Interpolation for Completely Positive Maps: Numerical Solutions
by
Călin Ambrozie(1) and Aurelian Gheondea(2)

Abstract

We present a few techniques to find completely positive maps between full matrix algebras taking prescribed values on given data, based on semidefinite programming, convex minimization supported by a numerical example, as well as representations by linear functionals. The particular case of commutative data is also discussed.

Key Words: Completely positive, interpolation, Choi matrix, quantum channel, semidefinite programming, convex minimization.

2010 Mathematics Subject Classification: 15B48, 15A72, 65K15, 81P45

1 Introduction

The present paper refers to a certain interpolation problem for completely positive maps that take prescribed values on given matrices, closely related to problems recently considered by C.-K. Li and Y.-T. Poon in [27], Z. Huang, C.-K. Li, E. Poon, and N.-S. Sze in [19], T. Heinosaari, M.A. Jivulescu, D. Reeb, and M.M. Wolf in [17] as well as G.M. D’Ariano and P. Lo Presti [13], D.S. Gonçalves et al. [16].

Let $M_n$ denote the $C^*$-algebra of all $n \times n$ complex matrices. In particular, positive elements (positive semidefinite matrices) in $M_n$ are defined. Recall that a matrix $A \in M_n$ is positive semidefinite if all its principal determinants are nonnegative. Let $M_n^+ \subset M_n$ denote the convex cone of all such matrices. Set $M_n^h := \{A \in M_n : A^* = A\}$ where $A^*$ denotes as usual the adjoint of $A$. A linear map $\varphi : M_n \rightarrow M_k$ is positive if $\varphi(M_n^+) \subset M_k^+$, namely it maps positive semidefinite matrices into positive semidefinite ones. Then $\varphi(A^*) = (\varphi(A))^*$ for every $A \in M_n$. We call $\varphi$ completely positive if $I_m \otimes \varphi : M_m \otimes M_n \rightarrow M_m \otimes M_k$ is positive for all $m \in \mathbb{N}$. An equivalent notion is that of positive semidefinite map, that is, for all $m \in \mathbb{N}$, all $h_1, \ldots, h_m \in \mathbb{C}^k$ and all $A_1, \ldots, A_m \in M_n$ we have $\sum_{i,j=1}^m \langle \varphi(A_i^*A_j)h_j, h_i \rangle \geq 0$. Let CP$(M_n, M_k)$ denote the convex cone of all completely positive maps $\varphi : M_n \rightarrow M_k$. If $\varphi : M_n \rightarrow M_k$ is completely positive then, cf. K. Kraus [23] and M.D. Choi [12], there are $n \times k$ matrices $V_1, V_2, \ldots, V_m$ with $m \leq nk$ such that

$$\varphi(A) = V_1^*AV_1 + V_2^*AV_2 + \cdots + V_m^*AV_m$$

for all $A \in M_n$ (1.1)

(and, of course, any map of the form (1.1) is completely positive). The representation (1.1) is called the Kraus representation of $\varphi$ and $V_1, \ldots, V_m$ are called the operation elements. The representation (1.1) of a given completely positive map $\varphi$ is non-unique, however the minimal number of the operation elements in the Kraus form of such a map $\varphi$ turns to be the rank of its Choi matrix $\Phi_\varphi$, cf. [12]. The next problem was raised by C.-K. Li and Y.-T.
Interpolation for Completely Positive Maps

Poon in [27], where a solution was given in case of commutative families of matrices \((A_\nu)_\nu, \quad (B_\nu)_\nu\).

**Problem A.** Given matrices \(A_\nu \in M_n\) and \(B_\nu \in M_k\) for \(\nu = 1, \ldots, N\), establish if there exist, and find \(\varphi \in \text{CP}(M_n, M_k)\) subject to the conditions

\[
\varphi(A_\nu) = B_\nu, \quad \text{for all } \nu = 1, \ldots, N. \tag{1.2}
\]

Other linear affine restrictions on \(\varphi\) may be added as well, like trace preserving etc. We can assume all \(A_\nu \in M_{n_\nu}, \quad B_\nu \in M_{k_\nu}\) (write also \(\varphi(A_\nu)^* = B_\nu^*\), replace \(A_\nu\) by \(A_\nu + A_\nu^*\) etc.).

In a first paper [5] we dealt with various necessary and/or sufficient conditions for the existence of solutions in terms of the density matrix, see our Theorems 2.4 and 2.5 in [5]. Most of results in this sense are related to Arveson’s Hahn-Banach type theorem [2] and various techniques of operator spaces, some of which being simplified in the present particular context by R.R. Smith and J.D. Ward [30]. In this article, we focus on methods, based on the Choi matrix \(\Phi_\varphi\), that can be numerically implemented in an efficient way.

