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Black hole solutions and Euler equation in Rastall and generalized Rastall theories of gravity

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Focusing on the special case of generalized Rastall theory, as a subclass of the non-minimal curvature-matter coupling theories in which the field equations are mathematically similar to the Einstein field equations in the presence of cosmological constant, we find two classes of black hole (BH) solutions including (i) conformally flat solutions and (ii) non-singular BHs. Accepting the mass function definition and by using the entropy contents of the solutions along with thermodynamic definitions of temperature and pressure, we study the validity of Euler equation on the corresponding horizons. Our results show that the thermodynamic pressure, meeting the Euler equation, is not always equal to the pressure components appeared in the gravitational field equations and satisfies the first law of thermodynamics, a result which in fact depends on the presumed energy definition. The requirements of having solutions with equal thermodynamic and Hawking temperatures are also studied. Additionally, we study the conformally flat BHs in the Rastall framework. The consequences of employing generalized Misner–Sharp mass in studying the validity of the Euler equation are also addressed.

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1. Introduction

The conservation of energy–momentum forms the backbone of the general relativity (GR) theory.[1,2] Its breakdown also helps us in providing an explanation for observations, confirming the current acceleration of the universe.[3] Indeed, GR is modified by admitting a non-minimal coupling between geometry and matter fields.[4–13] Theoretically, the origin of these attempts comes back to Rastall[4] who argued that the ordinary energy–momentum conservation law (OCL) can be violated in the curved spacetime, a hypothesis which leads to interesting cosmological and gravitational consequences.[14] Even whenever OCL is valid, and the non-minimal coupling has not happened, the existence of such ability in the structure of geometry and matter fields can provide proper description for the cosmic eras.[8]

Black hole (BH) solutions have a lot of significance from both theoretical and experimental perspectives.[1,2] For example, one can attribute the Hawking temperature to their horizon, and use the Misner–Sharp mass to write the gravitational field equations as a thermodynamic equation of state.[15–17] Therefore, in this approach, the energy definition has a crucial role,[15–17] a role which shows again itself in modeling the cosmic evolution.[19] It is also useful to mention that the Hawking temperature is not always equal to the thermodynamic temperature obtained from the thermodynamic definition of temperature (see Ref. 20 and references therein), and indeed, the deep connection between gravity and thermodynamics needs to be further studied.[15,20]

For a thermodynamic system with entropy S, energy E, pressure P and temperature T within the volume V, the Euler equation takes the form $E = TS - PV$.[21] The quality of validity of the Euler equation for BH solutions of various gravitational theories have not yet been fully studied.[15–17] In fact, the study of the quality of validity of the Euler equation can help us in achieving a better understanding of relation between thermodynamics and gravity, and also the concept of the quantities such as energy.

Although most of the known BH solutions are singular at their spatial origin, i.e. $r = 0$, it has been shown that there are also non-singular BH solutions in the framework of GR.[22] On the other hand, conformal flat spacetimes, for which the Weyl tensor is zero, has interesting properties[1,2] motivating physicists to study them.[28] Recently, some BH solutions have been derived in the Rastall theory and its generalized version[23–26] showing that the de Sitter spacetime, a conformal flat and non-singular spacetime, can be obtained as a vacuum solution in these theories.[23,26]

All of the above arguments motivate us to look for the conformally flat and non-singular BHs in the context of Rastall theory as well as in a special subclass of the generalized Rastall theory, which its field equations are similar to the Einstein equations with the cosmological constant. After addressing the field equations corresponding to the generalized Rastall theory and also the Rastall theory in Sec. 2, we obtain conformally flat BH solutions and investiagte the validity of the Euler equation in both mentioned frameworks in Sec. 3. In Sec. 4, we discuss non-singular BH solutions in the framework of a special subclass of the generalized Rastall theory. In Sec. 5, we present our conclusion. Throughout this work, we use the units so that $c = \hbar = k_{\mathrm{B}} = 1$, where k_{B} denotes the Boltzmann constant.

