Available online at www.sciencedirect.com
SCIENCE DIRECT®
J. Math. Anal. Appl. 322 (2006) 18–27
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Impulse functions over curves and surfaces and their applications to diffraction

Levent Onural

Department of Electrical and Electronics Engineering, Bilkent University, TR-06800 Bilkent, Ankara, Turkey

Received 23 March 2005
Available online 24 August 2005
Submitted by G. Chen

Abstract

An explicit preferred definition of impulse functions (Dirac delta functions) over lower-dimensional manifolds in $\mathbf{R}^N$ is given in such a way to assure uniform concentration per geometric unit of the manifold. Some related properties are presented. An application related to diffraction is demonstrated.

Keywords: Generalized functions; Distributions; Impulse functions; Curve impulses; Surface impulses; Diffraction

1. Introduction

The Dirac delta function $\delta(\mathbf{x})$ is well known (see, for example, [1, pp. 127–136] or [2, pp. 393–395]); and the mathematical definitions and the associated rigor can be found in the literature (see, for example, [3,4]). In summary, for $\mathbf{x} \in \mathbf{R}^N$, $\delta(\mathbf{x})$ is defined through an inner product of this function with other "good" functions as

$$\int \delta(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x} = f(\mathbf{0}). \tag{1}$$

E-mail address: onural@bilkent.edu.tr.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.07.012

Here the integral is a multiple integral whose order is the dimension of the space, N. Being defined by Eq. (1), the impulse function $\delta(\mathbf{x}-\xi)$ represents nothing but a concentration of some quantity over an infinitesimal N-dimensional volume around the point ξ.

It is natural to extend the definition to handle concentration of a quantity not only on a point, but also on a curve or a surface (hypersurface) defined in an N-dimensional space. Indeed, curve-impulses have been used in engineering and studied in mathematics, see [3–6]. Although the understanding is clear that these functions represent a concentration along the curve in a higher-dimensional space, the distribution of concentration per unit-length along the curve depends on the definition which is not uniform in the literature. Therefore, one of the purposes of this paper is to state a preferred definition.

2. Notation and mathematical preliminaries

We adopt the usual definitions of curves, surfaces, line integrals, and surface integrals, as commonly found in the literature. (See, for example, [7, Chapters 2 and 3].) We consider a lower-dimensional embedded orientable manifold S in $\mathbf{R}^N$. For notational simplicity, especially when we state the properties in Section 3, we adopt the same symbol S to represent the path of a curve, the trace of a surface, or the trace of a hypersurface in $\mathbf{R}^N$, depending on the application.

Let us consider the space $\mathbf{R}^N$. $f(\cdot)$ represents a function from N-dimensional real vectors to complex numbers, i.e., $f:\mathbf{R}^N \to \mathbf{C}$. We assume that S has all the properties which allow us to define integrals of f over S, whenever needed. The author prefers the notation $\delta_S(\mathbf{x})$ to describe a concentration over S in $\mathbf{R}^N$, where $\mathbf{x} \in \mathbf{R}^N$. With this notational preference, the conventional $\delta(\mathbf{x}-\xi)$ is also denoted as $\delta_S(\mathbf{x})$ where S is now the point (zero-dimensional subset) $\mathbf{x}=\xi$.

Following a similar approach as the definition given by Eq. (1), we can simply and immediately define impulse functions over S in $\mathbf{R}^N$ through an integral equation as

$$\langle \delta_S(\mathbf{x}), f(\mathbf{x})\rangle = \int_{\mathbf{R}^N} \delta_S(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x} = \int_S f(\mathbf{x})\, dS. \tag{2}$$

Integrals involving $\delta_S(\mathbf{x})$ in $\mathbf{R}^N$ reduce to line or surface integrals over S. More subtle than that is the potential benefits of converting a line or a surface integral to a regular integral over $\mathbf{R}^N$ via this definition.

