Josephson current between two $p$-wave superconducting nanowires in the presence of Rashba spin-orbit interaction and Zeeman magnetic fields

E. Nakhmedov$^{a,b}$, B.D. Suleymanli$^c$, O.Z. Alekperov$^a$, F. Tatardar$^{a,d}$, H. Mammadov$^b$, A.A. Konovko$^e$, A.M. Salesky$^e$, Yu.M. Shukrinov$^{f,g}$, K. Sengupta$^b$, B. Tanatar$^i$

$^a$ Institute of Physics of National Academy of Sciences of Azerbaijan, H. Javid ave. 133, Baku, AZ-1143, Azerbaijan
$^b$ Baku Engineering University, Hasan Aliyev str. 120, Ahscheron AZ-0101, Baku, Azerbaijan
$^c$ National Research Nuclear University MEPhI, 115409 Moscow, Russian Federation
$^d$ Khazar University, Mahsati str. 41, AZ 1096, Baku, Azerbaijan
$^e$ Faculty of Physics, Moscow State University, Leninskie Gory 1–2, 119991 Moscow, Russian Federation
$^f$ BLTP, JINR, Dubna, Moscow region, 141980, Russian Federation
$^g$ Dubna State University, Dubna, Moscow region, 141980, Russian Federation
$^h$ School of Physical Sciences, Indian Association for the Cultivation of Science, Jadavpur, Kolkata 700 032, India
$^i$ Physics Department, Bilkent University, 06800 Ankara, Turkey

A B S T R A C T

Josephson current between two one-dimensional nanowires with proximity induced $p$-wave superconducting pairing is calculated in the presence of Rashba spin-orbit interaction, in-plane and normal magnetic fields. We show that Andreev retro-reflection is realized by means of two different channels. The main contribution to the Josephson current gives a scattering in a conventional particle-hole channel, when an electron-like quasiparticle reflects to a hole-like quasiparticle with opposite spin yielding a current which depends only on the order parameters' phase differences $\phi$ and oscillates with $4\pi$ period. Second anomalous particle-hole channel, corresponding to the Andreev reflection of an incident electron-like quasiparticle to an hole-like quasiparticle with the same spin orientation, survives only in the presence of the in-plane magnetic field. The contribution of this channel to the Josephson current oscillates with $4\pi$ period not only with $\phi$ but also with orientational angle of the in-plane magnetic field $\theta$ resulting in a magneto-Josephson effect. In the presence of Rashba spin-orbit coupling (SOC) and normal-to-plane magnetic field $h$, a forbidden gap is shown to open in the dependence of Andreev bound state energies on the phases $\phi$ and $\theta$ at several values of SOC strength and magnetic field, where Josephson current seems to vanish. We present a detailed theoretical analysis of both DC and AC Josephson effects in such a system showing contributions from these channels and discuss experiments which can test our theory.

1. Introduction

A key model for realization of Majorana fermion (MF) in a condensed matter is a spinless $p$-wave superconductor (SC) [1,2]. Majorana zero modes are excitations at zero energy which are typically localized at interface of the topological-non-topological phases and spatially separated from one another. They emerge as electrically neutral fermions indistinguishable from their antiparticles in subgap quasiparticle excitation spectrum of a topological superconductor (TSC). In $s$-wave superconductors, the quasiparticle excitations at the stagnation points of the superconducting gap are indeed equal coherent superpositions of electrons and holes with opposite spins, and thus electrically neutral. Nevertheless, such a superposition of fermionic quasiparticles is not self-conjugate due to existence of spin. Therefore, MF can not appear in $s$-wave SC, and it is expected to occur in an effectively spinless $p$-wave SC [1]. Recently it was suggested that a topological (spinless $p$-wave) superconductivity can be effectively realized either in a spin-polarized normal metal or in a semiconductor nanowire with strong spin-orbit coupling under Zeeman magnetic field proximity-coupled to a conventional spin-singlet ($s$-wave) bulk SC [3,4]. In this case the magnetic field lifts one of the spin-polarized electron sub-band, transforming the system into effectively spinless $p$-wave superconductor.

In the above discussed $p$-wave superconductor models, where Majorana fermions emerge, spin of electrons is neglected. However, electrons in conventional materials are spin-half particles; thus, the notion of a spinless SC does not seem immediately relevant to real physical systems. Taking into account of an electronic spin in a $p$-wave superconductor results in an emergence of a Kramers pair of Majorana fermions at the ends of, e.g. a one-dimensional superconductor instead of a single Majorana at each end. Existence of the spin sensitive interactions like Rashba spin-orbit coupling and/or Zeeman magnetic field is expected to change the characteristics of the zero-energy edge states,
which can be observed, e.g. in Josephson current of a junction of p-wave superconductors or in zero-bias tunneling experiments of normal metal/p-wave superconductor junction.

Note that highly controlled zero-energy quasiparticles (Majorana fermions), produced in thin wire, can be utilized as a quantum information carrier qubit in quantum computer technology. MFs are exotic non-Abelian fermions obeying non-Abelian braiding statistics [5–7]. This unique property makes MF ideal for fault-tolerant quantum computation. These MFs typically arise at defect sites as Abrikosov vortex cores in bulk SCs, at interface of dielectric/TSC or normal metal/TSC, or at edges of TSC as localized excitations, and are topologically protected against any local perturbations like paramagnetic disorder. Essentially, two key points for the emergence of MFs are presented here: spin-orbit coupling (SOC) and superconducting proximity effect.

A neutral excitation in a superconductor has a special property owing to the inherent particle-hole symmetry of the material: it is bound to zero energy, so that there is no cost to occupy such a state. One dimensional (1D) topological superconductor supports a non-local fermionic mode comprising two Majorana zero modes localized at opposite ends of the chain and are separated by a distance that can be much larger than the superconducting coherence length.

Recent investigations have shown [8–11] that a proximity of semiconductor with s-wave superconductor induces not only s-wave but also p-wave superconducting pairing in the semiconductor. The pairing symmetry of a BCS-type two dimensional (2D) superconductor without inversion symmetry when the twofold degeneracy of the electron energy spectrum is lifted by SOC has been studied [8,12,13] to be a mixture of singlet and triplet symmetric. Reeg and Maslov have shown [10] by directly solving the fully quantum-mechanical Gor'kov equations that spin-triplet superconducting correlations are induced by Rashba SOC in both 1D and 2D proximity junctions via the proximity effect. Furthermore, the induced triplet component in 1D was shown to vanish when integrated over the momentum; this result is in agreement with Ref. [9]. The induced triplet amplitude in 2D was found to have an odd-frequency component that is isotropic in momentum.

Recently p-wave pairing has been suggested to appear in several materials, such as Sr$_2$RuO$_4$ crystal [14,15], heavy-fermion superconductors like UPt$_3$ [16], Bechgaard salts [17], lithium molybdenum purple bronze [18], and the Cr-based pnictide K$_2$Cr$_3$As$_4$ [19]. Furthermore, the spin-triplet pairing can also be created artificially in heterojunctions where the proximity effect from a conventional BCS superconductor induces superconducting correlations in a nearby nonsuperconducting material as it is suggested in above-mentioned references [8–10]. An extreme sensitivity to non-magnetic impurities [20], spin-susceptibility measurements by the nuclear-magnetic resonance and polarized neutron scattering below $T_c$ [21] support the spin-triplet scenarios in Sr$_2$RuO$_4$ crystal. The suggested triplet order parameter in (TMTSF)$_2$PF$_6$ organic superconductor explains why the upper critical magnetic field exceeds Pauli paramagnetic limit [22], Knight shift in NMR experiment [23] remains unchanged through the superconducting transition.

In this work we study Josephson junction (JJ) of two superconductors with p-wave pairing of spinful electrons separated by a $\delta$-function like insulator potential (see, Fig. 1). JJ consisting of p-wave superconductors in both sides of the barrier has been studied [24] for a $\delta$-like insulator potential between the superconductors in the absence of SOC and Zeeman magnetic field. Since the superconductor is in the topological phase, a fractional oscillation of Josephson current was obtained in Ref. [24]. The emergence of Majorana zero mode results in exotic Josephson effects [1,25–30], when the current flowing between two topological superconductors in the junction oscillates with a fractional periodicity $4\pi$ instead of $2\pi$ periodicity in a conventional Josephson junction. We also show that Josephson current oscillates with $4\pi$ fractional period by varying the angle $\theta$ between Zeeman magnetic field and quantum wire which is referred to as magneto-Josephson current. Recently it was shown [28–31] that a topological insulator edge, proximity coupled to an s-wave superconductor exhibits an exact superconductivity-magnetism duality, which maps the phase of the superconducting order parameter $\phi$ to the orientation of Zeeman magnetic field $\theta$. In this case, Josephson current is determined by the expression $j^J = \frac{\hbar I_J}{4e}$, where $I_J$ is the Hamiltonian of JJ. We show in this paper that the same duality exists in JJ of p-wave superconducting wires. Note that the spin-Josephson current has been shown [32–35] to flow throughout triplet superconductor-ferromagnet interface, which is $4\pi$ periodic with angle between the external Zeeman magnetic field and the ferromagnetic spin orientation as a manifestation of the Majorana modes. Therefore, the magneto-Josephson current seems to be identified with spin-Josephson effect.

