Some Open Problems Concerning Orthogonal Polynomials on Fractals and Related Questions

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Abstract

We discuss several open problems related to analysis on fractals: estimates of the Green functions, the growth rates of the Markov factors with respect to the extension property of compact sets, asymptotics of orthogonal polynomials and the Parreau-Widom condition, Hausdorff measures and the Hausdorff dimension of the equilibrium measure on generalized Julia sets.

1 Background and notation

1.1 Chebyshev and orthogonal polynomials

Let $K \subset \mathbb{C}$ be a compact set containing infinitely many points. We use $\| \cdot \|_{L^\infty(K)}$ to denote the sup-norm on $K$, $\mathcal{M}_n$ is the set of all monic polynomials of degree $n$. The polynomial $T_n(x)$ that minimizes $\|Q_n\|_{L^\infty(K)}$ for $Q_n \in \mathcal{M}_n$ is called the $n$-th Chebyshev polynomial on $K$.

Assume that the logarithmic capacity $\text{Cap}(K)$ is positive. We define the $n$-th Widom factor for $K$ by

$$W_n(K) := \|T_{n,K}\|_{L^\infty(K)/\text{Cap}(K)}^n.$$  

In what follows we consider probability Borel measures $\mu$ with non-polar compact support $\text{supp}(\mu)$ in $\mathbb{C}$. The $n$-th monic orthogonal polynomial $P_n(z; \mu) = z^n + \ldots$ associated with $\mu$ has the property

$$\|P_n(z; \mu)\|_{L^2(\mu)}^2 = \inf_{Q_n \in \mathcal{M}_n} \int |Q_n(z)|^2 \, d\mu(z),$$

where $\| \cdot \|_{L^2(\mu)}$ is the norm in $L^2(\mu)$. Then the $n$-th Widom-Hilbert factor for $\mu$ is

$$W_n^\mu(\mu) := \|P_n(z; \mu)\|_{L^2(\mu)}/(\text{Cap}(\text{supp}(\mu)))^n.$$

If $\text{supp}(\mu) \subset \mathbb{R}$ then a three-term recurrence relation

$$xP_n(x; \mu) = P_{n+1}(x; \mu) + b_{n+1}P_n(x; \mu) + a_n^2P_{n-1}(x; \mu)$$

is valid for $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The initial conditions $P_1(x; \mu) \equiv 0$ and $P_0(x; \mu) \equiv 1$ generate two bounded sequences $(a_n)_{n=1}^\infty, (b_n)_{n=1}^\infty$ of recurrence coefficients associated with $\mu$. Here, $a_n > 0, b_n \in \mathbb{R}$ for $n \in \mathbb{N}$ and

$$\|P_n(z; \mu)\|_{L^2(\mu)} = a_1 \cdots a_n.$$

A bounded two sided $\mathbb{C}$-valued sequence $(d_n)_{n=-\infty}^{\infty}$ is called \textit{almost periodic} if the set $\{d_n \in \mathbb{C} : n \in \mathbb{Z}\}$ is precompact in $l^\infty(\mathbb{Z})$. A one sided sequence $(c_n)_{n=1}^{\infty}$ is called almost periodic if it is the restriction of a two sided almost periodic sequence to $\mathbb{N}$. A sequence $(e_n)_{n=1}^{\infty}$ is called \textit{asymptotically almost periodic} if there is an almost periodic sequence $(e'_n)_{n=1}^{\infty}$ such that $|e_n - e'_n| \to 0$ as $n \to 0$.

The class of Parreau-Widom sets plays a special role in the recent theory of orthogonal and Chebyshev polynomials. Let $K$ be a non-polar compact set and $g_{C,K}$ denote the Green function for $\mathbb{C} \setminus K$ with a pole at infinity. Suppose $K$ is regular with respect to the Dirichlet problem, so the set $\mathcal{C}$ of critical points of $g_{C,K}$ is at most countable (see e.g. Chapter 2 in [9]). Then $K$ is said to be a \textit{Parreau-Widom set} if $\sum_{n=1}^{\infty} \delta_{C,K}(n) < \infty$. Parreau-Widom sets on $\mathbb{R}$ have positive Lebesgue measure. For different aspects of such sets, see [8, 15, 23].