2. Field Equations

In the generalized Rastall theory,[8] OCL is modified as

$$T^{\mu\nu}{}_{;\mu} = (\lambda R)^{;\nu}, \tag{1}$$

leading to the field equations

$$G_{\mu\nu} + \kappa\lambda g_{\mu\nu} R = \kappa T_{\mu\nu}, \tag{2}$$

where κ and λ denote the Rastall gravitational coupling constant and the Rastall parameter, respectively. Clearly, Eq. (1) indicates the validity of OCL does not necessarily mean that $\lambda = 0$ (OCL is met whenever $\lambda = \frac{\beta}{R}$ where β is an unknown constant). For this theory, the (anti)de Sitter spacetime can be obtained as the vacuum solution.[8] Now, let us consider the static spherically symmetric metric

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2\, d\Omega_2^2, \tag{3}$$

filled by a source satisfying OCL. In this manner, the field equation (2) take the form

$$G_{\mu\nu} = \kappa T_{\mu\nu} - \kappa\beta g_{\mu\nu} = \kappa(T_{\mu\nu} - \beta g_{\mu\nu}), \tag{4}$$

which obviously confirms that the (anti)-de Sitter spacetime is a vacuum solution meaning that the $\kappa\beta$ term plays the role of cosmological constant.[8]

Mathematically, the field equation (4) and the Einstein field equations in the presence of cosmological constant (EECC) have similar forms. In fact, for EECC, we have $G_{\mu\nu} = \kappa_E T_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa_E(T_{\mu\nu} - \frac{\Lambda}{\kappa_E} g_{\mu\nu})$, where κ_E is the Einstein gravitational coupling constant. Here, Λ represents the cosmological constant and $\frac{\Lambda}{\kappa_E} g_{\mu\nu}$ denotes its corresponding energy–momentum tensor. Comparing with Eq. (4), we find that the $\kappa\beta$ term is the counterpart of Λ in this formalism. Hence, we have $\rho = -p = \beta$ for the energy density (ρ) and pressure (p) of this source. However, there are four differences between Eq. (4) and EECC as (i) in Eq. (4), the $\kappa\beta$ term, playing the role of cosmological constant, is naturally arising from the ability of spacetime and matter fields to couple with each other in a non-minimal way (i.e. $\beta \neq 0$), and it has not manually been added to the field equations as in GR, (ii) the naturally arising cosmological constant of Rastall theory ($\kappa\beta$) is proportional to the gravitational coupling κ, a feature which does not exist in GR at least at the level of field equations, (iii) the Rastall gravitational coupling constant κ is not necessarily equal to that of the Einstein theory unless one sets $\lambda = 0$, and (iv) as we will show, the Misner–Sharp content of mass is different in the frameworks of EECC and Rastall theory. These differences lead to different thermodynamic features for the obtained solutions which are addressed in the paper.

Whenever $\lambda = \text{constant} \equiv \xi$, Eq. (1) reduces to the original Rastall hypothesis[4] in which

$$T^{\mu\nu}{}_{;\mu} = \xi R^{,\nu} \Rightarrow G_{\mu\nu} + \kappa_R \xi g_{\mu\nu} R = \kappa_R T_{\mu\nu}, \tag{5}$$

where κ_R denotes the original Rastall gravitational coupling which generally differs from κ and that of the Einstein theory.[4,8]

Investigation of the Misner–Sharp mass in the Rastall and generalized Rastall frameworks, as well as the Hawking temperature, horizon entropy and the validity of the first law of thermodynamics on the horizons of metric (3) have been studied in Refs. 16 and 17. For this propose, the Clausius relation, the first law of thermodynamics together with the $t-t$ and $r-r$ components of the field equations are used.[15–17]

3. Conformally Flat Black Hole Solutions

3.1. *Conformally flat BHs in the generalized Rastall theory*

In order to find the conformal flat solutions of (4), the Weyl tensor defined as

$$C_{ijkl} = R_{ijkl} - \frac{1}{n-1}\left(g_{ijkp}R^{p}{}_{l} + g_{ijpl}R^{p}{}_{k}\right) + \frac{1}{n(n-1)} g_{ijkl} R, \tag{6}$$

should be zero.[1] This gives the solution

$$f(r) = 1 - ar - br^2, \tag{7}$$

where a and b are arbitrary integration constants. Setting $a = 0$ and $b = \Lambda$, the line element represents the de Sitter spacetime. Then, this solution is a generalization to de Sitter solution and is also similar to the solution introduced by Kiselev, except possessing of a mass term.[18] The a parameter in (7) plays the role of a quintessence field in Ref. 18. Then, depending on the a and b parameter values, our metric function can represent a flat, quintessence and de Sitter spacetime, respectively. One may also consider the limiting behavior of this solution for small and large r values where each of the quintessence and cosmological constant fields dominate.