The first step in our approach is to firstly derive, in Subsection 2.1, an equivalent formulation in terms of existence of certain positive semidefinite matrices subject to linear affine restrictions, like the matrix \(X (= \Phi_\varphi)\) in Problem B. If the trace preserving condition is added to Problem A, that is, if \(\varphi\) must be a quantum channel, this fits Problem B since the additional constrained is just another linear one. In Subsection 2.2 we remind a method to solve Problem B by known techniques of semidefinite programming. In Subsection 2.3 we present another technique for such problems, via results in [3] and based on convex minimization, namely (in case of Problem A with \(A_\nu^* = A_\nu\) and \(B_\nu^* = B_\nu\)) of the function \(\text{tr}(e^{\sum_\nu A_\tau^\nu \otimes X_\nu}) - \text{tr}\sum_\nu B_\nu X_\nu\) over all matrices \(X_\nu \in M_{k_\nu}\) where \(\tau\) denotes the transposition and "tr" stands for the trace, see Theorem 2.4, (a). A numerical example illustrating this technique is performed in Subsection 2.4. Finally, in Subsection 2.6 we show that, under the commutation assumptions, the semidefinite programming problem we obtain here turns into a linear programming one, hence explaining the results in [27] from our perspective.

Let us mention that the positive semidefinite approach to Problem A has been previously observed also in [13], [16] and [17], in different formulation. In particular, the feasibility of (1.2) in \(\text{CP}(M_n, M_k)\) was already known to be characterized [17] by the positivity of the functional \(\sum_\nu A_\nu^\nu \otimes X_\nu \mapsto \text{tr}\sum_\nu B_\nu X_\nu\) (statement recovered also by Theorem 2.4, (b)). With respect to the cited works, our main results by the convex minimization approach to Problem A in subsections 2.3 and 2.4, as well as other topics like propositions 2.7, (b) and 2.8, are new.

## 2 Main results

Consider then the interpolation problem (1.2) for the given matrices \(A_\nu \in M_n\) and \(B_\nu \in M_k\) where \(\nu = 1, \ldots, N\). Firstly, we will translate it in terms of Choi matrices.

### 2.1 Equivalent setting of the problem

Let \(\{e_i^{(n)}\}_{i=1}^n\) be the canonical basis of \(\mathbb{C}^n\) \((n \in \mathbb{N})\). As usual, the linear space \(M_{n,k}\) of all \(n \times k\) matrices is identified with the vector space \(\mathcal{B}(\mathbb{C}^k, \mathbb{C}^n)\) of all linear transformations \(\mathbb{C}^k \to \mathbb{C}^n\) \((n, k \in \mathbb{N})\). Let \(\{E_i^{(n,k)}\} \mid i = 1, \ldots, n, \quad m = 1, \ldots, k\} \subset M_{n,k}\) be the matrix units.
In what follows we use the description of the mapping \( \varphi \) matrix that induces an affine, order preserving, continuous bijection between closed convex cones \( \varphi \) and its inverse \( \varphi^{-1} \) with all entries 0 except for the \((i, m)\)-th entry which is 1. If \( \Phi = [\varphi(E_{ij}^{(n)})]_{i,j=1}^n \), defined \( \Phi \) of size \( C. Ambrozie, A. Gheondea \)

\[
\varphi E_{ij}^{(n)} = \sum_{k,l=1}^n \varphi_{r,s}^k \varphi_{r,s}^l E_{ij}^{(k)} \otimes E_{ij}^{(l)}.
\]

and define \( \ell = \{1, \ldots, n\} \) and \( \ell' = \{m, \ldots, k\} \) consisting of \( nk \) elements, we can write \( \Phi \in M_{nk} \) and \( s = 1, \ldots, nk \). Set

\[
A_{\nu} = [a_{\nu,i,j}]_{i,j=1}^n = \sum_{i,j=1}^n a_{\nu,i,j} E_{ij}^{(n)} \quad \text{and} \quad B_{\nu} = [b_{\nu,m,l}]_{m,l=1}^n = \sum_{m,l=1}^n b_{\nu,m,l} E_{ml}^{(k)}.
\]

