X_m]$$

such that, for all $\alpha_1, \ldots, \alpha_m \in \mathbb{Z}$, we have

$$\theta \left( \sum_{i=1}^{m} \alpha_i p_i \right) = \sum_{j=1}^{n} \theta_j(\alpha_1, \ldots, \alpha_m) q_j.$$ 

Since composites of polynomial functions are polynomial, and since linear change of coordinates is polynomial, the defining condition is independent of the choices of bases $\{p_1, \ldots, p_m\}$ and $\{q_1, \ldots, q_n\}$. We shall be making use of a uniqueness principle: let us say that a subset $D$ of $P$ is dense in $P$ provided, for every finite subset $D_0 \subseteq D$, the subgroup generated by the complement $D - D_0$ has finite index in $P$. Two polynomial functions on $P$ that agree on a dense subset of $P$ are equal. We shall characterize $ten_G^F$ as the unique polynomial function $B(C, F) \to B(C, G)$ satisfying Eq. (8).

Yoshida [27, Section 3a] characterized tenduction for permutation sets in terms of certain hom-sets. Using ideas in Bouc [9, Section 6.7], the referee found a generalization of
this to fibred permutation sets. We shall be working with an explicit construction of the tenduction functor $\mathcal{A}\text{T}en^G_F$, but we shall also present the referee’s more systematic construction.

For the explicit construction, we begin by choosing an ordered left-transversal $\{t_1, \ldots, t_m\}$ for $F$ in $G$. Let $S_m$ denote the symmetric group of degree $m = |G : F|$. The elements of the wreath product $S_m \wr F$ are the tuples $(s; f_1, \ldots, f_m)$ where $s \in S_m$ and each $f_j \in F$. The group operation is given by

$$(s'; f'_1, \ldots, f'_m)(s; f_1, \ldots, f_m) = (s' s; f'_1 f_1, \ldots, f'_m f_m).$$

We embed $G$ in $S_m \wr F$ via the inclusion $g \mapsto (s(g); f_1(g), \ldots, f_m(g))$ where $gt_j = ts(g)j f_j (g)$. It is easy to check that, up to conjugacy in $S_m \wr F$, the embedding $G \hookrightarrow S_m \wr F$ is independent of the choice of the transversal $\{t_1, \ldots, t_m\}$.

Let $AX$ be a $A$-fibred $F$-set. The tensor product, over $A$, of $m$ copies of $AX$, denoted $A\text{T}en^G_F(AX) = \bigotimes^m AX$, is an $A$-fibred $S_m \wr F$-set such that, for $x_1, \ldots, x_m \in X$, we have

$$(s; f_1, \ldots, f_m)(x_1 \otimes \cdots \otimes x_m) = f_{x^{-1}_m} x_m \otimes \cdots \otimes f_{x^{-1}_1} x_1 x_m.$$ 

Observing that $a_1 x_1 \otimes \cdots a_m x_m = (a_1 \ldots a_m) x_1 \otimes \cdots \otimes x_m$ for $a_1, \ldots, a_m \in A$, we see that the action of $S_m \wr F$ on $A\text{T}en^G_F(AX)$ is independent of the choice of the set $X$ of representatives of the fibres of $AX$. Via the inclusion $G \hookrightarrow S_m \wr F$, we regard $A\text{T}en^G_F(AX)$ as an $A$-fibred $G$-set. We make $A\text{T}en^G_F$ become a functor, operating on maps by

$$(A\text{T}en^G_F(\alpha)(x_1 \otimes \cdots \otimes x_m) = \alpha(x_1) \otimes \cdots \otimes \alpha(x_m),$$

where $\alpha$ is a map with domain $AX$. As before, it is easy to see that $A\text{T}en^G_F(\alpha)$ is independent of the choice of the set $X$. Since the inclusion $G \hookrightarrow S_m \wr F$ is well-defined up to conjugacy, the functor $A\text{T}en^G_F$ is well-defined up to equivalence.