The problem of Josephson current through a junction of two superconductors with p-wave pairing of spinful electrons, that we are addressing in this paper, has been usually studied for spinless model [1], although many aspects of spinful p-wave symmetric JJ have been investigated [36–41] in the literature. Our paper is the first attempt where effects of Rashba spin-orbit interaction and Zeeman magnetic field on the Andreev bound state energy and Josephson current in the JJ produced by p-wave superconductors are analytically and numerically studied. In this paper we wish to understand how the SOC and external magnetic fields change the Josephson current in p-type JJ separated by a $\delta$-potential insulator. Similar effects have been studied in Ref. [30] for a junction consisting of s-wave SCs separated by $\delta$-potential thin insulator. Note that the Josephson current in the case of proximity induced both s- and p-wave pairings in JJ, as argued in several recent papers [8–11], can be found by summing up a corresponding result of Ref. [30] with that presented in this paper.

The main results of our paper are as follows. Andreev bound state energy and Josephson current are studied in a junction of the spinful p-wave superconductors. The fractional oscillation is shown to take place not only with varying order parameter phase difference, but also with the orientation of in-plane (the magnetic field $\mathbf{B}$ in Fig. 1) Zeeman magnetic field. We show in this paper that the Andreev reflection is realized by means of two channels, and clarify the origin of these channels. Apart from the conventional Andreev reflection, when a
quasi-particle reflects at the interface as a quasi-hole with opposite spin orientation, there appears a second anomalous particle-hole channel, corresponding to the Andreev reflection of an incident electron-like quasiparticle to a hole-like quasiparticle with the same spin orientation. This anomalous reflection channel is shown to survive only in the presence of the in-plane magnetic field, and contribution to the current in this channel oscillates with order parameter difference \( \varphi \) as well as with the in-plane magnetic field orientation angle \( \theta \). Although the magneto-Josephson effect has been predicted for JJ of \( s \)-wave superconductors, we show that this effect takes place in the JJ of \( p \)-wave superconductors too. Andreev’s retro-reflection provides in the absence of SOC and Zeeman magnetic field two subgap bound states symmetrically placed in the gap around the Fermi level. Rashba SOC and Zeeman magnetic field remove the spin degeneracy, which splits the subgap energy states. Now, instead of two states one gets four states, two quasi-electron states with positive energies above the Fermi level and two quasi-hole states with negative energies below Fermi level. We find the dependence of Josephson current expressions on Rashba spin-orbit coupling constant and external magnetic field anisotropically for several asymptotic cases as well as numerically for the general case. Zeeman magnetic field (\( \mathbf{h} \) in Fig. 1), which is normal to the wire’s plane, destroys the quasi-particle/ quasi-hole symmetry, and results in an asymmetric oscillation of the Josephson current. We show that Josephson current vanishes in some interval of the order parameter phase difference in the presence of Rashba spin-orbit interaction and Zeeman magnetic field, i.e. a forbidden gap is opened in the dependence of Josephson current on the phases at some definite values of SOC and magnetic fields. Since Josephson current is a “readout” tool of a qubit (Majorana fermion) in the quantum information technology, emergence of the fractional oscillation of the current with \( 4\pi \) period is the main manifestation of realization of Majorana fermion in the system. Dependence of Josephson current on the Rashba SOC constant and Zeeman magnetic field can be used to manipulate the quantum informations in the readout process.

In the absence of proximity induced superconductivity, 1D wire constitutes a Luttinger liquid in the presence of repulsive electron-electron interactions [42,43], where quasiparticle excitations are absent, the density of electronic states and the anti-ferromagnetic spin response display power-like temperature dependence. An influence of the Coulomb repulsion on a bulk superconductor has first been studied by Morel and Anderson [44] within Gorkov-Eliashberg formalism, which results in a reduction of the critical temperature of the superconducting transition. The same effect has also been calculated for quasi-2D layered superconductors [45]. Investigation of electron-electron repulsion within renormalization group and bosonization techniques on a 1D semiconductor in the proximity of a superconductor that supports Majorana edge states shows that the interaction reduces the induced superconducting gap, and localization length of Majorana edge states [46–48]. In our study we neglect the electron-electron repulsion effects assuming that the superconducting state and topological phase are established with renormalized physical parameters.

The paper is organized as follows. In the next Section formulation of the problem is presented. In Section III, Andreev bound state energy is calculated for different values of the external parameters as SOC constant, Zeeman energies for the magnetic fields normal to the junction \( h \) and the tilted magnetic field \( B \) lying in the junction plane and forming an angle \( \theta \) with SC wire. Section IV describes Josephson current as well as magneto-Josephson effect in the junction. ac-Josephson current and effects of SOC and magnetic field on the Shapiro step in the JJ of \( p \)-wave superconductors are studied in the last Section V.

### 2. Model and formulation of the problem

We consider a junction of two 1D nanowires of proximity induced \( p_x \)-wave pairing symmetry superconductivity, having the effective pairing potentials \( \Delta_L \) and \( \Delta_R \) on the left (L)- and right (R)-side of an insulating potential barrier separated two superconductors, in the presence of Rashba spin-orbital interaction and external Zeeman magnetic fields (see, Fig. 1). Hamiltonian for such a system reads

\[
\hat{H} = \hat{H}_{SC} + \hat{H}_b, \tag{1}
\]

where \( \hat{H}_{SC} \) is Hamiltonian of the nanowire in the presence of external magnetic fields and \( \hat{H}_b \) represents Rashba SOI. The former term is given by

\[
\hat{H}_{SC} = \int dx \sum_{\sigma, \sigma'} \left\{ \psi_{\sigma}^\dagger(x) \left\{ \left[ \epsilon_x^L + U(x) \right] \delta_{\sigma \sigma'} + \frac{1}{2} \mathbf{B}(x) \right\} \psi_{\sigma}(x) + \frac{1}{2} \mathbf{B}(x) \hat{\sigma} \right\} \psi_{\sigma'}^\dagger(x) \psi_{\sigma}(x) + \frac{1}{2} \mathbf{B}(x) \hat{\sigma} \right\} \psi_{\sigma'}^\dagger(x) \psi_{\sigma}(x) + h.c. \right\},
\]

where \( \epsilon_x \) is the kinetic energy as measured from the Fermi level \( \epsilon_F, \varphi(x) \) is the electron annihilation operator, \( \mathbf{h} \) and \( \mathbf{B} \) are external Zeeman magnetic fields in \( x \) direction and in the \( x-y \) plane respectively, \( \hat{\sigma}(x) \) is the Heaviside step function, and \( \delta_{\sigma x}, \delta_{\sigma y} \) and \( \delta_{\sigma z} \) denote Pauli and identity matrices respectively in spin space. Note that the magnetic field \( \mathbf{B} \) forms the angle \( \theta_{L} \) and \( \theta_{R} \) with superconductors correspondingly on the left- (L) and right-side (R) of the tunnel junction, which can be tuned externally. In what follows, we choose \( \mathbf{B} \) in the left side of the junction to be aligned along the wire (\( \theta_{L} = 0 \)) while in the right side it is chosen to make an angle \( \theta \) with \( \theta_{R} = \theta \). In Eq. (2), the triplet pairing potential \( \Delta(k) \) has a matrix structure and depends in momentum space on components of the momentum vector \( \mathbf{k} \), \( \Delta_{\mathbf{k}}(\mathbf{k}) \) in the right of the junction is chosen to have a phase difference \( \varphi \) compared to its left counterpart: \( \Delta_{\mathbf{k}}(\mathbf{k}) = \Delta(\mathbf{k}) \exp(\mathbf{i}\varphi) \) and \( \Delta_{\mathbf{k}}(\mathbf{k}) = \Delta(\mathbf{k}) \). The potential \( U(x) = U_{0}(x) \), located at \( x = 0 \), represents the barrier potential between two superconductors. The Hamiltonian of Rashba SOI can be written as

\[
\hat{H}_b = \sum_{\sigma, \sigma'} \int dx_2 \left\{ \psi_{\sigma}^\dagger(x_2) \left[ \epsilon_{\sigma} \delta_{\sigma \sigma'} + \frac{1}{2} \mathbf{B}(x_2) \hat{\sigma} \right] \psi_{\sigma}(x_2) \right\} \equiv \mathbf{h}(x_2), \tag{3}\]

where \( \epsilon_{\sigma} \) is the strength of Rashba SOI which is chosen to be the same for both wires.

