Filomat xx (yyyy), zzz–zzz
DOI (will be added later)

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Sequential Warped Products: Curvature and conformal vector fields

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Abstract. In this note, we introduce a new type of warped products called as sequential warped products to cover a wider variety of exact solutions to Einstein's field equation. First, we study the geometry of sequential warped products and obtain covariant derivatives, curvature tensor, Ricci curvature and scalar curvature formulas. Then some important consequences of these formulas are also stated. We provide characterizations of geodesics and two different types of conformal vector fields, namely, Killing vector fields and concircular vector fields on sequential warped product manifolds. Finally, we consider the geometry of two classes of sequential warped product space-time models which are sequential generalized Robertson-Walker space-times and sequential standard static space-times.

1. Introduction

O'Neill and Bishop defined warped product manifolds to construct Riemannian manifolds with negative sectional curvature[9]. Since then this notion has played some important roles in differential geometry as well as in physics because warped product space-time models are used to obtain exact solutions to Einstein's equation [1–3, 7, 8, 17, 20].

Doubly and multiply warped product manifolds are generalizations of (singly) warped product manifolds[13, 26, 27]. In this article, we define a new class of warped product manifolds, called as sequential warped products where the base factor of the warped product is itself a new warped product manifold. Sequential warped products can be considered as a generalization of singly warped products. There are many space-times where base, fiber or both are expressed as a warped product manifolds. Among many such examples, we would like to mention especially non-trivial ones such as Taub-Nut and stationary metrics (see [25]) also Schwarzschild and generalized Riemannian anti de Sitter $\mathbb{T}^2$ black hole metrics (see §3.2 of [5] for details). Moreover, some base conformal warped product space-times can be expressed as a sequential warped product (see [14]).

We first introduce fundamental definitions about the new concept and state some related remarks.

Definition 1.1. *Let M_i be three pseudo-Riemannian manifolds with metrics g_i for $i = 1, 2, 3$. Let $f : M_1 \to (0, \infty)$ and $h : M_1 \times M_2 \to (0, \infty)$ be two smooth positive functions on M_1 and $M_1 \times M_2$, respectively. Then the sequential*

2010 *Mathematics Subject Classification*. Primary 53C21, 53C25; Secondary 53C50, 53C80
Keywords. Warped product manifold, space-times, curvature, Killing vector fields, geodesics, concircular vector fields
Received: dd Month yyyy; Accepted: dd Month yyyy
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warped product manifold, denoted by $\left(M_1 \times_f M_2\right) \times_h M_3$, *is the triple product manifold* $\bar{M} = (M_1 \times M_2) \times M_3$ *furnished with the metric tensor*

$$\bar{g} = \left(g_1 \oplus f^2 g_2\right) \oplus h^2 g_3$$

The functions f *and* h *are called warping functions.*

Note that if (M_i, g_i) are all Riemannian manifolds for any $i = 1, 2, 3$, then the sequential warped product manifold $\left(M_1 \times_f M_2\right) \times_h M_3$ is also a Riemannian manifold.

Remark 1.2. *The warped product of the form* $M_1 \times_{f_1} \left(M_2 \times_{f_2} M_3\right)$ *furnished by the metric*

$$g = g_1 + f_1^2 \left(g_2 + f_2^2 g_3\right)$$

is called the iterated warped product manifold of the manifolds M_1, M_2 *and* M_3. *As a metric space, the iterated warped product manifold is equal to the sequential warped product* $\left(M_1 \times_f M_2\right) \times_h M_3$ *where* $f = f_1$ *and* $h = f_1 f_2$. *Similarly, a sequential warped product* $\left(M_1 \times_f M_2\right) \times_h M_3$ *with a separable function* $h : M_1 \times M_2 \to \mathbb{R}$ *is equal as a metric space to the iterated warped product manifold.*

Remark 1.3. *If the warping function* h *of the sequential warped product* $\left(M_1 \times_f M_2\right) \times_h M_3$ *is defined only on* M_1, *then we have a multiply warped product manifold* $M_1 \times_f M_2 \times_h M_3$ *with two fibers.*

Remark 1.4. *A multiply warped product manifold of the form* $M_1 \times_{f_1} M_2 \times_{f_2} M_3$ *is the sequential warped product manifold* $\left(M_1 \times_{f_1} M_2\right) \times_{f_2} M_3$ *equipped with the metric*

$$g = \left(g_1 + f_1^2 g_2\right) + f_2^2 g_3$$

where both f_1 *and* f_2 *are positive functions defined on* M_1.

Now, we would like to explain how to extend a generalized Robertson-Walker space-time and a standard static space-time within the framework of sequential warped products.

Let (M_i, g_i) be two n_i-dimensional Riemannian manifolds for any $i = 1, 2$. Suppose that I is an open, connected subinterval of $\mathbb{R}$ and dt^2 is the Euclidean metric tensor on I. Then

• An $(n_1 + n_2 + 1)-$ dimensional product manifold $I \times (M_1 \times M_2)$ furnished with the metric tensor

$$\bar{g} = -h^2 dt^2 \oplus \left(g_1 \oplus f^2 g_2\right) \tag{1}$$

is a sequential standard static space-time and is denoted by $\bar{M} = I_h \times \left(M_1 \times_f M_2\right)$ where $h : M_1 \times M_2 \to (0, \infty)$ and $f : M_1 \to (0, \infty)$ are two smooth functions.

Note that standard static space-times can be considered as a generalization of the Einstein static universe[2–4, 8, 12, 23, 24]. Obviously, one can obtain a standard static space-time from a sequential standard static space-time by taking M_2 to be a singleton.

• An $(n_1 + n_2 + 1)-$ dimensional product manifold $(I \times M_1) \times M_2$ furnished with the metric tensor

$$\bar{g} = -dt^2 \oplus h^2 \left(g_1 \oplus f^2 g_2\right), \tag{2}$$

is a sequential generalized Robertson-Walker space-time is denoted by $\bar{M} = I \times_h \left(M_1 \times_f M_2\right)$ where $h : I \to (0, \infty)$ and $f : M_1 \to (0, \infty)$ are two smooth functions.

Note that generalized Robertson-Walker space-times can be considered as a generalization of Robertson-Walker space-time [21, 22]. As in the case of sequential standard static space-times, one can obtain a generalized Robertson-Walker space-time from a sequential generalized Robertson-Walker space-time by taking M_2 to be the empty set a singleton.

In [25], there are many exact solutions of Einstein field equation where the space-time may be written of the form $I \times (M_1 \times M_2)$ with metrics of the form (1) or (2).

Notice also that $\mathbb{S}_1^n \times F$ or $\mathbb{H}_1^n \times F$ are standard models in string theory where F is a Calabi-Yau, Ricci flat Riemannian Manifold and $\mathbb{S}_1^n$ is the de Sitter and also $\mathbb{H}_1^n$ is the anti-de Sitter manifold both of which are warped product manifolds (see page 183 of [6]). Thus sequential warped product space-times play important role not only in the theory of general relativity but also in the string theory.