The referee’s alternative construction of $A\text{T}en^G_F$ avoids the need to choose any transversal for $F$ in $G$. Let $G_F$ denote $G$ regarded as an $F$-set, with (left) action such that an element $f \in F$ sends an element $k \in G_F$ to the element $kf^{-1}$. Allowing $F$ to act trivially on $A$, we form the set $A = \text{Hom}(G_F, A)$, which becomes an abelian group with pointwise multiplication. Let $\Delta$ be the kernel of the group epimorphism

$$A \ni \alpha \mapsto \prod_{g \in G} \alpha(g) \in A.$$ 

Via the action of $G$ on $G_F$ by left multiplication, we embed $G$ in the group $\Sigma = \text{Aut}(G_F)$. Consider the set $H = \text{Hom}_F(G_F, AX)$. We regard $H$ as a $\Sigma$-set such that $(\sigma \phi)(\sigma k) = \phi(k)$
where \( \sigma \in \Sigma \) and \( \phi \in \mathcal{H} \). We regard \( \mathcal{H} \) as an \( A \)-set such that \((\alpha \phi)(k) = \alpha(k)\phi(k)\) where \( \alpha \in A \). These two actions commute, so \( \Sigma \) and the group \( A/\Delta \cong A \) have commuting actions on the orbit space \( \Delta \setminus \mathcal{H} \). Moreover, the action of \( A \) on \( \Delta \setminus \mathcal{H} \) is free, so \( \Delta \setminus \mathcal{H} \) is an \( A \)-fibred \( \Sigma \)-set. By specialization, \( \Delta \setminus \mathcal{H} \) is an \( A \)-fibred \( G \)-set.

To show that \( A \) Ten \( GF \)(AX) \( \cong \Delta \setminus \mathcal{H} \) as \( A \)-fibred \( G \)-sets, let us return to the transversal \( \{t_1, \ldots, t_m\} \). As observed in Bouc [9, Proposition 6.11], there is an isomorphism \( \Sigma \cong S_m \wr F \) such that \( \sigma \leftrightarrow (s; f_1, \ldots, f_m) \) where \( \sigma(t_j) = ts_j f_j \). The embeddings of \( G \) in \( \Sigma \) and in \( S_m \wr F \) evidently commute with this isomorphism, so we may identify \( \Sigma \) with \( S_m \wr F \). We identify \( A \) with the direct product of \( m \) copies of \( A \) via the correspondence \( \alpha \leftrightarrow (\alpha(t_1), \ldots, \alpha(t_m)) \). We identify \( \mathcal{H} \) with the direct product of \( m \) copies of AX via the correspondence \( \phi \leftrightarrow (\phi(t_1), \ldots, \phi(t_m)) \).

It is easy to check that the bijection

\[
A \text{Ten}_F^G(AX) \ni a_1 x_1 \otimes \cdots \otimes a_m x_m \leftrightarrow \Delta(a_1 x_1, \ldots, a_m x_m) \in \Delta \setminus \mathcal{H}
\]

is an isomorphism of \( A \)-fibred \( S_m \wr F \)-sets and, perforce, an isomorphism of \( A \)-fibred \( G \)-sets.

Let us note that tenduction of fibred permutation sets is compatible with tenduction of modules. The latter form of tenduction is discussed in, for instance, Benson [1, Section 3.15]. Embedding \( A \) in the unit group of a commutative ring \( \Theta \), it is clear that \( A \) Ten \( G \)F(AX) commutes with tenduction \( \Theta F \)-MOD \( \rightarrow \Theta G \)-MOD via the linearization functors \( \text{Lin}_F \) and \( \text{Lin}_G \) (defined in Section 2).

**Lemma 9.1** (Mackey decomposition). *Given \( E \leq G \), then*

\[
\text{Res}_E^G \text{Ten}_F^G(AX) \cong \bigotimes_{E \leq G} \text{Res}_{E \cap F}^F \text{Res}_{E \cap F}^G \text{Con}_F^E(AX).
\]

**Proof.** Consider the action of \( E \) on the cosets of \( F \) in \( G \).

We now replace \( A \) with the supercyclic group \( C \). Recall that the transfer map \( t_F^G : G \rightarrow \overline{F} \) is the group homomorphism given by

\[
t_F^G(g) = \pi_F(f_1(g) \cdots f_m(g)),
\]

where \( \pi_F \) is the canonical projection from \( F \) to \( \overline{F} \). The kernel of \( t_F^G \) contains \( O(G) \), so we may regard \( t_F^G \) as a homomorphism \( G \rightarrow \overline{F} \).

