Generic initial ideals of modular polynomial invariants

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ABSTRACT

We study the generic initial ideals (gin) of certain ideals that arise in modular invariant theory. For all cases an explicit generating set is known we compute the generic initial ideal of the Hilbert ideal of a cyclic group of prime order for all monomial orders. We also consider the Klein four group and note that its Hilbert ideals are Borel fixed with certain orderings of the variables. In all situations we consider, it is possible to select a monomial order such that the gin of the Hilbert ideal is equal to its initial ideal. Along the way we show that gin respects a permutation of the variables in the monomial order.

1. Introduction

For a homogeneous ideal \( I \) in a polynomial ring, its generic initial ideal \( \text{gin}_{>}(I) \) with respect to a term order \( > \) is the ideal of initial monomials after a generic change of coordinates. The generic initial ideal measures how close to a \( > \)-segment ideal \( I \) can be made by a linear and invertible substitution. It encodes much information on the combinatorial, geometrical and homological properties of \( I \) and the associated variety and plays an important role in the computational aspects of commutative algebra and algebraic geometry. For instance generic initial ideals were used in Hartshorne’s proof of the connectedness of Hilbert schemes. Describing \( \text{gin}_{>}(I) \) is a very difficult task in general and despite their significance, there are relatively few classes of ideals for which generic initial ideals are explicitly computed. We refer the reader to [7] for a survey of results on this matter.

In this paper we study generic initial ideals that arise in invariant theory. We consider a finite dimensional module \( V \) of a group \( G \) over an infinite field \( F \). There is an induced action on the symmetric algebra \( F[V] := S(V^\ast) \) on \( V^\ast \). This is a polynomial algebra \( F[x_1, \ldots, x_n] \), where \( x_1, \ldots, x_n \) is a basis for \( V^\ast \). A classical object is the ring of invariants \( F[V]^G := \{ f \in F[V] \mid \sigma(f) = f \text{ for all } \sigma \in G \} \) which is a

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graded subalgebra of \(F[V]\). The ideal in \(F[V]\) generated by homogeneous invariants of positive degree is the Hilbert ideal of \(V\) and we denote it by \(H(V)\). The Hilbert ideal and its quotient in \(F[V]\) often contain information about the invariant ring itself. It plays an important role in finding generators of \(F[V]^G\) or obtaining degree bounds for them. When the characteristic of the field divides the order of the group, i.e., \(V\) is a modular module, the invariant ring is more complicated and difficult to obtain. Invariants are not known in general even in the simplest modular situation when \(G\) is a cyclic group of prime order. For this group we consider the cases where an explicit generating set is known for the Hilbert ideal, and we compute the generic initial ideals of these Hilbert ideals for all orders. It turns out that, with the upper triangular ordering of the variables, \(gin\) is equal to the initial ideal of the Hilbert ideal in these cases. A natural question one encounters in these computations is how \(gin\) changes when the variables in the monomial order are permuted. Since we did not find a source in the literature that addresses this question, we include a compact proof of the fact that \(gin\) is also permuted in the same way in Section 2. Case by case analysis and computations of the \(gin\) of the modular Hilbert ideals of a cyclic group of prime order are done in Section 3.

In the final section we note that any Hilbert ideal of the Klein four group in characteristic two is Borel fixed with the right choice of the monomial order and therefore \(gin\) is equal to the Hilbert ideal itself. We feel that these findings support Conjecture 13 which states that for a given module over characteristic \(p\) of a \(p\)-group there is always a choice of basis for the module and a monomial order such that \(gin\) of the Hilbert ideal is equal to its initial ideal.

We refer the reader to [8, §4] and to [3], [4] for more background on generic initial ideals and invariant theory, respectively.