The class of regular measures in the sense of Stahl-Totik can be defined by the following condition

$$\lim_{n \to \infty} W_n(\mu)^{1/n} = 1.$$

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For a measure $\mu$ supported on $\mathbb{R}$ we use the Lebesgue decomposition of $\mu$ with respect to the Lebesgue measure:

$$d\mu(x) = f(x)dx + d\mu_+(x).$$

Following [9], we define the Szegő class Sz($K$) of measures on a given Parreau-Widom set $K \subset \mathbb{R}$. Let $\mu_+$ be the equilibrium measure on $K$. By ess supp($\mu$) we denote the essential support of the measure, that is the set of accumulation points of the support. We have $\text{Cap}(\text{supp}(\mu)) = \text{Cap}(\text{ess supp}(\mu))$, see Section 1 of [21]. A measure $\mu$ is in the Szegő class of $K$ if

(i) $\text{ess supp}(\mu) = K$.
(ii) $\int \log f(x)d\mu_+(x) > -\infty$. (Szegő condition)
(iii) the isolated points $(x_n)$ of $\text{supp}(\mu)$ satisfy $\sum_n $ Cap$_{d,h}(x_n) < \infty$.

By Theorem 2 in [9] and its proof, (ii) can be replaced by one of the following conditions:

(ii') $\limsup_{s \to -\infty} W_2^2(\mu) > 0$. (Widom condition)
(ii'') $\liminf_{s \to -\infty} W_2^2(\mu) > 0$. (Widom condition 2)

One can show that any $\mu \in $ Sz($K$) is regular in the sense of Stahl-Totik.

1.2 Generalized Julia sets and $K(\gamma)$

Let $(f_n)_{n=0}^\infty$ be a sequence of rational functions with deg $f_n \geq 2$ in $\mathbb{C}$ and $F_n := f_n \circ f_{n-1} \circ \ldots \circ f_1$. The domain of normality for $(F_n)_{n=0}^\infty$ in the sense of Montel is called the Fatou set of $(f_n)$. The complement of the Fatou set in $\mathbb{C}$ is called the Julia set for $(f_n)$. We denote them by $F_{j(n)}$ and $J_{j(n)}$ respectively. These sets were considered first in [11]. In particular, if $f_n = f$ for some fixed rational function $f$ and $F_n$ is used instead. To distinguish the last case, the word autonomous is used in the literature.

Suppose $f_n(z) = \sum_{n=0}^N a_n \cdot z^n$ where $a_n \geq 2$ and $a_{n,k} \neq 0$ for all $n \in \mathbb{N}$. Following [7], we say that $(f_n)$ is a regular polynomial sequence (write $(f_n) \in Sz$) if positive constants $A_1,A_2,A_3$ exist such that for all $n \in \mathbb{N}$ we have the following three conditions:

$|a_{n,k}| \geq A_1$

$|a_{n,k}| \leq A_2 |a_{n,k-1}|$ for $j = 0,1,\ldots,d_n - 1$

For such polynomial sequences, by [7], $J_{j(n)}$ is a regular compact set in $\mathbb{C}$, so $\text{Cap}(J_{j(n)})$ is positive. In addition, $J_{j(n)}$ is the boundary of

$\Lambda_{j(n)}(\infty) := \{z \in \mathbb{C} : F_n(z) \text{ goes locally uniformly to } \infty\}$.