In the special case for permutation sets, the following formula is due to tom Dieck [12, Proposition 5.13.1], and it can also be found in Yoshida [27, Section 3b].
Lemma 9.2. Given a C-fibred G-set CX and a subelement (H, h) of G, then

\[ s_{H,h}^G[\text{Ten}_F^G(CX)] = \prod_{FgH \subseteq G} s_{FgH,H,fg(h)}^F[CX], \]

where \( t_g(h) = t_{fH}^H(s_h). \)

Proof. First suppose that \( H = G. \) The condition \( C_{x_1} = Cx \) determines a bijective correspondence between the fibres \( C_{x_1} \otimes \cdots \otimes x_m \) in \( \text{Ten}_F^G(CX) \) stabilized by \( G \) and the fibres \( Cx \) in \( CX \) stabilized by \( F. \) Suppose that \( C_{x_1} \otimes \cdots \otimes x_m \) is stabilized by \( G. \) Let \( \phi \) be the \( C \)-character of \( F \) such that \( f_{x_1} = \phi(f)x_1. \) Then

\[ g(x_1 \otimes \cdots \otimes x_m) = \phi(t_{gH}^H(h))x_1 \otimes \cdots \otimes x_m \]

for \( g \in G. \) We have shown that the assertion holds in the case \( H = G, \) indeed,

\[ s_{G,h}^G[\text{Ten}_F^G(CX)] = s_{F,hG}^F[CX]. \]

Suppose now that \( H < G. \) For \( k \in G, \) write \( L(k) = H \cap kF. \) Inductively, we may assume that the assertion holds when \( G \) is replaced by \( L(k). \) By Lemma 9.1,

\[ s_{H,h}^G[\text{Ten}_F^G(CX)] = \prod_{H \cap H \subseteq G} s_{H,h}^H[\text{Ten}_L^H(C_{L(k)}) \text{Res}_{L(k)}^F(kCX)] \]

\[ = \prod_{H \cap H \subseteq G} s_{L(k),H,tl_{H(k)}^H}^L[\text{Res}_{L(k)}^F(kCX)] \]

\[ = \prod_{H \cap H \subseteq G} s_{L(k),H,tl_{H(k)}^H}^L[kCX]. \]

The argument is completed by substituting \( g = k^{-1} \) and conjugating by \( g. \)

Still assuming that \( K \) has enough roots of unity, we define a function

\[ K_C^{\text{ten}_F^G} : KB(C, F) \to KB(C, G) \]

such that, given \( \xi \in KB(C, F) \) and \( (H, h) \in \text{el}(C, G), \) then the \((H, h)\)-coordinate of \( K_C^{\text{ten}_F^G}(\xi) \) is

\[ s_{H,h}^G(K_C^{\text{ten}_F^G}(\xi)) = \prod_{FgH \subseteq G} s_{FgH,H,fg(h)}^F(\xi). \]

At present, we have defined \( K_C^{\text{ten}_F^G} \) only in the case where \( K \) has enough roots of unity. The proofs of the following two results are to be interpreted only for such \( K. \) However, as soon as we have defined \( K_C^{\text{ten}_F^G} \) for arbitrary \( K, \) it will be obvious that the two results hold in general.
Lemma 9.3. The function \( K \) preserves products and satisfies Eq. (8).

Proof. The first part is immediate from the definition of \( K \). The second part follows from Lemma 9.2. \( \square \)

Theorem 9.4. The function \( K \) restricts to a polynomial function \( \tilde{Z} \) : \( B(C, F) \rightarrow B(C, G) \).

