COMPLETE INTERSECTION MONOMIAL CURVES AND THE COHEN-MACAULAYNESS OF THEIR TANGENT CONES

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Abstract. Let $C(n)$ be a complete intersection monomial curve in the 4-dimensional affine space. In this paper we study the complete intersection property of the monomial curve $C(n + wv)$, where $w > 0$ is an integer and $v \in \mathbb{N}^4$. Also we investigate the Cohen-Macaulayness of the tangent cone of $C(n + wv)$.

1. Introduction

Let $n = (n_1, n_2, \ldots, n_d)$ be a sequence of positive integers with $\gcd(n_1, \ldots, n_d) = 1$. Consider the polynomial ring $K[x_1, \ldots, x_d]$ in $d$ variables over a field $K$. We shall denote by $x^n$ the monomial $x_1^{n_1} \cdots x_d^{n_d}$ of $K[x_1, \ldots, x_d]$, with $u = (u_1, \ldots, u_d) \in \mathbb{N}^d$. The toric ideal $I(n)$ is the kernel of the $K$-algebra homomorphism $\phi : K[x_1, \ldots, x_d] \to K[t]$ given by

$$\phi(x_i) = t^{n_i} \text{ for all } 1 \leq i \leq d.$$

Then $I(n)$ is the defining ideal of the monomial curve $C(n)$ given by the parametrization $x_1 = t^{n_1}, \ldots, x_d = t^{n_d}$. The ideal $I(n)$ is generated by all the binomials $x^u - x^v$, where $u = v$ runs over all vectors in the lattice $\ker_0(n_1, \ldots, n_d)$ see for example, [16, Lemma 4.1]. The height of $I(n)$ is $d - 1$ and also equals the rank of $\ker_0(n_1, \ldots, n_d)$ (see [16]). Given a polynomial $f \in I(n)$, we let $f_*$ be the homogeneous summand of $f$ of degree $d$. We shall denote by $I(n)_*$ the ideal in $K[x_1, \ldots, x_d]$ generated by the polynomials $f_*$ for $f \in I(n)$.

Deciding whether the associated graded ring of the local ring $K[[t^{n_1}, \ldots, t^{n_d}]]$ is Cohen-Macaulay constitutes an important problem studied by many authors, see for instance [1], [6], [14]. The importance of this problem stems partially from the fact that if the associated graded ring is Cohen-Macaulay, then the Hilbert function of $K[[t^{n_1}, \ldots, t^{n_d}]]$ is non-decreasing. Since the associated graded ring of $K[[t^{n_1}, \ldots, t^{n_d}]]$ is isomorphic to the ring $K[x_1, \ldots, x_d]/I(n)_*$, the Cohen-Macaulayness of the associated graded ring can be studied as the Cohen-Macaulayness of the ring $K[x_1, \ldots, x_d]/I(n)_*$. Recall that $I(n)_*$ is the defining ideal of the tangent cone of $C(n)$ at 0.

The case that $K[[t^{n_1}, \ldots, t^{n_d}]]$ is Gorenstein has been particularly studied. This is partly due to the M. Rossi’s problem [13] asking whether the Hilbert function of a Gorenstein local ring of dimension one is non-decreasing. Recently, A. Oneto, F. Strazzanti and G. Tamone [12] found many families of monomial curves giving negative answer to the above problem. However M. Rossi’s problem is still open for a Gorenstein local ring $K[[t^{n_1}, \ldots, t^{n_d}]]$. It is worth to note that, for a complete intersection monomial curve $C(n)$ in the 4-dimensional affine space (i.e. the ideal $I(n)$ is a complete intersection), we have, from [14, Theorem 3.1], that if the minimal number of generators for $I(n)_*$ is three or four, then $C(n)$ has

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homogeneous elements in $I$.

He showed that there exists a positive integer $N$ such that, for all $w > N$, the Betti numbers of $I(n + wv)$ are periodic in $w$ with period $n_d - n_1$. The conjecture was proved by T. Vu [18]. More precisely, he showed that there exists a positive integer $N$ such that, for all $w > N$, the Betti numbers of $I(n + wv)$ are periodic in $w$ with period $n_d - n_1$. The bound $N$ depends on the Castelnuovo-Mumford regularity of the ideal generated by the homogeneous elements in $I(n)$. For $w > (n_d - n_1)^2 - n_1$ the minimal number of generators for $I(n + w(1, 1, 1))$ is periodic in $w$ with period $n_d - n_1$ (see [4]). Furthermore, for every $w > (n_d - n_1)^2 - n_1$ the monomial curve $C(n + w(1, 1, 1))$ has Cohen-Macaulay tangent cone at the origin, see [15]. The next example provides a monomial curve $C(n + w(1, 1, 1))$ which is not a complete intersection for every $w > 0$.

**Example 1.1.** Let $n = (15, 25, 24, 16)$, then $I(n)$ is a complete intersection on the binomials $x_1^2 - x_2^2$, $x_3^2 - x_4^2$ and $x_1x_2 - x_3x_4$. Consider the vector $v = (1, 1, 1, 1)$. For every $w > 85$ the minimal number of generators for $I(n + wv)$ is either 18, 19 or 20. Using CoCoA ([3]) we find that for every $0 < w < 85$ the minimal number of generators for $I(n + wv)$ is greater than or equal to 4. Thus for every $w > 0$ the ideal $I(n + wv)$ is not a complete intersection.

Given a complete intersection monomial curve $C(n)$ in the 4-dimensional affine space, we study (see Theorems 2.6, 3.2) when $C(n + wv)$ is a complete intersection. We also construct (see Theorems 2.8, 2.9, 3.4) families of complete intersection monomial curves $C(n + wv)$ with Cohen-Macaulay tangent cone at the origin.

Let $a_i$ be the least positive integer such that $a_i n_i \in \sum_{j \neq i} \mathbb{N} n_j$. To study the complete intersection property of $C(n + wv)$ we use the fact that after permuting variables, if necessary, there exists (see [14, Proposition 3.2] and also Theorems 3.6 and 3.10 in [10]) a minimal system of binomial generators $S$ of $I(n)$ of the following form:

(A) $S = \{x_1^{-a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4}, x_1^{a_1} x_2^{-a_2} x_3^{a_3} x_4^{a_4}, x_1^{a_1} x_2^{a_2} x_3^{-a_3} x_4^{a_4}, x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{-a_4}\}.$

(B) $S = \{x_1^{-a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4}, x_1^{a_1} x_2^{-a_2} x_3^{a_3} x_4^{a_4}, x_1^{a_1} x_2^{a_2} x_3^{-a_3} x_4^{a_4}, x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{-a_4}\}.$

In section 2 we focus on case (A). We prove that the monomial curve $C(n)$ has Cohen-Macaulay tangent cone at the origin if and only if the minimal number of generators for $I(n)$, is either three or four. Also we explicitly construct vectors $v_i$, $1 \leq i \leq 22$, such that for every $w > 0$, the ideal $I(n + wv_i)$ is a complete intersection whenever the entries of $n + wv_i$ are relatively prime. We show that if $C(n)$ has Cohen-Macaulay tangent cone at the origin, then for every $w > 0$ the monomial curve $C(n + wv_1)$ has Cohen-Macaulay tangent cone at the origin whenever the entries of $n + wv_1$ are relatively prime. Additionally we show that there exists a non-negative integer $w_0$ such that for all $w \geq w_0$, the monomial curves $C(n + wv_0)$ and $C(n + wv_{13})$ have Cohen-Macaulay tangent cones at the origin whenever the entries of the corresponding sequence $(n + wv_0)$ for the first family and $n + wv_{13}$ for the second are relatively prime. Finally we provide an infinite family of complete intersection monomial curves $C_m(n + wv_1)$ with corresponding local rings having non-decreasing Hilbert functions, although their tangent cones are not Cohen-Macaulay, thus giving a positive partial answer to M. Rossi’s problem.

In section 3 we study the case (B). We construct vectors $b_i$, $1 \leq i \leq 22$, such that for every $w > 0$, the ideal $I(n + wb_i)$ is a complete intersection whenever
the entries of $n + w_1b_1$ are relatively prime. Furthermore we show that there exists a non-negative integer $w_1$ such that for all $w \geq w_1$, the ideal $I(n + wb_{22})$, is a complete intersection whenever the entries of $n + wb_{22}$ are relatively prime.