The order parameter \( \Delta_{\mathbf{k}}(\mathbf{k}) \) for triplet pairing with \( S = 1 \) can be presented as [49,50]

\[
\Delta_{\mathbf{k}}(\mathbf{k}) = \Delta(\mathbf{k}) \mathbf{i} \sigma \mathbf{\cdot} \mathbf{k}_{\theta}, \tag{4}\]

where \( \mathbf{d}(\mathbf{k}) \) is odd function of \( \mathbf{k} \), \( \mathbf{d}(\mathbf{k}) = -\mathbf{d}(-\mathbf{k}) \), and can be expanded over spherical harmonics

\[
d_{\mathbf{\sigma}}(\mathbf{k}) = \sum_{m=-l}^{l} \hat{b}_{\mathbf{\sigma} m} \mathbf{Y}_{lm}(\mathbf{k}). \tag{5}\]

The quantum number \( l \) in Eq. (5) takes odd values \( 1, 3, \ldots \) corresponding to states of \( p, f, \ldots \) pairings. The coefficient \( \hat{b}_{\mathbf{\sigma} m} \) can be identified as a component of the superconducting order parameter with given \( l \) and \( m \). For a simple case of \( p \)-wave pairing for \( l = 1 \) and \( m = -1,0,1 \), \( \hat{d}_{\mathbf{\sigma}}(\mathbf{k}) \) can be expressed with the appropriate expressions for the spherical harmonics

\[
Y_{1-m} \propto (\mathbf{\hat{k}}_{\pm} - i \mathbf{\hat{k}}_{\mp}), \quad Y_{1,0} \propto \mathbf{\hat{k}}_{\pm}, \quad Y_{1,1} \propto (\mathbf{\hat{k}}_{\pm} + i \mathbf{\hat{k}}_{\mp}), \tag{6}\]

The \( p_x \)-wave symmetric order parameter \( \Delta_{\mathbf{k}}(\mathbf{k}_1) \) on the \( a = R, L \)-side of a junction between two superconductors aligned along the \( x \)-axis can be expressed as

\[
\Delta_{\mathbf{k}}(\mathbf{k}_1) = \Delta_{\mathbf{k}_1} \mathbf{k}_1, \tag{7}\]

where the subscript \( b \) signifies the left- or right-motion direction in one dimension.
It is advantageous to use a four-component field operator for bulk superconductor at the right ($a = R$) and left ($a = L$) side of the junction as,
\[
\Psi'^*_j(x) = \left(\Psi^{*, +}_R(x), \Psi^{*, -}_R(x), \Psi^{*, +}_L(x), \Psi^{*, -}_L(x)\right)
\] (8)

Here the third subscript of the annihilation operator (which we shall designate henceforth as $b$) labels the right ($b = +$) and left-moving ($b = -$) quasiparticles respectively while the index $a = R, L$ denotes either right ($R = -$) or left ($L = +$) superconductor. In terms of the field operator given by Eq. (8), the Hamiltonian (Eq. (1)) can be written as
\[
\mathcal{H}_0 = \sum_{j=R,L} \int dx \Psi'^*_j(x) \mathcal{H}_0 \Psi'^*_j(x)\] using the Pauli matrices $\sigma_i$ in spin- and $\tau_i$ in particle-hole spaces. From Eqs. (1) and (2), we find
\[
\mathcal{H}_0 = \varepsilon_{k,R} c_0 + \hbar \alpha_n \xi_k c_0 - i \hbar \alpha_n \xi_k c_0 + B \xi_k (\cos \theta + \sin \phi) + \Delta_0 (\tau_i \cos \phi - \tau_j \sin \phi) \sigma_i,
\] (9)
and $\mathcal{H}_L = \mathcal{H}_R (\varphi = 0; \varphi = 0)$. In Eq. (9), the energy spectrum of the electrons are linearized around the positive and negative Fermi momenta leading to $\xi_{k,R} = \hbar v_F \left(\frac{k_F^2}{2m} - k^2 \right)$, where $v_F$ is the Fermi energy.

Note that the Hamiltonian $\mathcal{H}_{0L}$ acquires a magnetism-superconductivity duality [29,31] in the absence of the kinetic term, implying that it becomes invariant under the transformation $[\Delta, \xi_a, \varphi, \tau_i] \to \mathcal{H}_{0L}$, and the existence of a magneto-Josephson effect in a topological insulator is known to be a result of this duality [29]. We shall see that for the system we study, the magneto-Josephson effect takes place even in the presence of the additional quadratic kinetic energy term of the electrons.

The energy spectrum of a quasiparticle in a 'bulk' quasi-1D superconductor is determined from the expression $\det(\mathcal{H}_{0L} - \mathcal{E}) = 0$, yielding
\[
(\xi_{k,R}^2 - v_F^2 k_x^2 + g^2 k_y^2 - h^2 - B^2 - |\Delta|^2)^2 + 4(E k v_F + h v_F k)^2 - 4 |\Delta|^2 (\xi_{k,R}^2 + B^2 + h^2) = 0.
\] (10)

This equation does not yield a simple analytic expression for $\mathcal{E}$, while it contains a linear in energy term, which is a result of an alignment of $\mathcal{H}$ and the effective magnetic field of the SOI $\mathcal{B}$. The linear in $E$ term vanishes for either $\mathcal{E} = 0$ or $\alpha = 0$, and Eq. (10) turns to quadratic equation for $\mathcal{E}^2$, which gives two symmetric dispersion branches for quasiparticles and quasi-holes. Note that square of the momentum $k^2$ is negative, where the subscripts indices indicate the spin branches, can be obtained from Eq. (10). The evident expression for $k^2$ is presented by Eq. (A2) in Appendix. Equation (10) is strongly simplified for different limiting cases, and yields the following expressions for the energy dispersion,
\[
\begin{align*}
E &= \begin{cases} \\
\sqrt{\xi_{k,R}^2 - v_F^2 k_x^2 + \sqrt{g^2 k_y^2 + h^2}} & \text{if} \quad \alpha = B = h = 0, \\
\sqrt{\xi_{k,R}^2 + g^2 k_y^2 + h^2} & \text{if} \quad \alpha = 0, B \neq 0, h \neq 0, \\
\xi_{k,R}^2 + |\Delta|^2 & \text{if} \quad h = 0, \alpha \neq 0, B \neq 0, \end{cases}
\end{align*}
\] (11)

where $s = \pm$ indicates the particle- and hole-branches of the spectrum. The energy levels of the Bogolyubov-de Gennes (BdG) quasiparticles lie in the gap, symmetric to the Fermi level. SOI and/or magnetic field $h$ split both electron and hole levels due to Rashba 'momentum-shifting' and/or Zeeman effect. The 'Fermi points' around + $k_F$ and − $k_F$ are split also due to these effects. At the same time, the magnetic field makes the energy dispersion asymmetric. Note that in our case, all energies are measured from the Fermi energy; thus the condition for realization of a conducting phase, $\Delta_0 \neq 0$. Instead, in the absence of a superconducting phase, $\Delta_0 = 0$. Instead, the absence of a superconducting phase, $\Delta_0 = 0$, these equations link a particle (hole) wave function $\Psi'^*_j(x)$ with hole (particle) wave function with opposite spin-polarized quasiparticle state $\Psi'^*_j(x)$ moving in the same direction provided that $\mathcal{B} = 0$. However, in the presence of in-plane magnetic field $\mathcal{B} \neq 0$ in the superconducting phase $\Delta_0 \neq 0$, Eqs. (13)-(16) connect a particle wave function $\Psi'^*_j(x)$ (a hole wave function $\Psi'^*_j(x)$) with hole (particle) wave function with the particle (hole) wave function with opposite spin-polarized quasiparticle state $\Psi'^*_j(x)$ moving in the same direction provided that $\mathcal{B} = 0$. However, in the presence of in-plane magnetic field $\mathcal{B} \neq 0$ in the superconducting phase $\Delta_0 \neq 0$. Note that the ratio $\Psi'^*_j(x)/\Psi'^*_j(x)$ is non-zero for $\mathcal{B} = 0$, which corresponds to the conventional Andreev reflection at the boundary of a superconductor with normal metal or insulator [24]. Eqs. (13)-(16) provide the following expressions for the essential ratios for arbitrary values of the parameters $a, \mathcal{B}, \mathcal{E}$.

\[
\begin{align*}
\eta_{\alpha,R}^{+ \pm} &= \frac{1}{\Delta_0} \left\{ E + iabkv_F - iabk + h - \frac{1}{\mathcal{B}^2(E - iabk)} \left( (E - h)^2 - (kv_F + gk)^2 + B^2 - |\Delta|^2 \right) \right\}, \\
\eta_{\alpha,R}^{- \mp} &= \frac{1}{\mathcal{B}^2} \left( (E - iabk)(E + iabk) - iabk h + \frac{1}{\mathcal{B}^2(E - iabk)} \left( (E + iabk^2 - iabk h + \frac{1}{\mathcal{B}^2(E - iabk)} \left( (E - h)^2 - (kv_F + gk)^2 + B^2 - |\Delta|^2 \right) \right) \right),
\end{align*}
\] (17)

where $\Delta_0 = b \Delta_0$ according to Eq. (7), and $\mathcal{B}_0 = \mathcal{B}_0 \exp(\varphi)$ and $\mathcal{B}_0 = \mathcal{B}_0 \exp(\varphi)$. Note that the expressions for $\eta_{\alpha,R}^{+ \pm}$ and $\eta_{\alpha,R}^{- \mp}$ can be obtained respectively from Eqs. (17), and (18) by replacing $\mathcal{B} \to -\mathcal{B}, \alpha \to -\alpha, \mathcal{E} \to h$, and by reversing the total sign of these expressions. According to these expressions the reflection channels, determined by the ratio $\eta_{\alpha,R}^{+ \pm}/\eta_{\alpha,R}^{- \mp}$ vanish in in-plane magnetic field $\mathcal{B} = 0$.