Equate the \((m, l)\) entries in the equality \( \varphi(A_{\nu}) = B_{\nu} \) to get \( \langle \varphi(A_{\nu}) e_i^{(k)}, e_m^{(k)} \rangle = b_{\nu,m,l} \), that is,

\[
\langle \varphi(\sum_{i,j=1}^n a_{\nu,i,j} E_{ij}^{(n)}) e_i^{(k)}, e_m^{(k)} \rangle = b_{\nu,m,l}
\]

and so

\[
\sum_{i,j=1}^n a_{\nu,i,j} \varphi(i,m)(j,l) = b_{\nu,m,l}.
\]

Write the equality from above using Kronecker’s symbol \( \delta_{p,q} \) \((= 1 \text{ if } p = q \text{ and } 0 \text{ if } p \neq q)\) as \( \sum_{(j,l') \in \ell} a_{\nu,i,j} \delta_{i,m'} \varphi(i,m')(j,l') = b_{\nu,m,l} \) where \((j, l')\) and \((i, m')\) run \( \{1, \ldots, n\} \times \{1, \ldots, k\} \), then set

\[
c(\nu, m, l)(j, l')(i, m') := a_{\nu,i,j} \delta_{i,m'} \varphi(i,m') = (A_{\nu}^\dagger)_{j,i}(E_{t,lm}^{(k)})_{j,l'} = (A_{\nu}^\dagger \otimes E_{t,lm}^{(k)})_{(j,l')(i, m')} = A_{\nu}^\dagger \otimes E_{t,lm}^{(k)}
\]

and define

\[
C(\nu, m, l) = [c(\nu, m, l)(j, l')(i, m')]_{(j,l') \in \ell} = A_{\nu}^\dagger \otimes E_{t,lm}^{(k)}
\]

that can be represented as an \( nk \times nk \) matrix \( C(\nu, m, l) \in M_{nk} \)

\[
C(\nu, m, l) = A_{\nu}^\dagger \otimes E_{t,lm}^{(k)} = [a_{\nu,j,i} E_{t,lm}^{(k)}]_{i,j=1}^n
\]

via the linear, isometric, order-preserving isomorphisms of \( \ell \)-algebras \( M_{nk} \equiv M_n \otimes M_k = M_n(M_k) \). We obtain, using \((2.5)-(2.7), \sum_{(j,l') \in \ell} c(\nu, m, l)(j, l')(i, m') \varphi(i,m')(j,l') = b_{\nu,m,l}\), namely \( \text{tr} (C(\nu, m, l) \Phi) = b_{\nu,m,l} \), that by \((2.8)\) we can write as well

\[
\text{tr} [(A_{\nu}^\dagger \otimes E_{t,lm}^{(k)}) \Phi] = \text{tr} (A_{\nu} \otimes E_{t,lm}^{(k)}) = b_{\nu,m,l}.
\]
This actually is a particular application of the next formula, easily checked following the lines from above, letting \( A = (a_{ij})_{i,j=1}^n = \sum_{i,j=1}^n a_{ij} E_{i,j}^{(n)} \) etc.:

\[
\varphi(A) = \left[ \text{tr} \left[ (A^T \otimes E_{i,m}^{(k)}) \Phi \right] \right]_{m,l=1}^k = \left[ \text{tr} \left[ (A \otimes E_{m,l}^{(k)}) \Phi^* \right] \right]_{m,l=1}^k \quad (A \in M_n).
\]  

(2.10)

Note that we have as well the formula \( \varphi(A) = \left[ \text{tr} \left[ (E_{i,m}^{(k)} \otimes A) D_p \right] \Phi \right] \) where \( D_p \) denotes the density matrix [5], for which we also omit the details. Conditions (2.2) on \( \varphi \) are thus equivalent to the equations (2.9) from above concerning \( \Phi \), via the formulas (2.6), (2.7) and (2.2), (2.3). Denote by \( i = (\nu, m, l) \) the generic triple consisting of arbitrary \( \nu = 1, \ldots, N \) and \( m, l = 1, \ldots, k \). Thus \( i \) runs a set of \( q := Nk^2 \) elements. We may write \( i = 1, \ldots, q \). Set also \( p = nk \). Write \( C(i) = C(\nu, m, l) \) \( (\in M_p) \) and \( b_i = b_{\nu, m, l} \); denote \( \Phi \) by \( X \), below.

With these notations, via (2.4) Problem A takes then the equivalent form from below.

