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Robust bilateral trade with discrete types

Kamyar Kargar[1] · **Halil Ibrahim Bayrak**[1] · **Mustafa Çelebi Pinar**[1]

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Abstract
Bilateral trade problem is the most common market interaction in which a seller and a buyer bargain over an indivisible object, and the valuation of each agent about the object is private information. We investigate the cases where mechanisms satisfying Dominant Strategy Incentive Compatibility (DIC) and Ex-post Individual Rationality (EIR) properties can exhibit robust performance in the face of imprecision in prior structure. We start with the general mathematical formulation for the bilateral trade problem with DIC, EIR properties. We derive necessary and sufficient conditions for DIC, EIR mechanisms to be Ex-post efficient at the same time. Then, we define a new property—Allocation Maximality—and prove that the Posted Price mechanisms are the only mechanisms that satisfy DIC, EIR and Allocation Maximal properties. We also show that Posted Price mechanism is not the only mechanism that satisfies DIC and EIR properties. The last part of the paper introduces different sets of priors for agents' types and consequently allows ambiguity in the problem framework. We derive robust counterparts and solve them numerically for the proposed objective function under box and ϕ-divergence ambiguity specifications. Results suggest that restricting the feasible set to Posted Price mechanisms can decrease the objective value to different extents depending on the uncertainty set.

Keywords Mechanism design · Robustness · Ambiguity · ϕ-Divergence

Mathematics Subject Classification 90C05 · 91B26

✉ Mustafa Çelebi Pinar
mustafap@bilkent.edu.tr

Kamyar Kargar
kamyar.kargar@bilkent.edu.tr

Halil Ibrahim Bayrak
halil.bayrak@bilkent.edu.tr

[1] Department of Industrial Engineering, Bilkent University, Ankara, Turkey

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1 Introduction

In general, mechanism design is about investigating the necessary and sufficient conditions to achieve desired social, environmental or economic outcomes under many assumptions such as individuals' self-interest and incomplete information. It can be said that mechanism design provides an optimization framework in strategic level. In the literature, mechanism design is referred to as a subfield of microeconomics and game theory but there is a distinct difference between game theory and mechanism design. While game theory looks for methods to predict the outcome of a given game, mechanism design takes the reverse path. In mechanism design, we start with a given desirable outcome and try to design a game which produces it. For example, let us consider a bargaining problem between a risk neutral seller and buyer over an indivisible object. Each individual's valuation about the object is assumed to be an independent random variable and private information. These two individuals will participate in some bargaining mechanism to make a decision about two important issues. Should the object be transferred from the seller to the buyer? If the answer is yes, then what is the transfer price? This well-known problem is referred to as "Bilateral Trading problem" in the mechanism design literature. One of the pioneering studies in bilateral trading problem was done by Myerson and Satterthwaite (1983). The authors show that when there exists a continuous common prior[1] over traders' valuations known to all participants, then it is impossible to have an *Ex-post efficient*[2] mechanism which satisfies the following three properties:

1. Bayesian Incentive Compatible:

A mechanism is Bayesian Incentive Compatible if truth telling is a Bayesian Nash equilibrium.

2. Interim Individual Rationality:

Interim Individual Rationality requires that each individual has nonnegative expected gains from the trade.

3. Budget Balancing:

There is no external funding source, and the payment made by the buyer equals to the payment received by the seller.

Later, Hagerty and Rogerson (1987) criticized this study in particular and mechanisms with common prior assumption in general for the following reasons: Most of the time, it is hard to derive exactly the traders' priors or it is possible that we encounter with a variety of priors over time. So the authors proposed an alternative mechanism which shows robust performance with respect to variations in prior structure.

In their mechanism, the Bayesian Incentive Compatible and Interim Individual Rationality properties are replaced with Dominant Strategy Incentive Compatibility (DIC) and Ex-post Individual Rationality (EIR), respectively. A mechanism is called Dominant Strategy Incentive Compatible if telling the truth is a weakly dominant

[1] The assumption that each state of the world is an independent draw from a commonly known distribution is called common prior assumption.

[2] The buyer gets the object if and only if the buyer's valuation is higher than the seller's.

strategy. Ex-post Individual Rationality means that regardless of the other agent's type, both traders find it beneficial to participate in the bargain.

When we look at the literature on bilateral trade problem with discrete types, we notice that most of the works focus on Bayesian Incentive Compatible, Interim Individually Rational mechanisms. When both agents have two types, Matsuo (1989) finds necessary and sufficient conditions on the agent beliefs so that Budget balanced, Ex-post efficient mechanism is possible. Othman and Sandholm (2009) draws samples with respect to different distributions to check the feasibility of Ex-post efficient bilateral trade. The authors conclude that as the cardinality of type set increases frequency of Ex-post efficiency decreases. Kos and Manea (2009) proves that there exists an Ex-post efficient, Ex-post Budget balanced mechanism if and only if a VCG-like mechanism does not run an expected deficit. The authors also consider the multiple buyers case and the effect of an additional buyer to the existence of Ex-post efficient mechanism. Lastly, the authors deal with the mechanism maximizing total gains from trade. Flesch et al. (2013) focus on Ex-post Individually Rational mechanisms and show that Ex-post efficiency is possible if the cardinality of the type set is less than or equal to five. One of the main results of Flesch et al. (2016) states that for any Ex-post efficient mechanism, there exists prior distributions such that it is also Bayesian Incentive Compatible and Interim Individually Rational. To the best of our knowledge, there are only two studies in the literature that consider DIC, EIR mechanisms with discrete types: Carroll (2017) and Pınar (2018). Carroll (2017) considers a non-trivial case when each agent has two types and shows that first-best welfare (Ex-post efficiency) is infeasible, while Pınar (2018) considers the robust trade mechanisms in the presence of an intermediary, i.e., when budget balance requirement is relaxed.

Recently, Vohra (2011, 2012) developed a linear programming approach to tackle problems in economics under discrete type spaces. His line of research was then followed by other researchers to investigate some celebrated problems in the literature. Bayrak and Pınar (2016) re-examines the optimal mechanism from Vohra (2012) and arrives at a conclusion that second price auction is suboptimal since the principal can do better with a slight modification. Koçyiğit et al. (2018) investigate maximizing the worst case revenue in an auction with single seller and multiple buyers where all agents are ambiguity-averse. Bayrak et al. (2017) consider the optimal mechanism for the ambiguity-averse principal utilizing costly inspection instead of monetary transfers.

Against this background, the purpose of present paper is to reconsider properties and results of robust mechanism design for bilateral trading problem under discrete framework, and various specifications for the set of priors. The main contributions and novelty of the present paper can be summarized as follows: Note that the all findings and results are for discrete type setting.

- We propose necessary and sufficient conditions so that Ex-post efficiency can be obtained together with DIC and EIR.
- We show by an example that Posted Price mechanisms are not the only DIC, EIR mechanisms, which is the case in continuous type space as proved by Hagerty and Rogerson (1987).

- We define a new property called Allocation Maximality and prove that the Posted Price mechanisms are the only mechanisms that satisfy DIC, EIR and Allocation Maximal properties.
- We consider ambiguity in the problem framework originating from different sets of priors for agents types. Then, robust counterparts from the perspective of an ambiguity-averse intermediary are derived, and related computational results are discussed.

The rest of the paper proceeds as follows: In the next section, we define the proposed problem and give the related assumptions and concepts. We then formulate the bilateral trade problem under DIC, EIR properties with discrete types. In Sect. 2, we also provide intuition about the necessary and sufficient conditions for a DIC, EIR mechanism to also be Ex-post efficient. In Sect. 3, the relations between the newly defined Allocation Maximal property and Posted Price mechanisms are scrutinized, and we prove that the Posted Price mechanisms are the only Allocation Maximal DIC, EIR mechanisms. In Sect. 4, we derive the robust counterparts for the bilateral trade problem while the intermediary wants to maximize seller's expected revenue. The proposed models consider ambiguity under box and ϕ-divergence-based sets, respectively. In Sect. 5, computational results are provided, and the performance of the proposed models is compared in terms of their objective function value. Finally, Sect. 6 concludes.