The following construction is from [12]. Let $\gamma := (\gamma_k)_{k=0}^\infty$ be a sequence provided that $0 < \gamma_k < 1/4$ holds for all $k \in \mathbb{N}$ and $\gamma_0 := 1$. Let $f_1(z) = 2(z - 1)/\gamma_1 + 1$ and $f_n(z) = \frac{1}{\gamma_n} (z^2 - 1) + 1$ for $n > 1$. Then $K(\gamma) := \cap_{n=0}^\infty F_n^{-1}([-1,1])$ is a Cantor set on $\mathbb{R}$. Furthermore, $F_1^{-1}([-1,1]) \subset F_2^{-1}([-1,1]) \subset [0,1]$ whenever $s > t$.

Also we use an expanded version of this set. For a sequence $\gamma$ as above, let $f_n(z) = \frac{1}{\gamma_n} (z^2 - 1) + 1$ for $n \in \mathbb{N}$. Then $K_1(\gamma) := \cap_{n=0}^\infty F_n^{-1}([-1,1]) \subset [-1,1]$ and $F_1^{-1}([-1,1]) \subset F_2^{-1}([-1,1]) \subset [-1,1]$ provided that $s > t$. If there is a $c$ with $0 < c < \gamma_k$ for all $k$ then $(f_n) \in Sz$ and $J_{j(n)} = K_1(\gamma)$, see [5]. If $\gamma_1 = \gamma_k$ for all $k \in \mathbb{N}$ then $K_1(\gamma)$ is an autonomous polynomial Julia set.

1.3 Hausdorff measure

A function $h : \mathbb{R} \to \mathbb{R}$ is called a dimension function if it is increasing, continuous and $h(0) = 0$. Given a set $E \subset \mathbb{C}$, its $h$-Hausdorff measure is defined as

$$\Lambda_h(E) = \liminf_{\delta \to 0} \left\{ \sum r_j : E \subset \bigcup B(z_j,r_j) \text{ with } r_j \leq \delta \right\},$$

where $B(z,r)$ is the open ball of radius $r$ centered at $z$. For a dimension function $h$, a set $K \subset \mathbb{C}$ is an $h$-set if $0 < \Lambda_h(K) < \infty$. To denote the Hausdorff measure for $h(t) = t^\alpha$, $\Lambda_\alpha$ is used. Hausdorff dimension of $K$ is defined as $\text{HD}(K) = \inf \{ \alpha \geq 0 : \Lambda_\alpha(K) = 0 \}$.

2 Smoothness of Green functions and Markov Factors

The next set of problems is concerned with the smoothness properties of the Green function $g_{C,K}$ near compact set $K$ and related questions. We suppose that $K$ is regular with respect to the Dirichlet problem, so the function $g_{C,K}$ is continuous throughout $\mathbb{C}$. The next problem was posed in [12].

**Problem 1.** Given modulus of continuity $\omega$, find a compact set $K$ such that the modulus of continuity $\omega(g_{C,K},\cdot)$ is similar to $\omega$.

Here, one can consider similarity either as coincidence of the values of moduli of continuity on some null sequence or in the sense of weak equivalence: $\exists C_1,C_2$ such that

$$C_1 \omega(\delta) \leq \omega(g_{C,K},\delta) \leq C_2 \omega(\delta)$$

for sufficiently small positive $\delta$.

We guess that a set $K(\gamma)$ from [12] is a candidate for the desired $K$ provided a suitable choice of the parameters. We recall that, for many moduli of continuity, the corresponding Green functions were given in [12], whereas the characterization of optimal smoothness for $g_{C,K}(\gamma)$ is presented in [[5], Th.6.3].
A stronger version of the above problem concerns with the pointwise estimation of the Green function:

**Problem 2.** Given modulus of continuity \( \omega \), find a compact set \( K \) such that

\[
C_1 \omega(\delta) \leq g_{C|K}(z) \leq C_2 \omega(\delta)
\]

for \( \delta = \text{dist}(z,K) \leq \delta_0 \), where \( C_1, C_2 \) and \( \delta_0 \) do not depend on \( z \).