2. The case (A)

In this section we assume that after permuting variables, if necessary, $S = \{x_1^{a_1} - x_2^{a_2}, x_3^{a_3} - x_4^{a_4}, x_1^{a_1}, x_2^{a_2} - x_3^{a_3} - x_4^{a_4}\}$ is a minimal generating set of $I(n)$. First we will show that the converse of [14, Theorem 3.1] is also true in this case.

Let $n_1 = \min\{n_1, \ldots, n_4\}$ and also $a_3 < a_4$. By [6, Theorem 7] a monomial curve $C(n)$ has Cohen-Macaulay tangent cone if and only if $x_1^n$ is not a zero divisor in the ring $K[x_1, \ldots, x_4]/I(n)$. Hence if $C(n)$ has Cohen-Macaulay tangent cone at the origin, then $I(n)_+: (x_1) = I(n)$. Without loss of generality we can assume that $a_3 < a_4$. In case that $u_2 > a_2$ we can write $u_2 = ga_2 + h$, where $0 \leq h < a_2$. Then we can replace the binomial $x_1^{u_1}x_2^{u_2} - x_3^{u_3}x_4^{u_4}$ in $S$ with the binomial $x_1^{u_1+ga_1}x_2^{h} - x_3^{u_3}x_4^{u_4}$. Without loss of generality we can also assume that $u_3 \leq a_3$.

**Theorem 2.1.** Suppose that $u_3 > 0$ and $u_4 > 0$. Then $C(n)$ has Cohen-Macaulay tangent cone at the origin if and only if the ideal $I(n)_+$ is either a complete intersection or an almost complete intersection.

**Proof.** $(\Longleftarrow)$ If the minimal number of generators of $I(n)_+$ is either three or four, then $C(n)$ has Cohen-Macaulay tangent cone at the origin.

$(\Longrightarrow)$ Let $f_1 = x_1^{u_1} - x_2^{u_2}$, $f_2 = x_3^{u_3} - x_4^{u_4}$, $f_3 = x_1^{u_1}x_2^{u_2} - x_3^{u_3}x_4^{u_4}$. We distinguish the following cases

(1) $u_2 < a_2$. Note that $x_1^{u_1+u_4} - x_2^{u_2}x_3^{u_3} - x_4^{u_4} \in I(n)$. We will show that $a_2 + u_2 \leq u_1 + u_2 + a_3 - u_3$. Suppose that $u_2 + u_2 + a_3 - u_3 < a_2 + u_4$, then $x_2^{u_2}x_3^{u_3} - x_4^{u_4} \in I(n)_+: (x_1)$ and therefore $x_2^{u_2}x_3^{u_3} - x_4^{u_4} \in I(n)$. Since $\{f_1, f_2, f_3\}$ is a generating set of $I(n)$, the monomial $x_2^{u_2}x_3^{u_3} - x_4^{u_4}$ is divided by at least one of the monomials $x_2^{u_2}$ and $x_3^{u_3}$. But $u_2 < a_2$ and $a_3 - u_3 < a_3$, so $a_3 < u_3 < a_1 + u_4 < u_1 + u_2 + a_3 - u_3$. Let $G = \{f_1, f_2, f_3, f_4 = x_2^{u_2}x_3^{u_3} - x_1^{u_1}x_2^{u_2}x_3^{u_3}\}$. We will prove that $G$ is a standard basis for $I(n)$ with respect to the negative degree reverse lexicographical order with $x_3 > x_4 > x_2 > x_1$. Note that $u_2 + u_3 < u_1 + u_2$, since $u_3 + u_4 \leq u_1 + u_3 + a_3 - a_4$ and also $a_3 - a_4 < 0$. Thus $LM(f_3) = x_2^{u_2}x_3^{u_3}$. Furthermore $LM(f_1) = x_2^{u_2}$, $LM(f_2) = x_3^{u_3}$ and $LM(f_4) = x_2^{u_2}x_3^{u_3}$. Therefore $NF(spoly(f_1, f_2)(G)) = 0$ as $LM(f_1)$ and $LM(f_2)$ are relatively prime, for $(i, j) \in \{(1, 2), (1, 3), (1, 4), (2, 4)\}$. We compute $spoly(f_2, f_3) = -f_4$, so $NF(spoly(f_2, f_3)(G)) = 0$. Next we compute $spoly(f_3, f_4) = x_1^{a_1}x_2^{u_2}x_3^{u_3} - x_1^{a_1}x_2^{u_2}x_3^{u_3}$. Then $LM(spoly(f_3, f_4)) = x_1^{a_1}x_2^{u_2}x_3^{u_3}$ and only $LM(f_2)$ divides $LM(spoly(f_3, f_4))$. Also $ecart(spoly(f_3, f_4)) = a_4 - a_3 = ecart(f_2)$. Then $spoly(f_2, spoly(f_3, f_4)) = 0$ and $NF(spoly(f_3, f_4)(G)) = 0$. By [8, Lemma 5.5.11] $I(n)_+$ is generated by the least homogeneous summands of the elements in the standard basis $G$. Thus the minimal number of generators for $I(n)_+$ is least than or equal to 4.

(2) $u_2 = a_2$. Note that $x_1^{u_1+u_2} - x_2^{u_2}x_3^{u_3} - x_4^{u_4} \in I(n)$. We will show that $a_2 + u_2 \leq u_1 + a_1 + a_3 - u_3$. Clearly the above inequality is true when $u_3 = a_3$. Suppose that $u_3 < a_3$ and $u_1 + a_1 + u_3 < u_4 + u_4$, then $x_3^{u_3} - x_4^{u_4} \in I(n)_+: (x_1)$ and therefore $x_3^{u_3} - x_4^{u_4} \in I(n)_+$. Thus $x_3^{u_3} - x_4^{u_4}$ is divided by $x_2^{u_2}$, a contradiction. Consequently $a_4 + u_4 \leq u_1 + a_1 + a_3 - u_3$. We will prove that $H = \{f_1, f_2, f_3 = x_1^{a_1+u_1} - x_3^{u_3}, f_6 = x_1^{a_1} - x_1^{u_1+u_2} - x_1^{a_1}x_3^{u_3-4}\}$ is a standard basis for $I(n)$ with respect to the negative degree reverse lexicographical order with $x_3 > x_4 > x_2 > x_1$. Here $LM(f_1) = x_2^{u_2}$, $LM(f_2) = x_3^{u_3}$, $LM(f_3) = x_0^{a_1}x_3^{u_3}$ and $LM(f_6) = x_4^{u_4}$. Therefore
By \([8, \text{Lemma 5.5.11}]\) and LM\((f_j)\) are relatively prime, for \((i, j) \in \{(1, 2), (1, 5), (1, 6), (2, 6)\}\). We compute spoly\((f_2, v_3) = -f_6\), therefore NF\((\text{spoly}(f_2, f_3)\}) = 0. Furthermore spoly\((f_2, f_6) = x_1^{u_1+v_1}x_3^{2} - x_1^{u_1+v_1}x_4^{2}\) and also LM\((\text{spoly}(f_5, f_6)) = x_1^{u_1+v_1}x_3^{2}\). Only LM\((f_2)\) divides LM\((\text{spoly}(f_2, f_3))\) and ecart\((\text{spoly}(f_2, f_6)) = a_4 - a_3 = \text{ecart}(f_2)\). Then spoly\((f_2, \text{spoly}(f_5, f_6)) = 0\) and therefore NF\((\text{spoly}(f_5, f_6)\}) = 0. By \([8, \text{Lemma 5.5.11}]\) \(I(\mathfrak{n})_v\) is generated by the least homogeneous summands of the elements in the standard basis \(H\). Thus the minimal number of generators for \(I(\mathfrak{n})_v\) is least than or equal to 4. \(\square\)

**Corollary 2.2.** Suppose that \(u_3 > 0\) and \(u_4 > 0\).

1. Assume that \(u_3 < 2\). Then \(C(\mathfrak{n})\) has Cohen-Macaulay tangent cone at the origin if and only if \(a_4 + u_3 \leq v_1 + u_2 + a_3 - u_3\).