**Problem B** Given \( C(i) \in M_p \) and numbers \( b_i \) \( (1 \leq i \leq q) \), find \( X \in M_p, X \geq 0 \), such that

\[
\text{tr} (C(i) X) = b_i \quad \text{for all } i = 1, \ldots, q.
\]  

(2.11)

Thus, the solvability of Problem A leads to the rather known topic of finding positive semidefinite matrices subject to linear affine conditions and, in particular, establishes whether such matrices do exist. These questions often occur and are dealt with by reliable numerical methods in the semidefinite programming, a few elements of which we sketch in what follows.

### 2.2 Solutions by means of semidefinite programming

Firstly, using \( \text{tr}(c^*) = \overline{\text{tr}(c)} = \text{tr}(dc) = \text{tr}(cd) \) and writing equation (2.11) in terms of \( C(i) + C(i)^* + i(C(i) - C(i)^*) \) we can assume all matrices \( C(i) \) to be selfadjoint. We can suppose, without loss of generality, that they are linearly independent over \( \mathbb{R} \). **Semidefinite programming** is concerned with minimization of linear functionals subject to the constraint that an affine combination of selfadjoint matrices is positive semidefinite: see in this sense [7], [9], [24], [26], [31], also [10], [15]. Roughly speaking, one sets \( a(x) = \sum x_i C(i) + a_0 \) for the given \( C(i) \) and a selfadjoint matrix \( a_0 \) that can be suitably chosen, here. Define then \( p^* = \inf \{ \sum b_i x_i : a(x) \geq 0 \} \) and \( q^* = \inf \{ -\text{tr}(a_0 X) : X \geq 0, \text{tr}(C(i) X) = b_i \forall i \} \).

A problem dual to (2.11) occurs now with respect to \( p^* \), namely to establish if there exist positive definite matrices of the form \( a(x) \). Standard algorithms exist to this aim, based on maximizing the minimal eigenvalue of \( a(x) \) in the variables \( x = (x_i)_i \), or on interior point methods using barrier functions [26]. In the case when either (2.11) has solutions \( X > 0 \), or the dual problem has solutions \( x \) with \( a(x) > 0 \), we have \( p^* = q^* \), see for instance [26], [31]. If both conditions hold, the optimal sets for \( p^* \) and \( q^* \) are nonempty. In this case for every \( \lambda \in (p^*, \overline{p}) \) there is a unique vector \( x^* = (x^*_i)_i \), the *analytic center* of this linear matrix inequality, such that \( a(x^*) > 0 \), \( \sum b_i x^*_i = \lambda \) and \( x^* \) minimizes the *logarithmic barrier function* \( \ln \det a(x)^{-1} \) over all \( x \) with \( \sum b_i x_i \) and \( a(x) > 0 \). It follows by the Lagrange multipliers method that \( \text{tr} (C(i) a(x^*)^{-1}) = \lambda b_i \forall i \), which gives a solution \( X = X_\star \) of (2.11), namely \( X_\star = \lambda^{-1} a(x^*)^{-1} \).

### 2.3 Solutions via a convex minimization technique

We show how to obtain solutions to Problems A, B by minimizing a certain convex function. Let \( C(i) \in M_p \) denote now arbitrary selfadjoint and linearly independent matrices and \( b_i \).
Theorem 2.1. [3]. The system of equations \( \text{tr}(C(t)X) = b_i \) \((i = 1, \ldots, q)\) admits solutions \( X > 0 \) if and only if the function \( V \) defined by (2.12) has a critical point (of minimum), that is, if and only if \( \lim_{x \to \infty} V(x) = +\infty \). In this case, if \( x^0 = (x_i^0) \), denotes the critical point of \( V \), we have also the (positive) particular solution

\[
X_0 = e^{\sum_i x_i^0 C(i)}.
\] (2.13)

Remark 2.2. (a) The function \( V \) given by (2.12) fulfills the conditions of application of the method of the conjugate gradients [11]. This yields, if problem (2.11) has solutions \( X > 0 \), a minimizing sequence of vectors \( x = (x_i) \), that is convergent to the critical point \( x^0 \) and so provides approximations \( \tilde{X}_0 = e^{\sum_i x_i C(i)} \approx X_0 \) of the solution (2.13), see Example 12 and Remark 11 in [3]. The gradient \( \nabla V = (\partial V/\partial x_i)_{i=1}^q \) of \( V \) is easily computed to this aim by

\[
\frac{\partial V}{\partial x}(x) = \text{tr} \left( C(k) e^{\sum_i x_i C(i)} \right) - b_i, \quad \text{see [3]}.\]

Various versions of Newton’s method that can be used as well to approximate \( x^0 \).