In this article, we study some geometric concepts such as curvature, geodesics, Killing vector fields and concircular vector fields on sequential warped products. In section 2, we derive covariant derivative formulas for sequential warped product manifolds. Then we derive many curvature formulas such as Ricci curvature and scalar curvature formulas. In section 3, we derive a characterization of two disjoint classes of conformal vector fields on sequential warped product manifolds. In the last section, we apply our results presented in Section 2 and Section 3, to sequential standard space-times and generalized Robertson-Walker space-times.

Before we begin to state our main results, we would like to fix notations used throughout the entire article.

Notation 1.5. *Let* $\bar{M} = \left(M_1 \times_f M_2\right) \times_h M_3$ *be a sequential warped product manifold with metric* $\bar{g} = \left(g_1 \oplus f^2 g_2\right) \oplus h^2 g_3$ *where* $f\colon M_1 \to (0,\infty)$ *and* $h\colon M_1 \times M_2 \to (0,\infty)$. *Then*

- $M = M_1 \times_f M_2$ *is a warped product with the metric tensor* $g = g_1 \oplus f^2 g_2$.
- $\mathrm{grad}^1 f$ *is the gradient of* f *on* M_1 *and* $\|\mathrm{grad}^1 f\|^2 = g_1(\mathrm{grad}^1 f, \mathrm{grad}^1 f)$.
- $\mathrm{grad} h$ *is the gradient of* h *on* M *and* $\|\mathrm{grad} h\|^2 = g(\mathrm{grad} h, \mathrm{grad} h)$.
- *The same notation is used to denote a vector field and its lift to the sequential warped product manifold.*

2. Curvature of Sequential Warped Product Manifolds

In this section, we will explore the geometry of sequential warped products of the form $\left(M_1 \times_f M_2\right) \times_h M_3$ by providing the covariant derivative, curvature tensor, Ricci and scalar curvature formulas. The proofs that are straightforward can be obtained by applying similar results on singly warped products twice.

Proposition 2.1. *Let* $\bar{M} = \left(M_1 \times_f M_2\right) \times_h M_3$ *be a sequential warped product manifold with metric* $\bar{g} = \left(g_1 \oplus f^2 g_2\right) \oplus h^2 g_3$ *and also let* $X_i, Y_i \in \mathfrak{X}(M_i)$ *for any* $i = 1, 2, 3$. *Then*

1. $\bar{\nabla}_{X_1} Y_1 = \nabla^1_{X_1} Y_1$
2. $\bar{\nabla}_{X_1} X_2 = \bar{\nabla}_{X_2} X_1 = X_1 (\ln f) X_2$
3. $\bar{\nabla}_{X_2} Y_2 = \nabla^2_{X_2} Y_2 - f g_2 (X_2, Y_2) \,\mathrm{grad}^1 f$
4. $\bar{\nabla}_{X_3} X_1 = \bar{\nabla}_{X_1} X_3 = X_1 (\ln h) X_3$
5. $\bar{\nabla}_{X_2} X_3 = \bar{\nabla}_{X_3} X_2 = X_2 (\ln h) X_3$
6. $\bar{\nabla}_{X_3} Y_3 = \nabla^3_{X_3} Y_3 - h g_3 (X_3, Y_3) \,\mathrm{grad} h$

Proposition 2.2. *Let* $\bar{M} = \left(M_1 \times_f M_2\right) \times_h M_3$ *be a sequential warped product manifold with metric* $\bar{g} = \left(g_1 \oplus f^2 g_2\right) \oplus h^2 g_3$ *and also let* $X_i, Y_i, Z_i \in \mathfrak{X}(M_i)$ *for any* $i = 1, 2, 3$. *Then*

1. $\bar{R}(X_1, Y_1) Z_1 = R^1 (X_1, Y_1) Z_1$
2. $\bar{R}(X_2, Y_2) Z_2 = R^2 (X_2, Y_2) Z_2 - \left\|\mathrm{grad}^1 f\right\|^2 [g_2 (X_2, Z_2) Y_2 - g_2 (Y_2, Z_2) X_2]$
3. $\bar{R}(X_1, Y_2) Z_1 = \dfrac{-1}{f} H_1^f (X_1, Z_1) Y_2$

4. $\bar{R}(X_1, Y_2) Z_2 = f g_2 (Y_2, Z_2) \nabla^1_{X_1} \mathrm{grad}^1 f$
5. $\bar{R}(X_1, Y_2) Z_3 = 0$
6. $\bar{R}(X_i, Y_i) Z_j = 0, i \neq j$
7. $\bar{R}(X_i, Y_3) Z_j = \dfrac{-1}{h} H^h \left(X_i, Z_j\right) Y_3, i, j = 1, 2$
8. $\bar{R}(X_i, Y_3) Z_3 = h g_3 (Y_3, Z_3) \nabla_{X_i} \mathrm{grad} h, i = 1, 2$
9. $\bar{R}(X_3, Y_3) Z_3 = R^3 (X_3, Y_3) Z_3 - \left\|\mathrm{grad} h\right\|^2 [g_3 (X_3, Z_3) Y_3 - g_3 (Y_3, Z_3) X_3]$

Now consider the Ricci curvature denoted by $\bar{\mathrm{Ric}}$ of a sequential warped product of the form $\left(M_1 \times_f M_2\right) \times_h M_3$.

Proposition 2.3. *Let $\bar{M} = \left(M_1 \times_f M_2\right) \times_h M_3$ be a sequential warped product manifold with metric $\bar{g} = \left(g_1 \oplus f^2 g_2\right) \oplus h^2 g_3$ and also let $X_i, Y_i, Z_i \in \mathfrak{X}(M_i)$ for any $i = 1, 2, 3$. Then*

1. $\bar{Ric}(X_1, Y_1) = Ric^1 (X_1, Y_1) - \dfrac{n_2}{f} H^f_1 (X_1, Y_1) - \dfrac{n_3}{h} H^h (X_1, Y_1)$
2. $\bar{Ric}(X_2, Y_2) = Ric^2 (X_2, Y_2) - f^\sharp g_2 (X_2, Y_2) - \dfrac{n_3}{h} H^h (X_2, Y_2)$
3. $\bar{Ric}(X_3, Y_3) = Ric^3 (X_3, Y_3) - h^\sharp g_3 (X_3, Y_3)$
4. $\bar{Ric}\left(X_i, Y_j\right) = 0, i \neq j$

   where $f^\sharp = f \Delta^1 f + (n_2 - 1) \left\|\mathrm{grad}^1 f\right\|^2$ *and* $h^\sharp = h \Delta h + (n_3 - 1) \left\|\mathrm{grad} h\right\|^2$

We now apply the last result to establish conditions for a sequential warped product to be Einstein.

