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On the discretization of Laine equations

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We consider the discretization of Darboux integrable equations. For each of the integrals of a Laine equation we constructed either a semi-discrete equation which has that integral as an n -integral, or we proved that such an equation does not exist. It is also shown that all constructed semi-discrete equations are Darboux integrable.

Keywords: Semi-discrete chain; Darboux integrability; x -integral, n -integral; discretization.

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1. Introduction

When considering hyperbolic type equations

$$u_{xy} = g(x, y, u, u_x, u_y) \quad (1.1)$$

one finds an important special subclass, so called Darboux integrable equations, that is described in terms of x - and y -integrals. Recall that a function $W(x, y, u, u_x, u_{xx}, \dots)$ is called a y -integral of equation (1.1) if $D_y W(x, y, u, u_x, \dots)|_{(1.1)} = 0$, where D_y represents the total derivative with respect to y (see [2] and [8]). An x -integral $\bar{W} = \bar{W}(x, y, u, u_y, u_{yy}, \dots)$ for equation (1.1) is defined in a similar way. Equation (1.1) is said to be Darboux integrable if it admits a nontrivial x -integral and a nontrivial y -integral.

The classification problem for Darboux integrable equations was considered by Goursat, Zhiber and Sokolov (see [2] and [8]). In his paper Goursat obtained a supposedly complete list of Darboux integrable equations of the form (1.1). A detailed discussion of the subject and corresponding references can be found in the survey [9].

Later Laine [7] published two Darboux integrable hyperbolic equations, which were absent in Goursat's list. The first equation found by Laine is

$$u_{xy} = u_x \left(\frac{\sqrt{u_y} + u_y}{u - y} + \frac{u_y}{u - x} \right). \quad (1.2)$$

It has a second order y -integral

$$W_1 = \frac{u_{xx}}{u_x} - \frac{1}{2}u_x \left(\frac{1}{u-y} + \frac{3}{u-x} \right) + \frac{1}{u-x} \quad (1.3)$$

and a third order x -integral

$$\begin{aligned} \bar{W} = & \left(u_{yyy} - \frac{u_{yy}^2}{2u_y} - u_{yy} \frac{1 + 5u_y^{\frac{1}{2}} + 4u_y}{u-y} \right) \left(u_{yy} - 2 \frac{u_y + 2u_y^{\frac{3}{2}} + u_y^2}{u-y} \right)^{-1} \\ & - \left(\frac{2u_y + 2u_y^{\frac{3}{2}} - 6u_y^2 - 10u_y^{\frac{5}{2}} - 4u_y^3}{(u-y)^2} \right) \left(u_{yy} - 2 \frac{u_y + 2u_y^{\frac{3}{2}} + u_y^2}{u-y} \right)^{-1}. \end{aligned} \quad (1.4)$$

The second equation found by Laine is

$$u_{xy} = 2 \left((u+X)^2 + u_x + (u+X) \sqrt{(u+X)^2 + u_x} \right) \left(\frac{\sqrt{u_y} + u_y}{u-y} - \frac{u_y}{\sqrt{(u+X)^2 + u_x}} \right). \quad (1.5)$$

It has a second order y -integral

$$\begin{aligned} W_2 = & \frac{u_{xx}}{2u_x} \left(1 - \frac{u+X}{\sqrt{(u+X)^2 + u_x}} \right) + u + \frac{(u+X)^2 + 2u_x}{\sqrt{(u+X)^2 + u_x}} \\ & - \frac{(u+X)^2 + u_x + (u+X) \sqrt{(u+X)^2 + u_x}}{u-y} \end{aligned} \quad (1.6)$$

and a third order x -integral (1.4). For the second equation Laine assumed X to be an arbitrary function of x . However Kaptsov (see [6]) has shown that X must be a constant function if equation (1.5) admits the integrals (1.6) and (1.4). Thus it can be assumed, without loss of generality, that $X = 0$.

One can also consider a semi-discrete analogue of Darboux integrable equations (see [1]). The notion of Darboux integrability for semi-discrete equations was developed by Habibullin (see [3]). For a function $t = t(n, x)$ of the continuous variable x and discrete variable n we introduce notations

$$t_k = t(n+k, x), \quad k \in \mathbb{Z}, \quad t_{[m]} = \frac{d^m}{dx^m} t(n, x), \quad m \in \mathbb{N}.$$

Then a hyperbolic type semi-discrete equation can be written as

$$t_{1x} = f(x, n, t, t_1, t_x). \quad (1.7)$$

A function F of variables x, n , and t, t_1, \dots, t_k is called an x -integral of equation (1.7) if $D_x F|_{(1.7)} = 0$. A function I of variables $x, n, t, t_{[1]}, \dots, t_{[m]}$ is called an n -integral of equation (1.7) if $DI|_{(1.7)} = I$, where D is a shift operator. Equation (1.7) is said to be Darboux integrable if it admits a nontrivial n -integral and a nontrivial x -integral. In what follows we consider the equalities $D_x F = 0$ and $DI = I$, which define x - and n -integrals F and I , only on solutions of the corresponding equations. For more information on semi-discrete Darboux integrable equations see [3], [4] and [5].