2. Assume that \(u_3 = 2\). Then \(C(\mathfrak{n})\) has Cohen-Macaulay tangent cone at the origin if and only if \(a_4 + u_4 \leq v_1 + a_1 + a_3 - u_3\).

**Theorem 2.3.** Suppose that either \(u_3 = 0\) or \(u_4 = 0\). Then \(C(\mathfrak{n})\) has Cohen-Macaulay tangent cone at the origin if and only if the ideal \(I(\mathfrak{n})_v\) is a complete intersection.

**Proof.** It is enough to show that if \(C(\mathfrak{n})\) has Cohen-Macaulay tangent cone at the origin, then the ideal \(I(\mathfrak{n})_v\) is a complete intersection. Suppose first that \(u_3 = 0\). Then \(\{f_1 = x_1^{u_1}, f_2 = x_3^{2} - x_4^{2}, f_3 = x_4^{2} - x_1^{2}\}\) is a minimal generating set of \(I(\mathfrak{n})\). If \(u_2 = a_2\), then \(\{f_1, f_2, x_3^{2} - x_1^{2+v_1}\}\) is a standard basis for \(I(\mathfrak{n})\) with respect to the negative degree reverse lexicographical order with \(x_3 > x_4 > x_2 > x_1\). By \([8, \text{Lemma 5.5.11}]\) \(I(\mathfrak{n})_v\) is a complete intersection. Assume that \(u_2 < a_2\). We will show that \(u_4 \leq u_1 + u_2\). Suppose that \(u_4 > u_1 + u_2\), then \(x_2^{u_2} \in I(\mathfrak{n})_v : (x_1)\) and therefore \(x_2^{u_2} \in I(\mathfrak{n})_v\). Thus \(x_2^{u_2}\) is divided by \(x_2^{2}\), a contradiction. Then \(\{f_1, f_2, f_3\}\) is a standard basis for \(I(\mathfrak{n})_v\) with respect to the negative degree reverse lexicographical order with \(x_3 > x_4 > x_2 > x_1\). Note that LM\((f_1) = x_3^{2}, \text{LM}(f_2) = x_3^{2}\) and LM\((f_3) = x_4^{2}\). By \([8, \text{Lemma 5.5.11}]\) \(I(\mathfrak{n})_v\) is a complete intersection. Suppose now that \(u_4 = 0\), so necessarily \(u_3 = a_3\). Then \(\{f_1, f_2, f_3 = x_4^{2} - x_1^{2}\}\) is a minimal generating set of \(I(\mathfrak{n})\). If \(u_2 = a_2\), then \(\{f_1, f_2, x_4^{2} - x_1^{2+v_1}\}\) is a standard basis for \(I(\mathfrak{n})_v\) with respect to the negative degree reverse lexicographical order with \(x_3 > x_4 > x_2 > x_1\). Thus, from \([8, \text{Lemma 5.5.11}]\), \(I(\mathfrak{n})_v\) is a complete intersection. Assume that \(u_2 < a_2\), then \(a_4 \leq u_1 + u_2\) and also \(f_1, f_2, f_3\) is a standard basis for \(I(\mathfrak{n})_v\) with respect to the negative degree reverse lexicographical order with \(x_3 > x_4 > x_2 > x_1\). From \([8, \text{Lemma 5.5.11}]\) we deduce that \(I(\mathfrak{n})_v\) is a complete intersection. \(\square\)

**Remark 2.4.** In case (B) the minimal number of generators of \(I(\mathfrak{n})_v\) can be arbitrarily large even if the tangent cone of \(C(\mathfrak{n})\) is Cohen-Macaulay, see [14, Proposition 3.14].

Given a complete intersection monomial curve \(C(\mathfrak{n})\), we next study the complete intersection property of \(C(\mathfrak{n} + u\mathfrak{v})\). Let \(M\) be a non-zero \(r \times s\) matrix, then there exist an \(r \times r\) invertible integer matrix \(U\) and an \(s \times s\) invertible integer matrix \(V\) such that \(UMV = \text{diag}(\delta_1, \ldots, \delta_m, 0, \ldots, 0)\) is the diagonal matrix, where \(\delta_i\) for all \(j = 1, 2, \ldots, m\) are positive integers such that \(\delta_i | \delta_{i+1}, 1 \leq i \leq m - 1, \) and \(m\) is the rank of \(M\). The elements \(\delta_1, \ldots, \delta_m\) are the invariant factors of \(M\). By \([9, \text{Theorem 3.9}]\) the product \(\delta_1 \delta_2 \cdots \delta_m\) equals the greatest common divisor of all non-zero \(m \times m\) minors of \(M\).

The following proposition will be useful in the proof of Theorem 2.6.

**Proposition 2.5.** Let \(B = \{f_1 = x_1^{b_1} - x_2^{b_2}, f_2 = x_3^{b_3} - x_4^{b_4}, f_3 = x_1^{v_1}x_2^{v_2} - x_3^{v_3}x_4^{v_4}\}\) be a set of binomials in \(K[x_1, \ldots, x_4]\), where \(b_i \geq 1\) for all \(1 \leq i \leq 4,\) at least one of \(v_1,\)
Proof. Consider the vectors $d_1 = (b_1, -b_2, 0, 0)$, $d_2 = (0, 0, b_3, -b_4)$ and $d_3 = (v_1, v_2, -v_3, -v_4)$. Clearly, $d_i \in \ker \mathbb{Z}(n_1, \ldots, n_4)$ for $1 \leq i \leq 3$, so the lattice $L = \sum_{i=1}^{3} \mathbb{Z}d_i$ is a subset of $\ker \mathbb{Z}(n_1, \ldots, n_4)$. Consider the matrix

$$M = \begin{pmatrix}
  b_1 & 0 & v_1 \\
  -b_2 & 0 & v_2 \\
  0 & b_3 & -v_3 \\
  0 & -b_4 & -v_4
\end{pmatrix}. $$

It is not hard to show that the rank of $M$ equals 3. We will prove that $L$ is saturated, namely the invariant factors $\delta_1, \delta_2$ and $\delta_3$ of $M$ are all equal to 1. The greatest common divisor of all non-zero $3 \times 3$ minors of $M$ equals the greatest common divisor of the integers $n_1, n_2, n_3$ and $n_4$. But $\gcd(n_1, \ldots, n_4) = 1$, so $\delta_1 \delta_2 \delta_3 = 1$ and therefore $\delta_1 = \delta_2 = \delta_3 = 1$. Note that the rank of the lattice $\ker \mathbb{Z}(n_1, \ldots, n_4)$ is 3 and also equals the rank of $L$. By [17, Lemma 8.2.5] we have that $L = \ker \mathbb{Z}(n_1, \ldots, n_4)$. Now the transpose $M^t$ of $M$ is mixed dominating. Recall that a matrix $P$ is mixed dominating if every row of $P$ has a positive and negative entry and $P$ contains no square submatrix with this property. By [5, Theorem 2.9] $I(n)$ is a complete intersection on the binomials $f_1, f_2$ and $f_3$. 

Theorem 2.6. Let $I(n)$ be a complete intersection ideal generated by the binomials $f_1 = x_1^{d_1} - x_2^{d_2}, f_2 = x_3^{a} - x_4^{b}$ and $f_3 = x_1^{w_1} x_2^{w_2} - x_3^{w_3} x_4^{w_4}$. Then there exist vectors $\nu_i, 1 \leq i \leq 22$, in $\mathbb{N}^4$ such that for all $w > 0$, the toric ideal $I(n + w \nu_i)$ is a complete intersection whenever the entries of $n + w \nu_i$ are relatively prime.