(b) In the particular case of our Problem A and for selfadjoint data, after replacing the matrices \( C = C(\nu, m, l) = A^*_{\nu} \otimes E_{l,m} \) by the symmetrized ones \( C + C^* \) etc., as described at the beginning of Subsection 2.3, setting \( X_\nu = [x_{\nu,m,l} + \pi_{\nu,l,m}]_{l,m=1}^k \in M_k \), the function \( V \) becomes

\[
V((x_\nu)_\nu) = \text{tr} \left( e^{\sum_{\nu=1}^N A^*_{\nu} \otimes X_\nu} \right) - \sum_{\nu} B_\nu X_\nu
\] (2.14)

since in this case \( \sum_i x_i C(i) = \sum_{\nu} A^*_{\nu} \otimes X_\nu \) and \( \sum_i b_i x_i = \sum_{\nu} B_\nu X_\nu \). Differentiating the right hand side of (2.14) obviously shows now that a critical point \( (X_\nu)_\nu \) provides a solution \( \varphi \) of (1.2) via formula (2.10) in which \( \Phi = \Phi^* = e^{\sum_{\nu} A_{\nu} \otimes X_\nu} \).

(c) A test [3] exists to check if the system \( \text{tr}(C(t)X) = b_i \) has no solutions \( X \geq 0 \) at all, too, namely \( \inf V = -\infty \) in which case the minimization provides, after some iterations, vectors \( x = (x_i) \), such that \( \sum_i x_i C(i) \geq 0 \) but \( \sum_i b_i x_i < 0 \) (see also Proposition 2.7.(a)). This also has a corresponding consequence for Problem A.

Remark 2.3. All approximations from a small neighborhood of the critical point of \( V \) can provide exact solutions \( X \) to Problem B as follows: fix a linear affine projection \( \pi \) onto the linear submanifold of \( M_k^h \) defined by the equations (2.11) and \( X^* = X \), then for an approximation \( \tilde{X}_0 \) of \( X_0 \) use \( X := \pi \tilde{X}_0 \) instead of \( \tilde{X}_0 \); since \( \tilde{X}_0 \approx X_0 \), \( X = \pi \tilde{X}_0 \approx \pi X_0 = X_0 \) and so \( X \approx X_0 (> 0) \) which implies \( X > 0 \) if the approximation \( \tilde{X}_0 \approx X_0 \) was good enough.

We summarize below the main topics stated in this Subsection.
Theorem 2.4. Let $A_{\nu} \in M_n$ and $B_{\nu} \in M_k$ for $\nu = 1, \ldots, N$ be selfadjoint matrices. Let $\mathbb{V}$ be the (strictly convex, smooth, real) function defined by

$$\mathbb{V}(X_{\nu}) = \text{tr}(e^{\sum_{\nu=1}^{N} A_{\nu} \otimes X_{\nu}}) - \text{tr} \sum_{\nu} B_{\nu} X_{\nu}$$

on the real linear space of all sets $(X_{\nu})_{\nu=1}^{N}$ consisting of selfadjoint matrices $X_{\nu} \in M_k$.

(a) The following two statements are equivalent:

(i) Problem $A$ has at least one solution $\varphi$ from the (dense) interior of $\text{CP}(M_n, M_k)$;

(ii) Function $\mathbb{V}$ has a (unique) critical point, namely its infimum $\inf \mathbb{V}$ is finite and attained.

In this case the critical point $(X_{\nu}^0)_{\nu}$ provides, by formula (2.10), a particular solution $\varphi_0$ of (1.2) whose Choi matrix $\Phi = \Phi_{\varphi_0}$ is given by $\Phi = e^{\sum_{\nu} A_{\nu} \otimes (X_{\nu}^0)^T} \, (> 0)$. Moreover, all close enough sets $(X_{\nu})_{\nu}$ provide solutions $\varphi$ with $\Phi_{\varphi} = \pi(e^{\sum_{\nu} A_{\nu} \otimes (X_{\nu})^T})$ as in Remark 2.3.

(b) The lack of any solution $\varphi \in \text{CP}(M_n, M_k)$ of (1.2) is characterized by the condition $\inf \mathbb{V} = -\infty$, that is equivalent also to the existence of sets $(X_{\nu})_{\nu}$ (some of which being always obtained when minimizing $\mathbb{V}$) such that $\sum_{\nu} A_{\nu} \otimes X_{\nu} \geq 0$ and $\text{tr} \sum_{\nu} B_{\nu} X_{\nu} < 0$.