2. Permutation of variables and \(gin\)

In this section we do not consider group actions so we let \(S\) (instead of \(F[V]\)) denote the polynomial ring \(F[x_1, \ldots, x_n]\) in \(n\) variables. Let \(I\) be a homogeneous ideal in \(S\). We fix a term order \(>\) on the set of monomials in \(S\). The largest monomial that appears in a polynomial \(f \in S\) is called the initial monomial of \(f\) and is denoted by \(In_>(f)\). We denote the ideal in \(S\) generated by the initial monomials of elements in \(I\) with \(In_>(I)\). Let \(\pi\) be a permutation of \(\{1, 2, \ldots, n\}\). Then \(\pi\) induces an isomorphism of \(S\) via \(\pi(x_i) = x_{\pi(i)}\). Note that this isomorphism sends monomials to monomials. Let \(\triangleright_{\pi}\) denote the term order such that

\[
\pi(M_1) \triangleright_{\pi} \pi(M_2) \text{ if and only if } M_1 > M_2.
\]

Let \(S_d\) denote the \(d\)-th homogeneous component of \(S\) and we consider the \(t\)-th exterior power \(\wedge^t S_d\) of \(S_d\). Recall that an element \(m_1 \wedge m_2 \cdots \wedge m_t\), where \(m_i\) is a monomial of degree \(d\) with \(m_1 > m_2 > \cdots > m_t\), is called a standard exterior monomial of \(\wedge^t S_d\) with respect to \(>\). One can order standard exterior monomials lexicographically: If \(m_1 \wedge m_2 \cdots \wedge m_t\) and \(w_1 \wedge w_2 \cdots \wedge w_t\) are two standard exterior monomials with respect to \(>\), then we set

\[
m_1 \wedge m_2 \cdots \wedge m_t > w_1 \wedge w_2 \cdots \wedge w_t
\]

if \(m_i > w_i\) for the smallest index \(i\) with \(m_i \neq w_i\). We denote the largest standard exterior monomial in the support of \(e \in \wedge^t S_d\) with \(In_{>}(e)\). Standard exterior monomials with respect to \(\triangleright_{\pi}\) are defined and ordered similarly.

Let \(\alpha \in GL_n(F)\). Then \(\alpha = (\alpha_{ij})\) induces a degree preserving isomorphism on \(S\) by \(x_j \mapsto \alpha_{ij} x_i + \alpha_{2j} x_2 + \cdots + \alpha_{nj} x_n\), for \(1 \leq j \leq n\). We consider the polynomial ring in the extended set of variables \(R = F[x_1, \ldots, x_n, \alpha_{ij}, |1 \leq i, j \leq n|]\). We extend \(\pi\) to \(R\) by letting \(\pi(\alpha_{ij}) = \alpha_{\pi(i)j}\).
Lemma 1. Let \( f \in S \) and \( \alpha \in GL_n(F) \). Consider \( \alpha(f) \) as a polynomial in \( x_1, \ldots, x_n \) with coefficients in \( F[\alpha_{ij}] \). Let \( M \) be a monomial in \( S \) that appears in \( \alpha(f) \) with coefficient \( c \in F[\alpha_{ij}] \). Then the coefficient of \( \pi(M) \) in \( \alpha(f) \) is \( \pi(c) \).

Proof. Note that \( \pi(\alpha(x_j)) = \pi(\sum_{1 \leq i \leq n} \alpha_{ij} x_i) = \sum_{1 \leq i \leq n} \alpha_{ij} \pi(x_i) = \alpha(x_j) \). Since both \( \pi \) and \( \alpha \) are ring homomorphisms it follows that \( \pi(\alpha(f)) = \alpha(f) \) for all \( f \in S \). We write \( \alpha(f) = \sum c_k M_k \), where \( c_k \in F[\alpha_{ij}] \) and \( M_k \) is a monomial in \( S \). Then we have

\[
\alpha(f) = \pi(\alpha(f)) = \sum \pi(c_k) \pi(M_k).
\]

Therefore we get \( \sum c_k M_k = \sum \pi(c_k) \pi(M_k) \). Since \( \pi \) is a permutation of the monomials in \( S \), the assertion of the lemma follows. \( \square \)

Definition 2. For each homogeneous ideal \( I \) there is a Zariski open set \( U \subseteq GL_n(F) \) and a monomial ideal \( J \) such that \( \ln_\alpha(I) = J \) for all \( \alpha \in U \). The ideal \( J \) is called the generic initial ideal with respect to \( > \) and is denoted \( \ln_\alpha(I) \), see [8, 4.13] and [5, 15.18].