Proof. By [11, Theorem 6] $n_1 = a_2(a_3 u_4 + u_3 a_4), n_2 = a_1(a_4 u_3 + u_3 a_4)$, $n_3 = a_4(a_1 u_2 + u_1 a_2), n_4 = a_3(a_1 u_2 + u_1 a_2)$. Let $v_1 = (a_2 a_3, a_1 a_3, a_2 a_4, a_2 a_3)$ and $B = \{1, f_1, f_2, f_3\}$. Then $w_1 w_2 a_3 = a_2(a_3 u_4 + u_3 a_4), n_2 + w_1 a_1 a_3 = a_1(a_3 u_4 + u_3 a_4), n_3 + w_2 a_4 a_3 = a_4(a_1 u_2 + (u_1 + w) a_2)$ and $n_4 + w_2 a_3 = a_3(a_1 u_2 + (u_1 + w) a_2)$. By Proposition 2.5 for every $w > 0$, the ideal $I(n + w \nu_1)$ is a complete intersection on $f_1, f_2$ and $f_3$ whenever $\gcd(n_1 + w a_3 a_4, n_2 + w a_1 a_3, n_3 + w a_4 a_3, n_4 + w a_3) = 1$. Consider the vectors $v_2 = (a_2 a_3, a_1 a_3, a_1 a_4, a_1 a_3), v_3 = (a_2 a_4, a_1 a_4, a_2 a_4, a_1 a_3), v_4 = (a_2 a_4, a_1 a_4, a_1 a_4, a_1 a_3), v_5 = (a_2(a_4 + a_3), a_1(a_3 + a_4), 0, 0)$ and $v_6 = (0, 0, a_1(a_1 + a_2), a_3(a_1 + a_2))$. By Proposition 2.5 for every $w > 0$, $I(n + w \nu_2)$ is a complete intersection on $f_1, f_2$ and $x_1^{w_1} x_2^{w_2} - x_3^{w_3} x_4^{w_4}$ whenever the entries of $n + w \nu_2$ are relatively prime, $I(n + w \nu_3)$ is a complete intersection on $f_1, f_2$ and $x_1^{w_1} x_2^{w_2} - x_3^{w_3} x_4^{w_4}$ whenever the entries of $n + w \nu_3$ are relatively prime, and $I(n + w \nu_4)$ is a complete intersection on $f_1, f_2$ and $x_1^{w_1} x_2^{w_2} - x_3^{w_3} x_4^{w_4}$ whenever the entries of $n + w \nu_4$ are relatively prime. Furthermore for all $w > 0, I(n + w \nu_5)$ is a complete intersection on $f_1, f_2$ and $x_1^{w_1} x_2^{w_2} - x_3^{w_3} x_4^{w_4}$ whenever the entries of $n + w \nu_5$ are relatively prime, and $I(n + w \nu_6)$ is a complete intersection on $f_1, f_2$ and $x_1^{w_1} x_2^{w_2} - x_3^{w_3} x_4^{w_4}$ whenever the entries of $n + w \nu_6$ are relatively prime. Consider the vectors $v_7 = (a_2(a_3 + a_4), a_1(a_3 + a_4), a_2 a_4, a_2 a_3), v_8 = (a_2(a_3 + a_4), a_1(a_3 + a_4), a_4(a_1 + a_2), a_3(a_1 + a_2), v_9 = (0, 0, a_2 a_4, a_2 a_3), v_{10} = (0, 0, a_2 a_4, a_1 a_4, a_1 a_4, a_1 a_3, a_3 a_1 + a_2), v_{11} = (0, 0, a_1 a_4, a_1 a_4, a_1 a_4, a_1 a_3, a_3 a_1 + a_2), v_{12} = (a_2 a_3 + a_4, a_1 a_3 + a_4, a_1 a_3, a_1 a_3), v_{13} = (0, 0, a_1 a_4, a_2 a_3), v_{14} = (a_2 a_3 + a_4, a_1 a_3 + a_4, a_1 a_3), 0, 0)$ and $v_{15} = (a_2 a_3 + a_4, a_1 a_3 + a_4, 0, 0)$. Using Proposition 2.5 we have that for all $w > 0, I(n + w \nu_i), 7 \leq i \leq 15$, is a complete intersection whenever the entries of $n + w \nu_i$ are relatively prime. For instance $I(n + w \nu_3)$ is a complete intersection on the binomials $f_1, f_2$ and $x_1^{w_1} x_2^{w_2} - x_3^{w_3} x_4^{w_4}$. Consider the vectors $\nu_{16} = (a_2 a_4, a_3 a_4, a_3 a_4, a_3 a_4, a_4(u_1 + u_2), a_3(a_1 + a_2))$, $\nu_{17} = (0, a_3 a_4 + a_4(u_1 + u_2))$. Theorem 2.9
Let $n = (93, 124, 195, 117)$, then $I(n)$ is a complete intersection on the binomials $x_1^2 - x_2^2$, $x_3^2 - x_4^2$ and $x_5^2 + 9x_6^2 - x_7^2x_8^2$. Here $a_1 = 4$, $a_2 = 3$, $a_3 = 3$, $a_4 = 5$, $u_1 = 9$, $u_2 = 3$, $u_3 = 2$ and $u_4 = 7$. Consider the vector $v_1 = (9, 12, 15, 9)$. For all $w \geq 0$ the ideal $I(n + wv_1)$ is a complete intersection on $x_1^2 - x_2^2$, $x_3^2 - x_4^2$ and $x_5^{d_2 + w}x_6^2 - x_7^2x_8^2$, whenever $\gcd(93 + 9w, 124 + 12w, 195 + 15w, 117 + 9w) = 1$. By Corollary 2.2 the monomial curve $C(n + wv_1)$ has Cohen-Macaulay tangent cone at the origin. Consider the vector $v_4 = (15, 20, 20, 12)$ and the sequence $n + 9v_4 = (228, 304, 375, 225)$. The toric ideal $I(n + 9v_4)$ is a complete intersection on the binomials $x_1^2 - x_2^2$, $x_3^2 - x_4^2$ and $x_5^{d_2 + 1}x_6^2 - x_7^2x_8^2$. Note that $x_1^2 - x_2^2x_3^2 \in I(n + 9v_4)$, so $x_1^2x_2^2 \in I(n + 9v_4)$, and also $x_2^2 \in I(n + 9v_4)$, and $x_2 \in C(n + 9v_4)$. If $C(n + 9v_4)$ has Cohen-Macaulay tangent cone at the origin, then $x_1^2 \in I(n + 9v_4)$, a contradiction. Thus $C(n + 9v_4)$ does not have a Cohen-Macaulay tangent cone at the origin.

**Theorem 2.8.** Let $I(n)$ be a complete intersection ideal generated by the binomials $f_1 = x_1^{a_1} - x_2^{a_1}$, $f_2 = x_3^{a_2} - x_4^{a_2}$ and $f_3 = x_5^{a_3}x_6^{a_4} - x_7x_8^{a_5}$. Consider the vector $d = (a_1a_3, a_2, a_3, a_4, a_2a_3)$. If $C(n)$ has Cohen-Macaulay tangent cone at the origin, then for every $w > 0$ the monomial curve $C(n + wd)$ has Cohen-Macaulay tangent cone at the origin whenever the entries of $n + wd$ are relatively prime.

**Proof.** Let $n = \min\{n_1, \ldots, n_d\}$ and also $a_3 < a_4$. Without loss of generality we can assume that $u_2 \leq a_3$ and $u_3 \leq a_3$. By Theorem 2.6 for every $w > 0$, the ideal $I(n + wd)$ is a complete intersection on $f_1$, $f_2$ and $f_3 = x_5^{a_1}x_6^{a_2} - x_7x_8^{a_5}$ whenever the entries of $n + wd$ are relatively prime. Note that $n_1 + wa_2a_3 = \min\{n_1 + wa_2a_3, n_2 + wa_3a_3, n_3 + wa_2a_4, n_4 + wa_2a_3\}$. Suppose that $u_3 > 0$ and $u_4 > 0$. Assume that $u_2 < a_2$. By Corollary 2.2 it holds that $a_3 + u_4 \leq u_1 + u_2 + a_3 - u_3$ and therefore $a_4 + (u_4 + w) \leq (u_1 + w) + u_2 + a_3 - u_3$. Thus, from Corollary 2.2 again $C(n + wd)$ has Cohen-Macaulay tangent cone at the origin. Assume that $u_2 = a_2$. Then, from Corollary 2.2, we have that $a_4 + u_4 \leq u_1 + a_1 + u_3 - u_3$ and therefore $a_4 + (u_4 + w) \leq (u_1 + w) + a_1 + a_3 - u_3$. By Corollary 2.2 $C(n + wd)$ has Cohen-Macaulay tangent cone at the origin.