For (a), use Theorem 2.1 together with Remark 2.2.(b). Note to this aim that the mapping (2.4) is one-one between the interiors of the closed convex cones $\text{CP}(M_n, M_k)$ and $M_{nk}^+$. For (b), use the results in [3] mentioned in Remark 2.2.(c).

2.4 A Numerical Example

We show how Theorem 2.4 applies to Problem A. Suppose one wishes to find $\varphi : M_2 \to M_2$ completely positive such that $\varphi(A_{\nu}) = B_{\nu}$ ($\nu = 1, 2$) for $A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

and $B_1 = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} 3.5 & 1.5 \\ 1.5 & 2.5 \end{bmatrix}$. Use to this aim the minimization method indicated by Remark 2.2. Formulas (2.6), (2.7) and (2.8) provide the matrices $C(i)$ for $i = (\nu, m, l)$ where $\nu, m, l = 1, 2$. Due to the symmetry equation (2.5) (or, equivalently, (2.11)) is equivalent to $\sum_{i=1}^{m} \pi_{\nu, i} \pi_{j, l}(i, m) = \pi_{\nu, j l}$ (or $\text{tr}(C(i)^* \Phi) = \pi_{\nu, j l}$), and so it is enough to consider (2.5) for those couples $(\nu, l)$ with $m \leq l$. That is, for each $\nu = 1, 2$ we have 3 equations corresponding to $(\nu, l) = (1, 1), (1, 2), (2, 2)$. The set $\{1, 2\} \times \{1, 2\}$ of indices $r, s$ like $(j, l), (j, l'), (i, m), (i, m')$ with $1 \leq i, j \leq n (= 2)$ and $1 \leq m, m', l, l' \leq k (= 2)$ from below is ordered lexicographically as $\{(1, 1), (1, 2), (2, 1), (2, 2)\} \equiv \{1, 2, 3, 4\}$. We represent the positive matrix $X = [a_{\nu, \beta}]_{\nu, \beta \in \{1, 2, 3, 4\}} \equiv \Phi = [\varphi_{r, s}]_{r, s}$ that we seek for and the given matrices $C(\nu, m, l)$ as follows: $C(1, 1, 1) = A_1^* \otimes E_{111}$, $C(1, 1, 2) = A_1^* \otimes E_{2, 1}$, etc. Numerically minimizing $\mathbb{V}$ gave us the matrix $\tilde{X}_0 \approx X_0$,

$$\tilde{X}_0 = \begin{bmatrix} 1.54993756 & -0.1694804138 & 0.4499571618 & 0.4047411695 \\ -0.1694804138 & 0.153477390 & -0.06572393508 & -0.1533566973 \\ 0.4499571618 & -0.06572393508 & 0.5249880063 & 0.6652436210 \\ 0.4047411695 & -0.1533566973 & 0.6652436210 & 1.326699194 \end{bmatrix}$$

which is positive and approximately satisfies (2.11). Let $\varphi$ be the map whose Choi matrix $\Phi = \Phi_{\varphi} = [\varphi_{r, s}]_{r, s}$ is $\tilde{X}_0 \equiv \Phi$. We got then an approximate solution to our present particular case of problem (1.2), namely $\varphi(A) = \left( \sum_{i, j=1}^{2} \Phi_{(i, m)(j, j)} a_{i, j} \right)_{m, l=1}^2 = \left( \text{tr}(A^* \otimes E_{lm}^{(2)}) \tilde{X}_0 \right)_{m, l=1}^2$.
for every $A = [a_{i,j}]_{i,j=1}^2 \in M_2$, see formula (2.10) (or (2.9)). For instance, we have
\[
\varphi(A_1)_{1,1} = \text{tr}((C(1,1,1)\tilde{X}_0) = 2(\tilde{X}_0)_{1,1} + (\tilde{X}_0)_{1,3} + (\tilde{X}_0)_{3,1} = 3.99978984 \approx 4 = (B_1)_{1,1},
\]
and $\varphi(A_1)_{1,2} = \text{tr}\left((A_1^* \otimes E_{2,1}^{(2)})\tilde{X}_0\right) = \text{tr}(C(1,1,2)\tilde{X}_0) = 0.0000564069 \approx 0 = (B_1)_{1,2}$, etc.

