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The stratified p-hub center and p-hub maximal covering problems

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ABSTRACT

Hub networks are the foundation of many transportation and distribution systems, and real-world hub networks often transport freight or passengers of different service classes. This paper introduces the stratified multiple allocation p-hub center and p-hub maximal covering problems where the traffic corresponding to each origin–destination (O/D) pair is divided into different strata each having a specific service level requirement. The problems are formulated as mixed-integer linear programming (MILP) models and efficient Benders decomposition algorithms are developed for solving large instances. Extensive computational experiments are conducted to demonstrate the efficiency of the proposed mathematical models and the solution algorithms. MILP formulations are also proposed for the generalized versions of the problems that include fixed set-up costs for hubs and hub arcs. Results indicate that the optimal sets of hub locations and hub arcs when considering different strata can be quite dissimilar to those of the traditional p-hub center or p-hub maximal covering problem, but are similar to those of hierarchical hub location problems. Furthermore, models are provided and solved for multi-modal stratified hub location problems with fixed setup costs for hubs and hub arcs. Optimal results show a wide range of network topologies that can be generated, as compared to the classical versions.

1. Introduction

Hub networks are the foundation of many transportation and distribution systems, and most hub location and hub network design research treats the origin–destination flows as receiving a single type of service, or being a single stratum (e.g., freight with common logistics and demand characteristics, or passengers of a single class of service). However, in many situations in practice, the service level required by different classes of commodities or customers are not the same. For example, some commodities/customers are sensitive to transportation time and need very fast delivery/transport, whereas other commodities/customers have more tolerance towards longer transport (slower delivery). Applications include the design of a hub network in postal and parcel services where the various types of service (regular, express, etc.) need to be provided for customers of different strata, passenger air transportation with different classes of passengers (first class, business class, economy class, etc.) and the distribution of perishable items where some require faster delivery (e.g., fresh fish) than others. Often the demand in different strata are willing to pay higher prices for better service, as for example, with more direct airline flights (from an origin to a destination) that have higher prices.

In hub-based many-to-many transportation systems, fundamental functions such as transshipment, sorting, consolidation, etc. are performed at hub facilities which serve as collection and distribution points between origin–destination (O/D) pairs. Due to flow concentration on network arcs (especially on the inter-hub connections), transportation costs in hub networks benefit from economies of scale. In hub location problems (HLPs) the objective is to locate a number of hub facilities and to allocate the demand

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points to the established hubs so that the O/D traffic is routed to achieve a desired objective (e.g., minimizing system-wide cost or maximizing the service level provided to the clients). Non-hub nodes (origins/destinations) can be allocated to the hubs according to either of two commonly adopted protocols: single or multiple allocation. If each non-hub node can be directly connected to only one hub, the network is called a single allocation hub network. As a generalization of this protocol, if each non-hub node is allowed to be directly connected to more than one hub, then we deal with a multiple allocation HLP. Incorporation of service level considerations in hub networks is very important, particularly when we deal with the delivery of time sensitive items or passenger transportation. One approach is to use hub center models where the objective is to minimize the maximum travel time between the O/D pairs. Alternatively, one can use maximal covering models, where the aim is to serve the O/D pairs in such a way that the fraction of traffic which is served within a specific allowable time threshold is maximized.

In this work we address the HLP from a service time perspective and introduce the stratified multiple allocation p-hub center problem (SpHCP) and stratified multiple allocation p-hub maximal covering problem (SpHMCP). We assume that the total traffic demand is divided into distinct strata, and the demand corresponding to each stratum has a distinct service level requirement. The goal of SpHCP is to locate hubs and route the traffic in such a manner that the weighted sum of the largest travel times between the O/D pairs associated with all the strata is minimized. SpHMCP, on the other hand, aims at maximizing the weighted sum of the covered demand values over all the strata given a specific coverage requirement for each stratum. In other words, the service level in our models is related to the travel times of the O/D traffic. In the case of the center problem, the lower the weighted sum of the maximum travel times is, the higher the service level is. For the maximal covering problem, the larger the weighted sum of the covered traffic is, the higher the service level is. To the best of our knowledge, this is the first work in the literature that tackles stratified hub location problems, and in particular the stratified p-hub center and stratified p-hub maximal covering problems. These problems are modeled as mixed-integer linear programs (MILPs) and exact solution approaches based on Benders decomposition are presented. Extensive computational experiments are conducted to study the effect of different input parameters on the optimal hub locations and traffic patterns and to demonstrate the efficiency of the proposed solution algorithms. Furthermore, we address generalized versions of the problems that include partial coverage, operational times at hubs, multi-modal transport, and fixed set-up costs for hubs and hub arcs and allowing an incomplete inter-hub network.

We propose MILP formulations to the generalized problems and analyze the obtained results. In this paper we focus on stratified hub location from a service level perspective (as in Albareda-Sambola et al. (2019)). Because of our focus on service objectives, we do not compute the cost for the obtained solutions; stratified hub location problems with a cost-oriented objective (e.g., the stratified p-hub median problem) are an interesting area for future research. Our basic stratified models address strategic transportation network design issues which allow different modes of transport as reflected by higher vehicle speeds between hubs (e.g., as from trucks and airplanes). In Section 6 we consider a model that explicitly include different modes but modeling mode-specific fixed costs for the hub nodes and hub arcs. Finally, as in most of hub location literature, we allow direct shipment between an origin and destination only when the origin and/or destination are hubs.

The remainder of the paper is structured as follows. Section 2 presents related papers from the literature. Mathematical formulations are presented in Section 3. The proposed Benders decomposition algorithms are developed in Section 4. Results from extensive computational experiments are given in Section 5. MILP formulations for the generalized problems with fixed set-up costs and their results are presented in Section 6. Conclusions as well as directions for future research are presented in Section 7.

2. Background

The HLP has become established as a rich territory of research within the area of transportation network design and facility location. Studies on the HLP have grown at a fast pace since 1980s and interested readers may refer to Alumur and Kara (2008), Campbell and O'Kelly (2012), Farahani et al. (2013), Contreras and O'Kelly (2019), Alumur et al. (2021) for review articles on the HLP.

The p-hub center problem (pHCP) is a service-oriented member of the family of HLPs that deals with determining the optimal location of p hubs and allocating the demand to these hubs in such a way that the maximum distance between all the O/D pairs is minimized. The path-based formulation by Campbell (1994) was the first mathematical model for the pHCP on a network. Kara and Tansel (2000) proposed a number of alternative formulations that have better computational performance, among which an arc-based formulation has the highest efficiency. Kara and Tansel (2001) extended the pHCP to incorporate synchronization issues on the operational level. The authors called this *the latest arrival hub location problem*. This problem was extended by Yaman et al. (2007) by allowing a path from a non-hub node to its hub to visit another non-hub node *en route*. Ernst et al. (2009) proposed ILP formulations for both the single and multiple allocation versions of the problem and proved that both problems are NP-Hard.

Sim et al. (2009) considered the p-hub center problem with stochastic travel times using normal distributions and employed chance constraints to limit the probability of excessive travel times. A two-phase algorithm based on branch-and-bound was presented by Meyer et al. (2009) for the single allocation p-hub center problem (USApHCP). Alumur et al. (2009) studied the single allocation incomplete hub center and hub covering problems. A combined p-center and network design problem was addressed in Contreras et al. (2012b), where facility location and underlying network design decisions were considered at the same time to minimize the maximum customer-facility travel time. Bashiri et al. (2013) presented a GA based heuristic to solve the fuzzy capacitated p-hub center problem. Yang et al. (2013) developed a hybrid PSO algorithm for the fuzzy p-hub center problem where the travel times were modeled using normal fuzzy vectors. Brimberg et al. (2017a) proposed a basic variable neighborhood search (BVNS) heuristic for the uncapacitated multiple allocation p-hub center problem (UMApHCP). In another work, Brimberg et al. (2017b) proposed a general variable neighborhood search (GVNS) algorithm for the USApHCP.

Similar to the p-hub center problem, the p-hub maximal covering problem (pHMCP) focuses on service-level considerations in the design of hub networks. The pHMCP aims at maximizing the covered demand by using a fixed number of hub facilities. In other words, in pHMCP it is not necessary to serve all O/D pairs as it is the case in profit maximizing HLPs (Taherkhani and Alumur, 2019). The models for both single and multiple allocations were proposed by Campbell (1994). A new mathematical formulation for the single assignment version of the problem was later proposed by Hwang and Lee (2012). Weng et al. (2006) developed a new model for the multiple allocation version and prove the NP-hardness of the problem. A tabu search (TS) based algorithm was developed by Çalık et al. (2009) for the single allocation hub covering problem with incomplete inter-hub networks. Jabalameli et al. (2012) proposed a simulated annealing (SA) algorithm for the single allocation p-hub maximal covering problem. Peker and Kara (2015) extended the p-hub maximal covering problem by considering partial coverage where it was possible to partially cover demand using gradual decay functions. Yıldız and Karaşan (2015) studied the regenerator location problem from a hub covering location perspective. A TS heuristic was proposed by Silva and Cunha (2017) for the uncapacitated single allocation p-hub maximal covering problem. Janković and Janković and Stanimirović (2017) proposed a GVNS heuristic for the uncapacitated r-allocation p-hub maximal covering problem (UrApHMCP) with binary coverage criterion. Efficient SA based algorithms were proposed for solving single and multiple allocation maximal covering HLPs under market competition in Ghaffarinasab et al. (2018).