Theorem 3. Let \( I \) be a homogeneous ideal. Then we have

\[
\ln_\alpha(I) = \pi(\ln_\alpha(I)).
\]

Proof. Consider a homogeneous component \( I_d \) of \( I \) with a basis \( f_1, \ldots, f_i \). Let \( m_{j_1}, \ldots, m_{j_i} \) be monomials in \( S \) that appear in \( \alpha(f_1), \ldots, \alpha(f_i) \) with coefficients \( c_{j_1}, \ldots, c_{j_i} \in F[\alpha_{ij}] \), respectively. Assume that \( c_{j_1} m_{j_1} \wedge \cdots \wedge c_{j_i} m_{j_i} \) is a multiple of the standard exterior monomial \( m_1 \wedge \cdots \wedge m_i \) (with respect to \( > \)) in \( \wedge^i S_d \). Call this multiple \( c \in F[\alpha_{ij}] \). Then by the previous lemma, \( \pi(m_{j_1}), \ldots, \pi(m_{j_i}) \) appear in \( \alpha(f_1), \ldots, \alpha(f_i) \) with coefficients \( \pi(c_{j_1}), \ldots, \pi(c_{j_i}) \in F[\alpha_{ij}] \), respectively. Since ranking of the monomials in \( > \) is preserved in \( \pi \) after we apply \( \pi \), it follows that the coefficient of the standard exterior monomial \( \pi(m_1) \wedge \cdots \wedge \pi(m_i) \) (with respect to \( >_\pi \) in \( \pi(c_{j_1}) \pi(m_{j_1}) \wedge \cdots \wedge \pi(c_{j_i}) \pi(m_{j_i}) \) is \( \pi(c) \). Therefore we have

\[
\alpha(f_1) \wedge \cdots \wedge \alpha(f_i) = \sum_{m_{j_1} \wedge \cdots \wedge m_{j_i} >_{\pi} m_1 \wedge \cdots \wedge m_i} c(m_{j_1}, \ldots, m_{j_i})(m_{j_1} \wedge \cdots \wedge m_{j_i})
\]

\[
= \sum_{m_{j_1} \wedge \cdots \wedge m_{j_i} >_{\pi} m_1 \wedge \cdots \wedge m_i} \pi(c(m_{j_1}, \ldots, m_{j_i})) \pi(m_{j_1} \wedge \cdots \wedge m_{j_i}).
\]

Since \( \pi \) is a permutation of variables in \( F[\alpha_{ij}] \), \( c(m_{j_1}, \ldots, m_{j_i}) \) is the zero polynomial if and only if \( \pi(c(m_{j_1}, \ldots, m_{j_i})) \) is the zero polynomial. So we have that if \( w_1 \wedge \cdots \wedge w_d \) is the largest exterior monomial (with respect to \( > \)) with the property that there is \( \alpha \in GL_n(F) \) with \( \ln_\alpha(\alpha(f_1) \wedge \cdots \wedge \alpha(f_i)) = w_1 \wedge \cdots \wedge w_d \), then \( \pi(w_1) \wedge \cdots \wedge \pi(w_d) \) is the largest exterior monomial (with respect to \( >_{\pi} \)) such that there exists \( \alpha \in GL_n(F) \) with \( \ln_\alpha(\alpha(f_1) \wedge \cdots \wedge \alpha(f_i)) = \pi(w_1) \wedge \cdots \wedge \pi(w_d) \). Since \( \ln_\alpha(I) \) and \( \ln_{\alpha_{i,j}}(I) \) are generated by \( w_1, \ldots, w_d \) and \( \pi(w_1), \ldots, \pi(w_d) \) (see [8, 4.14, 4.13]), respectively and \( d \) is arbitrary, the result follows. \( \square \)

Remark 4. If \( \ln_\alpha(I) \) satisfies some property with respect to \( > \), then \( \ln_{\alpha_{i,j}}(I) \) also satisfies that property after a permutation of variables. Since there are \( n! \) permutations of the variables, it also follows that the number of generic initial ideals of \( I \) is divisible by \( n! \).
Remark 5. There is a close connection with generic initial ideals and Borel fixed ideals (which we use in Section 3). This connection still exists after the permutation of variables in the monomial order but one needs to replace the standard Borel subgroup of upper triangular matrices with the non-standard Borel subgroup accommodating the permutation.