Suppose now that $u_3 = 0$. Then $\{f_1, f_2, f_3 = x_5^{a_1}x_6^{a_2} - x_7x_8^{a_5}\}$ is a minimal generating set of $I(n + wd)$. If $u_2 = a_2$, then $\{f_1, f_2, f_3 = x_5^{a_1}x_6^{a_2} - x_7x_8^{a_5}\}$ is a standard basis for $I(n + wd)$ with respect to the negative degree reverse lexicographical order with $x_3 > x_4 > x_2 > x_1$. Thus $I(n + wd)$ has a complete intersection and therefore $C(n + wd)$ has Cohen-Macaulay tangent cone at the origin. Assume that $u_2 < a_2$, then $u_4 \leq u_1 + u_3$ and therefore $u_4 + w \leq (u_1 + w) + u_2$. The set $\{f_1, f_2, f_3\}$ is a standard basis for $I(n + wd)$ with respect to the negative degree reverse lexicographical order with $x_3 > x_4 > x_2 > x_1$. Thus $I(n + wd)$ is a complete intersection and therefore $C(n + wd)$ has Cohen-Macaulay tangent cone at the origin. Assume that $u_4 = 0$, so necessarily $u_3 = a_3$. Then $\{f_1, f_2, f_3 = x_4^{a_1}x_5^{a_2} - x_1x_4^{a_3}x_5^{a_2}\}$ is a standard basis for $I(n + wd)$. If $u_2 = a_2$, then $\{f_1, f_2, f_3 = x_4^{a_1}x_5^{a_2} - x_1x_4^{a_3}x_5^{a_2}\}$ is a standard basis for $I(n + wd)$ with respect to the negative degree reverse lexicographical order with $x_3 > x_4 > x_2 > x_1$. Thus $I(n + wd)$ is a complete intersection and therefore $C(n + wd)$ has Cohen-Macaulay tangent cone at the origin. Assume that $u_2 < a_2$, then $a_4 \leq u_1 + u_2$ and therefore $a_4 + w \leq (u_1 + w) + u_2$. The set $\{f_1, f_2, f_3 = x_4^{a_1}x_5^{a_2} - x_1x_4^{a_3}x_5^{a_2}\}$ is a standard basis for $I(n + wd)$ with respect to
The negative degree reverse lexicographical order with $x_3 > x_4 > x_2 > x_1$. Thus $I(n + wd)_*$ is a complete intersection and therefore $C(n + wd)$ has Cohen-Macaulay tangent cone at the origin.

**Theorem 2.9.** Let $I(n)$ be a complete intersection ideal generated by the binomials $f_1 = x_1^{a_1} - x_2^{a_2}$, $f_2 = x_3^{a_3} - x_4^{a_4}$ and $f_3 = x_1^{a_1}x_2^{a_2} - x_3^{a_3}x_4^{a_4}$. Consider the vectors $d_1 = (0, 0, a_2a_4, a_2a_3)$ and $d_2 = (0, a_1a_4, a_1a_3)$. Then there exists a non-negative integer $w_0$ such that for all $w > w_0$, the monomial curves $C(n + wd_1)$ and $C(n + wd_2)$ have Cohen-Macaulay tangent cones at the origin whenever the entries of the corresponding sequence $(n + wd_1)$ and $(n + wd_2)$ are relatively prime.

**Proof.** Let $n_1 = \min\{n_1, \ldots, n_4\}$ and $a_3 < a_4$. Suppose that $u_2 \leq a_2$ and $u_3 \leq a_3$. By Theorem 2.6 for all $w > 0$, $I(n + wd_1)$ is a complete intersection on $f_1, f_2$ and $f_3 = x_1^{a_1}x_2^{a_2} - x_3^{a_3}x_4^{a_4}$ whenever the entries of $(n + wd_1)$ are relatively prime. Remark that $n_1 = \min\{n_1, n_2, n_3 + wa_2a_4, n_4 + wa_2a_3\}$. Let $w_0$ be the smallest non-negative integer greater than or equal to $u_3 + u_4 - u_1 - u_2 + a_1 - a_3$. Then for every $w > w_0$ we have that $a_4 + u_4 \leq u_1 + u_2 + a_3 - u_3$, so $u_0 + u_4 < u_1 + u_2 + u_3$. Let $G = \{f_1, f_2, f_3, f_5 = x_4^{a_4}u_4 - x_1^{a_1}x_2^{a_2}x_3^{a_3}u_3\}$. We will prove that for every $w > w_0$, $G$ is a standard basis for $I(n + wd_1)$ with respect to the negative degree reverse lexicographical order with $x_3 > x_4 > x_2 > x_1$. Note that $LM(f_1) = x_2^{a_2}$, $LM(f_2) = x_3^{a_3}$, $LM(f_3) = x_2^{a_2}x_4^{a_4}$ and $LM(f_4) = x_1^{a_1}x_2^{a_2}x_3^{a_3}$. Therefore $NF(spoly(f_1, f_2)) = 0$ as $LM(f_1)$ and $LM(f_2)$ are relatively prime, for $(i, j) \in \{(1, 2), (1, 4), (1, 5), (2, 5)\}$. We compute $spoly(f_2, f_3) = -f_5$, so $NF(spoly(f_2, f_3)) = 0$. Next we compute $spoly(f_4, f_5) = x_1^{a_1}x_2^{a_2}x_3^{a_3}$. Then $LM(spoly(f_4, f_5)) = x_1^{a_1}x_2^{a_2}x_3^{a_3}$ and only $LM(f_2)$ divides $LM(spoly(f_4, f_5))$. Also $ecart(spoly(f_4, f_5)) = a_4 - a_3 = ecart(f_2)$. Then $spoly(f_2, spoly(f_4, f_5)) = 0$ and $NF(spoly(f_4, f_5)) = 0$. Thus the minimal number of generators for $I(n + wd_1)$ is either three or four, so from [14, Theorem 3.1] for every $w > w_0$, $C(n + wd_1)$ has Cohen-Macaulay tangent cone at the origin whenever the entries of $(n + wd_1)$ are relatively prime.

By Theorem 2.6 for all $w > 0$, $I(n + wd_2)$ is a complete intersection on $f_1, f_2$ and $f_5 = x_1^{a_1}x_2^{a_2}x_3^{a_3} - x_4^{a_4}$ whenever the entries of $(n + wd_2)$ are relatively prime. Remark that $n_1 = \min\{n_1, n_2, n_3 + wa_1a_4, n_4 + wa_2a_3\}$. For every $w > w_0$ the set $H = \{f_1, f_2, f_4, x_4^{a_4} - x_1^{a_1}x_2^{a_2}x_3^{a_3}\}$ is a standard basis for $I(n + wd_2)$ with respect to the negative degree reverse lexicographical order with $x_3 > x_4 > x_2 > x_1$. Thus the minimal number of generators for $I(n + wd_2)_*$ is either three or four, so from [14, Theorem 3.1] for every $w > w_0$, $C(n + wd_2)$ has Cohen-Macaulay tangent cone at the origin whenever the entries of $(n + wd_2)$ are relatively prime.

**Example 2.10.** Let $n = (15, 25, 24, 16)$, then $I(n)$ is a complete intersection on the binomials $x_1^2 - x_3^2, x_2^2 - x_4^2$ and $x_1x_2 - x_3x_4$. Here $a_1 = 5, a_2 = 3, a_3 = 2, a_4 = 3, u_1 = 1, 1 \leq i \leq 4$. Note that $x_1^2 - x_1x_2x_3 \in I(n)$, so, from Corollary 2.2, $C(n)$ does not have a Cohen-Macaulay tangent cone at the origin. Consider the vector $d_1 = (0, 0, 0, 0, 6)$. For every $w > 0$ the ideal $I(n + wd_1)$ is a complete intersection on the binomials $x_1^2 - x_3^2, x_2^2 - x_4^2$ and $x_1^{i+1}x_2 - x_3x_4$ whenever $gcd(15, 25, 24 + 9w, 16 + 6w) = 1$. By Theorem 2.9 for every $w > 1$, the monomial curve $C(n + wd_1)$ has Cohen-Macaulay tangent cone at the origin whenever $gcd(15, 25, 24 + 9w, 16 + 6w) = 1$.