Benders decomposition (BD) (Benders, 1962) is a partitioning method for solving mixed integer programming (MIP) problems, and it has been successfully applied to several different variants of the HLP. de Camargo et al. (2008) solved the uncapacitated multiple allocation hub location problem (UMAHLP) using BD algorithms. Another BD algorithm was devised by de Camargo et al. (2009b) for the HLPs with flow-dependent discount factor. A generalized BD algorithm was developed for HLPs under congestion by de Camargo et al. (2009a, 2011). Large-scale instances of the UMAHLP were solved by using BD algorithms proposed by Contreras et al. (2011a). The same authors applied a BD algorithm for solving stochastic uncapacitated HLPs (Contreras et al., 2011b). An accelerated BD procedure was presented for an HLP in the context of urban transport and liner shipping network design in Gelareh and Nickel (2011) which was then extended to solve a multi-period HLP under budget constraints (Gelareh et al., 2015). Capacitated version of the HLPs were tackled by a BD algorithm in Contreras et al. (2012a). de Camargo et al. (2013) applied a BD algorithm for the many-to-many hub location routing problem. BD algorithm were also used to solve the tree of hubs location problem and the hub line location problem in de Sá et al. (2013, 2015), respectively. Another BD algorithm was devised by O'Kelly et al. (2015) for the HLP with price-sensitive demands. Fontaine and Minner (2014) proposed a BD solution approach for a traffic network design problem modeled as bilevel problems. Two BD procedures were proposed by Meraklı and Yaman (2016) for the robust uncapacitated multiple allocation p-hub median problem (UMApHMP) under polyhedral demand uncertainty. BD algorithms were proposed to solve the robust incomplete hub location problem with uncertain demands and hub fixed costs (de Sá et al., 2018a) and with uncertain travel times (de Sá et al., 2018b). Ghaffarinasab and Kara (2019) proposed BD algorithms for solving uncapacitated single allocation HLPs with fixed and variable number of hubs. A modified BD method was developed in Mokhtar et al. (2019) for solving the 2-allocation p-hub median problem which can also be generalized for solving other HLPs. Mahéo et al. (2019) addressed the design of a hub and shuttle public transit network and proposed an efficient procedure for solving it based on BD. Taherkhani et al. (2020) developed a BD procedure for solving the profit maximizing hub location problems. Ghaffarinasab (2020a) proposed an efficient BD algorithm for solving the UMApHCP. Najy and Diabat (2020) proposed a BD algorithm for a multiple allocation HLP with economies of scale and node congestion. Finally, Ghaffarinasab (2021) developed BD algorithms for the robust uncapacitated multiple allocation p-hub median problem.

Observing the literature, we conclude that carefully designed BD algorithms have been an effective solution approach for many HLPs. Furthermore, the stratified p-hub center and p-hub maximal covering problems have not been addressed before. In case of the HLP, the concept of providing multiple service levels has been used in some works (Campbell, 2009, 2013). However, these works focus on a cost minimization objective, while the service level considerations are addressed as constraints. In the classical facility location theory, the stratified p-center problem has been studied recently (Albareda-Sambola et al., 2019), where the authors developed and compared different formulations, valid inequalities and preprocessing techniques for this problem. In this work we introduce the stratified multiple allocation p-hub center and p-hub maximal covering problems, along with some generalizations, and develop efficient solution algorithms based on Benders decomposition for these problems.

3. Model formulations

Let $G = (N, A)$ be a graph with N and A respectively as the sets of nodes and arcs such that $A = \{(i,j) : i,j \in N, i \neq j\}$. We assume that all nodes in N are candidate sites for locating hub facilities. For all $i,j \in N$, we denote by d_{ij} the distance between nodes i and j. Distances are assumed to satisfy the triangle inequality, i.e., $d_{ij} \leq d_{ik} + d_{kj}$ for all $i,j,k \in N$. We further assume that the travel time between pair of nodes $(i,j) \in A$ is proportional to the distance between them, i.e., $\tau_{ij} = v d_{ij}$ where v is a positive scaling factor that can be viewed as the reciprocal of average speed. The number of hubs to be installed is a fixed value denoted by p. We assume that each O/D trip visits at least one hub. In basic stratified HLPs, as a result of the triangle inequality, the inter-hub network in an optimal solution is a complete network. We assume that a faster mode of transport is used on the inter-hub network, where the parameter α, between 0 and 1, is a speed-up factor on the transportation time over the inter-hub connections. Thus, while cost-oriented hub location models use α as a cost discount factor for inter-hub travel, we use α as a time discount factor, so the average inter-hub speed can be viewed as $1/(\alpha v)$. The total travel time for the O/D flow associated with pair $(i,j) \in A$ through hubs k and m in that order is calculated as:

$$c_{ijkm} = \tau_{ik} + \alpha \tau_{km} + \tau_{mj}. \tag{1}$$

In the proposed version of the problems, the O/D traffic in the network is partitioned into a set of strata with the possibility of more than one stratum for each O/D pair. Let S be the set of strata into which the O/D traffic are classified. We consider a collection of subsets $\{A^s\}_{s\in S}$ where $A^s \subseteq A$ is the set of O/D pairs at which stratum $s \in S$ is present. Let t^s_{ij} represent the traffic volume corresponding to O/D pair (i,j) in stratum $s \in S$. For each stratum $s \in S$, we assign a weight w^s which reflects the importance of that stratum and is used to weight the cost or coverage related to different strata in the objective function. We define the binary variable y_k as:

$$y_k = \begin{cases} 1, & \text{if node } k \text{ is selected as a hub,} \\ 0, & \text{otherwise.} \end{cases}$$

Also, the non-negative variable x^s_{ijkm} denotes the fraction of traffic associated with the O/D pair $(i,j) \in A^s$, that is routed via hubs k and m in stratum $s \in S$.

3.1. The stratified p-hub center formulation

In order to formulate the stratified p-hub center problem, we define z^s as the maximum travel time between the O/D pairs within stratum s. The problem consists of selecting p nodes as hubs and determining how the O/D flows will be routed via these hubs so that the weighted sum of the maximum travel times between O/D pairs within all strata is minimized. The MILP model for the multiple allocation stratified p-hub center problem (SpHCP) can be written as:

$$\min \sum_{s\in S} w^s z^s \tag{2}$$

$$\text{s.t.:} \sum_{k\in N} y_k = p \tag{3}$$

$$\sum_{k\in N}\sum_{m\in N} x^s_{ijkm} = 1 \qquad s \in S, (i,j) \in A^s \tag{4}$$

$$\sum_{m\in N} x^s_{ijkm} + \sum_{\substack{m\in N\\ m\neq k}} x^s_{ijmk} \le y_k \qquad s \in S, (i,j) \in A^s, k \in N \tag{5}$$

$$z^s \ge \sum_{k\in N}\sum_{m\in N} c_{ijkm} x^s_{ijkm} \qquad s \in S, (i,j) \in A^s \tag{6}$$

$$x^s_{ijkm} \ge 0 \qquad s \in S, (i,j) \in A^s, k,m \in N \tag{7}$$

$$z^s \in \mathbb{R} \qquad s \in S \tag{8}$$

$$y_k \in \{0,1\} \qquad k \in N \tag{9}$$

The objective function (2) minimizes the weighted sum of the maximum travel times between O/D pairs within all strata. Constraint (3) sets the number of installed hubs to be equal to p. Constraints (4) ensure that the entire flow corresponding to each O/D pair is routed via some pair of hubs. Constraints (5) state that the flows cannot go through an intermediate node unless that node is a hub. Constraints (6) calculate the maximum travel time between the O/D pairs within each stratum. Finally, (7), (8) and (9) determine the domains of the decision variables.