3. Gin of modular Hilbert ideals of cyclic groups of prime order

In this section, $G$ denotes a cyclic group of prime order $p$ and we assume that the characteristic of $F$ is also $p$. There are exactly $p$ indecomposable $G$-modules $V_1, \ldots, V_p$ over $F$ and each indecomposable module $V_n$ is afforded by a Jordan block of dimension $n$ with 1's on the diagonal. If $V$ is a direct sum of indecomposable modules $V_{n_1}, \ldots, V_{n_l}$, then we write $V = \oplus_{1 \leq j \leq k} V_{n_j}$. In the sequel we consider the action of the Borel subgroup on $F[V]$. We always assume that this action is compatible with the ranking of the variables, i.e., the action of a matrix in the Borel subgroup on the $i$-th variable in the monomial order is given by the $i$-th column of the matrix, see Remark 5. We compute all generic initial ideals of $H(V)$ for all cases for which an explicit generating set for $H(V)$ is known.

3.1. The monomial cases: $IV_2 \oplus mV_3$ and $V_4$

For $V = IV_2 \oplus mV_3$, we identify $F[V]$ with $F[x_i, y_j, z_t, \ 1 \leq i, j \leq l + m, \ l + 1 \leq t \leq l + m]$. For $1 \leq i \leq l$ \{x_i, y_i\} spans a copy of $V_2^*$ and \{x_i, y_i, z_i\} spans a copy of $V_3^*$ for $l + 1 \leq i \leq m + l$. In [9, 2.6] it is shown that $H(IV_2 \oplus mV_3)$ is generated by

$$L = \{x_i, y_i | 1 \leq i \leq l\} \cup \{x_i, y_i, z_i | 1 + l \leq i, j \leq m + l, \ i \leq j\}.$$

Proposition 6. There is a monomial order such that $H(IV_2 \oplus mV_3)$ is Borel fixed. Furthermore, there are $(2l + 3m)!$ generic initial ideals of $H(IV_2 \oplus mV_3)$ each of which is generated by $\pi(L)$ for some permutation $\pi$ of the variables in $F[IV_2 \oplus mV_3]$.

Proof. Let $> be a monomial order with $x_{m+l} > \cdots > x_1 > y_{m+l} > \cdots > y_1 > z_{m+l} > \cdots > z_{l+1}$ and let $\theta$ be a member of the non-standard Borel subgroup $\theta_{2l+3m}$ accommodating the ranking of the variables. Let $J$ denote the ideal generated by the subset $U = \{x_i | 1 \leq i \leq l + m\} \cup \{y_{i,j} | 1 + l \leq i, j \leq m + l, \ i \leq j\}$ of $L$. Notice that $J$ is a strongly stable ideal. It follows that $\theta$ sends $J$ into itself. On the other hand the remaining generators of $H(IV_2 \oplus mV_3)$ in $L \setminus U$ are pure powers of variables of degree $p$ and since we are in characteristic $p$, $\theta$ sends a $p$-th power of a variable to a combination of $p$-th powers of variables. But such a combination is in $H(IV_2 \oplus mV_3)$ as well since $H(IV_2 \oplus mV_3)$ contains all $p$-th powers of variables. It follows that $H(IV_2 \oplus mV_3)$ is Borel fixed and so we have $\text{gin}_p(H(IV_2 \oplus mV_3)) = H(IV_2 \oplus mV_3)$ by [2, 1.8]. The final assertion of the proposition follows from Theorem 3. \hfill $\square$

For $V = V_4$, we identify $F[V_4]$ with $F[x_1, x_2, x_3, x_4]$. From [9, 3.2] we get that $H(V_4)$ is generated by

$$M = \{x_1, x_2^2, x_2x_3^{p-3}, x_3^{p-1}, x_4^p\}.$$

Proposition 7. There is a monomial order such that $H(V_4)$ is Borel fixed. Furthermore, there are $4!$ generic initial ideals of $H(V_4)$ each of which is generated by $\pi(M)$ for some permutation $\pi$ of the variables in $F[V_4]$.