Theorem 2.4. *The sequential warped product $\left(M_1 \times_f M_2\right) \times_h M_3$ is Einstein with $\bar{Ric} = \lambda \bar{g}$ if and only if*

1. $Ric^1 = \lambda g_1 + \dfrac{n_2}{f} H^f_1 + \dfrac{n_3}{h} H^h$
2. $Ric^2 = \left(\lambda f^2 + f^\sharp\right) g_2 + \dfrac{n_3}{h} H^h$
3. *M_3 is Einstein with* $Ric^3 = \left(\lambda h^2 + h^\sharp\right) g_3$.

In [11], F. Dobarro and E. Lamí Dozo established a relationship between the scalar curvature of a warped product of the form $M \times_f N$ and that of its base and fiber manifolds M and N. In the following theorem we derive a quite different result for a sequential warped product manifold.

Theorem 2.5. *Let $\bar{M} = \left(M_1 \times_f M_2\right) \times_h M_3$ be a sequential warped product manifold with metric $\bar{g} = \left(g_1 \oplus f^2 g_2\right) \oplus h^2 g_3$ and let r_i be the scalar curvature of M_i, $i = 1, 2, 3$. Then the scalar curvature r of $\bar{M}$ is given by*

$$r = r_1 + \frac{r_2}{f^2} + \frac{r_3}{h^2} - \frac{2n_2}{f} \Delta^1 f - \frac{2n_3}{h} \Delta h - \frac{n_2 (n_2 - 1)}{f^2} \left\|\mathrm{grad}^1 f\right\|^2 - \frac{n_3 (n_3 - 1)}{h^2} \left\|\mathrm{grad} h\right\|^2$$

Proof. Let $\{e_1, e_2, ..., e_{n_1}\}$, $\{e_{n_1+1}, e_{n_1+2}, ..., e_{n_1+n_2}\}$ and $\{e_{n_1+n_2+1} + e_{n_1+n_2+2}, ..., e_n\}$ be three frames over M_1, M_2 and M_3 respectively. The scalar curvature r of M is given by

$$\begin{aligned}
r &= \sum_{i=1}^{n_1} \bar{\mathrm{Ric}} (e_i, e_i) + \frac{1}{f^2} \sum_{i=n_1+1}^{n_1+n_2} \bar{\mathrm{Ric}} (e_i, e_i) + \frac{1}{h^2} \sum_{i=n_1+n_2+1}^{n_1+n_2+n_3} \bar{\mathrm{Ric}} (e_i, e_i) \\
&= r_1 - \frac{n_2}{f} \Delta^1 f - \frac{n_3}{h} \sum_{i=1}^{n_1} H^h (e_i, e_i) + \frac{1}{f^2} r_2 - \frac{n_2}{f^2} f^\sharp - \frac{1}{f^2} \frac{n_3}{h} \sum_{i=n_1+1}^{n_1+n_2} H^h (e_i, e_i) \\
&\quad + \frac{1}{h^2} \left[r_3 - h^\sharp n_3\right] \\
&= r_1 + \frac{1}{f^2} r_2 + \frac{1}{h^2} r_3 - \frac{n_2}{f} \Delta^1 f - \frac{n_3}{h} \Delta h - \frac{n_2}{f^2} f^\sharp - \frac{n_3}{h^2} h^\sharp \\
&= r_1 + \frac{r_2}{f^2} + \frac{r_3}{h^2} - \frac{2n_2}{f} \Delta^1 f - \frac{2n_3}{h} \Delta h - \frac{n_2 (n_2 - 1)}{f^2} \left\|\mathrm{grad}^1 f\right\|^2 - \frac{n_3 (n_3 - 1)}{h^2} \left\|\mathrm{grad} h\right\|^2
\end{aligned}$$

□

Suppose that $\bar{M} = \left(M_1 \times_f M_2\right) \times_h M_3$ has a constant sectional curvature κ. Then the first item of Proposition (2.2) yields

$$\begin{aligned}\bar{\mathrm{R}}\left(X_1, Y_1\right) Z_1 &= \kappa\left\{g_1\left(X_1, Z_1\right) Y_1 - g_1\left(Y_1, Z_1\right) X_1\right\} \\ \bar{\mathrm{R}}\left(X_1, Y_1\right) Z_1 &= R^1\left(X_1, Y_1\right) Z_1\end{aligned}$$

Thus M_1 has a constant sectional curvature $\kappa_1 = \kappa$. The second item implies that

$$\begin{aligned}\bar{\mathrm{R}}\left(X_2, Y_2\right) Z_2 &= \kappa\left\{g\left(X_2, Z_2\right) Y_2 - g\left(Y_2, Z_2\right) X_2\right\} \\ &= \kappa f^2\left\{g_2\left(X_2, Z_2\right) Y_2 - g_2\left(Y_2, Z_2\right) X_2\right\} \\ \bar{\mathrm{R}}\left(X_2, Y_2\right) Z_2 &= R^2\left(X_2, Y_2\right) Z_2 - \left\|\mathrm{grad}^1 f\right\|^2\left\{g_2\left(X_2, Z_2\right) Y_2 - g_2\left(Y_2, Z_2\right) X_2\right\}\end{aligned}$$

Therefore, Shur's Lemma implies that M_2 has a constant sectional curvature κ_2 given by

$$\kappa_2 = \kappa f^2 + \left\|\mathrm{grad}^1 f\right\|^2$$

for $n_2 \geq 3$. Similarly, M_3 has a constant sectional curvature curvature κ_3 given by

$$\kappa_3 = \kappa h^2 + \left\|\mathrm{grad} h\right\|^2$$

for $n_3 \geq 3$.

Theorem 2.6. *Let $\bar{M} = \left(M_1 \times_f M_2\right) \times_h M_3$ be a sequential warped product manifold with metric $\bar{g} = \left(g_1 \oplus f^2 g_2\right) \oplus h^2 g_3$ and let $X_i, Y_i, Z_i \in \mathfrak{X}(M_i)$ for any $i = 1, 2, 3$. Assume that $\bar{M}$ has a constant sectional curvature κ. Then*

1. *M_1 has a costant sectional curvature $\kappa_1 = \kappa$,*
2. *M_2 has a costant sectional curvature $\kappa_2 = \kappa f^2 + \left\|\mathrm{grad}^1 f\right\|^2$ for $n_2 \geq 3$, and*
3. *M_3 has a costant sectional curvature $\kappa_3 = \kappa h^2 + \left\|\mathrm{grad} h\right\|^2$ for $n_3 \geq 3$.*

3. Conformal vector fields

Conformal vector fields have well-known geometrical and physical interpretations and have been studied for a long time by geometers and physicists on Riemannian and pseudo-Riemannian manifolds. Killing vector fields are conformal vector fields on (pseudo-) Riemannian manifolds that preserve metric, i.e, under the flow of a Killing vector field the metric does not change. The set of all Killing vector fields on a connected Riemannian manifold forms a Lie algebra over the set of real numbers.[7, 8, 16–19, 23]

In [12], the authors studied Killing vector fields of warped product manifolds specially on standard static space-times. They prove some global characterization of the Killing vector fields of a standard static space-time. More explicitly, they obtain a form of a Killing vector field on this class of space-times. Moreover, a characterization of the Killing vector fields on a standard static space-time with compact Riemannian parts and many other interesting results are given. In this section, we study the concept of conformal vector fields on sequential warped product manifolds.