A consequence of the definition given by Eq. (2) is that the concentration is uniform over S, i.e., the concentration has constant density per dS. It is easy to see this uniform distribution nature as

$$\int_{\mathbf{R}^N} \delta_S(\mathbf{x}) 1\, d\mathbf{x} = \int_S dS \tag{3}$$

which is equal to the total area A_S of S, and valid for any S, including smaller portions of a given S. A_S is finite if S has finite extent; otherwise it is infinite.

Now we will show that, for a p-dimensional manifold S, where $1 \leqslant p < N$, and for integrals over the manifold Q_P, where Q_P is a $(N-p)$-dimensional embedded orientable manifold which intersects S at an arbitrary intersection point P orthogonally, we have

$$\int\limits_{Q_P} \delta_S(\mathbf{x})\, dQ = 1; \tag{4}$$

we assume that Q_P has the necessary properties to define integrals over it.

Let P be a point on S. Let us define N curvelinear coordinates as N curves all in $\mathbf{R}^N$, such that the family of coordinate lines are smooth, and intersecting each other orthogonally at any $P \in S$. Neither the orthogonality, nor the smoothness is required for other points in $\mathbf{R}^N$ that are not in S. Let each coordinate be represented by a length variable l_i. Therefore, on S, $d\mathbf{l} = dl_1\, dl_2 \ldots dl_N$ represents the volume element dV which is also equal to $d\mathbf{x}$. (We assume that the directions of coordinates are chosen so that the possibility of $d\mathbf{l} = -d\mathbf{x}$ is eliminated.) Note that, the orthogonality of l_i's and their purely geometric length specification, and properly designated directions assure that the Jacobian between $\mathbf{x}$ and $\mathbf{l}$ is equal to one at any point on S. Let us order these curves so that the first p of them, $l_1, l_2, \ldots, l_p$, lie in S (describe S) and the rest, $l_{p+1}, \ldots, l_N$, do not lie in S. Therefore, $l_{p+1}, \ldots, l_N$ define the complement space Q of S in $\mathbf{R}^N$. So we can write

$$\begin{aligned}\int \delta_S(\mathbf{x})\, d\mathbf{x} &= \int \delta_S(\mathbf{x})\, dV = \int\limits_S \left(\int\limits_{Q_P} \delta_S(\mathbf{x})\, dQ \right) dS \\ &= \int\limits_S \left(\int\limits_{Q_P} \delta_S(\mathbf{x})\, dl_{p+1} \ldots dl_N \right) dl_1 \ldots dl_p,\end{aligned} \tag{5}$$

where $P = (l_1, l_2, \ldots, l_p)$ is the point on S. We would like to interpret $\int \delta_S(\mathbf{x})\, dQ$ further. Firstly, from Eq. (3) and the equation above, we must have, $\int_{Q_P} \delta_S(\mathbf{x})\, dQ = 1$. Since the manifold Q_P crosses the manifold S orthogonally at P, above result indicates that the integral of $\delta_S(\mathbf{x})$ as we cross the surface S orthogonally is uniform (independent of point P) and equal to 1.

Although the formal definition of the impulse function, $\delta_S(\mathbf{x})$, is given by Eq. (2), it may be useful to visualize these impulse functions as limits which wrap a "skin" over S, where the skin gets narrower as the mass in it gets squeezed and concentrated. First, let us remember that the 1D impulse function may also be reached as a limit

$$\delta_0(x) = \delta(x) = \lim_{\Delta \to 0} \begin{cases} 1/\Delta & \text{if } x \in [-\Delta/2, \Delta/2], \\ 0 & \text{else}, \end{cases} \tag{6}$$

which is an unnecessarily restrictive definition compared to Eq. (1), but surely provides an easier comprehension. Similarly, for the impulse over S, we can write a limit as,

$$\delta_S(\mathbf{x}) = \lim_{\Delta \to 0} \begin{cases} 1/\Delta & \text{if } \mathbf{x} \in S_\Delta, \\ 0 & \text{else}, \end{cases} \tag{7}$$

where S_Δ is a volume in $\mathbf{R}^N$ such that it includes the p-dimensional hypersurface S, and the "thickness" of this volume along the orthogonal direction to S is uniform, so that its

$(N-p)$-dimensional orthogonal cross-sections has a uniform measure Δ. Therefore, the impulse $\delta(\mathbf{x})$ represents a constant finite mass in S_Δ whose total volume is A_S and progressively squeezed to exist only on S as $\Delta \to 0$; mass per unit S is uniform throughout S and is equal to 1.