2 Problem statement

Suppose there is a risk neutral seller who owns an object and a risk neutral buyer who wishes to buy that object. Let i and j denote the value of the object to the seller and the buyer, respectively. These valuations are privately kept by traders. The value that each trader assigns to the object is called type of that trader. The type of each trader is an independent draw from the set $T = \{1, 2, \ldots, m\}$.[3] Variables p and x are defined to be trade probability and expected payment value, respectively, while g_{ij}^r is the probability mass function for the payment r conditional on the agents types i, j. A mechanism that is Dominant Strategy Incentive Compatible and Ex-post Individually Rational should satisfy the following system of nonlinear inequalities:

$$x_{ij} - ip_{ij} \geq x_{kj} - ip_{kj} \quad \forall i, j, k \in T \tag{1}$$
$$jp_{ij} - x_{ij} \geq jp_{ik} - x_{ik} \quad \forall i, j, k \in T \tag{2}$$
$$x_{ij} = p_{ij} \sum_r r g_{ij}^r \quad \forall i, j \in T \tag{3}$$
$$\sum_{r=i}^{j} g_{ij}^r = 1 \quad \forall i, j \in T \tag{4}$$
$$g_{ij}^r \geq 0 \quad \forall r, i, j \in T \tag{5}$$
$$p_{ij} \leq 1 \quad \forall i, j \in T \tag{6}$$
$$p_{ij} \geq 0 \quad \forall i, j \in T. \tag{7}$$

[3] We work with more general discrete type sets in Proposition 1. However, we prefer the simple type set T not to encumber the notation.

Note that a continuous analog of these constraints is also the starting point of Hagerty and Rogerson (1987). Obviously, constraints (6) and (7) ensure that trade probability is between zero and one. Constraint (3) calculates the expected payment from trade probability and payment distribution. Constraints (4) and (5) force g_{ij}^r variables to define a valid probability mass function. It is enough to consider g_{ij}^r variables for $i \leq r \leq j$ because we are interested in EIR mechanisms. Finally, constraints (1) and (2) represent the Dominant Strategy Incentive Compatibility for the seller and the buyer, respectively. These constraints ensure that reporting a different type other than the actual one will result in utility which is less than or equal to the case when the type is truthfully reported for all possible types. It is clear that we are only interested in the mechanisms in which the optimal strategy is to report truthfully. In order to have a linear system of inequalities we want to take out the g_{ij}^r variable and solve the problem over x_{ij} and p_{ij}. Note that x_{ij} variable should be zero if $p_{ij} = 0$, and otherwise x_{ij} is bounded below and above by ip_{ij} and jp_{ij}, respectively. Therefore, using the following system does not eliminate any EIR mechanisms and also gets rid of the nonlinear equality:

$$x_{ij} - ip_{ij} \geq x_{kj} - ip_{kj} \quad \forall i, j, k \in T \tag{1}$$
$$jp_{ij} - x_{ij} \geq jp_{ik} - x_{ik} \quad \forall i, j, k \in T \tag{2}$$
$$x_{ij} - ip_{ij} \geq 0 \quad \forall i, j \in T \tag{8}$$
$$jp_{ij} - x_{ij} \geq 0 \quad \forall i, j \in T \tag{9}$$
$$p_{ij} \leq 1 \quad \forall i, j \in T \tag{6}$$
$$p_{ij} \geq 0 \quad \forall i, j \in T. \tag{7}$$

Constraints (8) and (9) bound the expected payment variable so that it satisfies the EIR conditions. Given a mechanism satisfying the above system, one can easily find the set of all EIR payment distributions g_{ij}^r for all $p_{ij} > 0$ using the following system:

$$\sum_{r=i}^{j} r g_{ij}^r = x_{ij}/p_{ij} \quad \forall i, j \in T$$
$$\sum_{r=i}^{j} g_{ij}^r = 1 \quad \forall i, j \in T$$
$$g_{ij}^r \geq 0 \quad \forall r, i, j \in T.$$

Therefore, we continue our search for DIC, EIR mechanisms by considering the latter system. Next, we will look into the system of inequalities (2) and (9) which corresponds to the dual constraints of a shortest path problem:

$$jp_{ij} - jp_{ik} \geq x_{ij} - x_{ik} \quad \forall i, j, k \in T \tag{2}$$
$$jp_{ij} \geq x_{ij} \quad \forall i, j \in T. \tag{9}$$

This system is separable for each $i \in T$ so that we can consider each of them separately. Introducing a vertex for each type j and an arc between every successive type $(j+1, j)$ of length $jp_{ij} - jp_{ij+1}$, we will obtain the network in Fig. 1 for all $i \in T$ (also introduce a dummy node zero). Note that this network contains only a subset of the arcs defined by constraints (2) and (9). Thus, if the corresponding primal shortest path problem is unbounded, constraints (2) and (9) are infeasible. Then, we should not have any negative cost cycles in the network. Let us consider the length of the cycle $j \to j+1 \to j$:

$$(j+1)p_{ij+1} - (j+1)p_{ij} + jp_{ij} - jp_{ij+1} = p_{ij+1} - p_{ij} \geq 0.$$

A network with nonnegative cycle costs means that p_{ij} variable should be non-decreasing in $j \in T$. Besides, it can be shown that all shortest paths of the network are represented in the given figure. To see this, consider the length of $j \to j+1 \cdots \to k$ in the given network:

$$\begin{aligned}(j+1)p_{ij+1} - (j+1)p_{ij} + \cdots + kp_{ik} - kp_{ik-1} &= kp_{ik} - (j+1)p_{ij} - \sum_{l=j+1}^{k-1} p_{il} \\ &= kp_{ik} - kp_{ij} - \sum_{l=j+1}^{k-1} (p_{il} - p_{ij}),\end{aligned}$$

which is less than or equal to $kp_{ik} - kp_{ij}$, length of the arc (j,k), since p_{ij} variables are monotone increasing in j. Now we consider the path $j \to j-1 \ldots \to k$:

$$\begin{aligned}(j-1)p_{ij-1} - (j-1)p_{ij} + \cdots + kp_{ik} - kp_{ik+1} &= kp_{ik} - (j-1)p_{ij} + \sum_{l=k+1}^{j-1} p_{il} \\ &= kp_{ik} - kp_{ij} + \sum_{l=k+1}^{j-1} (p_{il} - p_{ij}),\end{aligned}$$

which is again less than or equal to $kp_{ik} - kp_{ij}$. Since this is true for all arcs, all shortest paths are represented in Fig. 1. We use this fact in the following manner: take $p_{i0} = 0$, $x_{i0} = 0$ and sum up the constraints corresponding to the shortest path from node 0 to j which is actually the tightest upper bound on x_{ij} variable:

$$\sum_{k=1}^{j} (kp_{ik} - kp_{ik-1}) = jp_{ij} - \sum_{k=1}^{j-1} p_{ik} \geq x_{ij}.$$

Similarly, by summing up the constraints corresponding to the shortest path from node j to 0, we will obtain:

$$\sum_{k=1}^{j} (k-1)(p_{ik-1} - p_{ik}) = -(j-1)p_{ij} + \sum_{k=1}^{j-1} p_{ik} \geq -x_{ij},$$

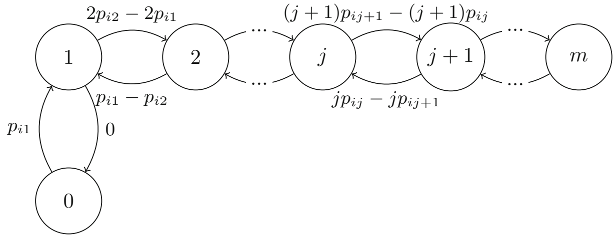

Fig. 1 Network of types where only the arcs between successive nodes are drawn

which turns out to be the tightest lower bound on x_{ij} implied by constraints (2) and (9). Our analysis on the dual shortest path problem for the buyer's DIC and EIR constraints led us to a relaxation as follows:

$$p_{im} \geq p_{im-1} \geq \cdots \geq p_{i2} \geq p_{i1} \quad \forall i \in T$$
$$jp_{ij} - \sum_{k=1}^{j-1} p_{ik} \geq x_{ij} \geq (j-1)p_{ij} - \sum_{k=1}^{j-1} p_{ik} \quad \forall i, j \in T.$$

Vohra (2011) made extensive use of this duality relation to transform the buyer's Bayesian Incentive Compatibility and Interim Individual Rationality constraints. He derives monotonicity of expected allocation variables and sets expected payment variables equal to their respective upper bounds. His model has an objective which maximizes total payments so that interchanging payment variables with their upper bounds is optimal. However, in the first part of the current study, we do not restrict our attention to any type of objective function in search of DIC, EIR mechanisms. Therefore, we also derive the implied lower bound and arrive at a relaxed formulation. Working with this relaxation proves to be useful for two reasons. First, Posted Price mechanism,[4] which is known to be DIC and EIR, can be formulated exactly by making a slight change in the relaxed formulation. Second, given any allocation rule, it either shows infeasibility or it narrows down the possible transfer rules that can be applied to have a DIC, EIR mechanism. We will make these cases clear using specific examples illustrated in Fig. 3.