In the most important case we get a problem of “two-sided Hölder” Green function, which was posed by A. Volberg on his seminar (quoted with permission):

**Problem 3.** Find a compact set \( K \) on the line such that for some \( \alpha > 0 \) and constants \( C_1, C_2 \), if \( \delta = \text{dist}(z,K) \) is small enough then

\[
C_1 \delta^\alpha \leq g_{C|K}(z) \leq C_2 \delta^\alpha.
\]

Clearly, a closed analytic curve gives a solution for sets on the plane.

If \( K \subset \mathbb{R} \) satisfies (1), then \( K \) is of Cantor-type. Indeed, if interior of \( K \) (with respect to \( \mathbb{R} \)) is not empty, let \( (a, b) \subset K \), then \( g_{C|K} \) has Lip 1 behavior near the point \((a+b)/2\). On the other hand, near endpoints of \( K \) the function \( g_{C|K} \) cannot be better than Lip 1/2.

By the Bernstein-Walsh inequality, smoothness properties of the Green functions are closely related with a character of maximal growth of polynomials outside the corresponding compact sets, which, in turn, allows to evaluate the Markov factors for the sets. Recall that, for a fixed \( n \in \mathbb{N} \) and (infinite) compact set \( K \), the \( n \)-th Markov factor \( M_n(K) \) is the norm of operator of differentiation in the space of holomorphic polynomials \( P_n \) with the uniform norm on \( K \). In particular, the Hölder smoothness (the right inequality in (1)) implies the Markov property of the set \( K \) (a polynomial growth rate of \( M_n(K) \)). The problem of inverse implication (see e.g. [20]) has attracted attention of many researches.

By W. Pleśniak [20], any Markov set \( K \subset \mathbb{R}^d \) has the extension property EP, which means that there exists a continuous linear extension operator from the space of Whitney functions \( \mathcal{S}(K) \) to the space of infinitely differentiable functions on \( \mathbb{R}^d \). We guess that there is some extremal growth rate of \( M_n \) which implies the lack of EP. Recently it was shown in [14] that there is no complete characterization of EP in terms of growth rate of the Markov factors. Namely, two sets were presented, \( K_i \) with EP and \( K_i \) without it, such that \( M_n(K_i) \) grows essentially faster than \( M_n(K_2) \) as \( n \to \infty \). Thus there exists non-empty zone of uncertainty where the growth rate of \( M_n(K) \) is not related with EP of the set \( K \).

**Problem 4.** Characterize the growth rates of the Markov factors that define the boundaries of the zone of uncertainty for the extension property.

### 3 Orthogonal polynomials

One of the most interesting problems concerning orthogonal polynomials on Cantor sets on \( \mathbb{R} \) is the character of periodicity of recurrence coefficients. It was conjectured in p.123 of [7] that if \( f \) is a non-linear polynomial such that \( J(f) \) is a totally disconnected subset of \( \mathbb{R} \) then the recurrence coefficients for \( \mu_{\gamma(f)} \) are almost periodic. This is still an open problem. In [6], the authors conjectured that the recurrence coefficients for \( \mu_{\gamma(f)} \) are asymptotically almost periodic for any \( \gamma \). It may be hoped that a more general and slightly weaker version of Bellissard’s conjecture can be valid.

**Problem 5.** Let \( (f_n) \) be a regular polynomial sequence such that \( J(f_n) \) is a Cantor-type subset of the real line. Prove that the recurrence coefficients for \( \mu_{\gamma(f_n)} \) are asymptotically almost periodic.

For a measure \( \mu \) which is supported on \( \mathbb{R} \), let \( Z_\mu : = \{ x : P_n(x; \mu) = 0 \} \). We define \( U_n(\mu) \) by

\[
U_n(\mu) : = \inf_{x \in \mathbb{R}^\nu} |x-x'|.
\]

In [17] Krüger and Simon gave a lower bound for \( U_n(\mu) \) depending on \( n \) where \( \mu \) is the Cantor-Lebesgue measure of the (translated and scaled) Cantor ternary set. In [16], it was shown that Markov’s inequality and spacing of the zeros of orthogonal polynomials are somewhat related.