A vector field ζon a Riemannian manifold (M, g) is conformal if

$$\mathcal{L}_\zeta g = \rho g \tag{3}$$

where $\mathcal{L}_\zeta$ is the Lie derivative in direction of the vector field ζ. Moreover, ζ is called a Killing vector field if $\rho = 0$. This is equivalent to say that ζ is Killing if

$$g\left(\nabla_X \zeta, Y\right) + g\left(X, \nabla_Y \zeta\right) = 0 \tag{4}$$

for any vector fields $X, Y \in \mathfrak{X}(M)$. By symmetry of the above equation, ζ is Killing if

$$g(\nabla_X \zeta, X) = 0 \tag{5}$$

for any vector field $X \in \mathfrak{X}(M)$.

From now on $\bar{M} = \left(M_1 \times_f M_2\right) \times_h M_3$ denotes a sequential warped product manifold with metric $\bar{g} = \left(g_1 \oplus f^2 g_2\right) \oplus h^2 g_3$.

Theorem 3.1. *A vector field $\zeta \in \mathfrak{X}\left(\left(M_1 \times_f M_2\right) \times_h M_3\right)$ is Killing if*

1. *ζ_i is Killing on M_i, for every $i = 1, 2, 3$*
2. $\zeta_1(f) = 0$
3. $(\zeta_1 + \zeta_2) h = 0$

Proof. The vector field $\zeta \in \mathfrak{X}(\bar{M})$ is Killing by equation (5) if and only if

$$\bar{g}(\bar{\nabla}_X \zeta, X) = 0$$

for any vector field $X \in \mathfrak{X}(\bar{M})$. It is clear that

$$\begin{aligned} \bar{g}(\bar{\nabla}_X \zeta, X) &= \bar{g}(\bar{\nabla}_{X_1}\zeta_1 + \bar{\nabla}_{X_1}\zeta_2 + \bar{\nabla}_{X_1}\zeta_3, X) \\ &\quad + \bar{g}(\bar{\nabla}_{X_2}\zeta_1 + \bar{\nabla}_{X_2}\zeta_2 + \bar{\nabla}_{X_2}\zeta_3, X) \\ &\quad + \bar{g}(\bar{\nabla}_{X_3}\zeta_1 + \bar{\nabla}_{X_3}\zeta_2 + \bar{\nabla}_{X_3}\zeta_3, X) \end{aligned}$$

Now using Proposition (2.1) we have

$$\begin{aligned} & \bar{g}(\bar{\nabla}_X \zeta, X) \\ = \; & \bar{g}\left(\nabla^1_{X_1}\zeta_1 + X_1(\ln f)\zeta_2 + X_1(\ln h)\zeta_3, X\right) \\ & + \bar{g}\left(\zeta_1(\ln f) X_2 + \nabla^2_{X_2}\zeta_2 - f g_2(\zeta_2, X_2)\,\mathrm{grad}^1 f + X_2(\ln h)\zeta_3, X\right) \\ & + \bar{g}\left(\zeta_1(\ln h) X_3 + \zeta_2(\ln h) X_3 + \nabla^3_{X_3}\zeta_3 - h g_3(\zeta_3, X_3)\,\mathrm{grad} h, X\right) \\ = \; & g_1\left(\nabla^1_{X_1}\zeta_1, X_1\right) + f^2 g_2\left(\nabla^2_{X_2}\zeta_2, X_2\right) + h^2 g_3\left(\nabla^3_{X_3}\zeta_3, X_3\right) \\ & + f\zeta_1(f) g_2(X_2, X_2) + h(\zeta_1 + \zeta_2)(h) g_3(X_3, X_3) \end{aligned}$$

From this equation one can easily deduce the result. □

The following result will enable us to discuss the converse of the above result.

Proposition 3.2. *A vector field $\zeta \in \mathfrak{X}\left(\left(M_1 \times_f M_2\right) \times_h M_3\right)$ satisfies*

$$\begin{aligned} (\mathcal{L}_\zeta g)(X, Y) &= \left(\mathcal{L}^1_{\zeta_1} g_1\right)(X_1, Y_1) + f^2\left(\mathcal{L}^2_{\zeta_2} g_2\right)(X_2, Y_2) + h^2\left(\mathcal{L}^3_{\zeta_3} g_3\right)(X_3, Y_3) \\ &\quad + 2f\zeta_1(f) g_2(X_2, Y_2) + 2h(\zeta_1 + \zeta_2)(h) g_3(X_3, Y_3) \end{aligned} \tag{6}$$

for any vector fields $X, Y \in \mathfrak{X}\left(\left(M_1 \times_f M_2\right) \times_h M_3\right)$.

Theorem 3.3. *Let $\zeta \in \mathfrak{X}\left(\left(M_1 \times_f M_2\right) \times_h M_3\right)$ be a Killing vector field. Then*

1. *ζ_1 is Killing on M_1,*
2. *ζ_2 is conformal on M_2 with conformal factor $-2\zeta_1(\ln f)$,*
3. *ζ_3 is conformal on M_3 with conformal factor $-2(\zeta_1 + \zeta_2)(\ln h)$.*

Proof. Consider equation (6). We have the following cases. By substituting $X = X_1$ and $Y = Y_1$, we obtain

$$\left(\mathcal{L}^1_{\zeta_1} g_1\right)(X_1, Y_1) = 0$$

and thus ζ_1 is Killing. Now, let $X = X_2$ and $Y = Y_2$, be then we have

$$\begin{aligned} 0 &= f^2\left(\mathcal{L}^2_{\zeta_2} g_2\right)(X_2, Y_2) + 2f\zeta_1(f)\, g_2(X_2, Y_2) \\ \left(\mathcal{L}^2_{\zeta_2} g_2\right)(X_2, Y_2) &= -2\zeta_1(\ln f)\, g_2(X_2, Y_2) \end{aligned}$$

and thus ζ_2 is conformal. Finally, if $X = X_3$ and $Y = Y_3$, then

$$\begin{aligned} 0 &= h^2\left(\mathcal{L}^3_{\zeta_3} g_3\right)(X_3, Y_3) + 2h(\zeta_1 + \zeta_2)(h)\, g_3(X_3, Y_3) \\ \left(\mathcal{L}^3_{\zeta_3} g_3\right)(X_3, Y_3) &= -2(\zeta_1 + \zeta_2)(\ln h)\, g_3(X_3, Y_3) \end{aligned}$$

and thus ζ_3 is conformal. □

Theorem 3.4. *Let $\zeta \in \mathfrak{X}\left(\left(M_1 \times_f M_2\right) \times_h M_3\right)$ be a vector field on a sequential warped product manifold. Assume that*

1. *ζ_i is conformal on M_i with factor ρ_i for each i,*
2. *$\rho_1 = \rho_2 + 2\zeta_1(\ln f)$,*
3. *$\rho_1 = \rho_3 + 2(\zeta_1 + \zeta_2)(\ln h)$.*

Then ζ is conformal on $\bar{M}$.