The next example gives a family of complete intersection monomial curves supporting M. Rossi’s problem, although their tangent cones are not Cohen-Macaulay. To prove it we will use the following proposition.

**Proposition 2.11.** [2, Proposition 2.2] Let $I \subseteq K[x_1, x_2, \ldots, x_d]$ be a monomial ideal and $I = (J, x^n)$ for a monomial ideal $J$ and a monomial $x^n$. Let $p(I)$ denote
the numerator $g(t)$ of the Hilbert Series for $K[x_1, x_2, \ldots, x_d]/I$. Then $p(I) = p(J) - t^{\deg(x^n)}p(J : \langle x^n \rangle)$.

**Example 2.12.** Consider the family $n_1 = 8m^2 + 6$, $n_2 = 20m^2 + 15$, $n_3 = 12m^2 + 15$ and $n_4 = 8m^2 + 10$, where $m \geq 1$ is an integer. The toric ideal $I(n)$ is minimally generated by the binomials

$$x_1^5 - x_2^2, x_3^2 - x_4^3, x_1^4 x_2^2 - x_3 x_4^2.$$  

Consider the vector $v_1 = (4, 10, 6, 4)$ and the family $n'_1 = n_1 + 4w$, $n'_2 = n_2 + 10w$, $n'_3 = n_3 + 6w$, $n'_4 = n_4 + 4w$ where $w \geq 1$ is an integer. Let $n' = (n'_1, n'_2, n'_3, n'_4)$, then for all $w \geq 0$ the toric ideal $I(n')$ is minimally generated by the binomials

$$x_1^5 - x_2^2, x_3^2 - x_4^3, x_1^4 x_2^2 - x_3 x_4^2 + w.$$  

whenever $\gcd(n'_1, n'_2, n'_3, n'_4) = 1$. Let $C_m(n')$ be the corresponding monomial curve. By Corollary 2.2 for all $w \geq 0$, the monomial curve $C_m(n')$ does not have Cohen-Macaulay tangent cone at the origin whenever $\gcd(n'_1, n'_2, n'_3, n'_4) = 1$. We will show that for every $w \geq 0$, the Hilbert function of the ring $K[[t^{n_1}, \ldots, t^{n_4}]]$ is non-decreasing whenever $\gcd(n'_1, n'_2, n'_3, n'_4) = 1$. It suffices to prove that for every $w \geq 0$, the Hilbert function of $K[x_1, x_2, x_3, x_4]/I(n')_*$ is non-decreasing whenever $\gcd(n'_1, n'_2, n'_3, n'_4) = 1$. The set

$$G = \{x_1^5 - x_2^2, x_3^2 - x_4^3, x_1^4 x_2^2 - x_3 x_4^2 + w, x_1^2 x_2^2 - x_3 x_4^2 + w, x_1 x_3^2 + w, x_2 x_4^2 - x_3 x_4^2 + w + 3\}$$

is a standard basis for $I(n')$ with respect to the negative degree reverse lexicographical order with $x_1 > x_2 > x_3 > x_4$. Thus $I(n')_*$ is generated by the set

$$\{x_1^2 x_3 x_4^4 + w + 3, x_1 x_2 x_3 x_4^2 + w + 2, x_3 x_4^2 + w + 2, x_2 x_4^2 + w + 2 + w\}.$$  

Also $\langle \text{LT}(I(n'))_* \rangle$ with respect to the aforementioned order can be written as

$$\langle \text{LT}(I(n'))_* \rangle = \langle x_1^2 x_3 x_4^4 + w + 3, x_1 x_2 x_3 x_4^2 + w + 2, x_3 x_4^2 + w + 2, x_2 x_4^2 + w + 2 + w \rangle.$$  

Since the Hilbert function of $K[x_1, x_2, x_3, x_4]/I(n')_*$ is equal to the Hilbert function of $K[x_1, x_2, x_3, x_4]/\langle \text{LT}(I(n'))_* \rangle$, it is sufficient to compute the Hilbert function of the latter. Let

$$J_0 = \langle \text{LT}(I(n'))_* \rangle, J_1 = \langle x_1^2 x_3 x_4^4 + w + 3, x_1 x_2 x_3 x_4^2 + w + 2, x_3 x_4^2 + w + 2, x_2 x_4^2 + w + 2 + w \rangle, J_2 = \langle x_1^2 x_3 x_4^4 + w + 3, x_1 x_2 x_3 x_4^2 + w + 2, x_3 x_4^2 + w + 2, x_2 x_4^2 + w + 2 + w \rangle, J_3 = \langle x_1^2 x_3 x_4^4 + w + 3, x_1 x_2 x_3 x_4^2 + w + 2, x_3 x_4^2 + w + 2, x_2 x_4^2 + w + 2 + w \rangle.$$  

Remark that $J_i = \langle J_{i+1}, g_i \rangle$, where $g_0 = x_1^2 x_3 x_4^4 + w$, $g_1 = x_1 x_2 x_3 x_4^2 + w$ and $g_2 = x_2 x_4^2 + w + 3$. We apply Proposition 2.11 to the ideal $J_i$ for $0 \leq i \leq 2$, so

$$p(J_i) = p(J_{i+1}) - t^{\deg(q_i)}p(J_{i+1} : \langle q_i \rangle).$$  

Note that $\deg(g_0) = 2m^2 + w + 2$, $\deg(g_1) = 2m^2 + w + 1$ and $\deg(g_2) = 2m^2 + w + 4$. In this case, it holds that $J_1 : \langle g_0 \rangle = \langle x_1 x_3 x_4^2 + w, x_2 x_3 x_4^2 + w, x_3 x_4^2 + w \rangle$, $J_2 : \langle g_1 \rangle = \langle x_1^2 x_3 x_4^2 + w, x_2 x_3 x_4^2 + w, x_3 x_4^2 + w \rangle$ and $J_3 : \langle g_2 \rangle = \langle x_1^2 x_3 x_4^2 + w, x_2 x_3 x_4^2 + w, x_3 x_4^2 + w \rangle$. We have that

$$p(J_3) = (1 - t)^3(1 + 3t + 4t^2 + \ldots + 4t^{4m^2 + 2w + 2} + 3t^{4m^2 + 2w + 3} + t^{4m^2 + 2w + 4}).$$  

Substituting all these recursively in Equation (2.1), we obtain that the Hilbert series of $K[x_1, x_2, x_3, x_4]/J_0$ is

$$\frac{1 + 3t + 4t^2 + \ldots + 4t^{2m^2 + w} + 3t^{2m^2 + w + 1} + t^{2m^2 + w + 2} + t^{2m^2 + w + 3} + t^{4m^2 + 2w + 2}}{1 - t}.$$  

Since the numerator does not have any negative coefficients, the Hilbert function of $K[x_1, x_2, x_3, x_4]/J_0$ is non-decreasing whenever $\gcd(n'_1, n'_2, n'_3, n'_4) = 1$. 

In this section we assume that after permuting variables, if necessary, $S = \{x_1^{a_1} - x_2^{a_2}, x_3^{a_3} - x_4^{a_4}, x_5^{a_5} - x_6^{a_6}, x_7^{a_7} - x_8^{a_8}, x_9^{a_9} - x_{10}^{a_{10}} x_{11}^{a_{11}} x_{12}^{a_{12}}\}$ is a minimal generating set of $I(n)$. Proposition 3.1 will be useful in the proof of Theorem 3.2.