3.2. The stratified p-hub maximal covering formulation

The sets, parameters, and decision variables used in formulating the stratified p-hub maximal covering problem are largely the same as the ones used for the stratified p-hub center model. However, we define two new parameters specific for this problem. Let β^s_{ij} denote the maximum allowable travel time (or coverage radius) for O/D pair $(i,j) \in A^s$ in stratum $s \in S$, and let a^s_{ijkm} be a binary covering parameter that takes the value of 1 if the traffic between nodes i and j in stratum s is covered and 0, otherwise. In other words, we have:

$$a^s_{ijkm} = \begin{cases} 1, & \text{if } c_{ijkm} \le \beta^s_{ij} \\ 0, & \text{otherwise} \end{cases} \qquad s \in S, (i,j) \in A^s, k,m \in N \tag{10}$$

Based on the above-mentioned parameters and notations, the MILP model for the stratified p-hub maximal covering problem (SpHMCP) can be written as:

$$\max \sum_{s\in S} w^s \left(\sum_{(i,j)\in A^s} \sum_{k\in N}\sum_{m\in N} a^s_{ijkm} t^s_{ij} x^s_{ijkm} \right) \tag{11}$$

$$\text{s.t.: (3), (4), (5), (7), (9)} \tag{12}$$

The objective function (11) maximizes the weighted sum of the covered traffic over all the strata.

We also propose a second formulation for SpHMCP which is more compact in terms of number of variables as compared with the above model. Let the binary variable u_{km} indicate whether both $k \in N$ and $m \in N$ are selected as hubs or not. Further, assume the variable v_{ij}^s represent the fraction of the O/D flow $(i,j) \in A^s$ in stratum $s \in S$ that is covered. Then we can write:

$$\max \sum_{s\in S}\sum_{(i,j)\in A^s} w^s t_{ij}^s v_{ij}^s \tag{13}$$

$$\text{s.t.:} \sum_{k\in N} y_k = p \tag{14}$$

$$u_{km} \le y_k \qquad \forall k,m \in N, (k<m) \tag{15}$$

$$u_{km} \le y_m \qquad \forall k,m \in N, (k<m) \tag{16}$$

$$v_{ij}^s \le \sum_{k\in H_{ij}^s} y_k + \sum_{(k,m)\in G_{ij}^s} u_{km} \qquad \forall s \in S, (i,j) \in A^s \tag{17}$$

$$0 \le v_{ij}^s \le 1 \qquad s \in S, (i,j) \in A^s \tag{18}$$

$$u_{km} \in \{0,1\} \qquad \forall k,m \in N, (k<m) \tag{19}$$

$$y_k \in \{0,1\} \qquad \forall k \in N \tag{20}$$

where $H_{ij}^s = \{k \in N : c_{ijkk} \le \beta_{ij}^s\}$ and $G_{ij}^s = \{(k,m) \in N \times N : k < m \;\; \& \;\; \min\{c_{ijkm}, c_{ijmk}\} \le \beta_{ij}^s \;\; \& \;\; \min\{c_{ijkk}, c_{ijmm}\} > \beta_{ij}^s\}$. Note that we can relax the integrality constraints on u_{km} variables as for any fixed binary solution for the y_k variables, the resulting LP always has an optimal solution with the u_{km} and v_{ij}^s variables binary corresponding to $u_{km} = \min\{y_k, y_m\}$ and $v_{ij}^s = \min\{1, \sum_{k\in H_{ij}^s} y_k + \sum_{(k,m)\in G_{ij}^s} u_{km}\}$. Therefore, we replace (19) with:

$$0 \le u_{km} \le 1 \qquad \forall k,m \in N, (k<m) \tag{21}$$

The above formulation can further be tightened by adding the following valid inequalities:

$$\sum_{\substack{m\in N\\ m>k}} u_{km} + \sum_{\substack{m\in N\\ m<k}} u_{mk} \le (p-1)y_k \qquad \forall k \in N \tag{22}$$

We call the models (11)–(12) and (13)–(18), (20)–(22) as MILP1 and MILP2, respectively and will use them as baseline for comparison with the proposed exact solution algorithms.

It is worth mentioning that when for every $s \in S$, $A^s = A$ (i.e., all O/D pairs are present in each stratum), then SpHCP and SpHMCP are equivalent to the original multiple allocation pHCP and pHMCP, respectively that belong the class of NP-hard problems (Ernst et al., 2009; Weng et al., 2006).

4. Solution algorithms

BD is an exact solution procedure for large-scale MIP models in which the problem is decomposed in such a way that the complicating variables form a master problem (MP) and the remaining variables constitute the subproblem (SP). The problem is then solved using a cutting plane approach where the cuts extracted from solving the SP are added to the MP. BD exploits the fact that computational burden of an MIP problem substantially increases with the problem size. Therefore, a divide-and-conquer scheme is employed to decompose a single large problem into smaller problems which can be solved more efficiently. Motivated by this fact, we apply Benders decomposition to SpHCP and SpHMCP. In classical implementation of Benders algorithm, we need to solve the master problem at each iteration. However, we use a modern implementation within a branch-and-cut (B&C) setting where the MP is solved in one attempt and the cuts are added on the fly whenever required by using the capabilities of the state-of-the-art solvers. We separate the Benders cuts as we visit a candidate integer solution in the B&C tree of the MP. By doing so, the computational effort needed to solve an integer problem at each iteration is considerably reduced.

4.1. Benders reformulation for the stratified p-hub center problem

By fixing the binary location variables vector as $\boldsymbol{y} = \hat{\boldsymbol{y}}$, the subproblem (SP) for SpHCP can be written as:

$$\min \sum_{s\in S} w^s z^s \tag{23}$$

$$\text{s.t.:} \sum_{k\in N}\sum_{m\in N} x_{ijkm}^s = 1 \qquad s \in S, (i,j) \in A^s \tag{24}$$

$$\sum_{m\in N} x_{ijkm}^s + \sum_{\substack{m\in N\\ m\ne k}} x_{ijmk}^s \le \hat{y}_k \qquad s \in S, (i,j) \in A^s, k \in N \tag{25}$$

$$z^s \ge \sum_{k\in N}\sum_{m\in N} c_{ijkm} x_{ijkm}^s \qquad s \in S, (i,j) \in A^s \tag{26}$$

$$x_{ijkm}^s \ge 0 \qquad s \in S, (i,j) \in A^s, k,m \in N \tag{27}$$

$$z^s \in \mathbb{R} \qquad s \in S \tag{28}$$

The subproblem (23)–(28) aims to minimize the weighted sum of the maximum travel time between O/D pairs within each stratum given that p hubs are installed in the network. Since every O/D flow within each stratum is routed via at least one open hub, there is at least one path (with a positive travel time) for routing the corresponding traffic. Hence, the SP is always feasible and bounded. Let σ^s_{ij}, π^s_{ijk}, and λ^s_{ij} be the dual variables associated with constraints (24), (25), and (26), respectively. Hence, the dual subproblem (DSP) for SpHCP solves the following linear programming problem:

$$\max \sum_{s\in S}\sum_{(i,j)\in A^s} \sigma^s_{ij} - \sum_{s\in S}\sum_{(i,j)\in A^s}\sum_{k\in N} \hat{y}_k \pi^s_{ijk} \tag{29}$$

$$\text{s.t.: } \sigma^s_{ij} - \pi^s_{ijk} - \pi^s_{ijm} \le c_{ijkm}\lambda^s_{ij} \qquad s \in S, (i,j) \in A^s, k, m \in N, (k \neq m) \tag{30}$$

$$\sigma^s_{ij} - \pi^s_{ijk} \le c_{ijkk}\lambda^s_{ij} \qquad s \in S, (i,j) \in A^s, k \in N \tag{31}$$

$$\sum_{(i,j)\in A^s} \lambda^s_{ij} = w^s \qquad s \in S \tag{32}$$

$$\pi^s_{ijk} \ge 0 \qquad s \in S, (i,j) \in A^s, k \in N \tag{33}$$

$$\lambda^s_{ij} \ge 0 \qquad s \in S, (i,j) \in A^s \tag{34}$$

$$\sigma^s_{ij} \in \mathbb{R} \qquad s \in S, (i,j) \in A^s \tag{35}$$

We can now write the master problem (MP) for SpHCP as follows:

$$\min \quad \theta \tag{36}$$

$$\text{s.t.: } \theta \ge \sum_{s\in S}\sum_{(i,j)\in A^s} \sigma^{s(v)}_{ij} - \sum_{s\in S}\sum_{(i,j)\in A^s}\sum_{k\in N} \pi^{s(v)}_{ijk} y_k \qquad v = 1,\dots,V \tag{37}$$

$$(3), (9) \tag{38}$$

$$\theta \ge 0 \tag{39}$$

in which $(\sigma^{(v)}, \pi^{(v)}, \lambda^{(v)})$ denotes the vth extreme point of the feasible solution space of the DSP. As noted above, with p installed hubs, the SP (23)–(28) is always feasible. Therefore, we do not need to add feasibility cuts to the MP.