Now, we will study the geodesic curves and their equations on a sequential warped product. In a sequential warped product of the form $\left(M_1 \times_f M_2\right) \times_h M_3$, as product manifold, a curve $\alpha(t)$ can be written as $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ with $\alpha_i(t)$ the projections of α into M_i for any $i = 1, 2, 3$.

Lemma 3.5. *Let $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ be a smooth curve on a sequential warped product of the form $\bar{M} = \left(M_1 \times_f M_2\right) \times_h M_3$ with metric $\bar{g} = \left(g_1 \oplus f^2 g_2\right) \oplus h^2 g_3$. Then α is a geodesic in $\bar{M}$ if and only if*

1. $\nabla^1_{\dot{\alpha}_1}\dot{\alpha}_1 = f\|\dot{\alpha}_2\|_2^2\, \mathrm{grad}^1 f + h\|\dot{\alpha}_3\|_3^2\, (\mathrm{grad} h)^T$ *on* M_1
2. $\nabla^2_{\dot{\alpha}_2}\dot{\alpha}_2 = -2\dot{\alpha}_1(\ln f)\,\dot{\alpha}_2 + h\|\dot{\alpha}_3\|_3^2\, (\mathrm{grad} h)^\perp$ *on* M_2
3. $\nabla^3_{\dot{\alpha}_3}\dot{\alpha}_3 = -2\dot{\alpha}_1(\ln h)\,\dot{\alpha}_3 - 2\dot{\alpha}_2(\ln h)\,\dot{\alpha}_3$ *on* M_3

Proof. Then $\alpha_i(t)$ is regular hence we can suppose $\alpha_i(t)$ is an integral curve of $\dot{\alpha}_i$ on M_i and so $\alpha(t)$ is an integral curve of $\dot{\alpha} = \dot{\alpha}_1 + \dot{\alpha}_2 + \dot{\alpha}_3$. Thus

$$\begin{aligned} \bar{\nabla}_{\dot{\alpha}}\dot{\alpha} &= \bar{\nabla}_{\dot{\alpha}_1}\dot{\alpha}_1 + \bar{\nabla}_{\dot{\alpha}_1}\dot{\alpha}_2 + \bar{\nabla}_{\dot{\alpha}_1}\dot{\alpha}_3 \\ &\quad + \bar{\nabla}_{\dot{\alpha}_2}\dot{\alpha}_1 + \bar{\nabla}_{\dot{\alpha}_2}\dot{\alpha}_2 + \bar{\nabla}_{\dot{\alpha}_2}\dot{\alpha}_3 \\ &\quad + \bar{\nabla}_{\dot{\alpha}_3}\dot{\alpha}_1 + \bar{\nabla}_{\dot{\alpha}_3}\dot{\alpha}_2 + \bar{\nabla}_{\dot{\alpha}_3}\dot{\alpha}_3 \end{aligned}$$

Now we apply Proposition (2.1) to get

$$\begin{aligned} \bar{\nabla}_{\dot{\alpha}}\dot{\alpha} &= \nabla^1_{\dot{\alpha}_1}\dot{\alpha}_1 + 2\dot{\alpha}_1(\ln f)\,\dot{\alpha}_2 + 2\dot{\alpha}_1(\ln h)\,\dot{\alpha}_3 \\ &\quad + 2\dot{\alpha}_2(\ln h)\,\dot{\alpha}_3 + \nabla^2_{\dot{\alpha}_2}\dot{\alpha}_2 - f g_2(\dot{\alpha}_2, \dot{\alpha}_2)\, \mathrm{grad}^1 f \\ &\quad + \nabla^3_{\dot{\alpha}_3}\dot{\alpha}_3 - h g_3(\dot{\alpha}_3, \dot{\alpha}_3)\, \mathrm{grad} h \end{aligned}$$

This equation implies that

$$\begin{aligned} \bar{\nabla}_{\dot{\alpha}}\dot{\alpha} &= \nabla^1_{\dot{\alpha}_1}\dot{\alpha}_1 - f g_2(\dot{\alpha}_2, \dot{\alpha}_2)\, \mathrm{grad}^1 f - h g_3(\dot{\alpha}_3, \dot{\alpha}_3)\, (\mathrm{grad} h)^T \\ &\quad + \nabla^2_{\dot{\alpha}_2}\dot{\alpha}_2 + 2\dot{\alpha}_1(\ln f)\,\dot{\alpha}_2 - h g_3(\dot{\alpha}_3, \dot{\alpha}_3)\, (\mathrm{grad} h)^\perp \\ &\quad + \nabla^3_{\dot{\alpha}_3}\dot{\alpha}_3 + 2\dot{\alpha}_1(\ln h)\,\dot{\alpha}_3 + 2\dot{\alpha}_2(\ln h)\,\dot{\alpha}_3 \end{aligned}$$

□

Theorem 3.6. *Let $\zeta \in \mathfrak{X}\left(\left(M_1 \times_f M_2\right) \times_h M_3\right)$ be a Killing vector field. Then $g(\zeta, X)$ is constant along the integral curve $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ of X if*

1. $\nabla^1_{X_1} X_1 = f \|\dot{\alpha}_2\|_2^2 \operatorname{grad}^1 f + h \|\dot{\alpha}_3\|_3^2 (\operatorname{grad} h)^T$ *on* M_1
2. $\nabla^2_{X_2} X_2 = -2X_1(\ln f) X_2 + h \|\dot{\alpha}_3\|_3^2 (\operatorname{grad} h)^\perp$ *on* M_2
3. $\nabla^3_{X_3} X_3 = -2X_1(\ln h) X_3 - 2X_2(\ln h) X_3$ *on* M_3.

Proof. The conditions (1-3) imply that $\alpha(t)$ is a geodesic and so $\nabla_X X = 0$ (see Lemma 3.5). Thus $g(\zeta, X)$ is constant along the integral curve of X. □

A vector field ζ on a Riemannian manifold M is called concircular vector field if

$$\nabla_X \zeta = \mu X$$

for any vector field X where μ is function defined on M. It is clear that

$$(\mathcal{L}_\zeta g)(X, Y) = 2\mu g(X, Y)$$

i.e. any concircular vector field is a conformal vector field. Concircular vector fields have many applications in geometry and physics[10]. A concircular vector field is sometimes called a closed conformal vector field.