Now, using the field equation (2), one can easily reach at

$$b = \frac{\kappa\beta}{3}, \quad \rho(r) = -p_r(r) = -2p_t(r) = \frac{2a}{\kappa r}, \tag{8}$$

as a solution in which $p_r(r)$ and $p_t(r)$ are the radial and transverse pressure components, respectively. We clearly see that the obtained solution (8) addresses an anisotropic fluid. As usual, $\rho(r)$ denotes the energy density, and respecting the weak energy condition requires to have $a\kappa > 0$. From the energy–momentum tensor components in (8), one also realizes that the quintessence field here has an average equation of state of $\omega_{\text{av}} = -\frac{2}{3}$, indicating a gravitationally repulsive fluid. Although the metric with (7) is well-behaved at the center ($r = 0$), the components of the corresponding $T_{\mu\nu}$ in (8) diverge at the $r \to 0$ limit, which signals a physical singularity. In fact, unlike the Weyl square, the Kretschmann invariant, the Ricci scalar and the Ricci square diverge at this limit. The $b = \frac{\kappa\beta}{3}$ relation is also interesting, because it is claiming that the value of the cosmological constant specifies b, meaning that the value of β affects the location of the spacetime horizons (the solutions of the $f(r_h) = 0$ equation). In fact, the (anti)-de Sitter sector of metric ($b \neq 0$) is cancelled in the field equation (4) by adopting the special value of β (and thus λ) satisfying the $b = \frac{\kappa\beta}{3}$ condition. For this solution, whenever $f(r) = 1 - ar$, OCL is met, and the energy–momentum tensor obtained in Eq. (8), satisfies the $G_{\mu\nu} = \kappa T_{\mu\nu}$ equations.

Here, instead of the generalized Misner–Sharp mass,[17] we use the mass function corresponding to the energy density ρ defined as[22]

$$m(r) = 4\pi \int_0^r \rho(r) r^2 \, dr, \tag{9}$$

and can be combined with the obtained energy density (8) to reach[22]

$$m(r) = 4\pi \int_0^r \rho(r) r^2 \, dr = \frac{4\pi a}{\kappa} r^2, \tag{10}$$

leading to

$$m(r_h) \equiv M = \frac{a}{\kappa} A, \tag{11}$$

for the mass circumscribed by the horizon located at r_h with area $A = 4\pi r_h^2$. In the generalized Rastall theory, the entropy (S)-area (A) relation is written as $S = \frac{2\pi}{\kappa} A$[17] combined with Eq. (10), to reach at $M = \frac{a}{2\pi} S$. Now, the thermodynamic temperature definition ($T = \frac{\partial M}{\partial S}$) yields $T = \frac{a}{2\pi}$ for the temperature of the energy source circumscribed by the radius r_h. Then, from the weak energy condition and the positivity of the entropy S, one finds that both a and κ parameters should be positive independently. Then, the positivity of the temperature $T = \frac{a}{2\pi}$ is guaranteed by these two physical conditions. This in turn means that the $f(r_h) = 0$ equation either has one solution located at $r_h = \frac{a - \sqrt{a^2 + 4b}}{-2b}$ for $b > 0$, or two solutions located at $r_h^{\pm} = \frac{a \pm \sqrt{a^2 + 4b}}{-2b}$ for $b < 0$ with the condition of $a^2 \geq |4b|$ for these solutions to be real. The above results give us the corresponding Euler equation as $M = TS$. Also, it may worth to mention that in a classical point of view, where there is no mass loss (no evaporation) for the BH, the positivity of the thermodynamic temperature as $T = \frac{\partial M}{\partial S}$ can be understood from the second law of thermodynamics. But including the mass loss due to the BH evaporation, and also demanding the second law of thermodynamics, the temperature by this definition will be negative. A rather similar situation happens when the Hawking horizon temperature is defined by the surface gravity, i.e. $T_H = \frac{K}{2\pi}$, which represents the gravitational acceleration as measured by the asymptotic observer for an infalling object to the BH. Positivity of temperature then means that gravity force is attractive and vice versa. However, for the inner horizon of Reissner–Nordstrom BH, one finds a negative surface gravity, representing a gravitational repulsion effect, and then a negative temperature. Thus, one may consider only the absolute values for both these definitions of temperature.