Now we devise an algorithm for solving the DSP by inspection (without using a standard solver) which results in substantial savings in the solution time. To this end, we first determine the optimal values of the λ^s_{ij} variables. Due to the maximization in the objective function of the DSP (29)–(35) and according to constraints (32), we can conclude that for each stratum $s \in S$, the value of variable λ^s_{ij} associated with the longest travel time for that stratum will take the value of w^s and the remaining variables will have 0 value in an optimal solution. We present a procedure for determining the optimal values of the λ^s_{ij} variables in Algorithm 1.

Algorithm 1 : Determining the optimal values for λ^s_{ij} variables

for all $s \in S$ **do**
  $v_{max} \leftarrow -\infty$
  $(o_s, d_s) \leftarrow (0, 0)$
  for all $(i,j) \in A^s$ **do**
    if $v_{max} < \min_{k,m\in H^1}\{c_{ijkm}\}$ **then**
      $(o_s, d_s) \leftarrow (i, j)$
      $v_{max} \leftarrow \min_{k,m\in H^1}\{c_{ijkm}\}$
    end if
  end for
  $\lambda^{s*}_{o_s d_s} \leftarrow w^s$
  for all $(i,j) \in A^s \setminus \{(o_s, d_s)\}$ **do**
    $\lambda^{s*}_{ij} \leftarrow 0$
  end for
end for

In the above algorithm, H^1 is the set of nodes that are selected as hubs at the current iteration.

Proposition 1. *The values obtained for λ^s_{ij} by Algorithm 1 are optimal.*

Proof. Let (o_s, d_s) be the O/D pair with the largest value of travel time within stratum $s \in S$ that is obtained using the installed hubs based on $\hat{y}$. Moreover, let A^s_0 denote the set of remaining O/D pairs (i.e., $A^s_0 = A^s \setminus \{(o_s, d_s)\}$). Therefore, for all $s \in S$, constraints (26) hold as equality for $(i,j) = (o_s, d_s)$ and hold as strict inequality for all $(i,j) \in A^s_0$. In other words, we can rewrite the constraints (26) as follows:

$$z^s = \sum_{k\in N}\sum_{m\in N} c_{o_s d_s km} x^s_{o_s d_s km} \qquad s \in S \tag{40}$$

$$z^s > \sum_{k\in N}\sum_{m\in N} c_{ijkm}x^s_{ijkm} \qquad s \in S, (i,j) \in A^s_0 \tag{41}$$

Due to complementary slackness conditions in duality theory, the dual variables associated with (41) are equal to zero (i.e., $\lambda^s_{ij} = 0$ for all $s \in S, (i,j) \in A^s_0$). Now, based on constraints (32), in any optimal solution of the DSP, we have:

$$\sum_{(i,j)\in A^s_0} \lambda^s_{ij} + \lambda^s_{o_s d_s} = w^s \qquad s \in S \tag{42}$$

that implies $\lambda^s_{o_s d_s} = w^s$. □

Having obtained the optimal values for the λ^s_{ij} variables, the DSP for SpHCP can now be reduced to the following linear programming problem:

$$\max \quad \sum_{s\in S}\sigma^s_{o_s d_s} - \sum_{s\in S}\sum_{k\in N}\hat{y}_k \pi^s_{o_s d_s k} \tag{43}$$

$$\text{s.t.: } \sigma^s_{o_s d_s} - \pi^s_{o_s d_s k} - \pi^s_{o_s d_s m} \le c_{o_s d_s km} w^s \qquad s \in S, k, m \in N, (k \ne m) \tag{44}$$

$$\sigma^s_{o_s d_s} - \pi^s_{o_s d_s k} \le c_{o_s d_s kk} w^s \qquad s \in S, k \in N \tag{45}$$

$$\pi^s_{o_s d_s k} \ge 0 \qquad s \in S, k \in N \tag{46}$$

$$\sigma^s_{o_s d_s} \in \mathbb{R} \qquad s \in S \tag{47}$$

Note that the reduced DSP corresponding to O/D pairs $(i,j) \in A^s_0$ has the optimal values $\sigma^s_{ij} = \pi^s_{ijk} = 0$ for all $(i,j) \in A^s_0, k \in N$ and hence these are not needed for deriving the optimality cuts. The model (43)–(47) is the dual of:

$$\min \sum_{s\in S}\sum_{k\in N}\sum_{m\in N} c_{o_s d_s km} w^s x^s_{o_s d_s km} \tag{48}$$

$$\text{s.t.: } \sum_{k\in N}\sum_{m\in N} x^s_{o_s d_s km} = 1 \qquad s \in S \tag{49}$$

$$\sum_{m\in N} x^s_{o_s d_s km} + \sum_{\substack{m\in N\\ m\ne k}} x^s_{o_s d_s mk} \le \hat{y}_k \qquad s \in S, k \in N \tag{50}$$

$$x^s_{o_s d_s km} \ge 0 \qquad s \in S, k, m \in N \tag{51}$$

This problem can be solved by inspection as a shortest path problem and since it is feasible and bounded, the corresponding optimal dual variables $\sigma^s_{o_s d_s}$ and $\pi^s_{o_s d_s k}$ for all $s \in S, k \in N$ can be determined using the complementary slackness conditions as explained by Contreras et al. (2011a). Our master problem can now be rewritten as follows:

(MP1-S*p*HCP)

$$\max \quad \theta \tag{52}$$

$$\text{s.t.: } \theta \ge \sum_{s\in S}\sigma^{s(v)}_{o_s d_s} - \sum_{s\in S}\sum_{k\in N}\pi^{s(v)}_{o_s d_s k} y_k \qquad v = 1,\dots,V \tag{53}$$

(3), (9), (39)

Note that for each stratum $s \in S$, dual variables corresponding to only one O/D pair that has the maximum travel time (i.e., (o_s, d_s)) are used in constructing the **MP1-S*p*HCP** as the values of the remaining dual variables are zero. Therefore, **MP1-S*p*HCP** can be solved much more efficiently than the original MP (36)–(39). Based on the fact that the SP for SpHCP can be disaggregated for each $s \in S$, we can assemble multiple cuts (one for each stratum) to be added to the MP together. Doing so, we can write a new multi-cut version of the master problem for SpHCP as follows:

(MP2-S*p*HCP)

$$\max \quad \sum_{s\in S}\theta^s \tag{54}$$

$$\text{s.t.: } \theta^s \ge \sigma^{s(v)}_{o_s d_s} - \sum_{k\in N}\pi^{s(v)}_{o_s d_s k} y_k \qquad s \in S, v = 1,\dots,V \tag{55}$$

(3), (9)

$$\theta^s \ge 0 \qquad s \in S \tag{56}$$

4.2. Benders reformulation for the stratified p-hub maximal covering problem

By fixing the binary location variables vector as $\boldsymbol{y} = \hat{\boldsymbol{y}}$, in the model MILP1 (11)–(12), the SP for SpHMCP can be written as:

$$\max \sum_{s\in S} w^s\left(\sum_{(i,j)\in A^s}\sum_{k\in N}\sum_{m\in N} a^s_{ijkm} t^s_{ij} x^s_{ijkm}\right) \tag{57}$$

$$\text{s.t.:} \sum_{k\in N}\sum_{m\in N} x^s_{ijkm} = 1 \qquad s\in S, (i,j)\in A^s \tag{58}$$

$$\sum_{m\in N} x^s_{ijkm} + \sum_{\substack{m\in N\\ m\neq k}} x^s_{ijmk} \le \hat{y}_k \qquad s\in S, (i,j)\in A^s, k\in N \tag{59}$$

$$x^s_{ijkm} \ge 0 \qquad s\in S, (i,j)\in A^s, k,m\in N \tag{60}$$

Let φ^s_{ij} and μ^s_{ijk} be the dual variable associated with constraints (58) and (59). The DSP for SpHMCP can then be formulated as follows:

$$\min \sum_{s\in S}\sum_{(i,j)\in A^s} \varphi^s_{ij} + \sum_{s\in S}\sum_{(i,j)\in A^s}\sum_{k\in N} \hat{y}_k \mu^s_{ijk} \tag{61}$$

$$\text{s.t.:}\ \varphi^s_{ij} + \mu^s_{ijk} + \mu^s_{ijm} \ge w^s a^s_{ijkm} t^s_{ij} \qquad s\in S, (i,j)\in A^s, k,m\in N, (k\neq m) \tag{62}$$

$$\varphi^s_{ij} + \mu^s_{ijk} \ge w^s a^s_{ijkk} t^s_{ij} \qquad s\in S, (i,j)\in A^s, k\in N \tag{63}$$

$$\mu^s_{ijk} \ge 0 \qquad s\in S, (i,j)\in A^s, k\in N \tag{64}$$

$$\varphi^s_{ij} \in \mathbb{R} \qquad s\in S, (i,j)\in A^s \tag{65}$$

We can now write the MP for SpHMCP as follows:

(MP1-SpHMCP)

$$\max \quad \gamma \tag{66}$$

$$\text{s.t.:}\ \gamma \le \sum_{s\in S}\sum_{(i,j)\in A^s} \varphi^{s(v)}_{ij} + \sum_{s\in S}\sum_{(i,j)\in A^s}\sum_{k\in N} \mu^{s(v)}_{ijk} y_k \qquad v=1,\dots,V \tag{67}$$

(3), (9)

$$\gamma \ge 0 \tag{68}$$

In order to solve the DSP for SpHMCP we propose a two-phase algorithm that derives the optimal values of dual variables by inspection, without using a standard solver. Our approach for breaking the subproblem into two phases is analogous to the idea of approximating the Pareto optimal cuts (Magnanti and Wong, 1981). In Phase I, we obtain the optimal value of the subproblem, whereas in Phase II, we strengthen the cut while preserving the optimality and feasibility of the solution.