The definition of the impulse functions over manifolds given in [3, vol. I, pp. 209–247] results in a unique but non-uniform concentration over a surface, and therefore, is different than the definition given in this paper.

We may now proceed to investigate some properties associated with $\delta_S(\mathbf{x})$.

3. Some properties of $\delta_S(\mathbf{x})$

1. Shift

Naturally,

$$\delta_S(\mathbf{x}-\mathbf{a}) = \delta_{S_\mathbf{a}}(\mathbf{x}), \tag{8}$$

where $S_\mathbf{a}$ is a new manifold obtained by translating S by adding the vector $\mathbf{a}$ to each point on it.

2. Sifting

$$\int_V \delta_S(\mathbf{x}-\mathbf{a}) f(\mathbf{x})\, d\mathbf{x} = \int_{S_\mathbf{a}} f(\mathbf{x})\, dS = \int_S f(\mathbf{x}+\mathbf{a})\, dS. \tag{9}$$

This property simply states that translating the manifold S by $\mathbf{a}$ will result in a surface integral over S of the backward translated function $f(\mathbf{x}+\mathbf{a})$.

3. Rotation

Let $\mathbf{R}$ be a rotation matrix, and let $S_\mathbf{R}$ be the manifold obtained by rotating each point on S; i.e., the coordinates $\mathbf{x}$ of a point P on S, is mapped to a point P' on $S_\mathbf{R}$ whose coordinates is $\mathbf{R}^{-1}\mathbf{x}$. Therefore, $\delta_S(\mathbf{R}\mathbf{x}) = \delta_{S_\mathbf{R}}(\mathbf{x})$. So,

$$\begin{aligned}\int_V \delta_{S_\mathbf{R}}(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x} &= \int_{S_\mathbf{R}} f(\mathbf{x})\, dS = \int_V \delta_S(\mathbf{R}\mathbf{x}) f(\mathbf{x})\, d\mathbf{x} = \int_V \delta_S(\mathbf{x}) f\big(\mathbf{R}^{-1}\mathbf{x}\big)\, d\mathbf{x} \\ &= \int_S f\big(\mathbf{R}^{-1}\mathbf{x}\big)\, dS. \end{aligned} \tag{10}$$

4. Affine transforms

Let $S_{\mathbf{A},\mathbf{b}}$ define a manifold which is obtained by mapping the coordinates $\mathbf{x}$ of each point on S, to a corresponding point on $S_{\mathbf{A},\mathbf{b}}$, with coordinates $\mathbf{A}^{-1}(\mathbf{x}-\mathbf{b})$. Furthermore, let $\delta_{S_{\mathbf{A},\mathbf{b}}}(\mathbf{x})$ be the impulse on $S_{\mathbf{A},\mathbf{b}}$. Then

$$\int_V \delta_{S_{\mathbf{A},\mathbf{b}}}(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x} = \int_{S_{\mathbf{A},\mathbf{b}}} f(\mathbf{x})\, dS = \int_V \delta_S(\mathbf{A}\mathbf{x}+\mathbf{b}) f(\mathbf{x})\, d\mathbf{x}$$
$$= \int_V \delta_S(\mathbf{x}) f\big(\mathbf{A}^{-1}(\mathbf{x}-\mathbf{b})\big)\frac{1}{|\mathbf{A}|}\, d\mathbf{x}$$
$$= \frac{1}{|\mathbf{A}|}\int_S f\big(\mathbf{A}^{-1}(\mathbf{x}-\mathbf{b})\big)\, dS. \tag{11}$$

5. Crossings

Let S_1 and S_2 be two different manifolds. Let us consider a crossing point of these manifolds, P. Then,

$$\int_V \delta_{S_1}(\mathbf{x})\delta_{S_2}(\mathbf{x})\, d\mathbf{x} = \int_{S_1} \delta_{S_2}(\mathbf{x})\, dS = \frac{1}{\sin\theta}, \tag{12}$$

where θ is the angle of crossing; if the crossing is orthogonal, $\sin\theta = 1$. For example, for crossings where both S_1 and S_2 are 1D, θ is the angle between the tangents to these curves at P. If one of the manifolds is 1D and the other one is 2D, θ is the angle between the tangent to the 1D manifold, and its projection onto the 2D manifold; as expected, $\sin\theta = 1$ for orthogonal crossings.