Besides of what we obtained till now for the special case of $b = \frac{\kappa\beta}{3}$, in which $\kappa\beta$ plays the role of cosmological constant, one can write the total solution of the field equation (2) as

$$\begin{aligned} \rho(r) = -p_r(r) &= \frac{1}{\kappa}\left(\frac{2a}{r} + 3b - \kappa\beta\right), \\ p_t(r) &= -\frac{1}{\kappa}\left(\frac{a}{r} + 3b - \kappa\beta\right), \end{aligned} \tag{12}$$

which clearly shows that at very large distances, we face a cosmological constant-like source. Again, the obtained solution (12) represents that the fluid supporting the geometry is anisotropic in general. Interestingly, in the asymptotic region, i.e. for $r \to \infty$, the fluid tends to be isotropic with $\rho(r) = -p_r(r) = -p_t(r) = \frac{3b}{\kappa} - \beta$. Then, for the asymptotic region, $\frac{3b}{\kappa} - \beta$ plays the role of cosmological constant. The latter can again establish a relation between the values of β, κ, b and the cosmological constant Λ (the current value of the dark energy density) as

$$\Lambda = 3b - \kappa\beta, \tag{13}$$

meaning that both b and β may contribute in forming Λ. Hence, if $b = \frac{\kappa\beta}{3}$ then $\Lambda = 0$ which is in agreement with the previous solution where in the (anti)-de Sitter sector of metric ($b \neq 0$) is cancelled in the field equation (4) by adopting the $b = \frac{\kappa\beta}{3}$ condition. We also see that, unlike the previous solution for which the $b = 0$ case does not cover the (anti)-de Sitter geometry, in the present solution even if we set $b = 0$, β can take on the role of cosmological constant. Equation (13) also implies that both the $\frac{b}{\kappa}$ and β terms have the same dimension, a result compatible with the outcome of previous solution based on the $b = \frac{\kappa\beta}{3}$ case.

In this situation, the mass circumscribed by the horizon is obtained as

$$m(r_h) \equiv \mathcal{M} = \frac{a}{\kappa}A + \frac{\Lambda}{\kappa}V, \tag{14}$$

where $V = \frac{4\pi}{3}r_h^3$. Combining this result with the $S = \frac{2\pi}{\kappa}A$ relation,[17] and bearing the thermodynamic definitions of temperature (T) and pressure (P) in mind, one reaches at

$$\begin{aligned} T &= \frac{\partial\mathcal{M}}{\partial S}\bigg|_{V=\text{constant}} = \frac{a}{2\pi}, \\ P &= -\frac{\partial\mathcal{M}}{\partial V}\bigg|_{S=\text{constant}} = -\frac{\Lambda}{\kappa}. \end{aligned} \tag{15}$$

It finally leads to the Euler equation $\mathcal{M} = TS - PV$. The Hawking temperature corresponding to metric (7) is also evaluated as[1,15]

$$T_h = \frac{\left|\frac{df(r)}{dr}\right|_{r=r_h}}{2\pi} = \frac{\sqrt{a^2 + 4b}}{2\pi}, \tag{16}$$

which indicates $T = T_h$ only if $b = 0$. Therefore, if we are looking for the solutions with the same Hawking and thermodynamic temperatures, then we have only one possibility as $b = 0$ and $a > 0$. Regarding (16), the condition of $a^2 \geq |4b|$ for $b < 0$ case is also required here for the temperature T_h to be real, similar to what we discussed after Eq. (11).