Theorem 3.7. *Let $\zeta \in \mathfrak{X}\left(\left(M_1 \times_f M_2\right) \times_h M_3\right)$ be a concircular vector field on $\bar{M} = \left(M_1 \times_f M_2\right) \times_h M_3$. Then each ζ_i is a non-zero concircular vector field on M_i for any $i = 1, 2, 3$ if and only if both f and h are constant functions.*

Proof. Using the definition of concircular vector fields and Theorem 2.1, we obtain that

$$\begin{aligned}
\nabla_X \zeta &= \nabla_{X_1}\zeta_1 + \nabla_{X_1}\zeta_2 + \nabla_{X_1}\zeta_3 + \nabla_{X_2}\zeta_1 + \nabla_{X_2}\zeta_2 + \nabla_{X_2}\zeta_3 + \nabla_{X_3}\zeta_1 + \nabla_{X_3}\zeta_2 + \nabla_{X_3}\zeta_3 \\
\mu X &= \nabla^1_{X_1}\zeta_1 + X_1(f)\zeta_2 + X_1(h)\zeta_3 + \zeta_1(f) X_2 + \nabla^2_{X_2}\zeta_2 - f g_2(X_2, \zeta_2) \operatorname{grad}^1 f \\
&\quad + X_2(h)\zeta_3 + \zeta_1(h) X_3 + \zeta_2(h) X_3 + \nabla^3_{X_3}\zeta_3 - h g_3(X_3, \zeta_3) \operatorname{grad} h
\end{aligned}$$

Suppose that both f and h are constant functions, then

$$\begin{aligned}
\nabla^1_{X_1}\zeta_1 - f g_2(X_2, \zeta_2) \operatorname{grad}^1 f - h g_3(X_3, \zeta_3)(\operatorname{grad} h)^T &= \mu X_1 \\
\nabla^2_{X_2}\zeta_2 + X_1(f)\zeta_2 + \zeta_1(f) X_2 - h g_3(X_3, \zeta_3)(\operatorname{grad} h)^\perp &= \mu X_2 \\
\nabla^3_{X_3}\zeta_3 + X_1(h)\zeta_3 + X_2(h)\zeta_3 + \zeta_1(h) X_3 + \zeta_2(h) X_3 &= \mu X_3
\end{aligned} \tag{7}$$

Now, suppose that both f and h are constant functions, then

$$\begin{aligned}
\nabla^1_{X_1}\zeta_1 &= \mu X_1 \\
\nabla^2_{X_2}\zeta_2 &= \mu X_2 \\
\nabla^3_{X_3}\zeta_3 &= \mu X_3
\end{aligned}$$

i.e., each ζ_i is concircular on M_i for $i = 1, 2, 3$. Conversely, we suppose that

$$\begin{aligned}
\nabla^1_{X_1}\zeta_1 &= \mu_1 X_1 \\
\nabla^2_{X_2}\zeta_2 &= \mu_2 X_2 \\
\nabla^3_{X_3}\zeta_3 &= \mu_3 X_3
\end{aligned}$$

Hence Equation 7 becomes

$$\begin{aligned}
\mu_1 X_1 - f g_2(X_2, \zeta_2) \operatorname{grad}^1 f - h g_3(X_3, \zeta_3)(\operatorname{grad} h)^T &= \mu X_1 \\
\mu_2 X_2 + X_1(f)\zeta_2 + \zeta_1(f) X_2 - h g_3(X_3, \zeta_3)(\operatorname{grad} h)^\perp &= \mu X_2 \\
\mu_3 X_3 + X_1(h)\zeta_3 + X_2(h)\zeta_3 + \zeta_1(h) X_3 + \zeta_2(h) X_3 &= \mu X_3
\end{aligned}$$

$$\begin{aligned}
\bar{\mu}_1 X_1 - f g_2(X_2, \zeta_2)\,\mathrm{grad}^1 f - h g_3(X_3, \zeta_3)(\mathrm{grad} h)^T &= 0 && (8)\\
\bar{\mu}_2 X_2 + X_1(f)\,\zeta_2 + \zeta_1(f)\,X_2 - h g_3(X_3, \zeta_3)(\mathrm{grad} h)^\perp &= 0 && (9)\\
\bar{\mu}_3 X_3 + X_1(h)\,\zeta_3 + X_2(h)\,\zeta_3 + \zeta_1(h)\,X_3 + \zeta_2(h)\,X_3 &= 0 && (10)
\end{aligned}$$

These equations must be satisfied by any arbitrary vector field X. Let us put $X_3 = 0$ in Equation 10, then

$$(X_1 + X_2)(h)\,\zeta_3 = 0$$

Since ζ_3 does not vanish, $(X_1 + X_2)(h) = 0$ for any vector field $X_1 + X_2$ and so h is constant. Now, Equations 8 and 9 become

$$\begin{aligned}
\bar{\mu}_1 X_1 - f g_2(X_2, \zeta_2)\,\mathrm{grad}^1 f &= 0\\
\bar{\mu}_2 X_2 + X_1(f)\,\zeta_2 + \zeta_1(f)\,X_2 &= 0
\end{aligned}$$

Similarly, we can prove that f is constant. □

The converse of the above result is considered in the following theorem.

Theorem 3.8. *A vector field $\zeta = \zeta_1 \in \mathfrak{X}\left(\left(M_1 \times_f M_2\right) \times_h M_3\right)$ is a concircular vector field if ζ_1 is a concircular vector field with factor $\mu_1 = \zeta_1(\ln f) = \zeta_1(\ln h)$.*

4. Geometry of Sequential Warped Product Space-times

We will state basic geometric formulas of two types sequential warped product space-times, namely sequential generalized Robertson-Walker and sequential standard static space-times. These results can be obtained by direct applications of the results presented in Section 2.

4.1. Sequential Generalized Robertson-Walker Space-times

Proposition 4.1. *Let $\bar{M} = \left(I \times_f M_2\right) \times_h M_3$ be a sequential generalized Robertson-Walker space-time with metric $g = \left(-dt^2 \oplus f^2 g_2\right) \oplus h^2 g_3$ and also let $X_i, Y_i \in \mathfrak{X}(M_i)$ for any $i = 2, 3$. Then*

1. $\bar{\nabla}_{\partial_t}\partial_t = 0$
2. $\bar{\nabla}_{\partial_t} X_i = \bar{\nabla}_{X_i}\partial_t = \frac{\dot{f}}{f} X_i, i = 2, 3$
3. $\bar{\nabla}_{X_2} Y_2 = \nabla^2_{X_2} Y_2 - f\dot{f} g_2(X_2, Y_2)\,\partial_t$
4. $\bar{\nabla}_{X_2} X_3 = \bar{\nabla}_{X_3} X_2 = X_2(\ln h)\,X_3$
5. $\bar{\nabla}_{X_3} Y_3 = \nabla^3_{X_3} Y_3 - h g_3(X_3, Y_3)\,\mathrm{grad} h$