6. Change of variables

Often it is desirable to write surface integrals as integrals over Cartesian coordinates. The associated change of variables can be achieved as usual. Therefore,

$$\int_V \delta_S(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x} = \int_S f(\mathbf{x})\, dS = \sum_i \int_{B_i} f(\mathbf{x}) \frac{dS}{d\mathbf{x}_m}\, d\mathbf{x}_m, \tag{13}$$

where the integral is an m-dimensional integral, and $\mathbf{x}_m$ is the m-dimensional subset of Cartesian coordinates $\mathbf{x}$; m is also the dimension of S. B is the projection of S onto $\mathbf{x}_m$. The summation in front of the integral takes care of the multiple crossings of S and the subspace orthogonal to B.

An example, which will be used later in an optical diffraction application, illustrates the notation.

Example. Let S be the path of the curve described by $g_1(\mathbf{x}) = 0$ in $\mathbf{R}^2$ where $g_1(\mathbf{x}) = x^2 + y^2 - k^2$, k is a constant. Therefore, S is a circle with radius k whose center is at the origin. Let $\delta_S(\mathbf{x})$ be the impulse on this circle. Therefore,

$$\int_{\mathbf{R}^2} \delta_S(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x} = \int_S f(\mathbf{x})\, dS, \tag{14}$$

where the integral on the right-hand side is now a line integral over the circle. The evaluation of this integral either on the x-axis, or on the y-axis, as usual, clarifies the adopted notation, as follows. The projection B, of circle S on x-axis (or y-axis) is a line segment. Therefore,

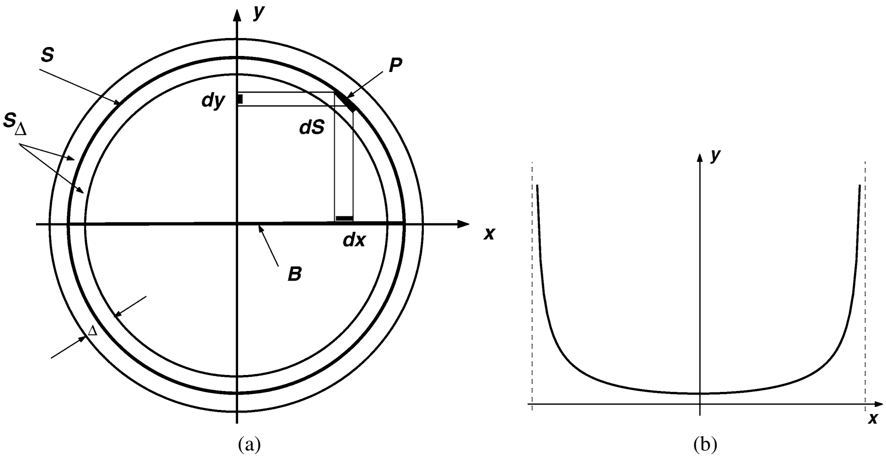

Fig. 1. (a) Corresponding notation for the impulse over a circle. (b) The non-uniform distribution of mass over the x-axis for this example; $f(x) = \sqrt{1 + \frac{x^2}{k^2 - x^2}}$.

$$\begin{aligned}\int_S f(x,y)\,dS &= \int_{-k}^{k} f\big(x, \sqrt{k^2-x^2}\big)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &\quad + \int_{-k}^{k} f\big(x, -\sqrt{k^2-x^2}\big)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= \int_{-k}^{k} f\big(\sqrt{k^2-y^2}, y\big)\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \\ &\quad + \int_{-k}^{k} f\big(-\sqrt{k^2-y^2}, y\big)\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy. \end{aligned} \tag{15}$$