The interest in the continuous and discrete Darboux integrable models is stimulated by exponential type systems. Such systems are connected with semi-simple and affine Lie algebras which have applications in Liouville and conformal field theories.

The discretization of equations from Goursat's list was considered by Habibullin and Zheltukhina in [5]. In the present paper we find semi-discrete versions of Laine equations (1.2) and (1.5). In particular we find semi-discrete equations that admit functions (1.3) or (1.6) as n -integrals, and show that these equations are Darboux integrable. This is the main result of our paper given in Theorem 1.1 and Theorem 1.2 below.

Theorem 1.1. *The semi-discrete chain (1.7), which admits a minimal order n -integral*

$$I_1 = \frac{t_{xx}}{t_x} - \frac{1}{2}t_x \left(\frac{1}{t - \varepsilon(n)} + \frac{3}{t - x} \right) + \frac{1}{t - x}, \quad (1.8)$$

where $\varepsilon(n)$ is an arbitrary function of n , is

$$t_{1x} = t_x \frac{(t_1 - x)}{(t - x)} B(n, t, t_1), \quad (1.9)$$

where B is a function of n, t, t_1 , satisfying the following equation

$$(t_1 - \varepsilon)(t_1 - \varepsilon_1) - 2(t - \varepsilon)(t_1 - \varepsilon_1)B + (t - \varepsilon)(t - \varepsilon_1)B^2 = 0. \quad (1.10)$$

Moreover, chain (1.9) admits an x -integral of minimal order 3.

Theorem 1.2. *The semi-discrete chain (1.7), which admits a minimal order n -integral*

$$I_2 = \frac{t_{xx}}{2t_x} \left(1 - \frac{t}{\sqrt{t^2 + t_x}} \right) + t + \frac{t^2 + 2t_x}{\sqrt{t^2 + t_x}} - \frac{t^2 + t_x + t\sqrt{t^2 + t_x}}{t - \varepsilon(n)}, \quad (1.11)$$

where $\varepsilon(n)$ is an arbitrary function of n , is

$$t_{1x} = 2A(tA - t_1)\sqrt{t^2 + t_x} + A^2t_x + 2tA(tA - t_1), \quad (1.12)$$

where A is a function of n, t, t_1 , satisfying the following system of equations

$$\begin{cases} A_t = \frac{-2t_1(t_1 - \varepsilon_1)A + (-\varepsilon + 2t)(t_1 - \varepsilon_1)A^2 - \varepsilon_1(t - 2\varepsilon)A^3}{2(t_1 - \varepsilon_1)(t - \varepsilon)(t_1 - tA)}, \\ A_{t_1} = \frac{\varepsilon(t_1 - \varepsilon_1) + (t - \varepsilon)(2t_1 - \varepsilon_1)A - 2t(t - 2\varepsilon)A^2}{2(t_1 - \varepsilon_1)(t - \varepsilon)(t_1 - tA)}. \end{cases} \quad (1.13)$$

Moreover, chain (1.12) admits an x -integral of minimal order 2.

The paper is organized as follows. In Sections 2 and 3 we give proofs of Theorems 1.1 and 1.2 respectively. In Section 4 we show that function (1.4) can not be a minimal order n -integral for any equation (1.7).

2. Proof of Theorem 1.1

Discretization by n -integral: Let us find $f(x, n, t, t_1, t_x)$ such that $DI_1 = I_1$, where I_1 is defined by (1.8). Equality $DI_1 = I_1$ implies

$$\frac{f_x + f_t t_x + f_{t_1} f + f_{t_x} t_{xx}}{f} - \frac{f}{2} \left(\frac{1}{t_1 - \varepsilon_1} + \frac{3}{t_1 - x} \right) + \frac{1}{t_1 - x} = \frac{t_{xx}}{t_x} - \frac{t_x}{2} \left(\frac{1}{t - \varepsilon} + \frac{3}{t - x} \right) + \frac{1}{t - x}, \quad (2.1)$$

where $\varepsilon = \varepsilon(n)$ and $\varepsilon_1 = \varepsilon(n+1)$.