Now, we also apply a similar approach to the seller's DIC, EIR constraints which can be written as:

$$ip_{kj} - ip_{ij} \geq x_{kj} - x_{ij} \quad \forall i, j, k \in T \tag{1}$$
$$-ip_{ij} \geq -x_{ij} \quad \forall i, j \in T. \tag{8}$$

Again consider these constraints as the dual of a shortest path problem. For all $j \in T$, this time we will obtain the network in Fig. 2. Dummy node $m+1$ is connected to

[4] In the Posted Price mechanism, the price of trade is posted by the planner and the agents trade at that price or do not trade at all.

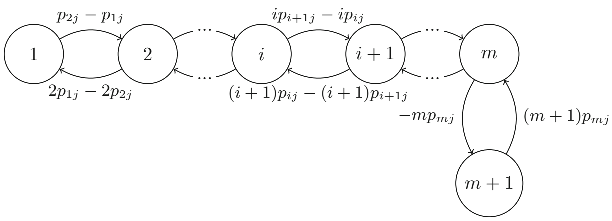

Fig. 2 Network of types for constraints (1) and (8)

node m, and p_{im+1}, x_{im+1} are equal to zero. After constructing the network, we utilize the same set of arguments in order to find the following set of inequalities:

$$\begin{aligned}
&p_{1j} \geq p_{2j} \geq \cdots \geq p_{m-1j} \geq p_{mj} \quad \forall j \in T \\
&\sum_{k=i+1}^{m} p_{kj} + ip_{ij} \leq x_{ij} \leq \sum_{k=i+1}^{m} p_{kj} + (i+1)p_{ij} \quad \forall i, j \in T.
\end{aligned}$$

No negative cost cycle argument requires p_{ij} to be monotone decreasing on i, and it can be shown that all shortest paths are contained in the given network. The only difference from the previous analysis is that we find the lower bound on x_{ij} by considering the path from node i to $m+1$ following the arcs in Fig. 2. Upper bound is given by the path from $m+1$ to i.

At this point, we introduce the relaxed formulation which should be satisfied by any DIC, EIR mechanism:

$$p_{im} \geq p_{im-1} \geq \cdots \geq p_{i2} \geq p_{i1} \quad \forall i \in T \tag{10}$$
$$p_{1j} \geq p_{2j} \geq \cdots \geq p_{m-1j} \geq p_{mj} \quad \forall j \in T \tag{11}$$
$$jp_{ij} - \sum_{k=1}^{j-1} p_{ik} \geq x_{ij} \geq (j-1)p_{ij} - \sum_{k=1}^{j-1} p_{ik} \quad \forall i, j \in T \tag{12}$$
$$\sum_{k=i+1}^{m} p_{kj} + ip_{ij} \leq x_{ij} \leq \sum_{k=i+1}^{m} p_{kj} + (i+1)p_{ij} \quad \forall i, j \in T \tag{13}$$
$$p_{ij} \leq 1 \quad \forall i, j \in T \tag{6}$$
$$p_{ij} \geq 0 \quad \forall i, j \in T. \tag{7}$$

A trivial solution of the above system is to set all trading probabilities to zero. Although we do not allow any trade in this mechanism, it satisfies the DIC and EIR conditions. Nobody is ex-post worse off by participating in the trade, and each trader's dominant strategy set contains reporting one's true type. We present three examples in Fig. 3 in order to investigate the relation between DIC, EIR mechanisms and the relaxed formulation, where $m = 5$. These examples only specify allocation rules, but we also need transfer rules to check if the mechanism satisfies DIC, EIR constraints or not. As we shall see below, the relaxed formulation helps us track down the DIC, EIR transfer rules.

Ex-post efficiency dictates that the trade should take place if and only if the buyer has a higher valuation than the seller. Example (a) in Fig. 3 illustrates an Ex-post

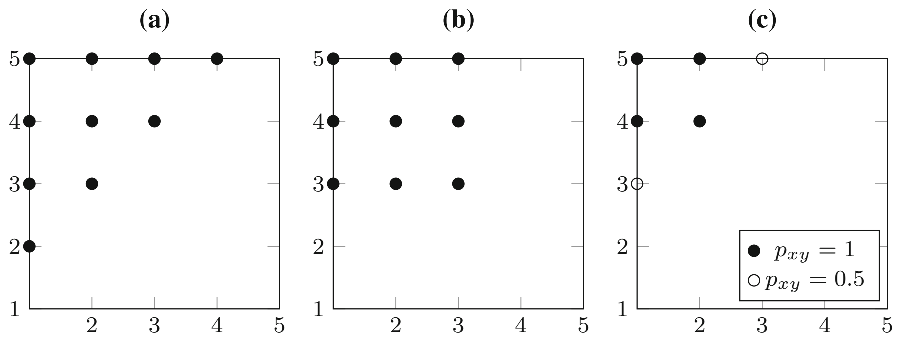

Fig. 3 Trade probabilities with different properties. **a** Ex-post efficient mechanism, **b** Posted Price mechanism, **c** neither Ex-post efficient nor Posted Price mechanism

efficient allocation where the tie break rule leaves the good to the seller. It is easy to check that Ex-post efficient mechanism (with any tie break rule) is not feasible in the relaxed formulation because of the constraints (12) and (13). Therefore, we can conclude that there does not exist any DIC, EIR and Ex-post efficient mechanism when both agents have type set $T = \{1, 2, 3, 4, 5\}$. However, this is not true in general, and the following proposition gives conditions using general discrete type sets T_b and T_s (not necessarily the first m integers), for buyer and seller, respectively, so that Ex-post efficiency can be obtained together with DIC and EIR. In order not to detract from the flow of the paper, we give the proof in "Appendix."

Proposition 1 *For finite type sets T_b and T_s with strictly positive elements, there exists a DIC, EIR, Ex-post efficient mechanism if and only if the convex hull of agents' efficient type sets which are defined as $T_b^* = \{b_k \in T_b | b_k > s_l$ for some $s_l \in T_s\}$ and $T_s^* = \{s_k \in T_s | s_k < b_l$ for some $b_l \in T_b\}$ have finite intersection.*

As an immediate result of this proposition, if the buyer and seller have a common type set $T = \{1, 2, \ldots, m\}$, which is the case in the current paper, Ex-post efficiency can be obtained when $m \leq 3$. In three types case, the posted price will be equal to 2 and efficient types will be $\{1, 2\}$ for the seller and $\{2, 3\}$ for the buyer. Adding an extra type 4 will result in efficient type sets $\{1, 2, 3\}$ and $\{2, 3, 4\}$ whose convex hulls have infinite intersection.

The other two examples in Fig. 3b, c, only specify allocation variables, but one can use the relaxed formulation to elicit transfer variables. When p_{ij} values of example (b) are written in the relaxed formulation, it is easy to see that the only feasible solution is setting x_{ij} equal to three whenever p_{ij} is equal to one. This is actually the Posted Price mechanism with price set to three and it is a DIC, EIR mechanism. Similarly, when we use p_{ij} values in example (c), we see that the relaxed formulation gives $x_{13} = 1$, $x_{24} = 3$, $x_{35} = 2$. For other transfer variables, we find following intervals, $x_{15} \in [2.5, 3.5]$, $x_{14} \in [2.5, 3]$, $x_{25} \in [3, 3.5]$. We use another characteristic from DIC mechanisms to find the unique solution in this case.