Now, let us consider the generalized Misner–Sharp mass (M_{MS}) confined to the horizon r_h of metric (3)[17]

$$M_{\mathrm{MS}} = \frac{4\pi}{\kappa}\left[r_h + \kappa\left(\int \lambda\left[\frac{d(r^2 f'(r))}{dr} - 2\left(1 - \frac{d(rf(r))}{dr}\right)\right]dr\right)_{r=r_h}\right]. \quad (17)$$

Here, "prime" denotes the derivative with respect to time. Using the $\lambda = \frac{\beta}{R}$ relation, in which $R = -\frac{r^2 f''(r) + 4rf'(r) - 2[1-f(r)]}{r^2}$, one can reach

$$M_{\mathrm{MS}} = \frac{4\pi}{\kappa}\left[r_h - \frac{\kappa\beta}{3}r_h^3\right] = \frac{S}{2\pi r_h} - \beta V, \quad (18)$$

where the $S = \frac{2\pi}{\kappa}A$ relation[17] has been used to obtain the last equality. For the thermodynamic temperature and pressure corresponding to (18), one finds

$$\begin{aligned} T_{\mathrm{MS}} &= \frac{\partial M_{\mathrm{MS}}}{\partial S}\bigg|_{V=\mathrm{constant}} = \frac{1}{2\pi r_h}, \\ P_{\mathrm{MS}} &= -\frac{\partial M_{\mathrm{MS}}}{\partial V}\bigg|_{S=\mathrm{constant}} = \beta, \end{aligned} \quad (19)$$

which again indicates that whenever the M_{MS} mass is considered, $\kappa\beta$ plays the role of cosmological constant. Also, we have $T_{\mathrm{MS}} = T_h$ only if $a = 0$, which is nothing but the (anti)-de Sitter spacetime for which the Cai–Kim temperature ($\frac{1}{2\pi r_h}$) is equal to the Hawking temperature.[27] Therefore, the above results suggest that the effects of b and a are stored in r_h, and the effect of β is appeared directly as the coefficient of volume (the thermodynamic pressure). At this step, one can realize another difference between the Einstein theory and our studied version of Rastall theory. Considering the vacuum case of EECC ($G_{\mu\nu} = -\Lambda g_{\mu\nu}$), one reaches at (anti)-de Sitter solution for (3) in which its Misner–Sharp mass is given by $\frac{r_h}{2G}$. This clearly differs from the $a = 0$ limit (or equally, the (anti)-de Sitter limit of (7)) of Eq. (18). Thus, the effect of cosmological constant is stored only in r_h, and this is another difference between Eq. (4) and EECC. Finally, it is worthwhile to mention that $P_{\mathrm{MS}} = P$ only if $b = 0$ leading to $T_h = T \neq T_{\mathrm{MS}} = 0$.

3.2. *Conformally flat BHs in the Rastall theory*

For this theory, the horizon entropy meets the relation $S = \frac{2\pi}{\kappa_R}A$ and the Newtonian limit leads to[16]

$$\kappa_R = \frac{4\gamma - 1}{6\gamma - 1}, \quad \xi = \frac{\gamma(6\gamma - 1)}{4\gamma - 1}, \quad (20)$$

in which $\gamma = \kappa_R \xi$, and we assumed $8\pi G = 1$. In this subsection, we use the index R to indicate that we work in the original Rastall framework. Inserting the conformally flat solution (7) into the field equation (5) and by using (20), one reaches

$$\begin{aligned} \rho(r) = -p_r(r) &= \frac{6\gamma - 1}{4\gamma - 1}\left[\frac{2a(1-3\gamma)}{r} + 3b(1-4\gamma)\right], \\ p_t(r) &= \frac{6\gamma-1}{4\gamma-1}\left[\frac{a(6\gamma-1)}{r} + 3b(4\gamma-1)\right], \end{aligned} \tag{21}$$

addressing us an anisotropic fluid. Now, following the recipe led to Eqs. (14) and (15), we obtain

$$\begin{aligned} m(r_h) \equiv \mathcal{M}_R &= \frac{a(1-3\gamma)(6\gamma-1)}{4\gamma-1} A - 3b(6\gamma-1)V, \\ T_R &= \left.\frac{\partial \mathcal{M}_R}{\partial S}\right|_{V=\text{constant}} = \frac{a(1-3\gamma)}{2\pi}, \\ P_R &= -\left.\frac{\partial \mathcal{M}_R}{\partial V}\right|_{S=\text{constant}} = 3b(6\gamma-1). \end{aligned} \tag{22}$$