Proof. Let $> be a monomial order such that $x_1 > x_2 > x_3 > x_4$ and let $\theta \in \theta_4$. Note that the ideal generated by the subset $M' = \{x_1, x_2^2, x_2x_3^{p-3}, x_3^{p-1}\}$ is strongly stable and hence $\theta$ sends this ideal into itself. Furthermore, the only other generator is a pure $p$-th power of a variable. Since $\theta$ sends a $p$-th power of a variable to a combination of $p$-th powers of variables and all $p$-th powers of variables are contained in...
$H(V_4)$, it follows that $H(V_4)$ is Borel fixed. So $\text{gin}_H(H(V_4)) = H(V_4)$, by [2, 1.8]. The final assertion of the proposition follows from Theorem 3 as in the previous case. □

3.2. The non-monomial case: $V_5$

We identify $F[V_5]$ with $F[x_1, x_2, x_3, x_4, x_5]$. In [9, 4.1] it is shown that

$$T = \{x_1, x_2^2, x_3^2 - 2x_2x_4 - x_2x_3, x_2x_3x_4, x_2x_4^{p-1}, x_3x_4^{p-1}, x_4^{p-1}, x_5^{p}\}$$

is a generating set for $H(V_5)$ for $p > 5$. We denote the generators of $H(V_5)$ in $T$ with $f_i$ for $1 \leq i \leq 8$ with $f_1 = x_1$ and $f_8 = x_5^p$. Let $\alpha = (\alpha_1, \ldots, \alpha_8)$ be an element in $F_5$. Define $C = \alpha_3^{-2}(2\alpha_2\alpha_3 - 2\alpha_2\alpha_4 - \alpha_2\alpha_3)$ and $D = \alpha_3^{-2}(-2\alpha_2\alpha_4)$. We describe a generating set for $\alpha(H(V_5))$.

**Lemma 8.** Assume the notation of the previous paragraph and that $p > 5$. Then $\alpha(H(V_5))$ is generated by

$$T' := \{x_1, x_2^2, x_3^2 + Cx_2x_3 + Dx_2x_4, x_2x_3x_4, x_2x_4^{p-1}, x_3x_4^{p-1}, x_4^{p-1}, x_5^p\}$$

**Proof.** For $1 \leq i \leq 8$, let $J_i$ denote the ideal generated by $\alpha(f_1), \ldots, \alpha(f_i)$. Since $\alpha$ is a ring homomorphism, $\alpha(H(V_5))$ is generated by $\alpha(f_1), \ldots, \alpha(f_8)$ and so we have $\alpha(H(V_5)) = J_8$. We also denote $x_3^3 + Cx_2x_3 + Dx_2x_4$ with $f_3$.

Note that, since $\alpha$ sends $x_1$ to a multiple of $x_1$ and $x_2$ to a linear combination of $x_1$ and $x_2$, we have that $J_3 = (x_1, x_2^2)$. On the other hand direct computation gives $\alpha(x_1^2 - 2x_2x_4 - x_2x_3) = (\sum_{1 \leq i \leq 3} \alpha_2x_i)^2 - 2(\sum_{1 \leq i \leq 4} \alpha_2x_i)\sum_{1 \leq i \leq 3} \alpha_3x_i + (\sum_{1 \leq i \leq 3} \alpha_3x_i)^2 \equiv \alpha_3^2f_3^2 \pmod{J_3}$. It follows that $J_3 = (x_1, x_2^2, f_3^2)$. We finish the proof by showing that $\alpha(f_i)$ is a scalar multiple of $f_1$ modulo $J_{i-1}$ for $4 \leq i \leq 8$. This gives $J_i = J_{i-1}$ for $4 \leq i \leq 8$ and hence $J_8 = \alpha(H(V_5))$ is generated by $T'$.