**Proposition 3.1.** Let $B = \{f_1 = x_1^{b_1} - x_2^{b_2}, f_2 = x_3^{b_3} - x_1^{c_1} x_2^{c_2}, f_3 = x_4^{b_4} - x_1^{d_1} x_2^{d_2} x_3^{d_3}\}$ be a set of binomials in $K[x_1, \ldots, x_4]$, where $b_i \geq 1$ for all $1 \leq i \leq 4$, at least one of $c_1, c_2$ is non-zero and at least one of $m_1, m_2$ and $m_3$ is non-zero. Let $n_1 = b_2 b_3 b_4, n_2 = b_1 b_3 b_4, n_3 = b_1 (b_1 c_2 + c_1 b_2), n_4 = m_3 (b_1 c_2 + b_2 c_1) + b_3 (b_1 m_2 + m_1 b_2)$. If $\gcd(n_1, \ldots, n_4) = 1$, then $I(n)$ is a complete intersection ideal generated by the binomials $f_1, f_2, f_3$.

**Proof.** Consider the vectors $d_1 = (b_1, -b_2, 0, 0)$, $d_2 = (-c_1, -c_2, b_3, 0)$ and $d_3 = (-m_1, -m_2, -m_3, b_4)$. Clearly $d_i \in \ker_2(n_1, \ldots, n_4)$ for $1 \leq i \leq 3$, so the lattice $L = \sum_{i=1}^{3} Z d_i$ is a subset of $\ker_2(n_1, \ldots, n_4)$. Let

$$M = \begin{pmatrix} b_1 & -c_1 & -m_1 \\ -b_2 & -c_2 & -m_2 \\ 0 & b_3 & -m_3 \\ 0 & 0 & b_4 \end{pmatrix},$$

then the rank of $M$ equals 3. We will prove that the invariant factors $\delta_1, \delta_2$ and $\delta_3$ of $M$ are all equal to 1. The greatest common divisor of all non-zero $3 \times 3$ minors of $M$ equals the greatest common divisor of the integers $n_1, n_2, n_3$ and $n_4$. But $\gcd(n_1, \ldots, n_4) = 1$, so $\delta_1 \delta_2 \delta_3 = 1$ and therefore $\delta_1 = \delta_2 = \delta_3 = 1$. Note that the rank of the lattice $\ker_2(n_1, \ldots, n_4)$ is 3 and also equals the rank of $L$. By [17, Lemma 8.2.5] we have that $L = \ker_2(n_1, \ldots, n_4)$. Now the transpose $M'$ of $M$ is mixed dominating. By [5, Theorem 2.9] the ideal $I(n)$ is a complete intersection on $f_1, f_2$ and $f_3$. \qed

**Theorem 3.2.** Let $I(n)$ be a complete intersection ideal generated by the binomials $f_1 = x_1^{a_1} - x_2^{a_2}$, $f_2 = x_3^{a_3} - x_1^{a_1} x_2^{a_2}$ and $f_3 = x_4^{a_4} - x_1^{a_1} x_2^{a_2} x_3^{a_3}$. Then there exist vectors $b_i$, $1 \leq i \leq 2$, in $\mathbb{N}^4$ such that for all $w > 0$, the toric ideal $I(n + w b_i)$ is a complete intersection whenever the entries of $n + w b_i$ are relatively prime.

**Proof.** By [11, Theorem 6.6] $n_1 = a_2 a_3 a_4, n_2 = a_1 a_3 a_4, n_3 = a_4 (a_1 u_2 + u_1 a_2)$ and consider the set $B = \{f_1, f_2, f_3 = x_4^{a_4} - x_1^{a_1} x_2^{a_2} x_3^{a_3}\}$. Then $n_1 + w a_2 a_3 = a_2 a_3 (a_1 + w), n_2 + w a_1 a_4 = a_1 a_3 (a_1 + w), n_3 + w (a_1 u_2 + u_1 a_2) = (a_1 + w)(a_1 u_2 + u_1 a_2)$ and $n_4 + w a_2 a_3 = v_3 (a_1 u_2 + a_2 u_1) + a_3 (a_1 v_1 + (v_1 + w) a_2)$. By Proposition 3.1 for every $w > 0$, the ideal $I(n + w b_1)$ is a complete intersection on $f_1, f_2$ and $f_3$ whenever the entries of $n + w b_1$ are relatively prime. Consider the vectors $b_2 = (a_2 a_3, a_1 a_2, u_1 a_2, a_1 a_2), b_3 = (a_2 a_3, a_1 a_3, a_1 u_2 + u_1 a_2, a_1 u_2 + u_1 a_2)$, $b_4 = (0, 0, a_1 (a_1 + a_2)), b_5 = (0, 0, 0, a_1 u_2 + a_2 u_1 + a_2 a_3)$ and $b_6 = (0, 0, 0, a_1 u_2 + a_2 u_1 + a_2 a_3)$. By Proposition 3.1 for every $w > 0$, $I(n + w b_2)$ is a complete intersection on $f_1, f_2$ and $x_4^{a_4} - x_1^{a_1} x_2^{a_2} x_3^{a_3}$ whenever the entries of $n + w b_2$ are relatively prime. Furthermore for every $w > 0$, $I(n + w b_3)$ is a complete intersection on $f_1, f_2$ and $x_4^{a_4} - x_1^{a_1} x_2^{a_2} x_3^{a_3}$ whenever the entries of $n + w b_3$ are relatively prime. Consider the vectors $b_7 = (a_2 a_3, a_1 a_3, a_1 u_2 + u_1 a_2, a_1 u_2 + u_1 a_2 + a_2 a_3), b_8 = (a_2 a_3, a_1 a_3, a_1 u_2 + u_1 a_2, a_1 u_2 + u_1 a_2 + a_2 a_3)$. \qed
$u_1a_2 + a_1a_3$, $b_{10} = (0, 0, 0, a_1u_2 + a_2u_1 + a_3(a_1 + a_2))$, $b_{11} = (a_2a_3, a_1a_3, a_1u_2 + u_1a_2, 0)$, $b_{12} = (0, 0, 0, a_2a_3)$, $b_{13} = (0, 0, 0, a_1a_3)$, $b_{14} = (0, 0, 0, a_1u_2 + a_2u_1)$ and $b_{15} = (a_2a_3, a_1a_3, a_1u_2 + u_1a_2, a_1a_2 + u_1a_2 + a_1(a_1 + a_2))$. Using Proposition 3.1 we have that for all $w > 0$, the ideal $I(n + wb_i)$, $7 \leq i \leq 15$, is a complete intersection whenever the entries of $n + wb_i$ are relatively prime. Finally consider the vectors $b_{16} = (a_3a_4, a_3a_4, a_3(u_1 + a_2), v_3(u_1 + a_2) + v_3(v_1 + v_2)), b_{17} = (0, a_3a_4, a_3u_2, a_3v_2 + a_3a_2), b_{18} = (a_3a_4, 0, a_3u_1, u_1v_3 + v_1a_3), b_{19} = (a_3a_4, a_3a_4, a_3v_3 + a_3v_1 + v_1a_2), b_{20} = (a_3a_4, a_3a_4, a_3v_3 + a_1v_2 + v_1a_2), b_{21} = (a_3a_4, a_3a_4, a_3v_3 + a_1v_2 + v_1a_2)$ and $b_{22} = (0, 0, a_4(a_1 + a_2), v_3(a_1 + a_2) + a_2(a_1 + a_2))$. It is easy to see that for all $w > 0$, the ideal $I(n + wb_i)$, $16 \leq i \leq 22$, is a complete intersection whenever the entries of $n + wb_i$ are relatively prime. For instance $I(n + wb_{22})$ is a complete intersection on the binomials $x_1^{10} - x_3^2, x_1^2 - x_1^{11}x_2$ and $x_1^{11} - x_1^2x_3^3$. 