By comparing the coefficients before t_{xx} in (2.1), we get $\frac{f_{t_x}}{f} = \frac{1}{t_x}$, which implies that $f = A(x, n, t, t_1)t_x$. We substitute this expression for f in (2.1) and get

$$\frac{A_x + A_t t_x + A_{t_1} A t_x}{A} - \frac{A t_x}{2} \left(\frac{1}{t_1 - \varepsilon_1} + \frac{3}{t_1 - x} \right) + \frac{1}{t_1 - x} = -\frac{t_x}{2} \left(\frac{1}{t - \varepsilon} + \frac{3}{t - x} \right) + \frac{1}{t - x}. \quad (2.2)$$

The above equation is equivalent to a system of two equations

$$\begin{cases} \frac{A_x}{A} + \frac{1}{t_1 - x} = \frac{1}{t - x}, \\ \frac{A_t}{A} + A_{t_1} - \frac{A}{2} \left(\frac{1}{t_1 - \varepsilon_1} + \frac{3}{t_1 - x} \right) = -\frac{1}{2} \left(\frac{1}{t - \varepsilon} + \frac{3}{t - x} \right). \end{cases} \quad (2.3)$$

The first equation of system (2.3) can be written as $\frac{\partial}{\partial x} (\ln |A| - \ln |t_1 - x| + \ln |t - x|) = 0$ which implies that

$$A(x, n, t, t_1) = \frac{t_1 - x}{t - x} B(n, t, t_1) \quad (2.4)$$

for some function B of variables n, t, t_1 . Substituting expression (2.4) for A into the second equation of system (2.3), we get

$$-\frac{1}{t - x} + \frac{B_t}{B} + \frac{B}{t - x} + \frac{B_{t_1}(t_1 - x)}{t - x} - \frac{B(t_1 - x)}{2(t - x)} \left(\frac{1}{t_1 - \varepsilon_1} + \frac{3}{t_1 - x} \right) = -\frac{1}{2} \left(\frac{1}{t - \varepsilon} + \frac{3}{t - x} \right). \quad (2.5)$$

Thus

$$(t - x) \frac{B_t}{B} + (t_1 - x) B_{t_1} - \frac{B}{2} \left(1 + \frac{t_1 - x}{t_1 - \varepsilon_1} \right) = -\frac{1}{2} \left(1 + \frac{t - x}{t - \varepsilon} \right). \quad (2.6)$$

We compare the coefficients before x and x^0 in (2.6) and obtain

$$\begin{cases} -\frac{B_t}{B} - B_{t_1} + \frac{B}{2(t_1 - \varepsilon_1)} = \frac{1}{2(t - \varepsilon)}, \\ \frac{t B_t}{B} + t_1 B_{t_1} - \frac{B}{2} - \frac{t_1 B}{2(t_1 - \varepsilon_1)} = \frac{-1}{2} - \frac{t}{2(t - \varepsilon)}, \end{cases} \quad (2.7)$$

which is equivalent to

$$\begin{cases} B_t = \frac{B(\varepsilon - 2t + t_1 - \varepsilon B + tB)}{2(t - \varepsilon)(t - t_1)}, \\ B_{t_1} = \frac{-\varepsilon_1 + t_1 + \varepsilon_1 B + tB - 2t_1 B}{2(t_1 - \varepsilon_1)(t - t_1)}. \end{cases} \quad (2.8)$$

The last system is compatible, that is $B_{tt_1} = B_{t_1t}$, if and only if equality (1.10) is satisfied.

Existence of an x -integral: Let us show that equation (1.9) where function B satisfies (1.10) has a finite dimensional x -ring. We have,

$$t_{1x} = \frac{t_1 - x}{t - x} B t_x, \quad t_{2x} = \frac{t_2 - x}{t - x} B B_1 t_x, \quad \text{and} \quad t_{3x} = \frac{t_3 - x}{t - x} B B_1 B_2 t_x, \quad (2.9)$$

where $B = B(n, t, t_1)$, $B_1 = B(n + 1, t_1, t_2)$ and $B_2 = B(n + 2, t_2, t_3)$. We are looking for a function $F(x, n, t, t_1, t_2, t_3)$ such that $D_x F = 0$, that is

$$F_x + F_t t_x + F_{t_1} t_{1x} + F_{t_2} t_{2x} + F_{t_3} t_{3x} = 0. \quad (2.10)$$

Thus

$$F_x + F_t t_x + F_{t_1} \frac{t_1 - x}{t - x} B t_x + F_{t_2} \frac{t_2 - x}{t - x} B B_1 t_x + F_{t_3} \frac{t_3 - x}{t - x} B B_1 B_2 t_x = 0, \quad (2.11)$$

which is equivalent to

$$\begin{cases} F_x = 0, \\ (t - x)F_t + (t_1 - x)B F_{t_1} + (t_2 - x)B B_1 F_{t_2} + (t_3 - x)B B_1 B_2 F_{t_3} = 0. \end{cases} \quad (2.12)$$

By comparing the coefficients of x^0 and x in the last equality we get the following system

$$\begin{cases} tF_t + t_1 B F_{t_1} + t_2 B B_1 F_{t_2} + t_3 B B_1 B_2 F_{t_3} = 0, \\ -F_t - B F_{t_1} - B B_1 F_{t_2} - B B_1 B_2 F_{t_3} = 0. \end{cases} \quad (2.13)$$