Equations (21) and (22) represent that in the present framework, the $3b(1-6\gamma)$ term denotes the thermodynamics pressure corresponding to the cosmological constant $3b(4\gamma-1)$. In the Rastall framework, by considering the generalized Misner–Sharp mass ($\mathcal{M}_R^{\mathrm{MS}}$) confined to radius r_h, expressed as[16]

$$\mathcal{M}_R^{\mathrm{MS}} = \frac{6\gamma-1}{2(4\gamma-1)}[(1-2\gamma)r_h + \gamma r_h^2 f'(r_h)], \tag{23}$$

and since $S = \frac{2\pi}{\kappa_R}A$,[16] one can get

$$\begin{aligned} \mathcal{M}_R^{\mathrm{MS}} &= \frac{1-\gamma(2+ar_h)}{2\pi r_h} S - 6b\lambda V, \\ T_R^{\mathrm{MS}} &= \left.\frac{\partial \mathcal{M}_R^{\mathrm{MS}}}{\partial S}\right|_{V=\text{constant}} = \frac{1-\gamma(2+ar_h)}{2\pi r_h}, \\ P_R^{\mathrm{MS}} &= -\left.\frac{\partial \mathcal{M}_R^{\mathrm{MS}}}{\partial V}\right|_{S=\text{constant}} = 6b\lambda. \end{aligned} \tag{24}$$

In summary, our results (Eqs. (11), (12), (14), (17) and Eqs. (20)–(22)) show that in both the Rastall theory and its special generalized case, the thermodynamic pressure obtained by accepting the mass definition (9) is equal to the pressure of the cosmological constant candidate appeared in the field equations, a result which is not obtained by employing the generalized Misner–Sharp mass. In addition, the obtained thermodynamic temperatures are not always equal to the Hawking temperature.

4. Non-Singular Black Hole Solutions

4.1. *Non-singular BHs in the generalized Rastall theory*

It is important recalling that in GR, as well as in extended theories of gravity (including the particular subclass of the Rastall theory which is the framework of this paper) an unsolved problem concerning BHs is the presence of a spacetime singularity in their core. Such a problem was present starting from the first historical papers concerning BHs[29–31] and was generalized in the famous paper by Penrose.[32] It is a common belief that this problem can be solved when a correct quantum gravity theory is obtained. For the sake of completeness, it is important recalling some issues which dominate the question about the existence or non-existence of BH horizons and singularities from both of the theoretical and observational points of view, and proposing some ways to remove BH singularities also at a classical level, i.e. without taking into account of a quantum gravity theory. Interesting alternatives to singular BHs are the so-called *eternally collapsing objects* (ECO), *magnetospheric eternally collapsing objects*, (MECO) and *nonlinear electrodynamics* (NLED) *objects*. An ECO is a gravitationally compact mass supported against gravity by an internal radiation pressure.[33] In its outer layers of mass, a plasma with some baryonic content is supported by a net outward flux of momentum via radiation.[33] Concerning MECOS, one can postulate that some physical reason as, why the existence of magnetic field should prevent formation of any event horizon, can emerge by considering contracting plasma which is threaded by a self-magnetic flux.[34–36] It is a good approximation to assume that the flux remains conserved. Even if it is not conserved, there should be a finite flux all the way.[34–36] Let us presume that the plasma ball collapses inside its event horizon. The region inside the event horizon is trapped, and this means, no lines of force can emanate out of the plasma ball, then a local observer siting at a radius larger than the Schwarzschild radius will not see any magnetic field.[34–36] Indeed, by no hair theorem, a neutral BH has no magnetic field. This means, the entire magnetic flux must vanish before the plasma ball can enter its event horizon.[34–36] Conversely, a plasma ball endowed with initial magnetic field cannot become a BH unless it can destroy its entire magnetic field. But why should the ball destroy its entire magnetic field to enter the event horizon, which in the BH folk lore, is a mere coordinate singularity which a comoving observer cannot notice at all. Hence, a plasma with initial magnetic field cannot form a BH.[34–36] This is consistent with a paper by the Nobel Laureate K. Thorne, who showed back in 1965 that pure magnetic energy would not collapse into a BH state.[37] On the other, it has been recently shown that NLED objects can remove BH singularities too. In Ref. 38, a particular solution of Einstein field equation for a model of star supported against self-gravity entirely by radiation pressure has been discussed. In such a solution trapped surfaces as defined in GR are not formed during a gravitational collapse, and hence the singularity theorem on BHs, as proposed in Ref. 32, cannot be applied. More in general, NLED effects turn out to be important as regard to the mass-radius relation,[38] which is maximum for a BH.