**Remark 2.5.** Semidefinite programming methods [26], [31] (or related, as in Subsection 2.4) can solve problems of type B for more sizeable dimensions $p (= nk)$ and $q (= Nk^2)$, and so allow to consider Problem A as well for somewhat larger integers $n, k, N$. The difficulty is to compute the (trace of the) exponential for the matrix $\sum x_i C(i) \in M_p$ (or its determinant, in the semidefinite programming approach) required for $V$ and $\nabla V$. Common techniques to this aim like eigenvalue decomposition require a time of order $p^3$. We need just linear functionals of this exponential, for which stochastic methods exist requiring a time $\sim p^2$, but they are better only for very large $p$ and provide rougher approximations [25]. See also [8], [22] for quantitative and software topics on the practicability range of the present methods.

**Remark 2.6.** Another question is to reduce the number of operation elements in the representation (1.1) of the solution $\varphi$ of (1.2), whenever possible — that is, to minimize the rank of $X$ in (2.11). The case of one term for instance corresponds to solutions $X \geq 0$ of rank one, namely to the existence of vectors $v \in \mathbb{C}^{nk}$ such that $\langle C(i)v, v \rangle = b_i$ for all $i$. A first easy step to rank reduction is to find the joint support $P$ of the symmetrizations $C = C(i) + C(i)^*$ etc. of the $\{C(i)\}$’s and consider solutions $X$ such that $X = PXP$. Indeed, set $K = \{h \in \mathbb{C}^{nk} : Ch = 0 \forall C\}$. Let $P$ be the orthogonal projection onto $K^\perp$. Then $C = CP$ and so $C = C^* = PC = PCP$, whence $\text{tr}(CX) = \text{tr}(CPX) \geq \text{tr}(CXP)$. Generally the question is difficult, for some possibilities of reducing the rank of $X$ see for instance Section II.13 in [7], or [6], [29].

### 2.5 Characterization in terms of linear functionals

By Theorem 2.5 from [5] or [30]), the solvability of (2.11) is described in terms of the map $
abla x_i C(i) \mapsto b_i x_i$. We recall this result in the form below, completed with a version (b) concerning the existence of strictly positive solutions; for the sake of completeness we sketch also the proof.

**Proposition 2.7.** Suppose that $C(i) \in M_p$ ($i = 1, \ldots, q$) are selfadjoint, linearly independent and their linear span contains strictly positive matrices. Then:

(a) There exist solutions $X \geq 0$ of the system of equations (2.11) if and only if $\sum x_i C(i) \geq 0$ for all $(x_i)$, such that $\sum x_i C(i) \geq 0$, namely, $\inf_{\sum x_i C(i) \geq 0} \sum b_i x_i \geq 0$.

(b) There exist solutions $X > 0$ of the system of equations (2.11) if and only if $\sum x_i C(i) > 0$ for all $(x_i)$, $\not= 0$ such that $\sum x_i C(i) \geq 0$, namely, $\inf_{\sum x_i C(i) \geq 0} \sum b_i x_i > 0$.

**Proof.** (a) Assume that $\inf_{\sum x_i C(i) \geq 0} \sum b_i x_i \geq 0$. The intersection of the closed convex cone of all positive semidefinite $p \times p$ matrices and the linear span $S$ of the $\{C(i)\}$’s contains a point that is interior to the cone, namely a positive matrix. The linear functional $l : \sum x_i C(i) \mapsto \sum b_i x_i$ is well defined, and $\geq 0$ on this intersection. By Mazur’s theorem, see for instance [1], [21], it has a linear extension $L$ to the space $M_p^n$ of all selfadjoint matrices in $M_p$, such that $LY \geq 0$ for all $Y \geq 0$ in $M_p^n$. Now $L$ has the form $LY = \text{tr}(XY)$ for some $X \in M_p^n$. Letting $Y = \langle \cdot, h \rangle h$ for an arbitrary vector $h \in \mathbb{CP}$ gives $\langle Xh, h \rangle \geq 0$. 


Hence $X \geq 0$. Since $L|_{S} = l$, for every $i$ we have $C(i) \in S$ and $\text{tr}(C(i)X) = LC(i) = lC(i) = b_i$. Conversely, suppose that there exists an $X \geq 0$ such that $\text{tr}(C(i)X) = b_i$ for all $i$. Then for every $(x_i)$, such that $\sum x_iC(i) \geq 0$, we have $\sum b_ix_i = \sum \text{tr}(C(i)X)x_i = \text{tr}(X \sum iC(i)) = \text{tr}(X^{1/2} \sum_i x_iC(i)X^{1/2}) \geq 0$.