Let \( \gamma = (\gamma_k)_{k=1}^\infty \) and \( n \in \mathbb{N} \) with \( n > 1 \) be given and define \( \delta_k = \gamma_0 \cdot \cdots \gamma_k \) for all \( k \in \mathbb{N}_0 \). Let \( s \) be the integer satisfying

\[
2s^{-1} \leq n < 2^s.
\]

By [2],

\[
\delta_{s+2} \leq U_n(\mu_{\gamma(s)}) \leq \frac{n^2}{4} \cdot \delta_{s+2}
\]

holds. In particular, if there is a number \( c \) such that \( 0 < c < \gamma_k < 1/4 \) holds for all \( k \in \mathbb{N} \) then, by [2], we have

\[
c^2 \cdot \delta_s \leq U_n(\mu_{\gamma(s)}) \leq \frac{n^2}{4c^2} \cdot \delta_s.
\]

By [13], at least for small sets \( K(\gamma) \), we have \( M_n(K(\gamma)) \sim 2/\delta_s \), where the symbol \( \sim \) means the strong equivalence.
**Problem 6.** Let $K$ be a non-polar compact subset of $\mathbb{R}$. Is there a general relation between the zero spacing of orthogonal polynomials for $\mu_K$ and smoothness of $g_{C,K}$? Is there a relation between the zero spacing of $\mu_K$ and the Markov factors?

As mentioned in section 1, the Szegö condition and the Widom condition are equivalent for Parreau-Widom sets. Let $K$ be a Parreau-Widom set. Let $\mu$ be a measure such that $\text{ess sup}(\mu) = K$ and the isolated points $\{x_n\}$ of $\text{sup}(\mu)$ satisfy $\sum_{n \in C \cup K} (x_n) < \infty$. Then, as it is discussed in Section 6 of [4], the Szegö condition is equivalent to the condition

$$\int_K \log(d\mu/d\mu_K) d\mu_K(x) > -\infty. \quad (3)$$

This condition is also equivalent to the Widom condition under these assumptions.

It was shown in [1] that $\sup_{n=1}^\infty W_n(\mu_K) \geq 1$ for non-polar compact $K \subset \mathbb{R}$. Thus the Szegö condition in the above form (3) and the Widom condition are related on arbitrary non-polar sets.

**Problem 7.** Let $K$ be a non-polar compact subset of $\mathbb{R}$ which is regular with respect to the Dirichlet problem. Let $\mu$ be a measure such that $\text{ess sup}(\mu) = K$. Assume that the isolated points $\{x_n\}$ of $\text{sup}(\mu)$ satisfy $\sum_{n \in C \cup K} (x_n) < \infty$. If the condition (3) is valid for $\mu$, is it necessarily true that the Widom condition or the Widom condition 2 holds? Conversely, does the Widom condition imply (3)?

It was proved in [10] that if $K$ is a Parreau-Widom set which is a subset of $\mathbb{R}$ then $(W_n(K))_{n=1}^\infty$ is bounded above. On the other hand, $(W_n(K))_{n=1}^\infty$ is unbounded for some Cantor-type sets, see e.g. [13].

**Problem 8.** Is it possible to find a regular non-polar compact subset $K$ of $\mathbb{R}$ which is not Parreau-Widom but $(W_n(K))_{n=1}^\infty$ is bounded? If $K$ has zero Lebesgue measure then it is true that $(W_n(K))_{n=1}^\infty$ is unbounded? We can ask the same problems if we replace $(W_n(K))_{n=1}^\infty$ by $(W_n^2(\mu_K))_{n=1}^\infty$ above.