Note that $x_2x_3^3 = x_2x_3^3 - x_2(Cx_3 + Dx_4) \in J_3$. Therefore, since $x_1, x_2^2 \in J_3$ as well, we have $\alpha(f_4) = \alpha(x_2x_3x_4) \equiv \alpha_2\alpha_3\alpha_4x_2x_3x_4 \pmod{J_3}$. We also have

$$\alpha(f_5) = \alpha(x_2x_4^{p-3}) \equiv \alpha_2\alpha_3x_2^{p-3} \pmod{J_5},$$

where the first equivalence uses $x_1 \in J_4$ and the second equivalence uses $x_2^2, x_2x_4^{p-3} \in J_4$ and $p > 5$. To compute $\alpha(f_6)$ we note the identities $x_2 = x_3^2, Cx_2x_3 = Cx_2x_3$, $x_2x_4^{p-1} = x_2x_4^{p-1}$, $x_3x_4^{p-1} = x_3x_4^{p-1}$, $x_4^{p-1} = x_4^{p-1}$, $x_5^p = x_5^p$, and the third equivalence because $x_2x_3, x_2x_4, x_2x_4^{p-4} \in J_5$. The final equivalence follows because $x_2^2, x_2x_3x_4, x_2^2, x_2x_4^{p-4} \in J_5$ and $p > 5$. For $f_7$ we have

$$\alpha(x_4^{p-1}) = (\alpha_4x_2 + \alpha_4x_3 + \alpha_4x_4)^{p-1} \equiv (\alpha_4x_2 + \alpha_4x_3 + \alpha_4x_4)^{p-1} \pmod{J_6},$$

where the first equivalence uses $x_1, x_2^2 \in J_5$ and we have the third equivalence because $x_2x_4^{p-3}, x_2x_4^{p-3}, x_2x_4^{p-4} \in J_5$. The final equivalence follows because $x_2^2, x_2x_3x_4, x_2^2, x_2x_4^{p-4} \in J_5$ and $p > 5$. For $f_7$ we have

$$\alpha(f_7) = \alpha_2^2x_5^p \pmod{J_7}$$

because $x_5^p \in J_7$ for $1 \leq i \leq 4$. □
Note that the sets $T$ and $T'$ differ by one polynomial only. For the simplicity of notation we set $f_3 = x_2^2 + C x_2 x_3 + D x_2 x_4$ and so that $\alpha(H(V_0))$ is generated by $f_i$ for $1 \leq i \leq 8$.

Fix a term order $\succ$ with $x_1 > \cdots > x_5$. We set

$$A = \{x_1, x_2^2, x_2 x_3, x_3^2, x_4, x_2 x_4^{p-4}, x_3 x_4^{p-3}, x_4^{p-1}, x_5^p\}$$

and

$$B = \{x_1, x_2^2, x_2 x_3, x_3^2, x_4, x_2 x_4^{p-5}, x_3 x_4^{p-3}, x_4^{p-1}, x_5^p\}.$$ 

We compute the Gröbner basis for $\alpha(H(V_0))$ for a special class of $C$.

**Lemma 9.** Let $\alpha = (\alpha_{ij}) \in f_{\alpha}$ such that $C \neq 0$ and assume that $p > 5$. Then $\text{In}_{\succ} (\alpha(H(V_0)))$ is generated by $A$ if $x_2^2 > x_2 x_4$ and by $B$ otherwise.

**Proof.** We recall that a Gröbner basis can be obtained by reduction of $S$-polynomials by polynomial division, see [1, §1.7]. Before we distinguish between two orders we collect a couple of more elements from $\alpha(H(V_0))$. The reduction of the $S$ polynomial $S(f_3, f_5)$ of $f_3$ with $f_5$ via $f_3, f_5, f_4$ is $x_2^2$ and the $S(f_3, f_4)$ is $x_3^2 x_4 + D x_2 x_4^2$. We denote $x_3^2$ and $x_2^2 x_4 + D x_2 x_4^2$ by $f_9, f_{10}$, respectively.