After diagonalization this system becomes

$$\begin{cases} F_t + \frac{B B_1 (t_2 - t_1)}{t - t_1} F_{t_2} + \frac{B B_1 B_2 (t_3 - t_1)}{t - t_1} F_{t_3} = 0, \\ F_{t_1} + \frac{B_1 (t - t_2)}{t - t_1} F_{t_2} + \frac{B_1 B_2 (t - t_3)}{t - t_1} F_{t_3} = 0. \end{cases} \quad (2.14)$$

We introduce vector fields

$$\begin{aligned} V_1 &= \frac{\partial}{\partial t} + \frac{B B_1 (t_2 - t_1)}{t - t_1} \frac{\partial}{\partial t_2} + \frac{B B_1 B_2 (t_3 - t_1)}{t - t_1} \frac{\partial}{\partial t_3}, \\ V_2 &= \frac{\partial}{\partial t_1} + \frac{B_1 (t - t_2)}{t - t_1} \frac{\partial}{\partial t_2} + \frac{B_1 B_2 (t - t_3)}{t - t_1} \frac{\partial}{\partial t_3}. \end{aligned} \quad (2.15)$$

and $V = [V_1, V_2]$. Then, we have

$$\frac{2(t - t_1)^2}{B_1} V = (t_1 - t_2 + B(t_2 - t + (t - t_1)B_1)) \frac{\partial}{\partial t_2} + B_2(t_1 - t_3 + B(t_3 - t + (t - t_1)B_1 B_2)) \frac{\partial}{\partial t_3}.$$

Direct calculation show that

$$[V_1, V] = \frac{3\varepsilon - 4t + t_1}{2(\varepsilon - t)(t - t_1)}V \quad \text{and} \quad [V_2, V] = \frac{3\varepsilon_1 + t - 4t_1}{2(\varepsilon_1 - t_1)(t_1 - t)}V. \quad (2.16)$$

Hence vector fields V_1 , V_2 and V form a finite-dimensional ring. By the Jacobi Theorem the system of three equations $V_1(F) = 0$, $V_2(F) = 0$, $V(F) = 0$ has a nonzero solution $F(t, t_1, t_2, t_3)$. The function $F(t, t_1, t_2, t_3)$ is an x -integral of equation (1.9).

3. Proof of Theorem 1.2

Discretization by n -integral: Let us find a function $f(x, n, t, t_1, t_x)$ such that $DI_2 = I_2$, where I_2 is given by (1.11). The equality $DI_2 = I_2$ implies that

$$\begin{aligned} \frac{f_x + f_t t_x + f_{t_1} f + f_{t_x} t_{xx}}{2f} \left(1 - \frac{t_1}{\sqrt{t_1^2 + f}} \right) - \frac{t_1^2 + f + t_1 \sqrt{t_1^2 + f}}{t_1 - \varepsilon_1} + t_1 + \frac{t_1^2 + 2f}{\sqrt{t_1^2 + f}} \\ = \frac{t_{xx}}{2t_x} \left(1 - \frac{t}{\sqrt{t^2 + t_x}} \right) - \frac{t^2 + t_x + t \sqrt{t^2 + t_x}}{t - \varepsilon} + t + \frac{t^2 + 2t_x}{\sqrt{t^2 + t_x}}, \end{aligned} \quad (3.1)$$

where $\varepsilon = \varepsilon(n)$ and $\varepsilon_1 = \varepsilon(n+1)$. Comparing the coefficients before t_{xx} in equality (3.1), we get

$$\frac{f_{t_x}}{f} \left(1 - \frac{t_1}{\sqrt{t_1^2 + f}} \right) = \frac{1}{t_x} \left(1 - \frac{t}{\sqrt{t^2 + t_x}} \right). \quad (3.2)$$

This can be written as

$$\frac{\partial}{\partial t_x} \ln \left(f \frac{\sqrt{f + t_1^2} + t_1}{\sqrt{f + t_1^2} - t_1} \right) = \frac{\partial}{\partial t_x} \ln \left(t_x \frac{\sqrt{t_x + t^2} + t}{\sqrt{t_x + t^2} - t} \right). \quad (3.3)$$

Thus

$$\sqrt{f + t_1^2} + t_1 = (\sqrt{t_x + t^2} + t)A(x, n, t, t_1), \quad (3.4)$$

where A is some function of variables x , n , t and t_1 . The last equality is equivalent to

$$f = (2A^2 t - 2At_1) \sqrt{t_x + t^2} + A^2 t_x + t(2A^2 t - 2At_1). \quad (3.5)$$