Proof. Let \((I_1, i_1), \ldots, (I_u, i_u)\) be a set of representatives of the \( F \)-classes of \( C \)-subelements of \( F \), and let \((J_1, j_1), \ldots, (J_v, j_v)\) be a set of representatives of the \( G \)-classes of \( C \)-subelements of \( G \). Consider an element \( \xi \in KB(C, F) \), and write

\[
\xi = \sum_{a=1}^{u} a' \in F \Rightarrow \text{ten}_{G}^{F}(\xi) = \sum_{b=1}^{v} b' \in F \Rightarrow \text{ten}_{G}^{F}(\xi).
\]

For any given \( \xi \), it is to be understood that each \( a' \in K \) and each \( b' \in K \). But, if we allow the coordinates \( a' \) to vary, then each \( b' \) becomes a function \( \alpha'_{a} : K \rightarrow K \), written \( \alpha'_{a} : (\alpha_{1}, \ldots, \alpha_{u}) \mapsto \alpha'_{a}(\alpha_{1}, \ldots, \alpha_{u}) \). The explicit formula in Lemma 9.2 shows that each \( \alpha'_{a} \) is a polynomial function whose coefficients are all 0 or 1. Let \((U_1, \mu_1), \ldots, (U_u, \mu_u)\) be a set of representatives of the \( F \)-classes of \( C \)-subcharacters of \( F \), and let \((V_1, \nu_1), \ldots, (V_v, \nu_v)\) be a set of representatives of the \( G \)-classes of \( C \)-subcharacters of \( G \). Write

\[
\xi = \sum_{a=1}^{u} a \in F \Rightarrow \text{ten}_{F}^{G}(\xi) = \sum_{b=1}^{v} b \in F \Rightarrow \text{ten}_{F}^{G}(\xi).
\]

The linear changes of coordinates from \( a_{a} \) to \( a'_{a} \) and from \( b'_{b} \) to \( b_{b} \) are expressed in Eqs. (3)–(6) in Section 5. Thence, each function \( (\alpha_{1}, \ldots, \alpha_{u}) \mapsto \beta_{b}(\alpha_{1}, \ldots, \alpha_{u}) \) is a polynomial function whose coefficients belong to some cyclotomic extension of \( \mathbb{Q} \). Since \( \beta_{b} \) takes rational values whenever its arguments are natural numbers, the coefficients of \( \beta_{b} \) are rational. Consider rational integers \( n, \alpha_{1}, \ldots, \alpha_{u} \) with \( n \) sufficiently large for our purposes. Given any rational prime \( p \), then \( p^{n} - \alpha_{1}, \ldots, p^{n} - \alpha_{u} \) are non-negative and \( p^{n} \) does not divide the denominators of any of the coefficients of \( \beta_{b} \). The integer \( \beta_{b}(p^{n} + \alpha_{1}, \ldots, p^{n} + \alpha_{u}) \) differs from \( \beta_{b}(\alpha_{1}, \ldots, \alpha_{u}) \) by a rational whose denominator is coprime to \( p \). Therefore, the denominator of \( \beta_{b}(\alpha_{1}, \ldots, \alpha_{u}) \) is coprime to \( p \). Since \( p \) is arbitrary, \( \beta_{b}(\alpha_{1}, \ldots, \alpha_{u}) \) is a rational integer. \( \square \)

In the proof of the theorem, we described \( \tilde{Z} \) as the polynomial function

\[
\tilde{Z}^{G}(\sum_{a=1}^{u} a_{a}[C_{\mu_a} F / U_a]) = \sum_{b=1}^{v} \beta_{b}(\alpha_{1}, \ldots, \alpha_{u})[C_{\nu_b} G / V_b].
\]

Furthermore, we proved that the coefficients of the polynomials \( \beta_{b} \) are rationals. Now letting \( K \) be arbitrary, we extend \( \tilde{Z}^{G} \) to a function \( K \) : \( KB(C, F) \rightarrow KB(C, G) \)
defined by the same formula. It is clear that, when \( K \) has enough roots of unity, the function \( K \) ten\( F \) is the same as before. It is also clear that Lemma 9.3 and Theorem 9.4 hold for arbitrary \( K \).