Lemma 1 *When all elements in finite type set T are strictly positive, any DIC mechanism has $x_{ij} = x_{kj}$ if and only if $p_{ij} = p_{kj}$ holds for all $i, j, k \in T$. Similarly, $x_{ij} = x_{ik}$ holds if and only if $p_{ij} = p_{ik}$ is satisfied for all $i, j, k \in T$.*

Proof Truthful reporting is a weakly dominant strategy if the following set of constraints are satisfied:

$$x_{ij} - ip_{ij} \geq x_{kj} - ip_{kj} \quad \forall i, j, k \in T \tag{1}$$
$$jp_{ij} - x_{ij} \geq jp_{ik} - x_{ik} \quad \forall i, j, k \in T. \tag{2}$$

For any pair of types $i, k \in T$, we have the following two constraints from inequality (1):

$$x_{ij} - ip_{ij} \geq x_{kj} - ip_{kj} \qquad \forall j \in T$$
$$x_{kj} - kp_{kj} \geq x_{ij} - kp_{ij} \qquad \forall j \in T.$$

If $x_{ij} = x_{kj}$ holds, we end up with $i(p_{kj} - p_{ij}) \geq 0$ and $k(p_{ij} - p_{kj}) \geq 0$. Then, for any $j \in T$, we should also have $p_{ij} = p_{kj}$ since all elements in T are strictly positive. Other parts can be proven similarly. □

The intuition behind Lemma 1 is that whenever one of these equalities holds, there is a profitable deviation for some type if the other equality does not hold. Therefore, transfer rule in example (c) should be $x_{15} = x_{14} = x_{25} = x_{24} = 3$. Along with this transfer rule, example (c) satisfies DIC, EIR constraints. Note that finding DIC, EIR transfer rules from the relaxed formulation is not generally easy.

Therefore, we found a DIC, EIR mechanism, example (c), which is not a Posted Price mechanism. Recall that according to Hagerty and Rogerson (1987) every DIC, EIR mechanism is a Posted Price mechanism when agents have continuous type space. Our example (c) showed that DIC, EIR constraints for the discrete type space are also satisfied by other solutions, a testimony to the discrepancy between continuous and discrete type space. In the following section, we will use the proposed relaxed formulation to show that Posted Price mechanisms can be formulated exactly.

3 Posted Price and Allocation Maximal mechanisms

In this section, we show that using the constraints of the relaxed formulation, we can formulate Posted Price mechanisms. We start our discussion by referring to the following set of inequalities as the final relaxed formulation (FRF). We get rid of transfer variables and use their upper and lower bounds given in (12) and (13) to come up with constraint (14). Obviously any DIC and EIR mechanism should satisfy FRF:

$$p_{im} \geq p_{im-1} \geq \cdots \geq p_{i2} \geq p_{i1} \quad \forall i \in T \tag{10}$$
$$p_{1j} \geq p_{2j} \geq \cdots \geq p_{m-1j} \geq p_{mj} \quad \forall j \in T \tag{11}$$
$$(j-i)p_{ij} \geq \sum_{k=1}^{j-1} p_{ik} + \sum_{k=i+1}^{m} p_{kj} \quad \forall i, j \in T \tag{14}$$
$$p_{ij} \leq 1 \quad \forall i, j \in T \tag{6}$$
$$p_{ij} \geq 0 \quad \forall i, j \in T. \tag{7}$$

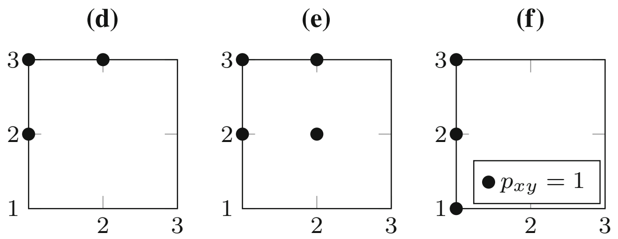

Fig. 4 Trade probabilities with different properties. **d** Ex-post efficient mechanism, **e** Posted Price mechanism with unique price 2, **f** Posted Price mechanism with unique price 1

First, we investigate another set of examples when $T = \{1, 2, 3\}$ in order to clarify the relation between DIC, EIR mechanisms and FRF. Monotonicity and bounding constraints for p variables are obviously satisfied for all three examples in Fig. 4. We will check the constraint (14) for Ex-post efficient example (d):

$$\begin{aligned}
&2p_{13} \geq p_{11} + p_{12} + p_{23} + p_{33} \quad \text{gives } 2 = 2 \\
&p_{12} \geq p_{11} + p_{22} \quad \text{gives } 1 \geq 0 \\
&p_{23} \geq p_{22} + p_{33} \quad \text{gives } 1 \geq 0.
\end{aligned}$$

Example (d) is a Posted Price mechanism with unique price two but its tie break rule awards the good to the seller unlike example (e). Posted Price mechanism in example (e) has another characteristic apart from being DIC, EIR, Ex-post efficient. It satisfies the constraint (14) with equality for all $i, j \in T$. It is easy to see that example (f) also satisfies the constraint (14) with equality and we cannot increase any p_{ij} variable without decreasing another one first. A mechanism with no trade also satisfies constraint (14) with equality, but we can increase p_{1m} as long as $m > 1$. When the cardinality of the type set gets bigger than three, we no longer have Ex-post Efficiency. However, in this case how much efficiency one can capture becomes a relevant question. To answer this question, we define the concept of Allocation Maximality and prove that a feasible mechanism in the FRF is Allocation Maximal only if it is a Posted Price mechanism.

Definition 1 An allocation rule, p^*, that is feasible in FRF is Allocation Maximal if and only if there does not exist any other mechanism, p, feasible in FRF such that $p_{ii} \geq p^*_{ii}$ for all $i \in T$ and $p_{kk} > p^*_{kk}$ for some $k \in T$.

In order to show the structure of Allocation Maximal mechanisms of FRF, we will need the following result.

Lemma 2 *The following two equations are equivalent for mechanisms feasible in FRF.*

$$(j - i)p_{ij} = \sum_{k=i}^{j-1} p_{ik} + \sum_{k=i+1}^{j} p_{kj} \quad \forall i, j \in T \tag{15}$$

$$p_{ij} = \sum_{k=i}^{j} p_{kk} \quad \forall i, j \in T. \tag{16}$$

Proof Firstly notice that we can change the constraint (14) with the following:

$$(j-i)p_{ij} \geq \sum_{k=i}^{j-1} p_{ik} + \sum_{k=i+1}^{j} p_{kj} \quad \forall i, j \in T.$$

We only need to consider p variables that satisfy $i \leq j$ in the right-hand side. This is because constraint (14) forces p_{ij} to be zero if $i > j$ is satisfied. Now we can continue with the proof.

Equivalence is obvious for the cases when i is greater than or equal to j since neither constraint is restrictive in this case. Therefore, we will consider remaining cases. Assume that (15) holds for all $i, j \in T$. We will use induction to show that if (15) holds, then (16) also holds. For the base case, $j = i + 1$, equivalence is simple:

$$p_{ij} = \sum_{k=i}^{j-1} p_{ik} + \sum_{k=i+1}^{j} p_{kj} = \sum_{k=i}^{j} p_{kk}.$$

Assume that (16) holds for all $i, j \in T$ such that $j \leq i + q$. Then, consider $j = i + q + 1$:

$$\begin{aligned}(q+1)p_{ij} &= \sum_{k=i}^{j-1} p_{ik} + \sum_{k=i+1}^{j} p_{kj} = \sum_{k=i}^{j-1}\sum_{l=i}^{k} p_{ll} + \sum_{k=i+1}^{j}\sum_{l=k}^{j} p_{ll} \\ &= \sum_{k=i}^{j-1}(j-k)p_{kk} + \sum_{k=i+1}^{j}(k-i)p_{kk} = (j-i)\sum_{k=i}^{j} p_{kk} \\ &= (q+1)\sum_{k=i}^{j} p_{kk}.\end{aligned}$$

Now assume that (16) holds for all $i, j \in T$. Then, we can rewrite the right-hand side of (15) as:

$$\begin{aligned}&\sum_{k=i}^{j-1} p_{ik} + \sum_{k=i+1}^{j} p_{kj} = \sum_{k=i}^{j-1}\sum_{l=i}^{k} p_{ll} + \sum_{k=i+1}^{j}\sum_{l=k}^{j} p_{ll} \\ &\quad = \sum_{k=i}^{j-1}(j-k)p_{kk} + \sum_{k=i+1}^{j}(k-i)p_{kk} = (j-i)\sum_{k=i}^{j} p_{kk} = (j-i)p_{ij}.\end{aligned}$$

□

Proposition 2 *An allocation rule that is feasible in FRF is Allocation Maximal if and only if p_{1m} is equal to one and $p_{ij} = \sum_{k=i}^{j} p_{kk}$ holds for all $i, j \in T$.*

Proof Assume that p is Allocation Maximal but equality (16) is not satisfied. Then, using Lemma 2, we also know equality (15) is not satisfied for some $i, j \in T$. We will show that we can increase some p_{ii} and still get feasibility in FRF which contradicts the Allocation Maximality of p.