We note from Eqs. (13)-(16) that the dependencies of these equations on $\mathcal{B}$ and $\varphi$ are completely removed by transforming the wave functions as
\[
\begin{align*}
\eta_{\alpha,R}^{+ \pm}(x) &\to e^{-i(\varphi + 0)/2} \eta_{\alpha,R}^{+ \pm}(x), e^{i(\varphi + 0)/2} \eta_{\alpha,R}^{+ \pm}(x), \\
e^{i(\varphi + 0)/2} \eta_{\alpha,R}^{- \mp}(x), e^{i(\varphi + 0)/2} \eta_{\alpha,R}^{- \mp}(x).
\end{align*}
\] (19)
In the transformed basis one has
\[ \eta_{\sigma,\delta}^{a,b} = e^{i\phi} \eta_{\sigma,\delta}^{a,b}, \]

where the ratio of wave functions in Eq. (21) depend on both \( \varphi \) and \( \theta \) while in Eq. (20) depends only on \( \varphi \). This suggests that the ratio (20) is responsible for the dependence of observable parameters on the order parameter phase difference \( \varphi \), whereas the ratio (21) is responsible for the dependence not only on \( \varphi \) but also on the magnetic field orientation angle \( \theta \). In what follows, we shall look for the local subgap Andreev bound states with \( \epsilon(k) < |A| \) for the Josephson junction of two nanowires described by Eq. (1).

### 3. Andreev bound states, josephson and magneto-Josephson effects

In order to obtain a solution for the Andreev bound states for the junction described by Hamiltonian (2) one follows the method used in Ref. [124]. The energy spectrum of an electron is split in the presence of Rashba SOC and/or Zeeman magnetic field, so that the Fermi level crosses the dispersion curve at four points, corresponding to right-moving \( k_F^+ \), \( k_F^- \) and left-moving \( -k_F^+ \), \( -k_F^- \) particles with oppositely polarized spin-states, and \( k_F^+ \neq k_F^- \) to \( 0 \). Furthermore, a condensate of the electron pairs in a superconducting state opens a gap around the Fermi level. We neglect here a difference between \( k_F^+ \) and \( k_F^- \), and take \( k_F^+ \approx k_F^- \). We assume that a transition occurs between the states with the same chirality. In order to obtain the wave function for \( L \) and \( R \) superconductors we superpose the wave functions for the left (-) and right (+) moving BdG quasiparticles with a given spin correspondingly around the Fermi level \( k_F \), and \( -k_F \) with arbitrary coefficients,

\[ \eta_{\alpha}(x) = e^{i\phi_{\alpha}/2} \left\{ \eta_{\sigma,\delta}(k_F) + e^{i\delta_{\sigma,\delta}} \eta_{\sigma,\delta}^*(k_F) \right\} \]

where \( \text{sgn}(a) = +(-) \) for \( a = \{ L,R \} \), and \( b \) assigns right- \((+)=+) \) and left-moving \((-)=-a\) quasiparticles. SOC and magnetic field remove the spin degeneracy in a quasiparticle \( \eta_{\alpha,a}(k) \) and a quasi-hole \( \eta_{\alpha,a}^*(k) \) wave functions, where the spin indices \( \sigma \) and \( \sigma' \) take independent values \( \sigma = \{ \uparrow, \downarrow \} \) and \( \sigma' = \{ \uparrow', \downarrow' \} \) and thereby split the wave functions written for the conventional superconductors [24].

Andreev bound state energies are obtained by imposing the usual boundary conditions on each component of these wave functions \( \eta_{\alpha}(x) \). For a barrier modeled by the delta function potential \( U(x) = U_0 \delta(x) \), the boundary conditions are provided at the merging point \( x = 0 \) of two superconductors as,

\[ \eta_{\alpha}(0) = \eta_{\alpha}(0), \quad \partial_x \eta_{\alpha} - \partial_x \eta_{\alpha} = k_F Z \eta_{\alpha}(0), \]

where \( Z = 2mU_0/k_F^2 \) and the transmission coefficient \( D \) is expressed through \( Z \) as \( D = 4/(Z^2 + 4) \).

Substituting Eq. (22) into the boundary conditions (23) one gets four linear homogeneous equations for each fixed spin indices \( \sigma \) and \( \sigma' \). These equations describe subgap states with energy \( |E| < \Delta \), and they are compatible if the determinant of the corresponding \( 4 \times 4 \) matrix in the front of the coefficients \( A_{\sigma,\sigma'} \) and \( B_{\sigma,\sigma'} \) is zero for each particle and hole spins’ orientation. The particle \( \eta_{\sigma,a,b} \) and hole \( \eta_{\sigma',a',b'} \) wave functions in Eq. (22) are determined from BdG equations (13)-(16) for right- or left-superconductor of the junction at \( x = 0 \) where \( U(x) = 0 \). The Andreev bound state energy for a transmission of the barrier through a particular channel is obtained by setting the determinant of these linear homogeneous equations equal to zero. Selection, e.g. \( \sigma = \downarrow \) for a quasiparticle spin and \( \sigma' = \uparrow \) for a quasi-hole spin, provides the following expression for the determinant \( F^+_1 \) (\( F^+_2 \)) which yields Andreev bound state energy in this scattering channel under the condition \( F^+_1(0) = 0 \) (\( F^+_2(0) = 0 \)),

\[ F^+_1 = \frac{1}{D} \eta_{\sigma,\delta} + \eta_{\delta,\sigma} \]

(24)

where

\[ F^+_1 = \left| \begin{array}{cc} \eta_{\sigma,\delta} - \eta_{\delta,\sigma} & \eta_{\sigma,\delta} - \eta_{\delta,\sigma} \\ \eta_{\sigma,\delta} - \eta_{\delta,\sigma} & \eta_{\sigma,\delta} - \eta_{\delta,\sigma} \end{array} \right| 

(1 - D) \left\{ \begin{array}{c} \eta_{\sigma,\delta} - \eta_{\delta,\sigma} \\ \eta_{\sigma,\delta} - \eta_{\delta,\sigma} \end{array} \right\} \]

(25)

Using Eq. (17) in this expression one gets an explicit expression for \( F^+_1 \)

\[ F^+_1 = \frac{4e^{-i\phi}}{|D|^2 M^2} \left\{ (E + h) M - 2B^2 E \right\}^2 - D \cos^2 \frac{\pi}{2} \]

\[ \left\{ (E + h) M - 2B^2 E \right\}^2 + |k(v_\sigma - \alpha) M + 2B^2 k|^2 \]

(26)

where \( M = (E \mp h)^2 + (v_\sigma \mp \alpha)^2 k^2 + B^2 - \Delta^2 \).

The expression for \( F^+_1 \), which differs from \( F^+_2 \) by changing the spin polarizations, is obtained from Eq. (26) by replacing \( \alpha \to -\alpha \) and \( M \to -M \). The solution of the equation \( F^+_1(E_1) = 0 \) for energy, where \( F^+_1 \) is given by Eq. (26), yields a contribution to the Andreev overlap energy \( E_1 \) in the particle-hole channel.