Figure 1(a) demonstrates the associated features of the impulse over the circle for this example: S, S_Δ, B and dS are illustrated for this case (using the interpretation associated with Eq. (7)). The uniform density curve impulse over the circle, converts the integral over $\mathbf{R}^2$ of a function multiplied by this impulse, into a line integral over the circle. If this line integral is converted to a regular integral over x, the associated correction of concentration per dx must be taken care of; and this is handled by

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{x^2}{y^2}} = \sqrt{1+\frac{x^2}{k^2-x^2}} = k/y$$

for this example. Actually, this is nothing but the projection of uniform concentration over the circle, onto x-axis. The concentration per dx, as a consequence of the projection of the uniform distribution over the circle is shown in Fig. 1(b).

Note that tempting notations like $\delta(g(x))$, which imply a concentration over the curve $g(x)=0$ are ambiguous without supporting explicit definitions, and therefore, potentially dangerous: note that, for the example above, using this version of notation we get, $\delta(x^2+y^2-k^2)$ which can also be written as $\delta(y\pm\sqrt{k^2-x^2})$ (or $\delta(x\pm\sqrt{k^2-y^2})$). And this may erroneously convert Eq. (14) to

$$\begin{aligned}\int_{\mathbf{R}^2}\delta_S(\mathbf{x})f(\mathbf{x})\,d\mathbf{x}&=\int_x\int_y\delta\big(y-\sqrt{k^2-x^2}\big)f(x,y)\,dy\,dx\\&\quad+\int_x\int_y\delta\big(y+\sqrt{k^2-x^2}\big)f(x,y)\,dy\,dx\\&=\int_x f\big(x,\sqrt{k^2-x^2}\big)+f\big(x,-\sqrt{k^2-x^2}\big)\,dx,\end{aligned}\tag{16}$$

if one interprets $\delta(y-\sqrt{k^2-x^2})$, for constant x, as a 1D unit impulse located at the point $y=\sqrt{k^2-x^2}$ on the y-axis. This obscures the differential term, $\sqrt{1+(dy/dx)^2}$, totally, and could lead to erroneous results if consistent corrections are omitted. The interpretation given in this paragraph, which results in the right-hand side of Eq. (16), implicitly assumes a non-uniform concentration over the circle, where this non-uniformity then cancels the differential term to have Eq. (16).

4. Application to diffraction

We have chosen the classical diffraction problem as an example. Please note that the purpose is not to provide the solution which has been known for a long time (see, for example, [2, Chapters 3 and 4]), anyway. Instead, the purpose is to show how the $\delta_S(\mathbf{x})$ function improves the structure of the problem formulation, provides a better understanding, and thus paves the way for elegant solutions for many associated problems.

The diffraction between two parallel planes can be computed as

$$\begin{aligned}\psi_{z_0}(x,y)&\triangleq\psi(\mathbf{x})\big|_{z=z_0}=\big(4\pi^2\big)\mathcal{F}_{2\mathrm{D}}^{-1}\Big\{A_{2\mathrm{D}}(k_x,k_y)e^{j(\sqrt{k^2-k_x^2-k_y^2})z_0}\Big\}=\psi_{z_0}(x,y)\\&=\mathcal{F}_{2\mathrm{D}}^{-1}\Big\{\mathcal{F}_{2\mathrm{D}}\big\{\psi_0(x,y)\big\}e^{j(\sqrt{k^2-k_x^2-k_y^2})z_0}\Big\},\end{aligned}\tag{17}$$

where $\psi_0(x,y)$ is the given 2D pattern on the (x,y)-plane located at $z=0$ (i.e., the 2D "object plane"), and $\psi_{z_0}(x,y)$ is the resultant diffraction pattern on a plane parallel to the object plane located at $z=z_0$.