First, notice that such a profile would have strictly positive difference, $j - i$. If difference is less than or equal to zero then equality (15) should be satisfied because of the monotonicity and non-negativity constraints. Then, we only need to consider profiles with $j - i > 0$. Consider the profile (x, y) which does not satisfy equality (15) and have the minimum difference, $y - x$, among all such profiles:

$$(y-x)p_{xy} > \sum_{n=x}^{y-1} p_{xn} + \sum_{n=x+1}^{y} p_{ny}.$$

Then, we know that (15) holds for all profiles (k, l) such that $(l - k) < (y - x)$. Using the induction argument from the proof of Lemma 2, we can show that equivalence holds for such profiles:

$$(l-k)p_{kl} = \sum_{n=k}^{l-1} p_{kn} + \sum_{n=k+1}^{l} p_{nl} \quad \forall k, l \in T \text{ such that } (l-k) < (y-x)$$
$$p_{kl} = \sum_{n=k}^{l} p_{nn} \quad \forall k, l \in T \text{ such that } (l-k) < (y-x).$$

For profile (x, y), we can write the following:

$$(y-x)p_{xy} > \sum_{n=x}^{y-1} p_{xn} + \sum_{n=x+1}^{y} p_{ny} = (y-x)\sum_{n=x}^{y} p_{nn}.$$

Then, using this result and constraint (14), we can conclude that:

$$p_{ij} > \sum_{n=i}^{j} p_{nn} \quad \forall i, j \in T \text{ such that } j \geq y \text{ and } i \leq x,$$

which means that $p_{1m} > \sum_{n=1}^{m} p_{nn}$. Now define $\epsilon = 1 - \sum_{n=1}^{m} p_{nn}$ so that we can exhibit a contradiction using p^* defined as follows:

$$p^*_{nn} = p_{nn} + \epsilon/m \quad \forall n \in T,$$
$$p^*_{ij} = \sum_{n=i}^{j} p^*_{nn} \quad \forall i, j \in T.$$

Because of the construction of p^*_{ij} variables, we know that monotonicity constraints hold and constraint (15) and (16) are satisfied with equality. Since $p^*_{ii} > p_{ii}$ for all $i \in T$, the existence of p^* contradicts the Allocation Maximality of p.

Now assume that p is Allocation Maximal, $p_{ij} = \sum_{n=i}^{j} p_{nn}$ holds for all $i, j \in T$ but p_{1m} is less than one. Then, we can construct a new allocation rule p^* that is feasible in FRF by increasing p_{nn} for all $n \in T$ by $\epsilon = (1 - p_{1m})/m$ as above. By Definition 1, p is not Allocation Maximal. This is a contradiction.

Now assume that we have an allocation rule that is feasible in FRF and it satisfies $p_{1m} = 1$ and $p_{ij} = \sum_{n=i}^{j} p_{nn}$ holds for all $i, j \in T$. Using Lemma 2, we also have equality (15) satisfied for all $i, j \in T$. Assume to the contrary that there exists a p^* feasible in FRF such that $p^*_{ii} \geq p_{ii}$ for all $i \in T$ and $p^*_{kk} > p_{kk}$ for some $k \in T$. Then, we have the following inequality:

$$\sum_{n=1}^{m} p^*_{nn} > \sum_{n=1}^{m} p_{nn} = p_{1m} = 1.$$

Using induction argument as in the proof of Lemma 2, one can also show that $p^*_{ij} \geq \sum_{n=i}^{j} p^*_{nn}$ should hold for any $i, j \in T$. Therefore, $p^*_{1m} \geq \sum_{n=1}^{m} p^*_{nn} > 1$, which means p^* is not feasible in FRF and this is a contradiction. □

We now show that all Allocation Maximal allocation rules in FRF are Posted Price mechanisms. We first need to define the Posted Price mechanism in general form. The seller (or the intermediary, if there is one) announces that he will post a price according to some distribution F and its probability mass function f. After observing the posted price, the buyer and the seller decide if they want to trade or not. Assuming that agents always favor trade more than status quo, we can write the Posted Price mechanism as:

$$p_{ij} = F(j) - F(i-1), \quad x_{ij} = \sum_{n=i}^{j} n f_n, \quad \forall i, j \in T.$$

In other words, trade probability, p_{ij}, is equal to the probability that posted price is in the set $\{i, i+1, \ldots, j-1, j\}$. Transfer value, x_{ij}, is equal to expected payment with respect to posted price probability mass function. The above definition of Posted Price mechanism allows the seller (intermediary) to pick a price distribution which will enable him to randomize the posted price he will announce.

Proposition 3 *A DIC, EIR mechanism is Allocation Maximal if and only if it is a Posted Price mechanism with the price mass function $\sum_{n=1}^{m} f(n) = 1$ where trade is preferred to status quo.*

Proof Assume that a DIC, EIR mechanism (p, x) is Allocation Maximal. Then, allocation rule p should be feasible in the FRF. By Proposition 2, we have $p_{1m} = 1$ and $p_{ij} = \sum_{n=i}^{j} p_{nn}$ holds for all $i, j \in T$. From constraints (12) and (13), we can write the following bounds for the transfer rule:

$$jp_{ij} - \sum_{k=i}^{j-1} p_{ik} \geq x_{ij} \geq \sum_{k=i+1}^{j} p_{kj} + ip_{ij}$$

$$j\sum_{n=i}^{j} p_{nn} - \sum_{k=i}^{j-1}\sum_{n=i}^{k} p_{nn} \geq x_{ij} \geq \sum_{k=i+1}^{j}\sum_{n=k}^{j} p_{nn} + i\sum_{n=i}^{j} p_{nn}$$

$$j\sum_{n=i}^{j} p_{nn} - \sum_{k=i}^{j-1}(j-k)p_{nn} \geq x_{ij} \geq \sum_{k=i+1}^{j} (k-i)p_{nn} + i\sum_{n=i}^{j} p_{nn}$$

$$\sum_{n=i}^{j} np_{nn} \geq x_{ij} \geq \sum_{n=i}^{j} np_{nn}.$$

We see that there is only one transfer rule feasible in the relaxation. This mechanism is equivalent to the following Posted Price mechanism with probability mass function f:

$$f_i = p_{ii} \quad \forall i \in T \Rightarrow p_{ij} = F(j) - F(i-1), \quad x_{ij} = \sum_{n=i}^{j} nf_n, \quad \forall i, j \in T.$$

Since p_{1m} is equal to one, we have $\sum_{n=1}^{m} f(n) = 1$. This mechanism awards the good to the buyer when both agents have the same type equal to the posted price. In other words, trade is preferred to status quo where seller keeps the good. Since we utilized Proposition 2 giving necessary and sufficient conditions, the proof is complete. □

Corollary 1 *The following system of equations is DIC-EIR implementable, and every feasible solution is a Posted Price mechanism where trade is preferred to status quo.*

$$x_{ij} = jp_{ij} - \sum_{k=i}^{j-1} p_{ik} \quad \forall i, j \in T \tag{17}$$

$$x_{ij} = ip_{ij} + \sum_{l=i+1}^{j} p_{lj} \quad \forall i, j \in T \tag{18}$$

$$p_{ij} \leq 1 \quad \forall i, j \in T \tag{6}$$

$$p_{ij} \geq 0 \quad \forall i, j \in T. \tag{7}$$

(10) , (11).

The proof directly follows from Lemma 2 and the definition of Posted Price mechanism. Restricting the allocation variables to be binary gives all Posted Price mechanisms with unique price where trade is preferred to status quo. Giving positive probability to more than one price might not be preferable due to practical concerns. Therefore, we will also investigate Posted Price mechanisms with a unique posted price and analyze its performance compared to Posted Price mechanism with not necessarily unique price in Sect. 5.

4 Bilateral trading under ambiguity

Until this point, we were interested in the general characteristics of DIC, EIR mechanisms. However, such analysis does not give specific information that a seller would need in practice. In order to specify the optimal trade probabilities and expected transfers, we need an objective function and an assumption about the priors. By relaxing the unique common prior assumption, which is commonly used in the literature, we introduce ambiguity into the problem framework. To deal with non-unique prior, we consider bilateral trading problem from the perspective of an ambiguity-averse seller.