Proposition 4.2. *Let $\bar{M} = \left(I \times_f M_2\right) \times_h M_3$ be a sequential generalized Robertson-Walker space-time with metric $\bar{g} = \left(-dt^2 \oplus f^2 g_2\right) \oplus h^2 g_3$ and also let $X_i, Y_i, Z_i \in \mathfrak{X}(M_i)$. Then*

1. $\bar{R}(\partial_t, \partial_t)\,\partial_t = \bar{R}(\partial_t, \partial_t)\,Z_j = \bar{R}(X_i, Y_i)\,Z_j = \bar{R}(\partial_t, Y_2)\,Z_3 = 0, i \neq j$
2. $\bar{R}(X_2, Y_2)\,Z_2 = R^2(X_2, Y_2)\,Z_2 + \dot{f}^2\left[g_2(X_2, Y_2)\,Y_2 - g_2(Z_2, Y_2)\,X_2\right],$
3. $\bar{R}(\partial_t, Y_2)\,\partial_t = \frac{\ddot{f}}{f} Y_2,$
4. $\bar{R}(\partial_t, Y_3)\,\partial_t = \dfrac{1}{f}\frac{\partial^2 h}{\partial t^2} Y_3, i, j = 1, 2$
5. $\bar{R}(\partial_t, Y_2)\,Z_2 = f\ddot{f} g_2(Y_2, Z_2)\,\partial_t$
6. $\bar{R}(X_2, Y_3)\,Z_2 = \dfrac{-1}{h} H^h(X_2, Z_2)\,Y_3,$
7. $\bar{R}(\partial_t, Y_3)\,Z_3 = h g_3(Y_3, Z_3)\,\nabla_{\partial_t}\mathrm{grad} h,$
8. $\bar{R}(X_2, Y_3)\,Z_3 = h g_3(Y_3, Z_3)\,\nabla_{X_2}\mathrm{grad} h$

9. $\bar{R}(X_3, Y_3) Z_3 = R^3 (X_3, Y_3) Z_3 - \left\|\mathrm{grad} h\right\|^2 [g_3 (X_3, Y_3) Y_3 - g_3 (Z_3, Y_3) X_3]$

Now we consider the Ricci curvature $\bar{\mathrm{Ric}}$ of a sequential generalized Robertson-Walker space-time of the form $\bar{M} = \left(I \times_f M_2\right) \times_h M_3$.

Proposition 4.3. *Let $\bar{M} = \left(I \times_f M_2\right) \times_h M_3$ be a sequential GRW space-time with metric $\bar{g} = \left(-dt^2 \oplus f^2 g_2\right) \oplus h^2 g_3$ and also let $X_i, Y_i, Z_i \in \mathfrak{X}(M_i)$. Then*

1. $\bar{Ric}(\partial_t, \partial_t) = \frac{n_2}{f} \ddot{f} + \frac{n_3}{h} \frac{\partial^2 h}{\partial t^2}$
2. $\bar{Ric}(X_2, Y_2) = Ric^2 (X_2, Y_2) - g_2 (X_2, Y_2) f^\sharp - \frac{n_3}{h} H^h (X_2, Y_2)$
3. $\bar{Ric}(X_3, Y_3) = Ric^3 (X_3, Y_3) - g_3 (X_3, Y_3) h^\sharp$
4. $\bar{Ric}\left(X_i, Y_j\right) = 0, i \neq j$

where $f^\sharp = -f\ddot{f} - (n_2 - 1) \dot{f}^2$ *and* $h^\sharp = h\Delta h + (n_3 - 1) \left\|\mathrm{grad} h\right\|^2$

A sequential GRW space-time $\bar{M} = \left(I \times_f M_2\right) \times_h M_3$ is Einstein if

$$\bar{\mathrm{Ric}} (X, Y) = \mu \bar{g} (X, Y)$$

We have the following cases. The first case is

$$\begin{aligned} \bar{\mathrm{Ric}} (\partial_t, \partial_t) &= \mu \bar{g} (\partial_t, \partial_t) \\ \frac{n_2}{f} \ddot{f} + \frac{n_3}{h} \frac{\partial^2 h}{\partial t^2} &= -\mu \end{aligned}$$

and the second case is

$$\mathrm{Ric}^2 (X_2, Y_2) - g_2 (X_2, Y_2) f^\sharp - \frac{n_3}{h} H^h (X_2, Y_2) = \mu f^2 g_2 (X_2, Y_2)$$

and so

$$\mathrm{Ric}^2 (X_2, Y_2) = \frac{n_3}{h} H^h (X_2, Y_2) + \left(\mu f^2 + f^\sharp\right) g_2 (X_2, Y_2)$$

and finally we have

$$\mathrm{Ric}^3 (X_3, Y_3) = \left(\mu h^2 + h^\sharp\right) g_3 (X_3, Y_3)$$

Theorem 4.4. *Let $\bar{M} = \left(I \times_f M_2\right) \times_h M_3$ be an Einstein sequential GRW space-time with metric $\bar{g} = \left(-dt^2 \oplus f^2 g_2\right) \oplus h^2 g_3$. Then,*

1. $\mu = -\left(\frac{n_2}{f} \ddot{f} + \frac{n_3}{h} \frac{\partial^2 h}{\partial t^2}\right)$
2. *(M_2, g_2) is Einstein with factor $\left(\mu f^2 + f^\sharp\right)$ if $H^h (X_2, Y_2) = 0$ for any $X_2, Y_2 \in \mathfrak{X}(M_2)$ and*
3. *(M_3, g_3) is Einstein with factor $\left(\mu h^2 + h^\sharp\right)$.*

Corollary 4.5. *Let $\bar{M} = \left(I \times_f M_2\right) \times_h M_3$ be an Einstein sequential GRW space-time with metric $\bar{g} = \left(-dt^2 \oplus f^2 g_2\right) \oplus h^2 g_3$ and factor μ. Then*

1. *$(\bar{M}, \bar{g})$ is Ricci flat if $n_2 h \ddot{f} + n_3 f \frac{\partial^2 h}{\partial t^2} = 0$,*
2. *(M_2, g_2) is Ricci flat if $\mu f^2 + f^\sharp = 0$ and $H^h (X_2, Y_2) = 0$ for any $X_2, Y_2 \in \mathfrak{X}(M_2)$ and*

3. (M_3, g_3) *is Ricci flat if* $\mu h^2 + h^\sharp = 0$.

The converse of the above theorem is considered in the following result.