We have completed the construction of the tenduction map \( K \) ten\( F \) where the coefficient ring \( R \) is either \( \mathbb{Z} \) or else an arbitrary field of characteristic zero. It is easy to see that, embedding \( C \) in the torsion unit group \( \Theta^\omega \) of a commutative ring \( \Theta \), then \( K \) ten\( F \) commutes with the tenduction map \( RR(\Theta F) \rightarrow RR(\Theta G) \) on the representation rings.

Let us show that the constructions do not work in the case where the coefficient ring \( R \) is an arbitrary integral domain of characteristic zero. We present the counter-example promised earlier in this section. Let \( p \) be a prime, let \( C = \mathbb{Z} \) and let \( F \) be the proper non-trivial subgroup of \( G \). Let \( \omega \) be a primitive \( 2p \)th root of unity, let \( R = \mathbb{Z}[\omega] \) and let \( K = \mathbb{Q}[\omega] \). The element \( \xi = \omega F/F \) is a torsion-unit of \( KB(C,G) \). We shall show that the element \( \zeta = K \) ten\( F \)(\( \xi \)) of \( KB(C,G) \) does not belong to \( KB(C,G) \). Using Lemma 9.2 and the coordinate transforms discussed in Section 5 (the special cases due to Gluck and Yoshida),

\[
\text{ten}^G_F(\alpha_1[F/F] + \alpha_2[F/F]) = \frac{(p\alpha_1 + \alpha_2)p - \alpha_2^p}{p^2}[G/F] + \frac{\alpha_2^p - \alpha_2}{p}[G/F] + \alpha_2[G/G].
\]

In particular, \( p\xi/\omega = (\omega^{p-1} - 1)[G/F] + [G/G] \). It suffices to show that the algebraic integer \( 1 - \omega^{p-1} \) is not divisible by \( p \). The case \( p = 2 \) is trivial. Assuming that \( p \) is odd, then \( \omega^{p-1} \) is a primitive \( p \)th root of unity. The product of the \( p - 1 \) Galois conjugates of \( 1 - \omega^{p-1} \) is \( p \). So at least one of the Galois conjugates is not divisible by \( p \). It follows that none of the Galois conjugates are divisible by \( p \), as required.

**Proposition 9.5** (Mackey formula). Let \( E \leq G \). As functions \( B(C,F) \rightarrow B(C,E) \) or, more generally, as functions \( KB(C,F) \rightarrow KB(C,G) \), we have

\[
\text{res}^G_E \text{ten}^G_F = \prod_{E \leq F \leq G} \text{ten}^F_{E \cap F} \text{res}^E_{E \cap F} \text{con}^F_E.
\]

**Proof.** The two specified functions on \( B(C,F) \) are equal because they are polynomial and because they agree on the dense subset of \( B(C,F) \) consisting of the isomorphism classes of \( C \)-fibred \( F \)-sets. The general case holds because any polynomial over \( \mathbb{Z} \) extends uniquely to a polynomial over \( K \).

Similar arguments show that the tenduction, restriction and conjugation functors and maps satisfy all the relations that characterize Mackey functors: idempotence, transitivity, and compatibility with conjugation. Furthermore, tenduction, restriction, and conjugation all preserve products. Therefore, regarding the ring \( B(C,G) \) as a module over the semi-ring \( \mathbb{N} \), with action \( n : \xi \mapsto \xi^n \), then \( B(C,\cdot) \), equipped with tenduction, restriction, and conjugation, is a Mackey functor over \( \mathbb{N} \). More to the point, we have proved the following theorem.
Theorem 9.6. The functor $B(C, -)^*$, defined on the category of inclusions of finite groups, and equipped with the tenduction, restriction and conjugation maps, is a Mackey functor over $\mathbb{Z}$. Furthermore, $B(C, -)^{\text{even}}$ and $B(C, -)^{\text{odd}}$ are Mackey subfunctors.

In particular, for an odd prime $p$, we have completed the realization of $B(C_p, G)^{\text{odd}}$ as a Mackey functor over the field $\mathbb{F}_p$.

References