Instead of formulating the problem as a transformation between the patterns over the two planes, as in Eq. (17), we can get a much better insight if the 3D volume diffraction pattern is considered in its entirety. Such an approach paves the way for efficient computational algorithms for diffraction between different geometries.

We will start with a scalar 3D optical plane wave which can be represented as $B(\mathbf{k})e^{j\mathbf{k}^T\mathbf{x}}$ for all $\mathbf{x} \in \mathbf{R}^3$. Therefore, the total 3D field, $\psi(x, y, z)$, as a consequence of superposition of all such plane waves is,

$$\psi(x, y, z) = \psi(\mathbf{x}) = \int_{\mathbf{k}} B(\mathbf{k})e^{j\mathbf{k}^T\mathbf{x}}\, d\mathbf{k} = \mathcal{F}_{3\mathrm{D}}^{-1}\{8\pi^3 B(\mathbf{k})\}. \tag{18}$$

Imposing the restriction that the wave is monochromatic, we get, $|\mathbf{k}| = k = 2\pi/\lambda$, where λ is the optical wavelength of the monochromatic signal. Therefore, $B(\mathbf{k}) = \delta_S(\mathbf{k})A(\mathbf{k})$ for the monochromatic wave, where S is the semi-sphere $|\mathbf{k}| = k$, with $k_z > 0$, assuming that the propagation is along the positive z direction. $A(\mathbf{k})$ is the amplitude of the monochromatic wave along the direction indicated by $\mathbf{k}$. So,

$$\psi(\mathbf{x}) = \int_{\mathbf{k}} \delta_S(\mathbf{k})A(\mathbf{k})e^{j\mathbf{k}^T\mathbf{x}}\, d\mathbf{k} = \mathcal{F}_{3\mathrm{D}}^{-1}\{8\pi^3 \delta_S(\mathbf{k})A(\mathbf{k})\} \int_S A(\mathbf{k})e^{j\mathbf{k}^T\mathbf{x}}\, dS. \tag{19}$$

Therefore, a surface integral over a semi-sphere is converted to a regular 3D integral using the impulse function over that surface (semi-sphere). What is achieved here is a new precise representation of the 3D optical field formulation, as given by Eq. (19), involving an impulse function over a surface (semi-sphere in this case).

Using the property 6 given in the previous section, and noting that

$$\frac{dS}{dk_x dk_y} = \frac{k}{k_z} = \frac{k}{\sqrt{k^2 - k_x^2 - k_y^2}},$$

we can convert the integrals of Eq. (19) to a regular integral over two variables k_x and k_y as

$$\psi(\mathbf{x}) = \int_B A(\mathbf{k})e^{j\mathbf{k}^T\mathbf{x}} \frac{k}{\sqrt{k^2 - k_x^2 - k_y^2}}\, dk_x\, dk_y, \tag{20}$$

where B is the projection of $\mathcal{S}$ onto (k_x, k_y)-plane. This projection is the disc $k_x^2 + k_y^2 < k^2$. Furthermore, since $k_x^2 + k_y^2 + k_z^2 = k^2$, where k is the constant wave number, we write,

$$\psi(\mathbf{x}) = \iint_B A\left(k_x, k_y, \sqrt{k^2 - k_x^2 - k_y^2}\right)e^{j[(k_x x + k_y y + \sqrt{k^2 - k_x^2 - k_y^2})z]} \times \frac{k}{\sqrt{k^2 - k_x^2 - k_y^2}}\, dk_x\, dk_y. \tag{21}$$

Now we can base the solution of Eq. (17) to this alternate approach by taking the cross-section of the 3D $\psi(\mathbf{x})$ at $z = 0$ and at $z = z_0$ planes and prove that the result given by Eq. (17) is true. Therefore,

$$\psi_0(x, y) \triangleq \psi(\mathbf{x})\big|_{z=0} = \iint_B A_{2\mathrm{D}}(k_x, k_y)e^{j(k_x x + k_y y)}\, dk_x\, dk_y, \tag{22}$$

where

$$A_{2D}(k_x, k_y) \triangleq A\left(k_x, k_y, \sqrt{k^2 - k_x^2 - k_y^2}\right)\frac{k}{\sqrt{k^2 - k_x^2 - k_y^2}}. \tag{23}$$

Diffraction pattern over the (x, y)-plane, $\psi(\mathbf{x})|_{z=0}$, usually represents the "object" in optics. As a consequence of Eq. (22), we see that,

$$\left(4\pi^2\right)A_{2D}(k_x, k_y) = \mathcal{F}_{2D}\left\{\psi_0(x, y)\right\}, \tag{24}$$

where the 2D Fourier transform is from (x, y)-domain to (k_x, k_y)-domain.