As in Gilboa and Schmeidler (1989), we maximize the worst case expected utility of the seller subject to DIC, EIR constraints. The bilateral trade problem with ambiguity-averse agents was also considered by De Castro and Yannelis (2010). The authors show that when all agents are ambiguity-averse, for some class of max–min preferences DIC, EIR mechanisms are Ex-post efficient. For other examples of mechanism design problems with ambiguity, we refer to Bose et al. (2006) and Pınar and Kızılkale (2017). In the following two sections, we consider two types of ambiguity specifications. The first set based on interval uncertainty is one of the most widely used polyhedral uncertainty sets in robust combinatorial optimization literature. Interval uncertainty sets have been applied for a variety of problems in the fields of economics, production, transportation, etc. The reader may refer to Kouvelis and Yu (2013) for use of robustness approach in different environments. The second set is constructed based on ϕ-divergence ambiguity sets which reflects distributional robustness. As the uncertainty set constructed around the nominal distribution covers all possible probability distributions in that range, the ϕ-divergence-based ambiguity region is in accordance with the DIC concept of robust mechanism design.

4.1 Bilateral trading mechanism under box ambiguity set

In this section, we derive the robust counterpart for bilateral trading problem under box ambiguity set. First let us write our objective function as follows:

$$\max_{x,p\in X}\left\{\min_{h\in U}\sum_{i,j} h_{ij}\left(x_{ij}-ip_{ij}\right)\right\}, \tag{19}$$

where h_{ij} is density of joint distribution of agents type, X contains the constraints acting on p and x depending on the model used, and U is a set of ambiguity for the prior h and defined as follows:

$$U=\left\{l_{ij}\le h_{ij}\le u_{ij}\ ,\ \sum_{i,j} h_{ij}=1\right\}.$$

In this step, we propose a linear programming model for the robust counterpart of this problem using Lagrangian duality. Let us consider the inner part of Eq. (19) separately as follows:

$$\begin{aligned}
&\min_{l_{ij}\leq h_{ij}\leq u_{ij}} \sum_{i,j} h_{ij}(x_{ij}-ip_{ij})\\
&\text{s.t:} \sum_{i,j} h_{ij}=1.
\end{aligned}$$

then the Lagrangian can be written as:

$$\mathcal{L}(h,\mu)=\sum_{i,j} h_{ij}(x_{ij}-ip_{ij})+\mu\left(\sum_{i,j} h_{ij}-1\right),$$

and the dual function is:

$$g(\mu)=\min_{h}\mathcal{L}(h,\mu)=-\mu+\min_{h}\sum_{i,j} h_{ij}(x_{ij}-ip_{ij}+\mu),$$

so the Lagrange dual problem is:

$$\max_{\mu} -\mu+\sum_{i,j}\left(l_{ij}(x_{ij}-ip_{ij}+\mu)_{+}+u_{ij}(x_{ij}-ip_{ij}+\mu)_{-}\right),$$

as a result we obtain the following optimization problem as the robust counterpart problem:

$$\begin{aligned}
&\max_{x,p\in X,\mu,a,b} \sum_{i,j} -\mu+l_{ij}a_{ij}-u_{ij}b_{ij}\\
&\text{s.t:}\quad x_{ij}-ip_{ij}+\mu=a_{ij}-b_{ij} \qquad \forall i,j\in T,\\
&\quad a_{ij},b_{ij}\geq 0 \qquad \forall i,j\in T.
\end{aligned}$$

4.2 Bilateral trading mechanism under ϕ-divergence ambiguity set

In this section, we derive robust counterpart for our objective function under ϕ-divergence-based ambiguity region. Using ϕ-divergence measures, we probabilistically ensure that the ambiguity set contains the true distribution with a desired level of confidence. This is the main advantage of ambiguity sets based on ϕ-divergence measures over those based on box ambiguity. The reader can refer to Bayraksan and Love (2015) and Ben-Tal et al. (2010) for other advantages and applications related to ϕ-divergence measures in robust optimization problems, specially in data-driven setting. The construction of the uncertainty region from the given data is out of scope

Table 1 ϕ-Divergence measures

Divergence measure	$\phi(t)$	$\phi^*(s)$	$I_\phi(h,g)$
Burg entropy	$-\log(t)+t-1$	$-\log(1-s),\ s<1$	$\sum_i g_i \log(\frac{g_i}{h_i})$
Kullback–Leibler	$t\ \log(t)-t+1$	e^s-1	$\sum_i h_i \log\left(\frac{h_i}{g_i}\right)$
χ^2-Distance	$\frac{1}{t}(t-1)^2$	$2-2\sqrt{1-s},\ s<1$	$\sum_i \frac{(h_i-g_i)^2}{h_i}$
Hellinger distance	$(\sqrt{t}-1)^2$	$\frac{s}{1-s}, s<1$	$\sum_i(\sqrt{h_i}-\sqrt{g_i})^2$

of this paper. However, we refer the interested reader to Ben-Tal et al. (2013) which explains how to obtain an approximate uncertainty set for probability vectors h around nominal distribution, $\hat{h}$, as confidence set of confidence level at least $(1-\alpha)$, for example. ϕ-divergence measures are commonly used to reflect the distance between two probability distributions and defined as follows:

The ϕ-divergence measure between two probability distributions $h=(h_1,\ldots,h_n)^T \geq 0$ and $g=(g_1,\ldots,g_n)^T \geq 0$ in $\mathbb{R}^{\mathrm{n}}$ is

$$I_\phi(h,g)=\sum_{i=1}^{n} h_i\ \phi\left(\frac{h_i}{g_i}\right),\ \phi\in\Phi,$$

where Φ is the class of all convex functions $\phi(t), t\geq 0$ such that $\phi(1)=0, 0\phi(0/0)=0$ and $0\phi(p/0)=\lim_{u\to\infty}\phi(u)/u$.

We suppose that h comes from an uncertainty set constructed around a prior which can be derived from historical data, forecasting, simulation, etc., and four well-known ϕ-divergence functionals are applied as a measure of distance. Table 1 shows their characteristics (see Ben-Tal et al. 2013 for other specifications and choices for ϕ). The reader may also refer to Pardo (2005) for detailed and comprehensive review on this subject.

Consider the following robust linear constraint:

$$(a+Bh)^T x \leq d \quad \forall h\in\mathcal{M}, \tag{20}$$

where $a\in\mathbb{R}^{\mathrm{n}}$, $B\in\mathbb{R}^{\mathrm{n\times m}}$, $d\in\mathbb{R}$ are given parameters; $h\in\mathbb{R}^{\mathrm{m}}$ is the uncertain parameter; $x\in\mathbb{R}^{\mathrm{n}}$ is the optimization vector and the uncertainty region $\mathcal{M}$ is given by

$$\mathcal{M}=\left\{h\in\mathbb{R}^{\mathrm{m}}|\ h\geq 0,\ e^T h=1,\ I_\phi(h,g)\leq\rho\right\}, \tag{21}$$

where ρ controls the ambiguity level. The large value of ρ means that our confidence in data is low, and small value for ρ indicates that we trust in data.

Ben-Tal et al. (2013) proves that:

Theorem 1 *A vector $x \in \mathbb{R}$ satisfies (20) with uncertainty region $\mathcal{M}$ such that $h \in \mathcal{M}$ if and only if there exist $\eta \in \mathbb{R}$ and $\lambda \in$ such that (x, λ, η) satisfies*

$$\begin{cases} a^T x + \eta + \rho\lambda + \lambda \sum_{i=1}^{m} h_i \phi^* \left(\frac{b_i^T x - \eta}{\lambda} \right) \leq d, \\ \lambda \geq 0. \end{cases}$$

In Theorem 1, b_i are the ith column of B and $\phi^* \colon \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is the conjugate function of ϕ which is defined as follows:

$$\phi^*(s) = \sup_{t \geq 0} \{st - \phi(t)\}.$$

Now let us reconsider the objective function of proposed problem with the uncertainty region defined by $\mathcal{M}$ as follows:

$$\max_{x,p \in X} \left\{ \min_{h \in \mathcal{M}} \sum_{i,j} h_{ij} \left(x_{ij} - i p_{ij} \right) \right\},$$

which is equal to:

$$\max_{x,p \in X, h \in \mathcal{M}, \beta} \left\{ \beta \mid \sum_{i,j} h_{ij} \left(x_{ij} - i p_{ij} \right) \geq \beta \right\}. \tag{22}$$