**Example 3.3.** Let $n = (231, 770, 1023, 674)$, then $I(n)$ is a complete intersection on the binomials $x_1^{10} - x_3^2, x_1^2 - x_1^{11}x_2^2$ and $x_1^{11} - x_1^2x_3^3$. Here $a_1 = 10, a_2 = 3, a_3 = 7, a_4 = 11, u_1 = 11, u_2 = 6, v_1 = 1, v_2 = 8$ and $v_3 = 1$. Consider the vector $b_{22} = (0, 0, 143, 104)$, then for all $w \geq 0$ the ideal $I(n + wb_{22})$ is a complete intersection on $x_1^{10} - x_3^2, x_1^2 - x_1^{11}x_2^2$ and $x_1^{11} - x_1^2x_3^3$ whenever $\gcd(231, 770, 1023, 674, 104w) = 1$. In fact, $I(n + wb_{22})$ is minimally generated by $x_1^{10} - x_3^2, x_1^2 - x_1^{11}x_2^2$ and $x_1^{11} - x_1^2x_3^3$. Remark that $231 = \min\{231, 770, 1023, 143w, 674, 104w\}$. The set $\{x_1^{10} - x_3^2, x_1^2 - x_1^{11}x_2^2, x_1^{11} - x_1^2x_3^3\}$ is a standard basis for $I(n + wb_{22})$ with respect to the negative degree reverse lexicographical order with $x_4 > x_3 > x_2 > x_1$. So $I(n + wb_{22})$, is a complete intersection on $x_1^2, x_1$ and $x_1^3$, and therefore for every $w \geq 0$ the monomial curve $C(n + wb_{22})$ has Cohen-Macaulay tangent cone at the origin whenever $\gcd(231, 770, 1023, 143w, 674, 104w) = 1$. Let $b_{16} = (77, 77, 187, 80)$. For every $w \geq 0$, $I(n + wb_{16})$ is a complete intersection on $x_1^{10} - x_3^2, x_1^2 - x_1^{11}x_2^2$ and $x_1^{11} - x_1^2x_3^3$ whenever $\gcd(231 + 77w, 770 + 77w, 1023 + 187w, 674 + 80w) = 1$. Note that $231 + 77w = \min\{231 + 77w, 770 + 77w, 1023 + 187w, 674 + 80w\}$. For $0 \leq w \leq 5$ the set $\{x_1^{10} - x_3^2, x_1^2 - x_1^{11}x_2^2, x_1^{11} - x_1^2x_3^3\}$ is a standard basis for $I(n + wb_{16})$ with respect to the negative degree reverse lexicographical order with $x_4 > x_3 > x_2 > x_1$. Thus $I(n + wb_{16})$, is minimally generated by $\{x_1^{10} - x_3^2, x_1^2, x_1^3\}$, so for $0 \leq w \leq 5$ the monomial curve $C(n + wb_{16})$ has Cohen-Macaulay tangent cone at the origin whenever $\gcd(231 + 77w, 770 + 77w, 1023 + 187w, 674 + 80w) = 1$. Suppose that there is $w \geq 6$ such that $C(n + wb_{16})$ has Cohen-Macaulay tangent cone at the origin. Then $x_1^2x_3^3 \in I(n + wb_{16}) \setminus \{x_1\}$ and therefore $x_1^2x_3^3 \in I(n + wb_{16})$. Thus $x_1^2x_3^3$ is divided by $x_1^5$, a contradiction. Consequently for every $w \geq 6$ the monomial curve $C(n + wb_{16})$ does not have Cohen-Macaulay tangent cone at the origin whenever $\gcd(231 + 77w, 770 + 77w, 1023 + 187w, 674 + 80w) = 1$.

**Theorem 3.4.** Let $I(n)$ be a complete intersection ideal generated by the binomials $f_1 = x_1^{a_1} - x_2^{a_2}$, $f_2 = x_3^{a_3} - x_1^{a_1}x_2^{a_2}$ and $f_3 = x_4^{a_4} - x_1^{a_1}x_2^{a_2}x_3^{a_3}$. Consider the vector $d = (0, 0, a_1 + a_2, v_3(a_1 + a_2) + a_3(a_1 + a_2))$. Then there exists a non-negative integer $w_1$ such that for all $w \geq w_1$, the ideal $I(n + wd)$, is a complete intersection whenever the entries of $n + wd$ are relatively prime.

**Proof.** By Theorem 3.2 for all $w \geq 0$, the ideal $I(n + wd)$ is minimally generated by $G = \{f_1, f_2 = x_3^{a_3} - x_1^{a_1}x_2^{a_2}, f_3 = x_4^{a_4} - x_1^{a_1}x_2^{a_2}x_3^{a_3}\}$ whenever the entries of $n + wd$ are relatively prime. Let $w_1$ be the smallest non-negative integer greater than or equal to $\max\{\frac{a_1 + a_2 - 2w_1}{2}, \frac{a_3 + a_1 - 2w_1}{2}\}$. Then $a_3 \leq w_1 + a_2 + 2w_1$ and $a_4 \leq v_1 + v_2 + v_3 + 2w_1$. It is easy to prove that for every $w \geq w_1$ the set $G$ is a standard basis for $I(n + wd)$ with respect to the negative degree reverse lexicographical order with $x_4 > x_3 > x_2 > x_1$. Note that $\operatorname{LM}(f_1)$ is either $x_1^{a_1}$
or \(x_2 a^2\). \(\text{LM}(f_1) = x_3 a^3\) and \(\text{LM}(f_3) = x_4 a^4\). By [8, Lemma 5.5.11] \(I(n + wd)\), is generated by the least homogeneous summands of the elements in the standard basis \(G\). Thus for all \(w \geq w_1\), the ideal \(I(n + wd)\), is a complete intersection whenever the entries of \(n + wd\) are relatively prime.

\[\Box\]

**Proposition 3.5.** Let \(I(n)\) be a complete intersection ideal generated by the binomials \(f_1 = x_1 a^1 - x_2 a^2, f_2 = x_3 a^3 - x_1 x_2 w_2\) and \(f_3 = x_4 a^4 - x_1 x_2 w_2\), where \(v_1 > 0\) and \(v_2 > 0\). Assume that \(a_2 < a_1, a_3 < a_1 + w_2, v_2 < a_2\) and \(a_1 + v_1 \leq a_2 - v_2 + a_4\). Then there exists a vector \(b\) in \(\mathbb{N}^4\) such that for all \(w \geq 0\), the ideal \(I(n + wb)\), is almost complete intersection whenever the entries of \(n + wb\) are relatively prime.

**Proof.** From the assumptions we deduce that \(v_1 + v_2 < a_4\). Consider the vector \(b = (a_2 a_3, a_1 a_3, a_1 w_2 + u_1 a_2, a_2 a_3)\). For every \(w \geq 0\) the ideal \(I(n + wb)\) is a complete intersection on \(f_1, f_2\) and \(f_4 = x_4 a^4 + w - x_1 w_2 x_2^2\) whenever the entries of \(n + wb\) are relatively prime. We claim that the set \(G = \{f_1, f_2, f_4, f_5 = x_4 a^4 + w - x_2 x_4 a^4\}\) is a standard basis for \(I(n + wb)\) with respect to the negative degree reverse lexicographical order with \(x_3 > x_2 > x_1 > x_4\). Note that \(\text{LM}(f_1) = x_2 a^2, \text{LM}(f_2) = x_3 a^3, \text{LM}(f_4) = x_1 w_2 x_2^2\) and \(\text{LM}(f_5) = x_1 x_4 w_2\). Also \(\text{spoly}(f_1, f_4) = -f_5\). It suffices to show that \(\text{NF}(\text{spoly}(f_4, f_5)) = 0\). We compute \(\text{spoly}(f_1, f_4) = x_2^2 x_4 a^4 + w - x_1 x_4 w_2\). Then \(\text{NF}(\text{spoly}(f_4, f_5)) = x_2^2 x_4 a^4 + w\) and only \(\text{LM}(f_1)\) divides \(\text{NF}(\text{spoly}(f_4, f_5))\). Moreover \(\text{ecart}(\text{spoly}(f_4, f_5)) = a_1 - a_2 = \text{ecart}(f_1)\). So \(\text{spoly}(f_1, \text{spoly}(f_4, f_5)) = 0\) and also \(\text{NF}(\text{spoly}(f_4, f_5)) = 0\). Thus

1. If \(a_1 + v_1 < a_2 - v_2 + a_4\), then \(I(n + wb)\) is minimally generated by \(\{x_2^2, x_3^3, x_1 w_2 x_2^2, x_1 x_4 w_2\}\).
2. If \(a_1 + v_1 = a_2 - v_2 + a_4\), then \(I(n + wb)\) is minimally generated by \(\{x_2^2, x_3^3, x_1 w_2 x_2^2, f_5\}\).

\[\Box\]

**References**


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