At a given iteration of the BD algorithm, we obtain an optimal solution $\hat{y}$ from MP. Let $H^1 = \{k\in N : \hat{y}_k = 1\}$ be the set of open hubs, and let $H^0 = \{k\in N : \hat{y}_k = 0\}$ be the set of closed hubs. Note that any feasible value of μ^s_{ijk} would be optimal when $k\in H^0$. Hence, we can solve the subproblem in two phases. In Phase I, we remove the variables μ^s_{ijk} and their corresponding constraints from the DSP associated with $k\in H^0$ and compute the values of the remaining variables by solving the following Phase I subproblem:

$$\min \sum_{s\in S}\sum_{(i,j)\in A^s} \varphi^s_{ij} + \sum_{s\in S}\sum_{(i,j)\in A^s}\sum_{k\in H^1} \mu^s_{ijk} \tag{69}$$

$$\text{s.t.:}\ \varphi^s_{ij} + \mu^s_{ijk} + \mu^s_{ijm} \ge w^s a^s_{ijkm} t^s_{ij} \qquad s\in S, (i,j)\in A^s, k,m\in H^1, (k\neq m) \tag{70}$$

$$\varphi^s_{ij} + \mu^s_{ijk} \ge w^s a^s_{ijkk} t^s_{ij} \qquad s\in S, (i,j)\in A^s, k\in H^1 \tag{71}$$

$$\mu^s_{ijk} \ge 0 \qquad s\in S, (i,j)\in A^s, k\in H^1 \tag{72}$$

$$\varphi^s_{ij} \in \mathbb{R} \qquad s\in S, (i,j)\in A^s \tag{73}$$

Note that the φ^s_{ij} variables are independent from k; accordingly, solving DSP-I results in obtaining the optimal values of all φ^s_{ij} variables. Hence, in Phase II, we find feasible values of μ^s_{ijk} for $k\in H^0$ with respect to constraints (62)–(63) to strengthen the cut. The model (69)–(73) is the dual of:

$$\min \sum_{s\in S}\left(\sum_{(i,j)\in A^s}\sum_{k\in H^1}\sum_{m\in H^1} c_{ijkm} w^s x^s_{ijkm}\right) \tag{74}$$

$$\text{s.t.:} \sum_{k\in H^1}\sum_{m\in H^1} x^s_{ijkm} = 1 \qquad s\in S, (i,j)\in A^s \tag{75}$$

$$\sum_{m\in H^1} x^s_{ijkm} + \sum_{\substack{m\in H^1\\ m\neq k}} x^s_{ijmk} \le 1 \qquad s\in S, (i,j)\in A^s, k\in H^1 \tag{76}$$

$$x^s_{ijkm} \ge 0 \qquad s\in S, (i,j)\in A^s, k,m\in H^1 \tag{77}$$

Constraints (76) are dominated by (75) for every $k\in H^1$ and thus can be removed entirely from the dual problem. Accordingly, it is optimal to set the dual variables associated with (76) to zero, that is, $\mu^s_{ijk} = 0$ for all $k\in H^1$. By replacing variables μ^s_{ijk} with zero in the DSP1 (69)–(73), we can obtain the optimal values for φ_{ij}^s as follows:

$$\varphi_{ij}^s = \max_{k,m\in H^1}\{w^s t_{ij}^s a_{ijkm}^s\} \qquad s\in S,(i,j)\in A^s. \tag{78}$$

Updating DSP by fixing the value of the computed variables and removing the already satisfied constraints, we get the following Phase II subproblem:

$$\min \sum_{s\in S}\sum_{(i,j)\in A^s}\sum_{k\in H^0}\mu_{ijk}^s \tag{79}$$

$$\text{s.t.: } \mu_{ijk}^s + \mu_{ijm}^s \geq B_{ijkm}^s \qquad s\in S,(i,j)\in A^s,k,m\in H^0,(k\neq m) \tag{80}$$

$$\mu_{ijk}^s \geq B_{ijkm}^s \qquad s\in S,(i,j)\in A^s,k\in H^0,m\in H^1 \tag{81}$$

$$\mu_{ijk}^s \geq B_{ijmk}^s \qquad s\in S,(i,j)\in A^s,k\in H^0,m\in H^1 \tag{82}$$

$$\mu_{ijk}^s \geq B_{ijkk}^s \qquad s\in S,(i,j)\in A^s,k\in H^0 \tag{83}$$

$$\mu_{ijk}^s \geq 0 \qquad s\in S,(i,j)\in A^s,k\in H^0 \tag{84}$$

where $B_{ijkm}^s = w^s a_{ijkm}^s t_{ij}^s - \varphi_{ij}^s$ for all $s\in S,(i,j)\in A^s,k,m\in N$. Note that if we only consider the constraints (83) and (84), we have:

$$\mu_{ijk}^s \leftarrow \max\{0,B_{ijkk}^s\} \qquad s\in S,(i,j)\in A^s,k\in H^0 \tag{85}$$

which represent the paths with only one closed hub $k\in H^0$. If we further consider the paths with one closed hub $k\in H^0$ and one open hub $m\in H^1$ by adding constraints (81) and (82), the optimal values will become:

$$\mu_{ijk}^s \leftarrow \max\left\{\mu_{ijk}^s,\max\{B_{ijkm}^s,B_{ijmk}^s\}\right\} \qquad s\in S,(i,j)\in A^s,k\in H^0,m\in H^1. \tag{86}$$

Finally, by adding constraints (80) we also consider the paths with two distinct closed hubs $k,m\in H^0,(k\neq m)$. In this case if the constraints (80) are violated, (i.e., if $\Delta = B_{ijkm}^s - \mu_{ijk}^s - \mu_{ijm}^s > 0$), we increase the sum of variables μ_{ijk}^s and μ_{ijm}^s in the left-hand side of (80) by Δ as follows:

$$\mu_{ijk}^s \leftarrow \mu_{ijk}^s + \omega\Delta$$

$$\mu_{ijm}^s \leftarrow \mu_{ijm}^s + (1-\omega)\Delta$$

in which ω is a constant value within the interval $[0,1]$.

The pseudo-code for the proposed two-phase procedure is shown in Algorithm 2.

Algorithm 2 : Two-phase procedure for determining the optimal values for φ_{ij}^s and μ_{ijk}^s variables

for all $s\in S$ **do**
  for all $(i,j)\in A^s$ **do**
    Phase I
    $\varphi_{ij}^s \leftarrow \max_{k,m\in H^1}\{w^s t_{ij}^s a_{ijkm}^s\}$
    for all $k\in H^1$ **do**
      $\mu_{ijk}^s \leftarrow 0$
    end for
    Phase II
    for all $k\in H^0$ **do**
      $\mu_{ijk}^s \leftarrow \max\{0,B_{ijkk}^s\}$
      for all $m\in H^1$ **do**
        $\mu_{ijk}^s \leftarrow \max\left\{\mu_{ijk}^s,\max\{B_{ijkm}^s,B_{ijmk}^s\}\right\}$
      end for
    end for
    for all $(k,m)\in H^0\times H^0,(m\neq k)$ **do**
      $\Delta \leftarrow B_{ijkm}^s - \mu_{ijk}^s - \mu_{ijm}^s$
      if $\Delta > 0$ **then**
        $\mu_{ijk}^s \leftarrow \mu_{ijk}^s + \omega\Delta$
        $\mu_{ijm}^s \leftarrow \mu_{ijm}^s + (1-\omega)\Delta$
      end if
    end for
  end for
end for