Let $T_N$ be a real polynomial of degree $N$ with $N \geq 2$ such that it has $N$ real and simple zeros $x_1 < \cdots < x_N$, and $N - 1$ critical points $y_1 < \cdots < y_{N-1}$ with $|T_N(y_i)| \geq 1$ for each $i \in \{1, \ldots, N - 1\}$. We call such a polynomial admissible. If $K = T_N^{-1}([-1, 1])$ for an admissible polynomial $T_N$ then $K$ is called a $T$-set. The following result is well known, see e.g. [22].

**Theorem 3.1.** Let $K = \bigcup_{n=1}^\infty [\alpha_n, \beta_n]$ be a union of $n$ disjoint intervals such that $\alpha_1$ is the leftmost end point. Then $K$ is a $T$-set if and only if $\mu_K([\alpha_1, \beta_1])$ is in $\mathbb{Q}$ for all $c \in \mathbb{R} \setminus K$.

For $K(\gamma)$, it is known that $\mu_K([0, \gamma]) \in \mathbb{Q}$ if $c \in \mathbb{R} \setminus K(\gamma)$, see Section 4 in [2].

**Problem 9.** Let $K$ be a regular non-polar compact subset of $\mathbb{R}$ and $\alpha$ be the leftmost end point of $K$. Let $\mu_K([\alpha, c]) \in \mathbb{Q}$ for all $c \in \mathbb{R} \setminus K$. What can we say about $K$? Is it necessarily a polynomial generalized Julia set? Does this imply that there is a sequence of admissible polynomials $(f_n)_{n=1}^\infty$ such that $(f_n^{-1}[-1, 1])_{n=1}^\infty$ is a decreasing sequence of sets such that $K = \bigcap_{n=1}^\infty f_n^{-1}[-1, 1]$?

## 4 Hausdorff measures

It is valid for a wide class of Cantor sets that the equilibrium measure and the corresponding Hausdorff measure on this set are mutually singular, see e.g. [18].

Let $\gamma = (\gamma_k)_{k=1}^\infty$ with $0 < \gamma_k < 1/32$ satisfy $\sum_{k=1}^\infty \gamma_k < \infty$. This implies that $K(\gamma)$ has Hausdorff dimension $0$. In [3], the authors constructed a dimension function $h$, that makes $K(\gamma)$ an $h$-set. Provided also that $K(\gamma)$ is not polar it was shown that there is a $C > 0$ such that for any Borel set $B$,

$$C^{-1} \cdot \mu_{K(\gamma)}(B) < \Lambda_{\gamma_k}(B) < C \cdot \mu_{K(\gamma)}(B)$$

and in particular the equilibrium measure and $\Lambda_{\gamma_k}$, restricted to $K(\gamma)$ are mutually absolutely continuous. In [14], it was shown that indeed these two measures coincide. To the best of our knowledge, this is the first example of a subset of $\mathbb{R}$ such that the equilibrium measure is a Hausdorff measure restricted to the set.

**Problem 10.** Let $K$ be a non-polar compact subset of $\mathbb{R}$ such that $\mu_K$ is equal to a Hausdorff measure restricted to $K$. Is it necessarily true that the Hausdorff dimension of $K$ is 0?

Hausdorff dimension of a probability Borel measure $\mu$ supported on $\mathbb{C}$ is defined by $\text{dim}(\mu) := \inf\{\text{HD}(K) : \mu(K) = 1\}$ where $\text{HD}(\cdot)$ denotes Hausdorff dimension of the given set. For polynomial Julia sets which are totally disconnected there is a formula for $\text{dim}(\mu_{\text{Jul}(f)})$, see e.g.p. 23 in [18] and p.176-177 in [20].

**Problem 11.** Is it possible to find simple formulas for $\text{dim}(\mu_{\text{Jul}(f)})$ where $(f_n)$ is a regular polynomial sequence?

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