Now we choose \( \sigma = \alpha' = \uparrow \) for the quasiparticle and quasi-hole spin orientations in order to get an evident expression for the determinant \( F^+_1 \) (\( F^+_2 \)) in the case of Andreev reflection in anomalous particle-hole reflection channel. Equation \( F^+_1 = 0 \) (\( F^+_2 = 0 \)) provides the condition to find the Andreev bound-state energy in this channel,

\[ \frac{F^+_1}{2} = \frac{1}{D} \left\{ \begin{array}{c} \eta_{\sigma,\delta} - \eta_{\delta,\sigma} \\ \eta_{\sigma,\delta} - \eta_{\delta,\sigma} \end{array} \right\} \left\{ \begin{array}{c} \eta_{\sigma,\delta} \eta_{\delta,\sigma} \end{array} \right\} \]

(28)

The expression for \( F^+_2 \) (for \( F^+_1 \)) is obtained from Eq. (25) by replacing all spin-down (all spin-up) with spin-up (spin-down). The evident expression for \( F^+_2 \) is obtained by using the ratio (18), which reads as,

\[ F^+_2 = 16B^2 k^2 \left\{ \begin{array}{c} \frac{E \frac{v_\sigma}{2} + \alpha k^2}{2} \\ \frac{\Delta^2 (E^2 + \alpha k^2)}{2} \end{array} \right\} \]

(29)

The expression for \( F^+_2(k) \) can be obtained from Eq. (29) by replacing \( \alpha \to -\alpha \), \( h \to -h \), \( \theta \to -\theta \), and \( M \to -M \). The general feature of the Andreev bound-state energy in the anomalous particle-hole reflection channel \( E_1 \) with the same spin orientation is that it takes non-zero values only in the presence of in-plane magnetic field \( B \). Therefore, it depends on the angle \( \theta \) between the junction and in-plane magnetic field. Oscillation of the Josephson current with \( \theta \) yields a fractional magneto-Josephson effect.

Andreev bound state energies and Josephson current, corresponding to different reflection channels, demonstrate completely different oscillation. The conditions \( F^+_1(E_1) = 0 \) and \( F^+_2(E_1) = 0 \) with Eq. (26) for
$F^*_B(E)$ provide contributions to the Andreev bound state energy in the particle-hole channel, which oscillates fractionally with the order parameters’ phases difference $\varphi$. Additional contributions to the energy come from the conditions $F^*_p(E_{11}) = 0$ and $F^*_B(E_{11}) = 0$ with $F^*_p$ given by Eq. (29), which arise only in the presence of an in-plane magnetic field $B$ and oscillate not only with $\varphi$ but also with $\theta$.

Below we calculate the Andreev bound state energies for several asymptotic cases.

3.1. Andreev bound state energy in the absence of in-plane magnetic field, $B = 0$.

Contribution to the Andreev bound state energy in the absence of in-plane magnetic field $B = 0$ comes only from the particle-hole reflection channel, determined by scattering amplitude Eq. (17), and the other channel vanishes under this condition. The evident expression for the bound state energy in the particle-hole channel is obtained from the equation $F^*_p(E_{11}) = 0$, where $F^*_p$ is given by Eq. (26). The general expression for $E_{11}$ when all the external parameters take non-zero values, $\alpha \neq 0$, $B \neq 0$ and $h \neq 0$, can be obtained from the expression (A1) in Appendix. By putting $B = 0$ in this equation and eliminating $k^2$ by replacing it with $E$ according to Eq. (A2) in the Appendix one gets the following equation after routine calculations,

$$\begin{align*}
\left[ (E_{11} + h)^2 - (1 - D \cos^2 \frac{\varphi}{2}) - (\frac{\nu_F - \alpha}{\nu_F + \alpha})^2 \right] & = 0, \\
4D \cos^2 \frac{\varphi}{2} & \left( (E_{11} + h)^2 (1 - D \cos^2 \frac{\varphi}{2}) (E_{11} + \alpha h)^2 - \Delta^2 \nu_F^2 \right) \\
- \left( \frac{\nu_F - \alpha}{\nu_F + \alpha} \right)^2 h^2 \Delta^2 D \cos^2 \frac{\varphi}{2} & = 0.
\end{align*}$$

This equation is fourth order in $E_{11}$, and it can be in principle solved analytically.

Equation (30) yields exact analytic solutions for $E_{11}$ in several asymptotic cases. This equation is further simplified for $h = 0, \alpha = 0, (B = 0)$, yielding

$$E_{11} \equiv E_{11} = \pm E_0 \equiv \pm \Delta \sqrt{D} \cos \frac{\varphi}{2},$$

which reproduces the well-known result [24] for the Andreev bound state energy of JJ with p-wave superconductors in the absence of magnetic field and spin-orbit interactions. This expression proposes the energy spectrum of quasi-electron and quasi-hole excitations, symmetrically located around the Fermi level in the gap.

In the case of $h = 0$ and $\alpha \neq 0 (B = 0)$, Rashba spin-orbit interaction splits both quasi-electron and quasi-hole spectra, and Eq. (30) yields four solutions for the bound state energy,

$$E = E_{11} = \pm \frac{\sqrt{D} |\Delta| (\nu_F + \alpha)}{2 |\nu_F + \alpha|^2} \cos \frac{\varphi}{2} - \frac{\nu_F^2}{|\nu_F + \alpha|^2} - \frac{\nu_F^2}{|\nu_F + \alpha|^2},$$

where $s = \pm$ assigns the electron and hole branches of the spectrum. Two solutions of this expression, corresponding to sign $(\pm)$ between $\nu_F$ and $\alpha$, coincide with Eq. (31), and do not depend on the strength of Rashba spin-orbit interaction. Nevertheless, the other two solutions depend on $\alpha$. The expression for Andreev bound state energy $E_{11}$, as mentioned above, is obtained by replacement of $\alpha \rightarrow -\alpha$ in the expression (32) written for $E_{11}$. Figs. 2a, b and Figs. 2c, d depict the dependence of $E = E_{11}$ and $E_{11}$ respectively on $\varphi$ for two different values of $\alpha$ when $\alpha = 0.2$ and $\alpha = 0.5$. The particle- and hole-like curves in the figures are depicted by red and blue colors correspondingly. According to Fig. 2, the branch of Andreev’s bound state energy, drawn by solid curves (red color for particle and blue color for hole level) do not depend on $\alpha$. Nevertheless, other two branches of $E_{11}$ (of $E_{11}$) decrease (increase) with increasing the strength of Rashba SOC. Note here that the parameters in all figures are given in a dimensionless form as $E \rightarrow E/\Delta, k \rightarrow (\nu_F \varphi)/\Delta, h \rightarrow h/\Delta, B \rightarrow B/\Delta$, and $\alpha \rightarrow \alpha/\nu_F$. In this limiting case, the quasi-electron and quasi-hole spectra are again symmetrically located around the Fermi level.

In the case of $\varphi = 0$ and $h \neq 0 (B = 0)$ Equation (30) yields the following expression for the Andreev bound state energy in this limit,

$$
E_{11}^* = -h \left[ 1 - 2D \cos^2 \frac{\varphi}{2} \right] + \nu_F \Delta \cos \frac{\varphi}{2} \left( \nu_F^2 - 4h^2 + 4Dh^2 \cos^2 \frac{\varphi}{2} \right) - \nu_F^2 \Delta^2 \cos \frac{\varphi}{2} \left( \nu_F^2 - 4h^2 \right).
$$

Andreev bound states are split again due to Zeeman effect. The dependence of $E_{11}^* \equiv E_{11}$ on $\varphi$ is depicted in Figs. 3a, b for two different values of $h$. Note that contribution to Andreev bound state energy $E_{11}$, found from the condition of $E_{11}^* = 0$, can be obtained again by replacing $h \rightarrow -h$ in Eq. (33). The dependence of $E_{11}$ on $\varphi$ for different values of $h$ is depicted in Figs. 3c, d for completeness. One pair of the subgap levels in Fig. 3 is drawn by solid curves in order to distinguish it from other pair which is shown by dashed curves. Furthermore, the particle...
(hole) levels are drawn by red (blue) color. The magnetic field destroys the time-reversal invariance, and shifts asymmetrically the energy spectra $E_{\uparrow\downarrow}(E_{\uparrow\downarrow})$ down (up). Two branches of solution (33), depicted in Fig. 3 by solid lines, differ from those given by Eq. (31) by shifting only the particle and hole pairs $E_{\uparrow\downarrow}(E_{\uparrow\downarrow})$ to the value of $-h$ $+h$), without changing their oscillation characteristics (see, Figs. 3a, b and c, d). As it is seen clearly from Figs. 3a, b, (Figs. 3c, d) the magnetic field reduces considerably the amplitude of the fractional oscillation for other two solutions of $E_{\uparrow\downarrow}(E_{\uparrow\downarrow})$, at the same time shifts down (up) asymmetrically the quasiparticle and quasi-hole spectra. One of the quasi-hole (quasiparticle) branch of $E_{\uparrow\downarrow}$ is pushed off from the gap at higher magnetic field when $h > h_{c} = 0.46\Delta$.