We first consider the case $x_2^2 > x_2 x_4$. Note then the set $A$ consists of $\text{In}_{\succ} (f_i)$ for $1 \leq i \leq 10, i \neq 4$ ($f_4 = C^{-1} (x_4 f_5 - f_{10})$ and $\text{In}_{\succ} (f_4)$ is divisible by $\text{In}_{\succ} (f_{10})$). Therefore it remains to show that this set of polynomials satisfies the Buchberger criterion. That is, the $S$-polynomial of any pair of polynomials $f_i, f_j$ with $1 \leq i, j \leq 10$ and $i, j \neq 4$ reduces to zero. Since the $S$-polynomial of two monomials is zero, it suffices to consider the $S$-polynomials involving either $f_5$ or $f_{10}$. We go through the pairs and write the polynomials in the order they appear in the polynomial division: $S(f_2, f_4)$ reduces to zero via $f_2, f_5, f_6, f_{10}$. Both $S$-polynomials $S(f_3, f_6)$ and $S(f_3, f_{10})$ reduce to zero via $f_3, f_5$ and $f_3, f_5, f_{10}$ and the $S$-polynomial $S(f_3, f_{10})$ reduces to zero via $f_2, f_3, f_5, f_{10}$. The $S$-polynomials $S(f_5, f_{10}), S(f_6, f_{10})$, $S(f_7, f_{10})$ reduce to zero at one step each via $f_5$, finally $S(f_5, f_{10})$ reduces to zero via $f_3$ and $f_{10}$. We have considered all pairs whose initial terms are not relatively prime. So the proof for this case is complete because the $S$-polynomial of two polynomials that have relatively prime initial terms reduces to zero.

Now we consider the case $x_2 x_4 > x_2^2$. Define $f_{11} = x_4^{p-6} f_{10} - D f_3 = x_3^2 x_4^{p-5}$. Since $D$ is a non-zero scalar, from this equality we have that $f_{11}$ for $1 \leq i \leq 11$ and $i \neq 4, 5, 10$ also generate $\alpha(H(V_0))$. Since the set $B$ consists of $\text{In}_{\succ} (f_i)$ for $1 \leq i \leq 11$ and $i \neq 4, 5$, it remains to show that this set satisfies the Buchberger criterion. As in the previous case we just need to check pairs whose initial terms are not relatively prime and one of the polynomials in the pair is not monomial. We note that the reductions of the $S$-polynomials involving $f_3$ in the previous case carry over to this case except $S(f_3, f_{10})$. To see this, first note that the missing initial monomial $\text{In}_{\succ} (f_5)$ is not used in these reductions. Also the initial monomials of $f_i$ for $1 \leq i \leq 10$ do not change with the change of the order except $f_{10}$. But in these reductions $f_{10}$ is not used in the polynomial division except at the last step. So the last non-zero remainder is a multiple of $f_{10}$ giving that this last polynomial division is also a reduction in the second order as well. Finally, $S(f_3, f_{11})$ reduces to zero via $f_5, f_3, f_{10}$. We finish the proof by checking the pairs involving $f_{10}$. We have that $S(f_3, f_{10})$ reduces to zero via $f_5, f_{10}$. The $S$-polynomial $S(f_2, f_{10})$ reduces to zero via $f_3, f_{10}$ and the $S$-polynomial $S(f_7, f_{10})$ reduces to zero via $f_3, f_{10}$. Both $S(f_5, f_{10})$ and $S(f_{10}, f_{11})$ reduce to zero via $f_9$. □

**Theorem 10.** Assume that $p > 5$. There are $2(5!)$ generic initial ideals of $H(V_0)$. Each of them is generated by $\pi(A)$ or $\pi(B)$, where $\pi$ is a permutation of the variables in $F[V_0]$. 