Theorem 4.6. *Let* $\bar{M} = \left(I \times_f M_2\right) \times_h M_3$ *be a sequential GRW space-time with metric* $\bar{g} = \left(-dt^2 \oplus f^2 g_2\right) \oplus h^2 g_3$. *Then* $(\bar{M}, \bar{g})$ *is Einstein with factor* μ *if*

1. $H^h(X_2, Y_2) = 0$ *for any* $X_2, Y_2 \in \mathfrak{X}(M_2)$,
2. (M_i, g_i) *is Einstein with factor* $\mu_i, i = 2, 3$,
3. $\mu_2 + f\ddot{f} + (n_2 - 1)\dot{f}^2 = \mu f^2$
4. $\mu_3 + h\dfrac{\partial^2 h}{\partial t^2} - (n_3 - 1)\left\|\mathrm{grad} h\right\|^2 = \mu h^2$
5. $\dfrac{n_2}{f}\ddot{f} + \dfrac{n_3}{h}\dfrac{\partial^2 h}{\partial t^2} = -\mu$

4.2. Sequential Standard Static Space-times

Theorem 4.7. *Let* $\bar{M} = \left(M_1 \times_f M_2\right) \times_h I$ *be a sequential standard static space-time with metric* $\bar{g} = \left(g_1 \oplus f^2 g_2\right) \oplus h^2\left(-dt^2\right)$ *and also let* $X_i, Y_i \in \mathfrak{X}(M_i)$. *Then*

1. $\bar{\nabla}_{X_1} Y_1 = \nabla^1_{X_1} Y_1$
2. $\bar{\nabla}_{X_1} X_2 = \bar{\nabla}_{X_2} X_1 = X_1(\ln f) X_2$
3. $\bar{\nabla}_{X_2} Y_2 = \nabla^2_{X_2} Y_2 - f g_2(X_2, Y_2)\,\mathrm{grad}^1 f$
4. $\bar{\nabla}_{X_i}\partial_t = \bar{\nabla}_{\partial_t} X_i = X_i(\ln h)\partial_t, i = 1, 2$
5. $\bar{\nabla}_{\partial_t}\partial_t = h\,\mathrm{grad} h$

Theorem 4.8. *Let* $\bar{M} = \left(M_1 \times_f M_2\right) \times_h I$ *be a sequential standard static space-time with metric* $\bar{g} = \left(g_1 \oplus f^2 g_2\right) \oplus h^2\left(-dt^2\right)$ *and also let* $X_i, Y_i, Z_i \in \mathfrak{X}(M_i)$. *Then*

1. $\bar{R}(X_1, Y_1) Z_1 = R^1(X_1, Y_1) Z_1$
2. $\bar{R}(X_2, Y_2) Z_2 = R^2(X_2, Y_2) Z_2 - \left\|\mathrm{grad}^1 f\right\|^2 \left[g_2(X_2, Y_2) Y_2 - g_2(Z_2, Y_2) X_2\right]$
3. $\bar{R}(X_1, Y_2) Z_1 = \dfrac{-1}{f} H^f_1(X_1, Z_1) Y_2$
4. $\bar{R}(X_1, Y_2) Z_2 = f g_2(Y_2, Z_2) \nabla^1_{X_1} \mathrm{grad}^1 f$
5. $\bar{R}(X_1, Y_2)\partial_t = \bar{R}(\partial_t, \partial_t)\partial_t = \bar{R}(X_i, Y_i) Z_j = 0, i \neq j$
6. $\bar{R}(X_i, \partial_t) Z_j = \dfrac{-1}{h} H^h\left(X_i, Z_j\right)\partial_t, i, j = 1, 2$
7. $\bar{R}(X_i, \partial_t)\partial_t = -h\nabla_{X_i}\mathrm{grad} h, i = 1, 2$

Now consider the Ricci curvature $\bar{\mathrm{Ric}}$ of a sequential standard static space-time of the form $\left(M_1 \times_f M_2\right) \times_h I$.

Theorem 4.9. *Let* $\bar{M} = \left(M_1 \times_f M_2\right) \times_h I$ *be a sequential standard static space-time with metric* $\bar{g} = \left(g_1 \oplus f^2 g_2\right) \oplus h^2\left(-dt^2\right)$ *and also let* $X_i, Y_i, Z_i \in \mathfrak{X}(M_i)$. *Then*

1. $\bar{Ric}(X_1, Y_1) = Ric^1(X_1, Y_1) - \dfrac{n_2}{f} H^f_1(X_1, Y_1) - \dfrac{1}{h} H^h(X_1, Y_1)$
2. $\bar{Ric}(X_2, Y_2) = Ric^2(X_2, Y_2) - g_2(X_2, Y_2) f^\sharp - \dfrac{1}{h} H^h(X_2, Y_2)$
3. $\bar{Ric}(\partial_t, \partial_t) = h\Delta h$

4. $\bar{Ric}\left(X_i, Y_j\right) = 0, i \neq j$
   where $f^\sharp = f\Delta^1 f + (n_2 - 1)\left\|\mathrm{grad}^1 f\right\|^2$.

A sequential standard static space-time $\left(M_1 \times_f M_2\right) \times_h I$ is Einstein with factor μ if

$$\bar{\mathrm{Ric}}(X, Y) = \mu \bar{g}(X, Y) \tag{11}$$

In this case

$$\mu = -\frac{\Delta h}{h}$$

But taking the trace of equation (11) we get that

$$\mu = \frac{r}{n_1 + n_2 + 1}$$

where r is the scalar curvature i.e.

$$r = -\frac{\Delta h}{h}(n_1 + n_2 + 1)$$

Moreover,

$$\mathrm{Ric}^1(X_1, Y_1) - \frac{n_2}{f} H_1^f(X_1, Y_1) - \frac{1}{h} H^h(X_1, Y_1) = \mu g_1(X_1, Y_1)$$

and

$$\mathrm{Ric}^2(X_2, Y_2) - g_2(X_2, Y_2) f^\sharp - \frac{1}{h} H^h(X_2, Y_2) = \mu f^2 g_2(X_2, Y_2)$$

Corollary 4.10. *Let* $\bar{M} = \left(M_1 \times_f M_2\right) \times_h I$ *be an Einstein sequential standard static space-time with metric* $\bar{g} = \left(g_1 \oplus f^2 g_2\right) \oplus h^2\left(-dt^2\right)$. *Then the scalar curvature* r *of* $\bar{M}$ *is given by*

$$r = -\frac{\Delta h}{h}(n_1 + n_2 + 1)$$

Corollary 4.11. *Let* $\bar{M} = \left(M_1 \times_f M_2\right) \times_h I$ *be an Einstein sequential standard static space-time with metric* $\bar{g} = \left(g_1 \oplus f^2 g_2\right) \oplus h^2\left(-dt^2\right)$. *Then*

1. (M_1, g_1) *is Einstein with factor* μ *if* $n_2 h H_1^f(X_1, Y_1) - f H^h(X_1, Y_1) = 0$,
2. (M_2, g_2) *is Einstein with factor* $\mu f^2 + f^\sharp$ *if* $H^h(X_2, Y_2) = 0$

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