The general case for $B = 0$, but $a \neq 0$ and $h \neq 0$ is calculated numerically according to Eq. (30) writing this equation in the dimensionless parameters such as $\tilde{E} = E/\Delta$, $\tilde{k} = (kv_0)/\Delta$, $\tilde{h} = h/\Delta$, $\tilde{B} = B/\Delta$, and $\tilde{\alpha} = \alpha/v_{\gamma}$. Fig. 4 shows the dependence of $E_{\uparrow\downarrow}$ on $\varphi$ for $D = 0.3$ and $a = 0.4$ with different values of $\tilde{h}$, $\varphi = 0.3; 0.5001; 0.506$; and $0.5185$ (the parameters in all figures are given without tilde). One of the quasi-electron and quasi-hole pair of the spectrum, depicted by solid (red and blue) lines in Fig. 4, shifts down with increasing the magnetic field $h$ without changing the form and amplitude. The amplitude of the other quasi-electron and quasi-hole branch of $E_{\uparrow\downarrow}$ (drawn by dashed red and blue curves) decreases, and the form of the curves is deformed with increasing the magnetic field $h$. At $\tilde{h} > h_{c} = 0.5001$ a forbidden gap appears in the spectrum, i.e. as it is seen in Fig. 4c the quasi-electron and quasi-hole states disappear for some values of the order parameter phase difference $\varphi$. The quasiparticle and quasi-hole states, shown by dashed (blue and red) curves in Fig. 4 vanish by further increasing the magnetic field at $\tilde{h} > h_{c} = 0.51921$.

For completeness, $E_{\uparrow\downarrow}$ vs. $\varphi$ dependence is calculated also for $h = 0.3; 0.5001; 0.506$; and $0.5185$ under the condition of $B = 0$, $a = 0.4$ and $D = 0.3$, which is depicted in Fig. 5. As expected, $E_{\uparrow\downarrow}$ behaves like $E_{\uparrow\downarrow}$, i.e. the magnetic field shifts up one of the quasiparticle and quasihole pair, drawn by solid red and blue curves in Fig. 5 without changing the amplitude and form. The other pair, presented by dashed blue and red curves in Fig. 5 is deformed and amplitude decreases with increasing the magnetic field $h$. For $h > 0.5001$ a forbidden gap is opened (see, Fig. 5 b) in the spectrum. This branch (dashed curves in Fig. 5 c, d) squeezes and disappears for $h > h_{c} = 0.91124$.

3.2. Andreev bound state energy in the presence of in-plane magnetic field $B \neq 0$

In the presence of the in-plane magnetic field $B$ both reflection channels described by Eqs. (17)- (20) give contributions to Andreev bound state energy $E$. The expressions for general dependencies of $E_{\uparrow\downarrow}$ and $E_{\uparrow\downarrow}$ on $a, B, h$ can be obtained from the Equations (A1) and (A3) presented in Appendix after replacement of $k^{2}$ by $E$ according to Eq. (A2). These equations can be solved analytically for energy in several asymptotic cases. Note that main contribution to Andreev bound state energy still gives the conventional particle-hole channel.

The case of $a = 0$, $B = 0$, and $h \neq 0$. In this limiting case an interference between SOC-induced effective magnetic field and $h$ vanishes, and hence the energy spectrum depends on the modulus of total magnetic field according to Eq. (11) as $H = \sqrt{B^{2} + h^{2}}$. The expression (A1) for $E_{\uparrow\downarrow}$ is strongly simplified in this limiting case, and substitution of $k^{2}$ from Eq. (11) into this expression yields,

$$E_{\uparrow\downarrow}^{\uparrow\downarrow} = \pm \sqrt{B^{2} + h^{2} + s\Delta/\sqrt{D}} \cos \frac{\theta}{2},$$

(34)

where $s = \pm$.

The Andreev bound state energy in the anomalous particle-hole channel $E_{\uparrow\downarrow}$ can be found in this limiting case from the general expression given by Eq. (A4) yielding,

$$E_{\uparrow\downarrow}^{\uparrow\downarrow} = \pm \sqrt{B^{2} + h^{2} + s\Delta/\sqrt{D}} \cos \frac{\theta - \varphi}{2}.$$  

(35)

This expression differs from that given by (34) for $E_{\uparrow\downarrow}$ by dependence of cosine function not only on $\varphi$ but also on $\theta$.

The Andreev bound state energies $E_{\uparrow\downarrow}$ and $E_{\up\downarrow}$ are obtained from (34) and (35) by replacements $\alpha \rightarrow -\alpha$, $h \rightarrow -h$, $\varphi \rightarrow -\varphi$. Therefore, one can write, e.g. for $E_{\up\downarrow}$

$$E_{\up\downarrow} = \pm \sqrt{B^{2} + h^{2} + s\Delta/\sqrt{D}} \cos \frac{\theta + \varphi}{2}.$$  

(36)

The case of $h = 0$, $\alpha = 0$ and $B \neq 0$. The expression $E_{\up\downarrow}^{\up\downarrow} = 0$ can be simplified for $h = 0$ and $\alpha = 0$, $B \neq 0$, Routine calculations yield the following expression to determine $E_{\up\downarrow}$,

$$E_{\up\downarrow}^{\up\downarrow} = \mp \frac{v_{\gamma}^{2}}{2\Delta^{2}} \left[ (v_{\gamma}^{2} + \Delta^{2} - E_{\up\downarrow}^{\up\downarrow}) - \frac{v_{\gamma}^{2}B^{2}}{2\Delta^{2}} \right]$$

$$= \mp \frac{v_{\gamma}^{2}}{2\Delta^{2}} \left[ (E_{\up\downarrow}^{\up\downarrow} - B^{2} + \Delta^{2}) - B^{2}(v_{\gamma}^{2} - \alpha) \right]$$

$$= \mp \frac{v_{\gamma}^{2}}{2\Delta^{2}} \left[ \left( E_{\up\downarrow}^{\up\downarrow} + \Delta^{2} - B^{2} \right)^{2} + 4E_{\up\downarrow}^{\up\downarrow}\Delta^{2} \right]$$

$$= \mp \frac{v_{\gamma}^{2}}{2\Delta^{2}} \left[ \left( E_{\up\downarrow}^{\up\downarrow} - B^{2} + \Delta^{2} \right)^{2} + 4E_{\up\downarrow}^{\up\downarrow}\Delta^{2} \right]$$

(37)

One can put the expression for $k^{2}$ from (A2) and solve numerically this equation for $E_{\up\downarrow}$. Fig. 6 shows the dependence of $E_{\up\downarrow}$ on different values of the in-plane magnetic field $B$ for particular value of $\alpha = 0.3$ and $h = 0$. One quasiparticle and quasi-hole pair in the spectrum, depicted in blue (dashed lines) is enlarged and is partially pushed off from the
gap with increasing the in-plane magnetic field $B$ at $B > B_c \approx 0.55$. On the other hand the pair, depicted in red in Fig. 6, is narrowed with increasing $B$, and the gap is opened in the spectrum at $B > B_c \approx 0.95$. Further increase in $B$ makes this branch of the spectrum again regular at $B > 1.063$.

The dependence of $E_{1\uparrow\downarrow}$ on $\phi$ in the anomalous particle-hole channel for non-zero values of the external parameters $B$ and $\alpha$ but for $h = 0$ is depicted in Fig. 7 for $\alpha = 0.2, h = 0, D = 0.3$ (a) $B = 0.2$ and (b) $B = 0.5$. The quasi-electron (quasi-hole) dispersion at $h = 0$ is shifted to higher (lower) values with increasing $B$ and/or $\alpha$ without changing the shape and symmetry of the energy spectrum.

In the case of $h = 0, \alpha = 0$ and $B = 0$ out-of-plane magnetic field $h$ destroys a particle-hole symmetry in the spectrum. Dependence of $E_{1\uparrow\downarrow}$ and $E_{1\uparrow\downarrow}$ on $\phi$ in the particle-hole channel is depicted in Figs. 8 and 9 for finite $B = 0.4$ and different values of $h$, $h = 0.2, h = 0.4, h = 0.6$, and $h = 0.8$. The magnetic field seemly does not change the amplitude and structure one of the quasiparticle and quasi-hole energy pair, drawn by blue and dashed curves in Figs. 8 and 9. These bound state energies are shifted along the energy axis only. Instead, the magnetic fields strongly change other quasiparticle and quasi-hole pair, presented by red and solid curves in Figs. 8 and 9. This pair of the bound state energy becomes reduced in amplitude with increasing the magnetic field. At $h = h_c \approx 0.623$ a forbidden gap appears in the spectrum, and $h_c$ increases with increasing $B$.

Numerical calculation of the Andreev bound state energy $E_{1\uparrow\downarrow}$ in the anomalous particle-hole channel is shown in Fig. 10. The dashed (red) and blue) curves, corresponding to the spin-up branches of the bound energy, move away each other with increasing the magnetic fields.

Instead the solid (blue and red) curves, corresponding to the spin-down branch’s of the spectrum, is slightly narrowed with increasing the magnetic field.