Also, there is another interesting proposal which concerns the possibility to replace the Schwarzschild singularity with a de Sitter vacuum. This proposal started from an idea by Sakharov,[39] who considered a negative density in the equation of state for super-high density, and by Gliner,[40] who interpreted such a negative density as corresponding to a vacuum and suggested that it could be the final state of the gravitational collapse. This approach has been recently considered in Ref. 22.

Let us follow Ref. 22 and consider the energy density

$$\rho(r) = a \exp\left(-\frac{r^3}{b^3}\right), \tag{25}$$

where a and b are unknown constants in general. Note that a and b parameters here are different than in the previous solution. In this manner, solving Eq. (4), one reaches at

$$f(r) = 1 - \frac{C_1}{r} - \frac{\kappa\beta}{3}r^2 + \frac{\kappa a b^3}{3r}\exp\left(-\frac{r^3}{b^3}\right), \tag{26}$$

and

$$p_r(r) = -\rho(r), \quad p_t(r) = \left(\frac{3r^3}{2b^3} - a\right)\rho(r), \tag{27}$$

which is an isotropic source. For this solution, the Ricci square and Ricci scalar are well-behaved at the $r \to 0$ limit. Moreover, the Weyl and Riemann squares are also well-behaved at the $r \to 0$ limit whenever

$$C_1 = \frac{b^3}{3\kappa a}, \quad \kappa = \pm\frac{1}{a}, \tag{28}$$

where regarding (26), C_1 appears as the mass parameter. Here, the positivity of both the a and b parameters are guaranteed by the weak energy condition and by the positivity of the mass parameter C_1. Moreover, bearing the $S = \frac{2\pi}{\kappa}A$[17] relation in mind and noting that entropy should be positive, the possibility of $\kappa = -\frac{1}{a}$ is ruled out.

To show that the solution (26) with constraints $C_1 = \frac{b^3}{3}$ and $\kappa = \frac{1}{a}$ (28) is regular everywhere, one can use the following Eddington–Finkelstein coordinate transformation

$$du = dt + \frac{dr}{1 - \frac{\beta}{3a}r^2 - \frac{b^3}{3r}\left(1 - \exp\left(-\frac{r^3}{b^3}\right)\right)}, \tag{29}$$

to write the metric (26) as

$$ds^2 = -\left(1 - \frac{\beta}{3a}r^2 - \frac{b^3}{3r}\left(1 - \exp\left(-\frac{r^3}{b^3}\right)\right)\right)du^2 + 2\,du\,dr + r^2\,d\Omega^2, \tag{30}$$

representing that the solution is non-singular everywhere (in both $r = 0$ and $r = r_h$). One notes that due to the existence of the non-minimal coupling, the asymptotic nature of the solutions (26) (or (30)) is different than[22] for large r values which coincides with the Schwarzschild solution and for small r values behaves like the de Sitter solution. The solution (26) has the same internal nature as in Ref. 22 due to the considered specific vacuum stress–energy–momentum tensor in (25) and (27), but possesses a de Sitter asymptote, instead of Schwarzschild, with a cosmological term $\frac{\kappa\beta}{3}$ induced by the non-minimal coupling property of the background theory. In this manner, $f(r_h) = 0$ has a solution at $r_h = b$ if

$$\beta = a\left(\frac{3 - b^2(1 - e^{-1})}{b^2}\right), \tag{31}$$

This is a relation between the non-minimal coupling parameter β (or equally λ) and the BH's properties (or equally the energy source). In both sides of $r_h = b$ horizon, we have $f(r < b) > 0$ and $f(r > b) < 0$. This means that the metric changes its signature from $(-, +, +, +)$ to $(+, -, +, +)$ at $r = b$.