We substitute f given by (3.5) into equality (3.1), use (3.4) and equality

$$\sqrt{f + t_1^2} - t_1 = \frac{f(\sqrt{t_x + t^2} - t)}{At_x}$$

to get

$$\frac{1}{\sqrt{t_x + t^2} \sqrt{f + t_1^2}} \left(\Lambda_1 t_x^2 + \Lambda_2 t_x \sqrt{t_x + t^2} + \Lambda_3 t_x + \Lambda_4 \sqrt{t_x + t^2} + \Lambda_5 t^2 \right) = 0, \quad (3.6)$$

where

$$\Lambda_i = \alpha_{i1} A_x + \alpha_{i2} A_t + \alpha_{i3} A_{t_1} + \alpha_{i4}, \quad 1 \leq i \leq 5 \quad (3.7)$$

and

$$\begin{aligned}\alpha_{11} &= 0, \alpha_{12} = 1, \alpha_{13} = A^2, \alpha_{14} = \frac{A}{t - \varepsilon} - \frac{A^3}{t_1 - \varepsilon_1}, \\ \alpha_{21} &= 0, \alpha_{22} = t - \frac{t_1}{A}, \alpha_{23} = -3t_1A + 3tA^2, \alpha_{24} = \frac{-t_1 + 2tA}{t - \varepsilon} + \frac{2t_1A^2 - 3tA^3}{t_1 - \varepsilon_1} + A^2 - A, \\ \alpha_{31} &= 1, \alpha_{32} = t^2, \alpha_{33} = 2t_1^2 + 5t^2A^2 - 6t_1tA, \\ \alpha_{34} &= \frac{-t_1t + t(t + 2\varepsilon)A}{t - \varepsilon} + \frac{-5t^2A^3 + 4t_1tA^2 - t_1^2A}{t_1 - \varepsilon_1} + t_1 + 2tA^2 - t_1A, \\ \alpha_{41} &= t - \frac{t_1}{A}, \alpha_{42} = 0, \alpha_{43} = 4t^3A^2 - 6t_1t^2A + 2t_1^2t, \\ \alpha_{44} &= \frac{2\varepsilon t^2A + \varepsilon t t_1}{t - \varepsilon} + \frac{-4t^3A^3 + 4t_1t^2A^2 - t_1^2tA}{t_1 - \varepsilon_1} + 2t^2A^2 - t_1tA, \\ \alpha_{51} &= 1, \alpha_{52} = 0, \alpha_{53} = 2t_1^2 + 4t^2A^2 - 6t_1tA, \\ \alpha_{54} &= \frac{-t_1t + 2\varepsilon t}{t - \varepsilon} + \frac{-4t^2A^3 + 4t_1tA^2 - t_1^2A}{t_1 - \varepsilon_1} + t_1 + 2tA^2 - t_1A.\end{aligned}$$

We can solve the overdetermined system of linear equations $\Lambda_i = 0, i = 1, 2 \dots 5$, with respect to A_x, A_t, A_{t_1} and obtain

$$\begin{cases} A_x = 0, \\ A_t = -\frac{A}{t - \varepsilon} + \frac{A^2}{2(t_1 - tA)} \left(\frac{A\varepsilon_1}{t_1 - \varepsilon_1} - \frac{\varepsilon}{t - \varepsilon} \right), \\ A_{t_1} = \frac{A}{t_1 - \varepsilon_1} - \frac{1}{2(t_1 - tA)} \left(\frac{A\varepsilon_1}{t_1 - \varepsilon_1} - \frac{\varepsilon}{t - \varepsilon} \right). \end{cases} \quad (3.8)$$

By direct calculations one can check that $A_{tt_1} = A_{t_1t}$, so the above system has a solution.

Existence of an x -integral: We are looking for a function $F(t, t_1, t_2)$ such that $D_x F = 0$ that is

$$F_t t_x + F_{t_1} t_{1x} + F_{t_2} t_{2x} = 0, \quad (3.9)$$

where t satisfies equation (1.7) with function f given by (3.5). We use

$$t_{1x} = A^2(t, t_1)t_x + 2A(t, t_1)(tA(t, t_1) - t_1)(\sqrt{t_x + t^2} + t)$$

and

$$\sqrt{f + t_1^2} = (\sqrt{t_x + t^2} + t)A - t_1,$$

to get

$$t_{2x} = A^2(t, t_1)A^2(t_1, t_2)t_x + 2(\sqrt{t_x + t^2} + t)(tA(t, t_1) - t_1)A(t, t_1)A^2(t_1, t_2) + \\ 2(\sqrt{t_x + t^2} + t)(t_1A(t_1, t_2) - t_2)A(t, t_1)A(t_1, t_2).$$

By substituting these expressions for t_{1x} and t_{2x} into equality (3.9) and comparing the coefficients of $\sqrt{t_x + t^2}$, t_x and t_x^0 , we obtain the following system of equations

$$\begin{cases} 2A(t, t_1)(tA(t, t_1) - t_1)F_{t_1} + 2A(t, t_1)A(t_1, t_2)(tA(t, t_1)A(t_1, t_2) - t_2)F_{t_2} = 0, \\ F_t + A^2(t, t_1)F_{t_1} + A^2(t, t_1)A^2(t_1, t_2)F_{t_2} = 0, \\ 2tA(t, t_1)(tA(t, t_1) - t_1)F_{t_1} + 2tA(t, t_1)A(t_1, t_2)(tA(t, t_1)A(t_1, t_2) - t_2)F_{t_2} = 0. \end{cases}$$