Using Theorem 1 and Table 1, we can derive the robust counterpart for (22) with different divergence measures as follows:

Burg entropy

$$\max_{x,p \in X, \lambda \geq 0, \eta} \left\{ -\eta - \rho\lambda - \lambda \sum_{i,j} \left(h_{ij} \left(-\log \left(1 - \left(\frac{-\left(x_{ij} - i p_{ij} \right) - \eta}{\lambda} \right) \right) \right) \right) \right\},$$

Kullback–Leibler

$$\max_{x,p \in X, \lambda \geq 0, \eta} \left\{ -\eta - \rho\lambda - \lambda \sum_{i,j} \left(h_{ij} \left(e^{\left(\frac{-\left(x_{ij} - i p_{ij} \right) - \eta}{\lambda} \right)} - 1 \right) \right) \right\},$$

Table 2 Results for models without ambiguity

m	h-Distribution	$\mathrm{OF}_3(x^*)$	OF_2	OF_1
5	Uniform	0.480 (4)	0.480	0.500
	Normal	0.448 (5)	0.448	0.456
10	Uniform	0.840 (7)	0.840	0.861
	Normal	0.942 (8)	0.942	0.953
15	Uniform	1.222 (11)	1.222	1.237
	Normal	1.263 (11)	1.263	1.280
20	Uniform	1.592 (14)	1.592	1.609
	Normal	1.557 (14)	1.557	1.573

χ^2-distance:

$$\max_{x,p\in X,\lambda\geq 0,\eta} \left\{ -\eta - \rho\lambda - \lambda \sum_{i,j} \left(h_{ij} \left(2 - 2\sqrt{1 - \left(\frac{-(x_{ij} - ip_{ij}) - \eta}{\lambda} \right)} \right) \right) \right\},$$

Hellinger distance

$$\max_{x,p\in X,\lambda\geq 0,\eta} \left\{ -\eta - \rho\lambda - \lambda \sum_{i,j} \left(h_{ij} \left[\frac{\left(\frac{-(x_{ij}-ip_{ij})-\eta}{\lambda}\right)}{1 - \left(\frac{-(x_{ij}-ip_{ij})-\eta}{\lambda}\right)} \right] \right) \right\}.$$

We solve these models numerically, and the results are reported and discussed in the next section.

5 Computational results

In this section, we present the computational results related to the problems with the objective functions discussed in Sect. 4. For each problem, we construct three models with different constraint sets. Model 1 is the general model for robust bilateral trading model and considers the constraints (1), (2) and (6)–(9). We construct Model 2 by considering the constraints given in Corollary 1. This set of constraints lead to Posted Price mechanisms. In Model 3, we consider the same constraints as in Model 2 but p_{ij}'s are defined as binary variables and as a result Model 3 is even tighter than Model 2. This modification results in Posted Price mechanism with unique price which is more applicable. We consider these three models in our computational study to investigate how objective function value is changed if we want to apply the Posted Price mechanism.

In each table, first column is labeled with "m" which denotes the cardinality of set T. The second column entitled "h-distribution" specifies the distribution that h comes from. We consider two types of distributions for this purpose, "Uniform" stands for the uniform distribution such that $h_{ij} = 1/m^2$ and "Normal" refers to the normal

Table 3 Results for models under box ambiguity

m	h-Distribution	r	$\mathrm{OF}_3(x^*)$	OF_2	OF_1
5	Uniform	0.5	0.240 (4)	0.240	0.250
		0.25	0.360 (4)	0.360	0.375
		0.1	0.432 (4)	0.432	0.450
	Normal	0.5	0.224 (5)	0.224	0.228
		0.25	0.336 (5)	0.336	0.342
		0.1	0.403 (5)	0.403	0.410
10	Uniform	0.5	0.420 (8)	0.420	0.431
		0.25	0.630 (8)	0.630	0.646
		0.1	0.756 (8)	0.756	0.775
	Normal	0.5	0.471 (8)	0.471	0.477
		0.25	0.707 (8)	0.707	0.715
		0.1	0.848 (8)	0.848	0.858
15	Uniform	0.5	0.611 (11)	0.611	0.619
		0.25	0.917 (11)	0.917	0.928
		0.1	1.100 (11)	1.100	1.114
	Normal	0.5	0.631 (11)	0.631	0.640
		0.25	0.947 (11)	0.947	0.960
		0.1	1.137 (11)	1.137	1.152
20	Uniform	0.5	0.796 (14)	0.796	0.804
		0.25	1.194 (14)	1.194	1.207
		0.1	1.433 (14)	1.433	1.448
	Normal	0.5	0.778 (14)	0.778	0.787
		0.25	1.167 (14)	1.167	1.180
		0.1	1.401 (14)	1.401	1.416

distribution with $N \sim (\frac{m}{2}, (\frac{m}{8})^2)$. The last three columns provide objective function values for Models 3, Model 2 and Model 1, respectively. The value between parenthesis in the "$\mathrm{OF}_3(x^*)$" column is the unique price that has to be posted in Model 3 at optimality. The problem instances were formulated in GAMS 23.3.3 and solved using BARON (Tawarmalani and Sahinidis 2005) and COINIPOPT (Wächter and Biegler 2006) solvers.

In Table 2, we give results for the problem without ambiguity. This helps us to have a clear insight about the behavior of the problem with ambiguity.

In Table 3, the results for the problem under box ambiguity set are illustrated. The "r" column defines the range of the interval by specifying the upper and lower bounds using the following formulae: $u_{ij} = h_{ij}(1+r)$ and $l_{ij} = h_{ij}(1-r)$. We set three values of 0.1, 0.25 and 0.5 for "r" which reflect low, medium and high ambiguity, respectively. Results suggest that it is optimal for Posted Price mechanisms to have unique price.

Results for the problem under different ϕ-divergence measures are summarized in Tables 4, 5, 6 and 7. The column ρ is the same parameter introduced in (21) which

Table 4 Results for models under Burg Entropy divergence measure

m	h-Distribution	ρ	$\mathrm{OF}_3(x^*)$	OF_2	OF_1
5	Uniform	0.1	0.168 (4)	0.173	0.196
		0.01	0.358 (4)	0.358	0.378
		0.001	0.439 (4)	0.439	0.459
	Normal	0.1	0.146 (4)	0.169	0.195
		0.01	0.318 (5)	0.328	0.346
		0.001	0.404 (5)	0.404	0.419
10	Uniform	0.1	0.284 (7)	0.302	0.323
		0.01	0.620 (7)	0.622	0.642
		0.001	0.766 (7)	0.767	0.788
	Normal	0.1	0.326 (7)	0.343	0.361
		0.01	0.692 (8)	0.695	0.710
		0.001	0.858 (8)	0.858	0.869
15	Uniform	0.1	0.400 (11)	0.427	0.448
		0.01	0.883 (11)	0.892	0.911
		0.001	1.107 (11)	1.107	1.123
	Normal	0.1	0.412 (10)	0.439	0.461
		0.01	0.918 (11)	0.921	0.940
		0.001	1.146 (11)	1.146	1.164
20	Uniform	0.1	0.461 (12)	0.552	0.572
		0.01	1.154 (14)	1.159	1.177
		0.001	1.444 (14)	1.444	1.460
	Normal	0.1	0.423 (12)	0.539	0.559
		0.01	1.124 (14)	1.127	1.146
		0.001	1.409 (14)	1.409	1.427

determines the uncertainty region around h. The three values that ρ can take are 0.1, 0.01 and 0.001, which correspond to high, medium and low ambiguity, respectively.

As to be expected, the first observation is that as the ambiguity increases, we see that the objective function value decreases for all models and instances. Similarly, when ambiguity decreases, the difference between objective function values in all models also decreases and in low level of ambiguity the objective function values for Model 2 and Model 3 are equal in most cases. This valuable result means that when we encounter low level of ambiguity the proposed "Posted Price mechanism with unique price" which is quite common practice can provide a solution without significant loss of profit. We also observe that in the absence of ambiguity Model 2 and Model 3 provide the same solution which means that the Posted Price mechanisms with unique price are the optimal mechanisms. However, this is not the case for the models with ambiguity.