4. Equilibrium josephson- and magneto-Josephson current

The current flowing though the quasiparticle and quasi-hole states $J_{\pm}$ in the simplest case of $B = h = 0$ but $\alpha \neq 0$ can be obtained from the tunneling energy given by Eq. (32). For $J_{\uparrow\uparrow}$, when the Andreev bound state energy becomes

$$E_{1\uparrow\uparrow}^+ = s\sqrt{\Delta}\cos \frac{\phi}{2},$$

with $s = \pm$ assigning the quasiparticle and quasi-hole pair, one gets,

$$J_{\uparrow\uparrow}^+ = -\frac{e\sqrt{\Delta}}{h} \pm \sin \frac{\phi}{2}. \tag{39}$$

For other particle-hole pair of the bound state energy Eq. (32)

$$E_{1\uparrow\downarrow}^+ = s\sqrt{\Delta} \frac{\sqrt{\frac{\gamma}{\gamma + \alpha^2}}}{\sqrt{1 - D_+ \frac{4\gamma\alpha^2}{\gamma + \alpha^2}}} \cos \frac{\phi}{2}, \tag{40}$$

Josephson energy $J_{\uparrow\downarrow}^+$ reads as

$$J_{\uparrow\downarrow}^- = -\frac{e\sqrt{\Delta}}{h} \frac{\sqrt{\frac{\gamma}{\gamma + \alpha^2}}}{\sqrt{1 - D_+ \frac{4\gamma\alpha^2}{\gamma + \alpha^2}}} \cos \frac{\phi}{2} \sin \frac{\phi}{2}. \tag{41}$$

In thermodynamic equilibrium at temperature $T$ the total contribution of the Andreev bound states to the Josephson current can be

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**Fig. 6.** Andreev bound state energy $E_{1\uparrow\downarrow}$ in the particle-hole channel for $h = 0, D = 0.3$ and $\alpha = 0.3$. (a) $B = 0.3$, (b) $B = 0.5$, (c) $B = 0.83$, (d) $B = 1.0548$, (e) $B = 1.063$, and (f) $B = 1.1$.

**Fig. 7.** Andreev bound state energy $E_{1\uparrow\downarrow}$ in the anomalous particle-hole channel for $h = 0, D = 0.3$ and $\alpha = 0.2$. (a) $B = 0.2$, (b) $B = 0.5$.

**Fig. 8.** Andreev bound state energy $E_{1\uparrow\downarrow}$ in the particle-hole channel for $D = 0.3, \alpha = 0.3$, and $B = 0.4$ (a) $h = 0.1$, (b) $h = 0.3$, (c) $h = 0.4$, and (d) $h = 0.6$.

**Fig. 9.** Andreev bound state energy $E_{1\uparrow\downarrow}$ in the particle-hole channel for $D = 0.3, \alpha = 0.3$, and $B = 0.4$ (a) $h = 0.1$, (b) $h = 0.3$, (c) $h = 0.4$, and (d) $h = 0.6$.
calculated according to the expression

\[
I = \frac{2e}{h} \sum_{n=\pm} \frac{\Delta E_{n}^{\alpha}}{\Delta \varphi} \sin \left( \frac{\sqrt{\Delta} \cos \frac{\varphi}{2}}{2k_{B}T} \right)
\]

where the expression of \(E_{n}^{\alpha}\) is presented by Eqs. (38) and (40). Taking into account the expression for the energies one gets for Josephson current in the simplest case when \(\alpha = 0\)

\[
I = \frac{e}{h} \sqrt{\Delta} \sin \varphi \left( \tanh \frac{\sqrt{\Delta} \cos \frac{\varphi}{2}}{2k_{B}T} \right)
\]

\[
- \frac{2e}{h} \left( \frac{h D \sin \varphi - \frac{\sqrt{\Delta}}{2} \cos \frac{\varphi}{2} D^{2} + 4h^{2}D \cos \frac{\varphi}{2}}{2} \right)
\]

\[
\tan \frac{h (1 - 2D \cos \frac{\varphi}{2}) + \sqrt{\Delta}D \cos \frac{\varphi}{2}}{2k_{B}T}
\]

\[
- \frac{2e}{h} \left( \frac{\sqrt{\Delta}}{2} \sin \frac{\varphi}{2} \cos \frac{\varphi}{2} D^{2} + 4h^{2}D \cos \frac{\varphi}{2} \right)
\]

\[
\tanh \frac{h (1 - 2D \cos \frac{\varphi}{2}) + \sqrt{\Delta}D \cos \frac{\varphi}{2}}{2k_{B}T}
\]

\[
+ \frac{e}{h} \sqrt{\Delta} \frac{\sin \varphi}{2}
\]

\[
\left[ \tan \frac{h + \sqrt{\Delta} \cos \frac{\varphi}{2}}{2k_{B}T} + \tan \frac{h + \Delta \sqrt{\Delta} \cos \frac{\varphi}{2}}{2k_{B}T} \right].
\]

For example, for \(p^-\) wave JJs with \(\alpha = h = 0\), where \(E = \pm \Delta \sqrt{\Delta} \cos (\varphi/2)\) according to Eq. (31), this procedure leads to

\[
I_{j AC}^{\pm} = I_{0} \sqrt{\Delta} \cos (\varphi_{0}/2)\]

\[
\frac{I_{j AC}^{\pm}}{I_{0}} = \frac{\pm I_{0} \sqrt{\Delta} \cos (\varphi_{0}/2)}{I_{0}}
\]

where \(I_{0} = 2e\Delta/h\) and we have used the identity \(\exp[i \sin \varphi] = \sum \exp[i \varphi \alpha \exp[i \alpha]]\). Eq. (50) reflects the fact that for \(p^-\) wave junction one has Shapiro steps at \(\varphi_{0}/\alpha = 2n\) for integer \(n\); the odd Shapiro steps are absent. The width of the step corresponding to \(n = n_{0}\) is given by

\[
\Delta I_{0} = |I_{0} \sqrt{\Delta} \cos (\varphi_{0}/2)|
\]

where we have used the fact that the maximum width of the step occurs at \(\varphi_{0} = \varphi_{0}^{\pi} = \pi\). The dependence of the width of the step on \(\alpha_{i}\) for different values of \(\alpha_{i} = 0, 1, 2, 3\) is presented in Fig. 11.

Next, we apply this procedure for the case where \(B = h = 0\) but \(\alpha \neq 0\). The Andreev bound state energy is given by Eq. (52) and consists of four branches, i.e., each electron and hole branch is split into two states. One of these split states, corresponding to the \(+\) sign (Eq. 52) are independent of \(\alpha\).

For these two states, the AC Josephson current can be easily shown to be given by Eq. (50); the corresponding Shapiro step width is given by Eq. (51). In contrast, the energy dispersion of the other two branches with designated by \(-\) sign (Eq. 52) depend on the ratio \(\alpha/\varphi_{0}\) and can be rewritten as

\[
E_{\pm} = \frac{\Delta \alpha_{\beta} \sqrt{\Delta} \cos (\varphi_{0}/2)}{\sqrt{1 - D(1 - \beta^{2}) \cos^{2}(\varphi_{0}/2)}}
\]

where \(\beta = (1 - \alpha/\varphi_{0})/(1 + \alpha/\varphi_{0})\). We note that these branches do not
The dependence of the Shapiro step width for p-wave JJs at $\alpha = B = h = 0$ and $a \neq 0$ on the amplitude of the ac-voltage $\alpha_0 = 2eV/(\eta_0 n_0)$ is plotted in Fig. (54) for two different values of the parameter $D_0$. (a) $D_0 = 1/8 < 1/6$ at $D = 0.5$ and $\beta = 0.105/\sigma$, corresponding to $\theta_{m}^{n} = \pi$ and (b) $D_0 = 7/40 > 1/6$ at $D = 0.5$ and $\beta = 0.103$. The curves in each figure are drawn by (1) solid red line, (2) green dot-dashed curve at $n_0 = 1$, (3) blue dashed curve at $n_0 = 2$, (4) brown dotted curve at $n_0 = 3$, and (5) black double dot-dashed curve at $n_0 = 4$. For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.

The dependence of the Shapiro step width on the amplitude of the spin-orbit coupling in these JJs from the Shapiro step width is maximum. This value needs to be numerically determined for the present case by minimizing Eq. (58) at $\omega = 2n_0\omega$ with respect to $\theta_{m}^{n_0}$.

Next we consider the case where $\alpha = B = 0$ and $h = 0$. Here the energy dispersion corresponds to four branches as can be seen from Eq. (33). We first consider the branches corresponding to $E_{c}^{c}(s)$ with $s = \pm$. The corresponding Shapiro step width is given by

$$\Delta E_{c}^{c}(n_0) = \frac{\alpha_{0}^{2} - 4h^2 + 4Dh}{\Delta^2 - 4h^2 + 2hD + 2hD\sum_{n}J_{n}^{0}(\alpha_{0})\sin[\varphi_{0}^{n} + (\omega - n_0\omega)t]/2}$$

(58)

where $m_0$ is an integer and $\theta_{m}^{n_0}$ denotes the value of $\varphi_{0}^{n_0}$ for which the step width is maximum. This value needs to be numerically determined for the present case by minimizing Eq. (58) at $\omega = 2n_0\omega$ with respect to $\varphi_{0}^{n_0}$.