To check for the existence of a solution we transform the above system to its row reduced form

$$\begin{cases} F_t + \frac{A^2(t, t_1)A(t_1, t_2)(t_2 - t_1A(t_1, t_2))}{tA(t, t_1) - t_1}F_{t_2} = 0, \\ F_{t_1} + \frac{A(t_1, t_2)(t_2 - tA(t, t_1)A(t_1, t_2))}{-tA(t, t_1) + t_1}F_{t_2} = 0. \end{cases} \quad (3.10)$$

The corresponding vector fields

$$V_1 = \frac{\partial}{\partial t} + \frac{A^2(t, t_1)A(t_1, t_2)(t_2 - t_1A(t_1, t_2))}{tA(t, t_1) - t_1} \frac{\partial}{\partial t_2}, \\ V_2 = \frac{\partial}{\partial t_1} + \frac{A(t_1, t_2)(t_2 - tA(t, t_1)A(t_1, t_2))}{-tA(t, t_1) + t_1} \frac{\partial}{\partial t_2}$$

commute, that is $[V_1, V_2] = 0$, provided A satisfies system (3.8). Thus by the Jacobi theorem, system (3.10) has a solution. To solve the system define a function $E(t, t_1, t_2)$ by

$$E_t = \frac{A^2}{tA - t_1}, E_{t_2} = \frac{1}{A_1(t_1A_1 - t_2)}, E_{t_1} = \frac{t_2 - tAA_1}{(tA - t_1)(t_1A_1 - t_2)} + \frac{1}{t_1 - \varepsilon_1}E,$$

where $A = A(t, t_1)$ and $A_1 = A(t_1, t_2)$.

One can check that $E_{tt_1} = E_{t_1t}$ and $E_{t_1t_2} = E_{t_2t_1}$, so such a function E exists. Function E is a first integral of the first equation of system (3.10). We write system (3.10) using new variables

$$\tilde{t} = t, \tilde{t}_1 = t_1, \tilde{t}_2 = E(t, t_1, t_2)$$

and obtain

$$\begin{cases} \tilde{F}_{\tilde{t}} = 0 \\ \tilde{F}_{\tilde{t}_1} + \frac{\tilde{t}_2}{\tilde{t}_1 - \varepsilon_1} \tilde{F}_{\tilde{t}_2} = 0. \end{cases} \quad (3.11)$$

Therefore one of the x -integrals is $F(t, t_1, t_2) = E(t, t_1, t_2)/(t_1 - \varepsilon(n+1))$ where function E defined above.

4. Nonexistence of a chain (1.7) admitting the minimal order n -integral (1.4)

Let us find a function $f(x, n, t, t_1, t_x)$ such that equation (1.7) has the n -integral

$$I = \frac{t_{xxx} - \frac{t_{xx}^2}{2t_x} - t_{xx} \frac{1+5\sqrt{t_x}+4t_x}{t-x} - \frac{2t_x+2t_x\sqrt{t_x}-6t_x^2-10t_x^2\sqrt{t_x}-4t_x^3}{(t-x)^2}}{t_{xx} - \frac{2t_x+4t_x\sqrt{t_x}+2t_x^2}{t-x}}.$$

We have,

$$t_{1x} = f(x, n, t, t_1, t_x),$$

$$t_{1xx} = f_x + f_t t_x + f_{t_1} f + f_{t_x} t_{xx},$$

$$t_{1xxx} = (f_{xx} + f_{xt} t_x + f_{xt_1} f + f_{xt_x} t_{xx}) + t_x (f_{xt} + f_{tt} t_x + f_{tt_1} f + f_{tt_x} t_{xx}) + f_t t_{xx}$$

$$+ f(f_{xt_1} + f_{tt_1} t_x + f_{t_1 t_1} f + f_{t_1 t_x} t_{xx}) + f_{t_1} (f_x + f_t t_x + f_{t_1} f + f_{t_x} t_{xx})$$

$$+ t_{xx} (f_{xt_x} + f_{tt_x} t_x + f_{t_1 t_x} f + f_{t_x t_x} t_{xx}) + f_{t_x} t_{xxx}.$$

Equality $DI = I$ is equivalent to $J := L(DL)(DI - I) = 0$, where $L = \sqrt{2}t_x(t-x)\{t_{xx}(t-x) - 2t_x(\sqrt{t_x}+1)^2\}$. We have,

$$J = \Lambda_1 t_{xxx} + \Lambda_2 t_{xx}^3 + \Lambda_3 t_{xx}^2 + \Lambda_4 t_{xx} + \Lambda_5,$$

where Λ_k , $1 \leq k \leq 5$, are some functions of variables x, n, t, t_1, t_x . In particular,

$$\frac{\Lambda_1}{2(t-x)(t_1-x)t_x f} = 2(t-x)f(1+\sqrt{f})^2 - 2(t_1-x)t_x f_x(1+\sqrt{t_x})^2 - (t_1-x)(t-x)(f_x + f_t t_x + f_{t_1} f),$$