In Table 8, we summarize the amount of profit loss in percentage caused by the application of the Posted Price mechanism. The "Uncertainty set" column specifies the considered uncertainty set. The "Min", "Max" and "Avg." labels stand for the

Table 5 Results for models under Kullback–Leibler divergence measure

m	h-Distribution	ρ	$\mathrm{OF}_3(x^*)$	OF_2	OF_1
5	Uniform	0.1	0.125 (4)	0.138	0.165
		0.01	0.352 (4)	0.352	0.373
		0.001	0.438 (4)	0.438	0.458
	Normal	0.1	0.107 (4)	0.142	0.170
		0.01	0.312 (4)	0.322	0.341
		0.001	0.403 (5)	0.403	0.418
10	Uniform	0.1	0.204 (7)	0.234	0.259
		0.01	0.609 (7)	0.612	0.633
		0.001	0.765 (7)	0.765	0.786
	Normal	0.1	0.249 (7)	0.270	0.293
		0.01	0.681 (8)	0.684	0.700
		0.001	0.857 (8)	0.857	0.868
15	Uniform	0.1	0.283 (10)	0.328	0.351
		0.01	0.866 (11)	0.876	0.894
		0.001	1.105 (11)	1.105	1.121
	Normal	0.1	0.295 (10)	0.337	0.362
		0.01	0.900 (11)	0.904	0.924
		0.001	1.144 (11)	1.144	1.162
20	Uniform	0.1	0.363 (13)	0.421	0.444
		0.01	1.132 (14)	1.138	1.157
		0.001	1.441 (14)	1.441	1.458
	Normal	0.1	0.353 (12)	0.413	0.435
		0.01	1.101 (14)	1.106	1.125
		0.001	1.407 (14)	1.407	1.424

minimum, maximum and average profit loss in percentage, respectively, considering the instances presented in Tables 3, 4, 5, 6 and 7. The "Unique Posted Price" column represents the difference between objective function values of Model 3 and Model 1, and the "Posted Price" column provides the difference between objective function values of Model 2 and Model 1. For example, in the uncertainty set defined by Burg Entropy divergence measure, on average we lose 7.2% of the objective function value for optimal DIC, EIR mechanism if we insist on a Posted Price mechanism with unique price.

6 Conclusion

In this study, we focused on the robust bilateral trade problem with discrete types. First, we formulated a general model for DIC, EIR mechanisms and considered its relaxation which proved to be useful in two different ways. Given any allocation rule, the relaxation can be used to find transfer rules that give DIC, EIR mechanisms. Besides,

Table 6 Results for models under χ^2-distance divergence measure

m	h-Distribution	ρ	$\mathrm{OF}_3(x^*)$	OF_2	OF_1
5	Uniform	0.1	0.200 (4)	0.200	0.277
		0.01	0.394 (4)	0.394	0.414
		0.001	0.451 (4)	0.451	0.471
	Normal	0.1	0.226 (4)	0.241	0.254
		0.01	0.356 (5)	0.360	0.378
		0.001	0.417 (5)	0.417	0.430
10	Uniform	0.1	0.442 (7)	0.449	0.469
		0.01	0.685 (7)	0.687	0.707
		0.001	0.787 (7)	0.788	0.809
	Normal	0.1	0.492 (8)	0.506	0.521
		0.01	0.766 (8)	0.766	0.779
		0.001	0.883 (8)	0.883	0.894
15	Uniform	0.1	0.626 (10)	0.640	0.660
		0.01	0.983 (11)	0.986	1.004
		0.001	1.141 (11)	1.141	1.156
	Normal	0.1	0.646 (11)	0.660	0.681
		0.01	1.019 (11)	1.020	1.038
		0.001	1.180 (11)	1.180	1.198
20	Uniform	0.1	0.802 (14)	0.831	0.850
		0.01	1.283 (14)	1.284	1.302
		0.001	1.487 (14)	1.487	1.504
	Normal	0.1	0.786 (14)	0.809	0.829
		0.01	1.251 (14)	1.251	1.269
		0.001	1.453 (14)	1.453	1.470

constraints of the relaxation can be used to formulate Posted Price mechanisms which are DIC and EIR. On the other hand, we show that Ex-post Efficiency can be obtained together with DIC and EIR if and only if convex hull of agents' efficient type sets have finite intersection. When agents share the same type set with cardinality larger than or equal to four, Ex-post efficiency is infeasible but one can consider Allocation Maximal mechanisms. We showed that the Posted Price mechanisms are not the only DIC, EIR mechanisms but they are the only ones satisfying Allocation Maximality together with DIC, EIR. Lastly, we introduced different sets of priors and considered the problem in the shoes of ambiguity-averse intermediary. To manage the ambiguity in the probability distribution of agents types, we derived robust counterparts for the proposed objective function under box and ϕ-divergence ambiguity specifications. We also examined the performance of the proposed robust models based on an extensive numerical study.

Table 7 Results for models under Hellinger distance divergence measure

m	h-Distribution	ρ	$\mathrm{OF}_3(x^*)$	OF_2	OF_1
5	Uniform	0.1	0.066 (4)	0.087	0.106
		0.01	0.309 (4)	0.309	0.329
		0.001	0.422 (4)	0.422	0.442
	Normal	0.1	0.056 (4)	0.094	0.113
		0.01	0.272 (4)	0.284	0.306
		0.001	0.385 (5)	0.386	0.403
10	Uniform	0.1	0.107 (7)	0.147	0.166
		0.01	0.531 (7)	0.535	0.556
		0.001	0.735 (7)	0.736	0.757
	Normal	0.1	0.138 (7)	0.172	0.191
		0.01	0.592 (8)	0.602	0.618
		0.001	0.823 (8)	0.823	0.834
15	Uniform	0.1	0.150 (9)	0.205	0.223
		0.01	0.754 (10)	0.764	0.784
		0.001	1.060 (11)	1.060	1.077
	Normal	0.1	0.155 (10)	0.210	0.229
		0.01	0.780 (11)	0.789	0.809
		0.001	1.098 (11)	1.098	1.116
20	Uniform	0.1	0.192 (12)	0.263	0.280
		0.01	0.939 (14)	0.992	1.011
		0.001	1.382 (14)	1.382	1.399
	Normal	0.1	0.188 (12)	0.256	0.274
		0.01	0.951 (14)	0.964	0.985
		0.001	1.349 (14)	1.349	1.360

Table 8 Profit loss in percentage for different models

Uncertainty set	Unique Posted Price (OF_3)			Posted Price (OF_2)		
	Min	Max	Avg.	Min	Max	Avg.
Box	1.0	4.0	1.8	1.0	4.0	1.8
Burg Entropy	1.1	25.1	7.2	1.1	13.3	3.9
Kullback–Leibler	1.2	37.1	9.2	1.2	16.5	4.8
χ^2	1.1	27.8	4.6	1.1	27.8	3.6
Hellinger	1.0	50.4	14.2	1.0	18.0	5.5

Appendix

Proof of Proposition 1

Proof Assume that there exists a DIC, EIR and Ex-post Efficient mechanism (p^*, x) but convex hull of sets T_b^* and T_s^* have infinite intersection. Then, there exist $b_j \in T_b^*$

and $s_i \in T_s^*$ such that b_j is strictly less than s_i. By definition of efficient type sets, there exist types $s_l \in T_s$ and $b_k \in T_b$ satisfying $s_l < b_j$ and $b_k > s_i$. Then, we can write $s_l < b_j < s_i < b_k$ so that $p_{lj} = p_{lk} = p_{ik} = 1$ holds. We know from Lemma 1 that $x_{lj} = x_{lk} = x_{ik}$ should also hold in order to satisfy DIC constraints. Given all this information, let us check EIR constraints. We see that $b_j \geq x_{lj} \geq s_l$ and $b_k \geq x_{ik} \geq s_i$ cannot be satisfied together with $x_{lj} = x_{ik}$ since we have $b_j < s_i$. Hence, there is no transfer rule we can use together with p^* to have a DIC, EIR mechanism. This is a contradiction.

Now we start from efficient type sets T_b^* and T_s^* whose convex hulls have finite intersection. If both efficient type sets are empty, we have a trivial case $b_m \leq s_1$ where seller always values the good more. Then, any Posted Price mechanism imposes Ex-post Efficiency. In the non-trivial case, both sets are non-empty and minimum type, $\underline{b}$, in T_b^* should be bigger than or equal to maximum type, $\bar{s}$, in T_s^*. Here, any Posted Price mechanism with unique price $x \in [\bar{s}, \underline{b}]$ will be Ex-post efficient. Since all Posted Price mechanisms are DIC, EIR, the proof is complete. □

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