Finally, we treat the case $B, h = 0$ and $\alpha = 0$. Here the Andreev
bound states are given by Eq. (34) and (35). For $E = E_{\uparrow \downarrow}^{\pm} (x)$ (Eq. (34)), there are four branches. For $E = E_{\uparrow \downarrow}^{\pm}$ both the branches are above the Fermi energy if $\sqrt{B^2 + k^2} \geq \Delta \sqrt{B}$. In this case, there is no Josephson current contribution from these branches. Similarly, for $E = E_{\uparrow \downarrow}^{\pm}$, the same condition results in both the branches being below the Fermi level. In this case, their contribution to the Josephson current cancel each other. Thus, the contribution to the Josephson current from $E_{\uparrow \downarrow}$ occurs only when $\sqrt{B^2 + k^2} < \Delta \sqrt{B}$. However, even in this case, the contribution to the Josephson currents from the positive ($E^+$) and negative ($E^-$) branches cancel each other and the net Josephson current vanishes. A similar results can be easily deduced for $E = E_{\uparrow \downarrow}^{\pm}$ (Eq. (35)) and $E_{\uparrow \downarrow}$ (Eq. (36)) branches.

6. Conclusion

In this paper we study the Josephson current between 1D nanowires of $p$-wave superconductors separated by an insulating barrier in the presence of Rashba SOI and the magnetic fields $B$ and $h$. The presence of the SOI and Zeeman magnetic fields enlarges the standard two-component BdG system equations to four-component system equations (13)-(16) for new BdG wave vector $\eta_+(x) = (\eta_{+1}(x), \eta_{+1}(x), \eta_{-1}(x), \eta_{-1}(x))$. The BdG equations (13)-(16) coincide with the standard BdG equations in the absence of the in-plane magnetic field $B$, which provide only one relation $\eta_{e,h,b}/\eta_{e,h,b}$ between quasiparticle and quasi-hole states, where $\sigma = \uparrow, \downarrow$ and $\tau = \pm$. Instead, the BdG equations (13)-(16) in the presence of in-plane magnetic field $B$ and Rashba SOI provide these relations $\eta_{e,h,b}/\eta_{e,h,b}$, $\eta_{s,b}/\eta_{s,b}$, and $\eta_{s,b}/\eta_{s,b}$, and $\eta_{s,b}/\eta_{s,b}$ between the quasiparticle and quasi-hole states, corresponding to new Andreev scattering channels. We studied in this paper two of these scattering channels in detail by generalizing the method of Ref. [24] for study of Josephson junction with $\delta$-function insulator between two $p$-wave superconductors to systems with SOI and Zeeman fields. Moreover, we have demonstrated the existence of magneto-Josephson effect in these systems. We note that although the existence of the magneto-Josephson effect in a topological superconductor has been predicted recently [29,52,53], the question of whether this effect is observable in superconducting junctions with $p$-wave superconductors and the presence of SOI was not addressed before. We have predicted in the paper new Andreev-type reflection channels, when a quasiparticle reflects to a quasi-hole state with the same spin orientation, which are responsible to the magneto-Josephson effect.

It is necessary to note that $p$-wave JJ can be experimentally fabricated either by growing a semiconductor-insulator-semiconductor 1D wire with strong SOC in each semiconductor on a substrate of a $p$-wave superconductor or by contacting two thick superconductors with spin-triplet pairing through an insulator layer between them. In experiments to date the Josephson effect has been studied only in junctions between Sr$_2$RuO$_4$ (or UPt$_3$) and a conventional superconductor [54–56]. By fabricating Nb/Ru/Sr$_2$RuO$_4$ superconducting junction between s-symmetric Nb- and $p$-symmetric Sr$_2$RuO$_4$-superconductors and by analyzing asymmetry, hysteresis and noise in the $I-V$ dependence of the junction the authors concluded [55] that the multicomponent order-parameter is realized in Sr$_2$RuO$_4$ consistent with the chiral $p$-wave state. All these experimental facts show that SC/Insulator/SC or SC/Normal Metal/SC can be fabricated by using, e.g. Sr$_2$RuO$_4$ superconductor in order to investigate effects of SOI and Zeeman magnetic field on Josephson junction and Shapiro step. Note that Rashba SOI should be created due to deformation potential between the substrate and the superconducting wires grown on the substrate.

In conclusion, we have studied Josephson effect in a junction between two $p$-wave 1D nanowires in the presence of SOI and Zeeman magnetic fields. We have analyzed the Josephson current in these junctions and provided analytic expressions of the Andreev bound states in several limiting cases. We have also demonstrated the existence of magneto-Josephson effect in these junctions, and studied the dependence of the Shapiro step width in AC Josephson effect on the SOI strength. Our theoretical predictions are shown to be verifiable by straightforward experiments on this systems.

CRediT authorship contribution statement


Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Andreev bound state energies at $B \neq 0$

Andreev bound state energies are obtained from the conditions $F_{\uparrow \downarrow}^{\pm}(E_{\uparrow \downarrow}) = 0$, $F_{\uparrow \downarrow}^{\pm}(E_{\uparrow \downarrow}) = 0$ corresponding to two different channels, and also from the conditions, obtained by interchanging the spin orientations as $F_{\uparrow \downarrow}^{\pm}(E_{\uparrow \downarrow}) = 0$, $F_{\uparrow \downarrow}^{\pm}(E_{\uparrow \downarrow}) = 0$.

Andreev bound state energy $E_{\uparrow \downarrow}$ is obtained from the condition $F_{\uparrow \downarrow}^{\pm}(E_{\uparrow \downarrow}) = 0$ with the expression (26) yielding

$$
\begin{align*}
&[(E_{\uparrow \downarrow}+h)((E_{\uparrow \downarrow}+h)^2-k^2(v\mp \alpha)^2+B^2-\Delta^2)-2B^2E_{\uparrow \downarrow}][1-D \cos^2 \frac{\varphi}{2}] \\
&[k(v\mp \alpha)((E_{\uparrow \downarrow}+h)^2-k^2(v\mp \alpha)^2+B^2-\Delta^2)+2kB^2\varphi^2D \cos^2 \frac{\varphi}{2}] = 0.
\end{align*}
$$

(A1)

This expression depends apart from the parameters $\alpha$, $k$, and $h$ also on the momentum $k^2$. Expression for $k^2$, obtained from the energy spectrum (10), reads

$$
\begin{align*}
k^2 &= ((v\mp \alpha)^2-(E^2-h^2-B^2-\Delta^2)(v\mp \alpha)^2+2(E_{\uparrow \downarrow}+ah^2) + 2\Delta(v\mp \alpha)^2 + \frac{1}{2}[(E^2-h^2-B^2-\Delta^2)(v\mp \alpha)^2-2(E_{\uparrow \downarrow}+ah^2) + 2\Delta(v\mp \alpha)^2 - 2(E_{\uparrow \downarrow}+ah^2) + 2\Delta(v\mp \alpha)^2 - 2(E_{\uparrow \downarrow}+ah^2) + 2\Delta(v\mp \alpha)^2]^{1/2}. \\
\end{align*}
$$

(A2)

Elimination of $k^2$, by substituting it from (A2) into Eq. (A1), yields a general expression to find $E_{\uparrow \downarrow}$, which is not easy to solve exactly.

The condition $F_{\uparrow \downarrow}^{\pm}(E_{\uparrow \downarrow}) = 0$ with the expression (29) yields
$k^2(\mathbf{E}_2 + \alpha h)^2 - \Delta^2(E_{11}^2 + \alpha^2 k^2) + \Delta' D(E_{11}^2 + \alpha^2 k^2)\cos^2 \frac{\varphi - \theta}{2} = 0. \tag{A3}$

Routine calculations, after substitution of $k^2$ from Eq. (A2) to this equation, result in

$$\left\{(v_0^2 - \alpha^2)E_{11}^2\Delta^2\left(1 + D \cos^2 \frac{\varphi - \theta}{2}\right) - \left(E_{11}^2 + \Delta^2 - B^2 - h^2\right)\left[(E_{11}^2 v_F + \alpha h)^2 - \alpha^2 \Delta^2\left(1 - D \cos^2 \frac{\varphi - \theta}{2}\right)\right]\right)^2 - 4E_{11}^2 \Delta D \cos^2 \frac{\varphi - \theta}{2} \left((E_{11}^2 v_F + \alpha h)^2 - \alpha^2 \Delta^2\left(1 - D \cos^2 \frac{\varphi - \theta}{2}\right)\right) = 0 \tag{A4}$$

This equation can be solved numerically for a general case when $\alpha, B, h \neq 0$. The expression for $F_{11}'(E_0) = 0$ is obtained from Eq. (A4) by interchanging $\varphi \rightarrow -\varphi$, $h \rightarrow -h$, and $\theta \rightarrow -\theta$.

**Supplementary material**

Supplementary material associated with this article can be found, in the online version, at 10.1016/j.physc.2020.1353753