Similarly, on $z = z_0$ plane, we get,

$$\begin{aligned}\psi_{z_0}(x, y) \triangleq \psi(\mathbf{x})\big|_{z=z_0} &= \left(4\pi^2\right)\mathcal{F}_{2D}^{-1}\left\{A_{2D}(k_x, k_y)e^{j(\sqrt{k^2-k_x^2-k_y^2})z_0}\right\} = \psi_{z_0}(x, y) \\ &= \mathcal{F}_{2D}^{-1}\left\{\mathcal{F}_{2D}\left\{\psi_0(x, y)\right\}e^{j(\sqrt{k^2-k_x^2-k_y^2})z_0}\right\},\end{aligned} \tag{25}$$

where the 2D Fourier transform is between the (x, y)- and (k_x, k_y)-domains. Thus we reach the result of Eq. (17).

The approach presented in this paper gives a better understanding of the diffraction formulation compared to the plane computational procedure given by Eq. (17), because now we know the amplitudes, $B(\mathbf{k})$ of all 3D plane waves (the 3D spectrum) which superpose to form the object $\psi_0(x, y)$ and its diffraction $\psi_{z=z_0}(x, y)$; therefore, we know the entire field $\psi(\mathbf{x})$ as given by Eqs. (18), (19), or (20); and furthermore, using the developed $\delta_S(\mathbf{x})$ concept and the associated definitions, we are able to communicate, with great ease, that $B(\mathbf{k}) = \delta_S(\mathbf{k})A(\mathbf{k})$.

5. Conclusion

Though it is clear that the impulse functions (Dirac delta-functions) over lower-dimensional manifolds in $\mathbf{R}^N$ represent concentration (of mass) over a curve or a surface (including hypersurfaces for higher-dimensional spaces), how that concentration varies over that curve or surface is an issue. The proposed definition overcomes this problem by assuring uniform distribution per length (1D manifolds); the variation is uniform per area for 2D manifolds, and per unit geometric measure for higher-dimensional manifolds.

As a consequence of this definition, inner products of these functions and an arbitrary function is converted to a line or a surface integral. This observation, in the reverse direction, gives a tool to convert rather difficult surface integrals arising from the very nature of the problem to be handled as easier regular integrals involving the defined impulse functions at higher dimensions. This feature is demonstrated by modeling the 3D scalar optical wave propagation using the defined impulse functions. It is further shown that the well-known diffraction problem is a special case of the presented model. Therefore, we conclude that the adoption of these impulse functions, paves the way for solving rather more difficult diffraction problems.

References

[1] A.V. Oppenheim, A.S. Wilsky, S.H. Nawab, Signals and Systems, second ed., Prentice Hall, 1996, pp. 127–136.
[2] J.W. Goodman, Introduction to Fourier Optics, second ed., McGraw–Hill, 1996, pp. 393–395.
[3] I.M. Gel'fand, G.E. Shilov, Generalized Functions, vols. 1–5, Academic Press, 1964.
[4] P. Kanwal, Generalized Functions: Theory and Technique, second ed., Birkhäuser, 1998.
[5] A. Papoulis, Probability, Random Variables, and Stochastic Processes, McGraw–Hill, 1965, pp. 444–445.
[6] S. Mudaliar, Some properties of Green's function of waves at boundaries, J. Electromagn. Waves Appl. 16 (2002) 1661–1679.
[7] S. Hoffman, Advanced Calculus, Prentice Hall, 1970.