$$\Lambda_2 = (t-x)^2(t_1-x)^2\{ff_{t_x} - t_x f_{t_x}^2 + 2t_x f f_{t_x t_x}\},$$

$$\frac{\Lambda_3}{(t-x)(t_1-x)} = (t-x)f[4f^{3/2} + 2f^2 + (x-t_1)f_x + f(2+(x-t_1)f_{t_1})] + 10(x-t_1)t_x^{3/2} f f_{t_x}$$

$$+ t_x[10(t-x)f^{3/2} f_{t_x} + 2(t-x)(t_1-x)f_x f_x + 4(t-x)f^2(2f_{t_x} + (x-t_1)f_{t_1 t_x})]$$

$$+ t_x f(2(t-t_1)f_{t_x} + (t-x)(x-t_1)(3f_t + 4f_{xt_x}))$$

$$- 2(t_1-x)t_x^2[2f(2f_{t_x} - f_{t_x t_x} + (t-x)f_{tt_x}) + f_{t_x}(f_{t_x} + (x-t)f_t)]$$

$$- 4(f_{t_x}^2 - 2f f_{t_x t_x})(t_1-x)t_x^{5/2} - 2(f_{t_x}^2 - 2f f_{t_x t_x})(t_1-x)t_x^3.$$

Equality $\Lambda_2 = 0$ implies that $ff_{t_x} - t_x f_{t_x}^2 + 2t_x f f_{t_x t_x} = 0$, thus

$$\frac{f^2}{f_{t_x}} \frac{\partial}{\partial t_x} \left\{ \frac{t_x f_{t_x}^2}{f} \right\} = 0.$$

Hence, $\frac{t_x f_{t_x}^2}{f} = A^2(x, n, t, t_1)$ for some function A depending on x, n, t, t_1 only. Therefore, $\frac{f_{t_x}}{\sqrt{f}} = \frac{A}{\sqrt{t_x}}$ and hence $\frac{\partial}{\partial t_x} \{ \sqrt{f} - A\sqrt{t_x} = 0 \}$. We have,

$$\sqrt{f} = A\sqrt{t_x} + B$$

where $A = A(x, n, t, t_1)$ and $B = B(x, n, t, t_1)$. We substitute $f = A^2 t_x + 2AB\sqrt{t_x} + B^2$ into $\Lambda_1 = 0$ and get

$$\alpha_1 t_x^2 + \alpha_2 t_x^{3/2} + \alpha_3 t_x + \alpha_4 \sqrt{t_x} + \alpha_5 = 0.$$

We solve the system of equations $\alpha_k = 0, 1 \leq k \leq 5$, and obtain $B = 0$, that is

$$\begin{cases} A_x = \frac{B}{2A} B_t - \frac{3}{2} A B B_{t_1} + \frac{2(t_1 - x)B + A\{2(t - t_1) + 6(t - x)B + 3(t - x)B^2\}}{2(t - x)(x - t_1)}, \\ A_t = \frac{A}{2B} B_t + \frac{A^3}{2B} B_{t_1} + \frac{A\{2(t_1 - x)A + 2(x - t_1)B - (t - x)A^2(2 + B)\}}{2(t - x)(x - t_1)B}, \\ A_{t_1} = -\frac{1}{2AB} B_t - \frac{A}{2B} B_{t_1} + \frac{2(x - t_1) + (t - x)A(2 + 3B)}{2(t - x)(x - t_1)B}, \\ B_x = -B^2 B_{t_1} - \frac{B(1+B)^2}{t_1 - x}. \end{cases} \quad (4.1)$$

We substitute $f = A^2 t_x + 2AB\sqrt{t_x} + B^2$ into $\Lambda_3 = 0$ and get

$$\beta_1 t_x^3 + \beta_2 t_x^{5/2} + \beta_3 t_x^2 + \beta_4 t_x^{3/2} + \beta_5 t_x + \beta_6 \sqrt{t_x} + \beta_7 = 0.$$

We solve the system of equations $\beta_k = 0, 1 \leq k \leq 7$, and obtain $B = 0$, or

$$\begin{cases} A_x = \frac{3B}{8A} B_t - \frac{23}{24} A B B_{t_1} + \frac{21(t_1 - x)B + A\{16(t - t_1) + 51(t - x)B + 23(t - x)B^2\}}{24(t - x)(x - t_1)}, \\ A_t = \frac{3A}{8B} B_t + \frac{3A^3}{8B} B_{t_1} + \frac{A\{7(t_1 - x)A + 8(x - t_1)B - (t - x)A^2(7 + 3B)\}}{8(t - x)(x - t_1)B}, \\ A_{t_1} = -\frac{3}{8AB} B_t - \frac{3A}{8B} B_{t_1} + \frac{7(x - t_1) + (t - x)A(7 + 11B)}{8(t - x)(x - t_1)B}, \\ B_x = -B^2 B_{t_1} - \frac{B(1+B)^2}{t_1 - x}. \end{cases} \quad (4.2)$$