(b) Assume that $\inf_{x: \sum x_iC(i) \geq 0, \|x\|=1} \sum b_ix_i > 0$ ($\|\|$ is any norm). Proceed as above, except we need the following fact: given a finite dimensional real space $M$, a linear subspace $S$ and a closed convex cone $C \subset M$ such that $C \cap (-C) = \{0\}$, any linear functional $l$ on $S$ such that $ls > 0$ for all $s \neq 0$ from $S \cap C$ has a linear extension $L$ to $M$ such that $Lm > 0$ for all $m \neq 0$ from $C$. This is a known consequence of the Hahn-Banach, Mazur type theorems, see for instance [4]. The necessity follows as in the case (a). \[\Box\]

2.6 The case of commutative data

In this case Problem A was shown by Theorem 2.1 in [27] to be equivalent to solving a system of nonhomogeneous linear equations in nonnegative variables. This result can be explained from the present perspective, too. Firstly, by the commutativity assumption we can suppose without loss of generality that all $A_\nu, B_\nu$ are diagonal. For any matrix $u = [u_{ij}]_{i,j}$, set $\tilde{u} = [\delta_{ij}u_{ij}]_{i,j}$. Then by Proposition 2.8, in the equations (2.9) we can replace any positive semidefinite $\Phi$ (= $C_\varphi$ for a solution $\varphi$ of (1.2)) by the (positive semidefinite) diagonal matrix $\tilde{\Phi}$ (= $C_\psi$ for another solution $\psi$). This leads indeed to the problem of finding $nk$ nonnegative numbers, namely the diagonal entries of $\tilde{\Phi}$, satisfying the system of $Nk$ equations: $\text{tr}[(A_\nu \otimes E^{(k)}_{m,l})\tilde{\Phi}] = B_{\nu,l,m}$ for $1 \leq \nu \leq N$ and $1 \leq l \leq k$.

Proposition 2.8. If $A_\nu, B_\nu$ are diagonal, $X \in M_{nk}$ and $\text{tr}[(A_\nu \otimes E^{(k)}_{m,l})X] = B_{\nu,l,m}$, then we have also the equality $\text{tr}[(A_\nu \otimes E^{(k)}_{m,l})\tilde{X}] = B_{\nu,l,m}$ (both terms of which are null if $m \neq l$).

Proof. Represent $X \in M_{nk} \equiv M_n \otimes M_k$ as $X = \sum Y_\mu \otimes Z_\mu$ with $Y_\mu \in M_n$ and $Z_\mu \in M_k$. Using the easily checked formula $u \otimes v = \tilde{u} \otimes \tilde{v}$, we obtain $\tilde{X} = \sum \tilde{Y}_\mu \otimes \tilde{Z}_\mu$. Hence, the equality in the conclusion holds for $l \neq m$ by inserting $\tilde{X}$ in the left hand side, then using the formula $\text{tr}(u \otimes v) = \text{tr}(u)\text{tr}(v)$ and the equalities $\text{tr}(E^{(k)}_{m,l}Z_\mu) = 0$, $B_{\nu,l,m} = 0$. To prove it also for $l = m$, use again $\text{tr}(u \otimes v) = \text{tr}u\text{tr}v$ to write the desired conclusion in the form $\sum \text{tr}(A_\nu\tilde{Y}_\mu)Z_{\mu,l,l} = B_{\nu,l,l}$. This is equivalent, by means of the equalities $A_\nu = A_\nu$ and the formula $\text{tr}(\tilde{u}\tilde{v}) = \text{tr}(\tilde{u}\tilde{v})$, to $\text{tr}[(A_\nu \otimes E^{(k)}_{m,l})Y_\mu \otimes Z_\mu] = B_{\nu,l,l}$, that is the case $l = m$ of (2.9) and so holds true by hypotheses. \[\Box\]

Acknowledgements. First named author partially supported by grants RVO:67985840. Both authors supported by a grant of the Romanian National Authority for Scientific Research, CNCSIS UEFISCDI, project number PN-II-ID-PCE-2011-3-0119.

References


Received: 30.01.2017
Revised: 13.03.2017
Accepted: 19.03.2017

(1) Institute of Mathematics of the Romanian Academy  
PO Box 1-764, RO 014700 Bucharest, Romania  
and  
Institute of Mathematics of the Czech Academy  
Zitna 25, 11567 Prague 1, Czech Republic  
E-mail: Calin.Ambrozie@imar.ro

(2) Department of Mathematics, Bilkent University  
06800 Bilkent, Ankara, Turkey  
and  
Institute of Mathematics of the Romanian Academy  
PO Box 1-764, RO 014700 Bucharest, Romania  
E-mail: aurelian@fen.bilkent.edu.tr  
A.Gheondea@imar.ro