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Proof. Let $\alpha = (\alpha_{ij}) \in \mathbb{B}_5$. If $C = 0$, then from Lemma 8 we get that the monic generators of $\text{In}_{>}(\alpha(H(V_5)))$ of degree at most two are $x_1 x_2^2 x_3^2$ if $x_3^2 > x_2 x_4$ and are $x_1 x_2^2, x_2 x_4$ otherwise. Note that $x_2 x_3 \notin \text{In}_{>}(\alpha(H(V_5)))$ in both cases and so $\text{In}_{>}(\alpha(H(V_5)))$ fails to be Borel fixed because either $x_2 x_3$ or $x_2 x_4$ lies in $\text{In}_{>}(\alpha(H(V_5)))$. It follows that $\text{gin}_{>}(H(V_5)) \neq \text{In}_{>}(\alpha(H(V_5)))$ if $C = 0$ because generic initial ideals are always Borel fixed, see for instance [5, §15]. On the other hand, by the previous lemma all other members of $\mathbb{B}_5$ generate the same initial ideal. But it is a standard fact that at least one element in $\mathbb{B}_5$ generates the generic initial ideal ([5, 15.18]) and so $\text{gin}_{>}(H(V_5)) = \text{In}_{>}(\alpha(H(V_5)))$ for $\alpha \in \mathbb{B}_5$ satisfying $C \neq 0$. This initial ideal is generated by $A$ or $B$ depending how $>$ compares $x_2 x_4$ and $x_3$. The final assertion of the theorem follows from Theorem 3. \(\square\)

Remark 11.

1. The computation of $\text{gin}_{>}(H(V_5))$ for $p = 5$ is very similar and there are only small changes in the generating set. We include the details in the version we submit to arXiv preprint server.

2. The identity matrix satisfies that $C = 0$. Therefore from the proof of the theorem we get that $\text{gin}_{>}(H(V_5)) = \text{In}_{>}(\alpha(H(V_5)))$ for all monomial orders with $x_1 > \cdots > x_3$.

4. A proposal of a conjecture

We first note that Hilbert ideals of the Klein four group over characteristic two is Borel fixed with the right ordering of the variables in the monomial order.

Proposition 12. Let $G$ denote the Klein four group and let $W$ be a $G$-module over characteristic two not containing the regular representation as a summand. Then there is choice for a basis for $W^*$ and a ranking of variables in $F[W]$ such that $H(W)$ is Borel fixed for all monomial orders compatible with this ranking. In particular $\text{gin}_{>}(H(V_5)) = H(W)$ for all such orders $>$. \(\square\)

Proof. Generators of $H(W)$ have been studied in [10] and [6]. From [6, Proposition 16, 17] we see that there is a basis for $W^*$ such that $H(W)$ is generated by first, second and forth powers of these basis elements (recall that $F[W]$ is a polynomial ring in these basis elements). Say $F[W] = F[x_1, \ldots, x_n]$ and $H(W)$ is generated by $x_i^{a_i}$, where $a_i \in \{1, 2, 4\}$ for $1 \leq i \leq n$. Let $> >$ be a monomial order such that $x_i > x_j$ whenever $a_i < a_j$. Since we are in characteristic two and $a_i$ is a 2-th power, a member of the non-standard Borel subgroup (see Remark 5) sends $x_i^{a_i}$ to some combination of $x_i$-th powers of variables of higher or equal rank. By the choice of $>$, this combination is also in $H(W)$, so $H(W)$ is Borel fixed. Therefore, $\text{gin}_{>}(H(W)) = H(W)$ by [2, 1.8]. \(\square\)

Reviewing the data we have collected so far, the monomial Hilbert ideals of the cyclic group of prime order are Borel fixed (assuming $x_1 > x_2 > \cdots$) and so the gin of the Hilbert ideal is equal to the Hilbert ideal itself, see Propositions 6 and 7. The non-monomial case, with the same ordering, still satisfies $\text{gin}_{>}(H(V_5)) = \text{In}_{>}(H(V_5))$, see Remark 11. Together with the previous proposition on the Klein four group we feel that there is enough ground for the following conjecture.

Conjecture 13. Let $G$ be a $p$-group and $V$ be a $G$-module over a field $F$ of characteristic $p$. Then there is a choice of a basis for $V^*$ and a monomial order $>$ on the monomials in $F[V]$ such that $\text{gin}_{>}(H(V)) = \text{In}_{>}(H(V))$. 

References