We equate expressions for A_x and A_t from (4.1) and (4.2) and find

$$\begin{cases} B_t = -\frac{A\{2(t_1 - x)B + A((t - t_1) + (t - x)B)\}}{2(t - x)(x - t_1)B}, \\ B_{t_1} = \frac{t - t_1 + 3(t - x)B + 2(t - x)B^2}{2(t - x)(x - t_1)B}. \end{cases} \quad (4.3)$$

Then, it follows from (4.1) that

$$\begin{cases} A_x = \frac{(t_1 - x + (t - x)A)B}{2(t - x)(x - t_1)}, \\ A_t = \frac{A((t_1 - x)A + (x - t)A^2 + 2(x - t_1)B)}{2(t - x)(x - t_1)B}, \\ A_{t_1} = \frac{x - t_1 + (t - x)A(1 + 2B)}{2(t - x)(x - t_1)B}, \\ B_x = \frac{B(t + t_1 - 2x + (t - x)B)}{2(t_1 - x)(x - t)}. \end{cases} \quad (4.4)$$

Equality $A_{tt_1} - A_{t_1t} = 0$ becomes $\frac{(t_1 - x)^2 - (t - x)^2 A^3}{(t - x)^2 (t_1 - x)^2 B} = 0$, thus

$$A^3 = \frac{(t_1 - x)^2}{(t - x)^2}. \quad (4.5)$$

Equality $A_{xt_1} - A_{t_1x} = 0$ becomes $\frac{-(t_1 - x)^2 + (t - x)^2 A(1 + B)^2}{(t - x)^2 (t_1 - x)^2 B} = 0$, thus

$$A(1 + B)^2 = \frac{(t_1 - x)^2}{(t - x)^2}. \quad (4.6)$$

Equality $A_{xt} - A_{tx} = 0$ becomes $\frac{(t_1 - x)^2 (A - B)^2 - (t - x)^2 A^3}{(t - x)^2 (t_1 - x)^2 B} = 0$. It implies that

$$\frac{A^3}{(A - B)^2} = \frac{(t_1 - x)^2}{(t - x)^2}, \quad (4.7)$$

or $A = B$, that leads to $A = B = 0$ and $f = 0$. It follows from (4.5) and (4.7) that $A - B = 1$ or $A - B = -1$. It follows from (4.5) and (4.6) that $1 + B = A$ or $1 + B = -A$. This gives rise to four possibilities:

- 1) $A - B = 1$;
- 2) $A - B = 1$ and $A + B = -1$ which gives $A = 0$, $B = -1$ and therefore $f = 1$;
- 3) $A - B = -1$ and $A - B = 1$ which is an inconsistent system;
- 4) $A - B = -1$ and $A + B = -1$ which gives $A = -1$, $B = 0$ and therefore $f = t_x$.

We have to study case 1) only. In this case we get $B = A - 1$ and equation $\sqrt{t_{1x}} = A\sqrt{t_x} + B$ becomes $\sqrt{t_{1x}} + 1 = A(\sqrt{t_x} + 1)$, that can be written as well as

$$(\sqrt{t_{1x}} + 1)^3 = A^3(\sqrt{t_x} + 1)^3. \quad (4.8)$$

Due to (4.5), our equation (4.8) becomes

$$\frac{(\sqrt{t_{1x}} + 1)^3}{(t_1 - x)^2} = \frac{(\sqrt{t_x} + 1)^3}{(t - x)^2}.$$

The last equation admits an n -integral $I = \frac{(\sqrt{t_x} + 1)^3}{(t - x)^2}$ of order one.

Let us consider case $B = 0$. We write $DI - I = 0$ for the chain $t_{1x} = C(x, n, t, t_1)t_x$ and get

$$\Lambda_1 t_{xxx} + \Lambda_2 t_{xx}^2 + \Lambda_3 t_{xx} + \Lambda_4 = 0$$

where $\Lambda_k = \Lambda_k(x, n, t, t_1, t_x)$, $1 \leq k \leq 4$. Equation $\Lambda_1 = 0$ implies

$$\alpha_1 t_x + \alpha_2 \sqrt{t_x} + \alpha_3 = 0$$

where $\alpha_k = \alpha_k(x, n, t, t_1)$, $1 \leq k \leq 3$. In particular, $\alpha_2 = 4C(-(t_1 - x) + (t - x)\sqrt{C})$. Since $\alpha_2 = 0$, we have $C = (t_1 - x)^2(t - x)^{-2}$. The chain becomes $t_{1x} = (t_1 - x)^2(t - x)^{-2}t_x$. It admits the n -integral $I = (t - x)^{-2}t_x$ of order one.

Therefore, if equation (1.7) admits n -integral (1.4) then (1.4) is not a minimal order integral.

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