Note that as in the case of the stratified p-hub center problem, the SP for SpHMCP can be separated for each $s\in S$. Nevertheless, our preliminary computational tests showed that this scheme for cut disaggregation does not perform well in terms of solution time reduction. On the other hand, it can be seen that the SP can also be separated for each O/D pair $(i,j) \in A$. However, as shown by de Camargo et al. (2008), if we add $|A|$ cuts per iteration of the Benders algorithm, the time saved as a result of the reduction in the number of iterations cannot compensate for the increased computational time for solving the master problem, even for instances of small sizes. Therefore, rather than adding $|A|$ cuts at each iteration, we can aggregate the information and add cuts associated with subsets of O/D pairs. In particular, for each node $i \in N$, we can add a cut corresponding to the O/D pairs originating from node i. Therefore, we can separate the subproblem into $|N|$ independent subproblems, one for each node $i \in N$. Hence, the master problem for the multi-cut version of the proposed Benders reformulation can be stated as follows:

(MP2-SpHMCP)

$$\max \quad \sum_{i\in N} \gamma_i \tag{87}$$

$$\text{s.t.: } \gamma_i \le \sum_{s\in S}\sum_{j\in N} \varphi_{ij}^{s(v)} + \sum_{s\in S}\sum_{j\in N}\sum_{k\in N} \mu_{ijk}^{s(v)} y_k \qquad v = 1,\dots,V \tag{88}$$

(3), (9)

$$\gamma_i \ge 0 \qquad i \in N. \tag{89}$$

For each $i \in N$, an upper bound UB_i on the value of γ_i can be obtained by allowing every node to act as a hub as follows:

$$UB_i = \sum_{s\in S}\sum_{j\in N} a_{ijij}^s w^s t_{ij}^s$$

Applying these upper bounds on the value of the γ_i variables has shown to be effective in reducing the solution time of the algorithm.

5. Computational experiments

In this section, we present first in Section 5.1 some optimal solutions to illustrate how the stratified demand impacts the optimal hub network, including hub locations and traffic routing. Then in Section 5.2 we provide extensive computational results for problems with up to 200 nodes and 10 strata.

5.1. An illustrative example

This section presents an illustrative example to show the effect of having different strata with varying levels of importance on the optimal hub network. To this end, we use the CAB data set (O'Kelly, 1987) which is based on air passenger traffic data and assume that we have three classes of passengers, called *economy class*, *business class*, and *first class*, that have different levels of willingness to pay and hence are of different importance levels for the airline. The first class passengers are very sensitive to the service level and hence desire their travel time to be as short as possible. They also provide the highest profit margin to the airline company. In contrast, the economy class passengers are not very sensitive to travel time and tend to pay lower prices for their trips. The business class passengers are assumed to be moderately sensitive to both travel time and cost. We assume that economy class passengers exist for all O/D pairs between the 25 cities of the CAB data set. However, we generate the O/D matrices for the business class and first class passengers randomly as shown in Fig. 1, so that business class and the first class passengers exist for only 50.33% and 20.66% of the O/D pairs, respectively. The demand matrices are all symmetric and the weights (w_s) for the economy, business, and first class strata are set to be 0.1, 0.3, and 0.6, respectively. For our illustration, we show optimal solutions with the number of hubs (p) equal to 4 and the discount factor (α) at 0.6 (i.e., inter-hub travel speed is 67% faster than the speed not between two hubs).

We first solve the uncapacitated p-hub center problem for each stratum separately, and then solve the problem considering all the strata together as SpHCP. The optimal solutions (hubs and hub arcs) for individual strata as well as the aggregate problem are depicted in Fig. 2. We report the objective function values for the four different problems as a quadruplet under each map. The first number is the objective function value for the p-hub center problem considering only the O/D pairs present in the first stratum given the set of hubs shown in the corresponding map. The second and the third numbers show the objective function values for the second and the third strata, respectively. Finally, the last number gives the objective function value for the stratified p-hub center problem consisting of all the three strata using the given weights.

As can be seen in these maps, when we consider only the first stratum (i.e., the economy class passengers; Fig. 2-a) the optimal location of hubs are at nodes 1 (Atlanta), 2 (Baltimore), 12 (Los Angeles), and 23 (Seattle). For the business class passengers (Fig. 2-b), the optimal hub set consists of four other cities: 5 (Cincinnati), 8 (Denver), 18 (Philadelphia), and 24 (Tampa). For the first class passengers (Fig. 2-c), the optimal hub set includes three new cities: 6 (Cleveland), 7 (Dallas-Fort Worth), and 22 (San Francisco), along with 8 (Denver) which was also optimal for business class passengers. Finally, the optimal solution for the stratified p-hub center problem (Fig. 2-d) includes cities 6 (Cleveland), 8 (Denver), 16 (New Orleans), and 22 (San Francisco) as the hub nodes. We observe that the optimal hub set for SpHCP is very similar to the optimal set of hubs for the first class passengers, as three out of four hubs (i.e., Cleveland, Denver, and San Francisco) are the same in the two solutions. This is due to large weight value of the third stratum ($w_3 = 0.6$) compared to the first two strata ($w_1 = 0.1$, $w_2 = 0.3$).

Since economy class passengers exist for all O/D pairs between the 25 cities of the CAB data set, the solution for the economy class passengers (Fig. 2-a) is identical to that of the traditional p-hub center problem (pHCP). Hence, we can see that the optimal

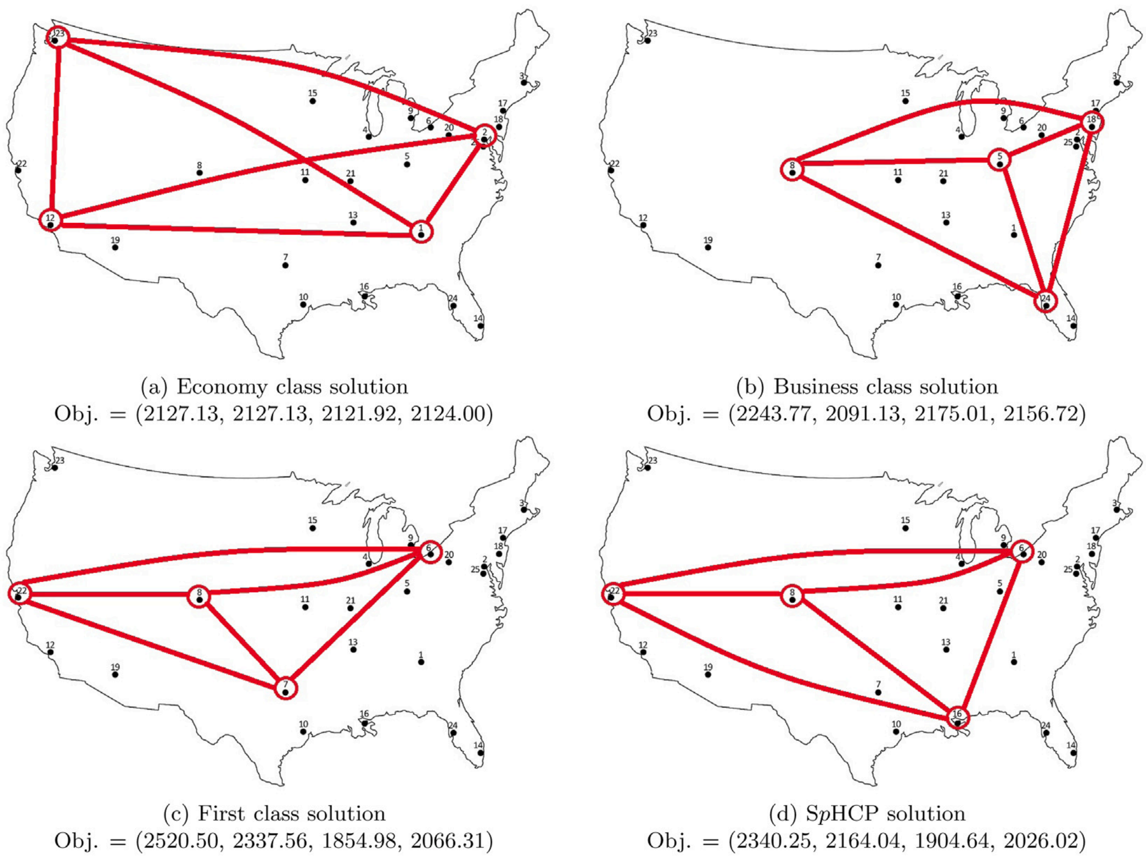
0	1	1	1	1	0	0	1	1	0	1	0	0	0	1	0	0	1	1	1	0	1	0	1	0
1	0	0	0	0	0	1	0	1	0	1	1	1	0	1	0	1	0	0	1	0	1	0	1	0
1	0	0	0	0	1	0	0	0	1	0	1	0	0	0	1	1	1	1	0	0	0	0	1	0
1	0	0	0	0	1	1	1	0	0	1	1	0	0	1	0	1	0	0	0	1	1	1	0	0
1	0	0	0	0	0	1	1	0	0	1	0	1	1	1	1	1	0	1	1	0	1	0	0	0
0	0	1	1	0	0	0	1	0	1	1	0	0	0	1	0	0	0	0	0	0	0	1	1	1
0	1	0	1	1	0	0	1	0	1	1	0	1	1	0	1	0	1	0	0	1	1	0	0	1
1	0	0	1	1	1	1	0	0	0	1	0	1	0	1	1	0	0	1	1	0	1	0	0	1
1	1	0	0	0	0	0	0	0	1	1	1	0	0	0	1	0	1	0	1	0	1	1	1	1
0	0	1	0	0	1	1	0	1	0	1	0	1	1	0	1	0	0	0	0	0	1	1	1	0
1	1	0	1	1	1	1	1	1	1	0	1	0	0	0	1	1	1	0	1	0	1	1	0	0
0	1	1	1	0	0	0	0	1	0	1	0	1	1	1	1	0	1	0	1	1	1	1	1	1
0	1	0	0	1	0	1	1	0	1	0	1	0	0	0	0	1	0	1	1	0	0	1	1	1
0	0	0	0	1	0	1	0	0	1	0	1	0	0	0	1	1	0	0	0	1	1	0	0	1
1	1	0	1	1	1	0	1	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	1	0
0	0	1	0	1	0	1	1	1	1	1	1	0	1	0	0	0	1	1	1	1	0	1	0	0
0	1	1	1	1	0	0	0	0	0	1	0	1	1	0	0	0	1	1	0	1	1	1	1	1
1	0	1	0	0	0	1	0	1	0	1	1	0	0	0	1	1	0	1	1	0	1	1	0	1
1	0	1	0	1	0	0	1	0	0	0	0	1	0	1	1	1	1	0	0	0	0	1	0	1
1	1	0	0	1	0	0	1	1	0	1	1	1	0	0	1	0	1	0	0	1	0	1	0	0
0	0	0	1	0	0	1	0	0	0	0	1	0	1	0	1	1	0	0	1	0	0	0	1	0
1	1	0	1	1	0	1	1	1	1	1	1	0	1	0	0	1	1	0	0	0	0	1	1	1
0	0	0	1	0	1	0	0	1	1	1	1	1	0	0	1	1	1	1	1	0	1	0	0	1
1	1	1	0	0	1	0	0	1	1	0	1	1	0	1	0	1	0	0	0	1	1	0	0	0
0	0	0	0	0	1	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	0	0