Now, without considering $\kappa = \frac{1}{a}$, using Eq. (25) and the area-entropy relation $S = \frac{2\pi}{\kappa}A$,[17] one reaches

$$m(b) \equiv \tilde{M} = 4\pi\int_0^b \rho(r)r^2\,dr = \frac{\kappa ba}{6\pi}S - \frac{a}{e}V, \tag{32}$$

as the BH's mass confined to the horizon located at $r_h = b$ where $V = \frac{4\pi}{3}b^3$ and $A = 4\pi b^2$. The thermodynamic temperature and pressure can be obtained as

$$\begin{aligned} T &= \frac{\partial\tilde{M}}{\partial S}\bigg|_{V=\text{constant}} = \frac{\kappa ab}{6\pi}, \\ P &= -\frac{\partial\tilde{M}}{\partial V}\bigg|_{S=\text{constant}} = \frac{a}{e}, \end{aligned} \tag{33}$$

respectively. We see that the temperature of the system is positive only if $\kappa = \frac{1}{a}$ which leads to $T = \frac{b}{6\pi}$, and hence, the Euler equation takes the form of $\tilde{M} = TS - PV$. For this solution, we also find $T_h = \frac{b|1-\frac{6}{b^2}-4e^{-1}|}{6\pi}$ as the Hawking temperature which is equal to the thermodynamic temperature (33) only if $1 - \frac{6}{b^2} - 4e^{-1} = \pm 1$. This constraint yields $b^2 = \frac{3}{1-2e^{-1}}$. Equation (31) has also an interesting solution for $f(r_h) = 0$ located at $b = \frac{3}{1-e^{-1}}$ for which β, and thus λ, will be zero. In this situation, the generalized Rastall theory reduces to the Einstein theory $(\kappa = 8\pi G)$[17] in the absence of cosmological constant. Hence, Eq. (28) implies $\kappa = \frac{1}{a} = 8\pi G$ whenever $\beta = 0$ which is in agreement with Ref. 22.

Bearing Eq. (17) in mind, one can easily see Eq. (18) and thus Eq. (19) are still valid. It is due to this fact that these results are independent of $f(r)$, a property which comes from the $\beta = \lambda R$ constraint. Of course, it should be noted that for the above solution, r_h differs from that of the conformal BHs, and it is obtained by finding out the zeros of Eq. (26). It is also easy to check Eq. (26) is valid for EECC if we change $\kappa\beta$ and κ as Λ and $8\pi G$, respectively. Hence, since the Misner–Sharp mass of the Einstein framework takes the $\frac{r_h}{2G}$ form, once again, we can see that the Misner–Sharp mass of the obtained solution is different in the frameworks of the considered generalized Rastall theory and EECC.

4.2. *Non-singular BHs in the Rastall theory*

Here, we do not focus on finding out the non-singular BHs solutions, corresponding to Eq. (25), in the Rastall framework, due to the complexity of the field equations. This issue requires further investigations that the results of which will be reported as an independent research work. It is also useful to mention that the Gaussian BHs, in which $\rho \sim \exp(-r^2)$, have previously been studied.[24,25]

5. Conclusion

Focusing on a special subclass of the generalized Rastall theory,[8] whose field equations are mathematically similar to EECC, as well as on the original Rastall theory, we show that the ability of spacetime to couple with the matter fields in a non-minimal way can either play the role of cosmological constant, see (4), or affect it, see (13) and (21), depending on the energy source supporting the geometry. The conformally flat BHs have also been derived, and some of their properties were studied in both frameworks. Additionally and only in the framework of field equations (4), a non-singular BH has been obtained. Relations between the parameters of obtained non-singular BH, those of the energy source and the non-minimal coupling were also addressed. We found out that the ability of spacetime to non-minimally couple with the matter fields can affect the location of the horizons of spacetime even whenever OCL is valid ($\beta = \text{constant} \neq 0$).

The Euler equations corresponding to the solutions have also been derived. It is shown that for the metric (3), if the integral (9) and $S = \frac{2\pi}{\kappa}A$ are accepted as the mass[22] and entropy[17] of BH, respectively, then the thermodynamic pressure and temperature, satisfying the Euler equation, are not always equal to the pressure components obtained using the field equations, and also the Hawking temperature, respectively, a result also valid in the Einstein theory (the $\beta = 0$ and $\kappa = 8\pi G$ limits of the obtained relations).[20] The requirements of having solutions with the same thermodynamic and Hawking temperatures have also been studied.

The consequences of considering the generalized Misner–Sharp mass have also been investigated, and the corresponding Euler equations have been derived in both frameworks. It seems that the thermodynamic pressure obtained by accepting the mass definition (9) is equal to the pressure of the cosmological constant candidate appeared in the field equations compared with the situation in which the corresponding generalized Misner–Sharp mass is employed. In this situation, the obtained thermodynamic temperature is not always equal to the Hawking temperature.

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