Business class

0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	1	1	1	0	0	1	0	0	1
0	0	0	0	0	0	0	0	1	0	0	0	0	1	1	0	0	1	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	1	0	0	1	0	1	0	1	1
0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	1	0	0	1	0	0	1	0	1	0	1	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	1	0	0	0
0	1	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	1	0	1	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	1	0	0
1	0	1	0	1	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	1	0	1	0	1	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	1	0	1	0	1	1	0	0
0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
1	1	0	1	0	0	1	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	1	1	0
1	0	1	0	0	0	0	0	0	0	0	0	1	1	0	1	0	0	0	0	0	0	0	0	1
1	1	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0
1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	1
0	0	1	0	0	0	1	0	0	0	0	0	1	0	0	1	0	0	1	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	1	0	1	1	0	1	0	1	1	0	0	0	0	1	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	1	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	1
1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	1	0

First class

Fig. 1. O/D matrices for the business class and the first class passengers.



(a) Economy class solution
Obj. = (2127.13, 2127.13, 2121.92, 2124.00)

(b) Business class solution
Obj. = (2243.77, 2091.13, 2175.01, 2156.72)

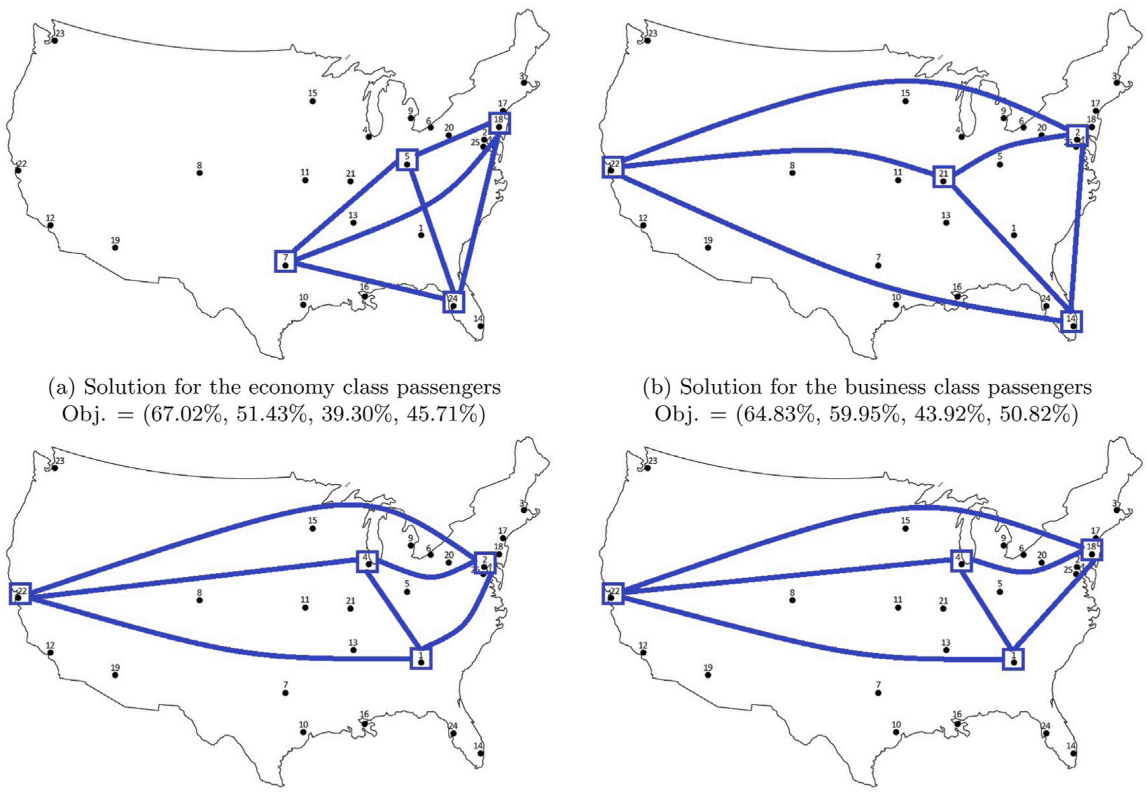
(c) First class solution
Obj. = (2520.50, 2337.56, 1854.98, 2066.31)

(d) SpHCP solution
Obj. = (2340.25, 2164.04, 1904.64, 2026.02)

Fig. 2. Optimal solutions for SpHCP with $p = 4$ and $\alpha = 0.6$.

hub network (i.e., set of hub locations and hub arcs) for the stratified p-hub center problem (Fig. 2-d) is very different from that of the pHCP (Fig. 2-a), as none of the four optimal hub locations for SpHCP are optimal hubs for the pHCP.

Note that using the optimal network for a single class of passengers is 2%-6.5% worse (in terms of the objective for the stratified model) than using the optimal solution for the stratified model (in Fig. 2-d). While the stratified solution is not optimal for any one class of passengers (e.g., it is 10% worse for economy passengers than the optimal network for economy passengers alone in 2-a), it represents a balance of the competing forces from the individual optimal networks Note that both Figs. 2-a and 2-d include traffic between all O/D pairs, yet the optimal hub locations are routes are quite different. For example, for O/D pair (3,10) the travel path in Fig. 2-a, which reflects an equal weighting for all O/D pairs, is 3 → 2 → 1 → 10, while the path in Fig. 2-d is 3 → 6 → 16 → 10.



(a) Solution for the economy class passengers
Obj. = (67.02%, 51.43%, 39.30%, 45.71%)

(b) Solution for the business class passengers
Obj. = (64.83%, 59.95%, 43.92%, 50.82%)

(c) Solution for the first class passengers
Obj. = (59.09%, 49.55%, 53.36%, 52.79%)

(d) Solution for SpHMCP
Obj. = (59.01%, 50.48%, 53.36%, 53.06%)

Fig. 3. Optimal solutions for SpHMCP with $p = 4$ and $\alpha = 0.6$.

Interestingly, node 6 which is a hub in SpHCP (Fig. 2-d) and for the first class passengers (Fig. 2-c) has no originating or terminating demand of first-class passengers; yet it is an effective hub for the six rather clustered nodes to its right (east) (nodes 2, 3, 17, 18, 20, and 25) and nearby nodes 5 and 9. The travel paths for O/D pair (3,10) are also different in Figs. 2-c and 2-d (3 → 18 → 5 → 10 and 3 → 6 → 7 → 10, respectively), in part due to the sparse demand patterns for business and first-class passengers (See Fig. 1).

The results for the maximal covering case are presented in Fig. 3. Here, we have assumed the covering radius of 1000, 800, and 600 for the economy class, the business class, and the first class passengers, respectively. Here also we can observe that due to large value of the weight corresponding to the third stratum, the optimal hub set for SpHMCP is very similar to the optimal set of hubs for the first class passengers. As can be seen, three out of four hubs (i.e., Atlanta (1), Chicago (4), and San Francisco (22)) are the same in the two solutions. However, as shown in the hub center case, the optimal set of hub locations for the stratified p-hub maximal covering problem (Fig. 3-d) is quite dissimilar to that of the traditional p-hub maximal covering problem (Fig. 3-a). Note that using the optimal network for a single class of passengers is 0.05%–14% worse (in terms of the objective for the stratified model SpHMCP) than using the optimal solution for the stratified model (in Fig. 3-d). While the stratified solution provides optimal coverage for the first class passengers, it provides 12%–16% worse coverage for the economy and business class passengers, respectively, compared to the optimal network for that class of passengers alone. Note that the optimal set of hubs for SpHMCP is quite different from that of SpHCP. As can be seen in Figs. 2-d and 3-d from the four installed hubs in each solution, only the hub 22 is common in the two solutions.

5.2. Numerical results

In this section, we conduct an extensive set of computational tests to demonstrate the efficiency of the proposed BD algorithms. We use three well-known data sets available in the literature of the HLP, namely the CAB, the TR, and the AP data sets. The CAB data set is from air passenger traffic data between 25 US cities in 1970 (O'Kelly, 1987). The TR data set is based on the cargo flows between 81 cities of Turkey (Tan and Kara, 2007), whereas the Australia Post (AP) data set (Ernst and Krishnamoorthy, 1996) includes 200 nodes in the postal network of Sydney, Australia. In our experiments, we use instances of different sizes as $|N| \in \{25, 50, 75, 100, 150, 200\}$ from the AP data set. We set the values of the parameter α at four levels as: $\alpha \in \{0.2, 0.4, 0.6, 0.8\}$. Furthermore, four values are used as the number of open hubs: $p \in \{2, 3, 4, 5\}$.

We generate a total of $|S| = 10$ independent strata for each problem instance. For each stratum $s \in S$, similar to Albareda-Sambola et al. (2019), we go through the following steps to determine the O/D pairs which are included in that stratum. First, a random number $\xi^s_{ij} \in (0, 1)$ is generated for each $(i, j) \in A$ and $s \in S$. Then we generate another random number $q^s \in [0, 1)$. If

Table 1
Results for the Stratified p-hub center problem with the CAB data set.

p	α	Opt	Hubs	MILP	BD1		BD2	
				CPU(s)	CPU(s)	# cuts	CPU(s)	# cuts
2	0.2	2021.47	5, 22	262.07	0.10	14	0.11	100
	0.4	2331.78	5, 8	211.10	0.04	11	0.06	120
	0.6	2531.47	8, 21	125.91	0.03	12	0.04	80
	0.8	2663.28	6, 8	113.91	0.03	18	0.03	103
3	0.2	1758.90	9, 16, 22	286.71	0.05	28	0.09	261
	0.4	2022.50	5, 12, 23	311.23	0.08	27	0.07	160
	0.6	2239.26	8, 20, 24	291.91	0.05	28	0.04	143
	0.8	2499.39	6, 8, 24	90.94	0.05	34	0.06	183
4	0.2	1601.98	12, 13, 23, 25	352.02	0.09	45	0.15	249
	0.4	1768.61	9, 12, 16, 23	335.25	0.05	34	0.09	314
	0.6	2102.38	12, 16, 20, 23	314.08	0.06	48	0.07	290
	0.8	2420.21	6, 8, 23, 24	237.20	0.06	51	0.05	230
5	0.2	1283.80	11, 12, 23, 24, 25	301.26	0.08	48	0.07	152
	0.4	1596.65	11, 12, 14, 18, 23	314.48	0.07	50	0.11	267
	0.6	1909.05	1, 18, 19, 22, 23	212.39	0.05	41	0.09	279
	0.8	2270.80	6, 12, 22, 23, 24	72.53	0.03	25	0.03	135
Average				239.56	0.06		0.07	

$q^s < \xi^s_{ij}$, we set $(i,j) \in A^s$ and otherwise, $(i,j) \notin A^s$. Furthermore, if $(i,j) \in A^s$, the corresponding traffic volume t^s_{ij} is set to be the same as the associated traffic value in the original demand matrix of the respective data sets. The traffic values (t^s_{ij}) are then scaled in such a way that the sum of traffic associated with each stratum equals 1. We assume that the corresponding weights for the strata are equal (i.e., the weight of each stratum is 1/10). For the maximal covering problem, the covering radius β^s_{ij} is set as $\beta^s_{ij} = 500 + 100s, \forall (i,j) \in A^s$ and $s = 1, \dots, 10$ for the CAB data set. For the TR and the AP data sets, the corresponding values are $\beta^s_{ij} = 500 + 50s$ and $\beta^s_{ij} = 2000 + 2000s$, respectively. We use the value of parameter ω as 0.5 in the proposed two-phase procedure for solving the subproblems in all of our experiments.

The proposed mathematical models as well as the Benders algorithms are coded in JAVA and are solved by CPLEX version 12.10. All the experiments have been run on a computer with Intel(R) Core(TM) i3-3220 CPU of 3.30 GHz and 16 GB of RAM, using the Microsoft Windows 7 operating system. The Benders algorithm is implemented within a branch-and-cut framework, where the master problem is solved on a single tree and the optimality cuts are added one at a time using the lazy constraint callback function available in CPLEX. The time limit of five hours is used in our experiments.

5.2.1. Results for SpHCP

Table 1 shows the results obtained by solving the stratified p-hub center problem with the CAB data set. The number of hubs and the value of the discount factor are presented in the columns entitled p and α, respectively. The next two columns show the optimal objective function value and the set of opened hubs. The next columns show the solution results for the MILP model and the two Benders decomposition algorithms proposed for solving the stratified p-hub center problem. In the first Benders algorithm (BD1), we add a single cut at a time to the master problem as in **MP1-SpHCP**. The second algorithm (BD2) uses the cut disaggregation scheme and adds $|S|$ cuts at a time to the MP as in **MP2-SpHCP**. The computational efficiency of different solution approaches can be compared in terms of solution times which are presented (in seconds) for each solution method in the columns labeled as "CPU(s)". The number of Benders cuts added in BD1 and BD2 until the optimal solution or the best solution obtained within the time limit is presented in the columns labeled as "# cuts". The cuts are added whenever an incumbent solution is found and the cut pool is managed by the solver. The results reported in Table 1 show that the proposed BD algorithms are capable of obtaining the optimal solution for all the CAB instances in a fraction of a second, whereas the average time taken by CPLEX to solve the MILP models is around 240 s. As expected, for a fixed value of p, as the value of the discount factor (α) increases, the optimal objective value increases; and for a given value of the discount factor, as p increases the longest transportation time decreases.

Table 2 shows the results obtained by solving the stratified p-hub center problem with the TR data set. Due to the large size of the instances in the TR data set, the MILP model could not be solved using the standard solver due to excessive memory requirements. This is indicated by the word "Memory" in Table 2. Nevertheless, as can be observed, both of the proposed Benders algorithms solve the instances in short computational times. The average time spent by the single-cut and multi-cut algorithms are around 10 and 20 s, respectively.

In order to present the solution results for the AP data set, we divide the corresponding instances into three groups based on their sizes. The instances with 25 and 50 nodes ($|N| = 25$ and 50) are grouped together and called small instances. The instances of 75 and 100 nodes are categorized as medium instances and the ones with 150 and 200 nodes are called large instances. The results obtained by solving the stratified p-hub center problem for the small instances of the AP data set are reported in Table 3. Note that only two of the instances with 50 nodes could be solved using the standard solver within the allowed time limit of 5 h. For the instances not solved within the time limit we present the gap percentage between the corresponding upper and lower bounds which

Table 2
Results for the Stratified p-hub center problem with the TR data set.

p	α	Opt	Hubs	MILP	BD1		BD2	
				CPU(s)	CPU(s)	# cuts	CPU(s)	# cuts
2	0.2	1394.50	26, 62	Memory	0.34	28	0.34	210
	0.4	1534.20	12, 26	Memory	0.10	22	0.19	229
	0.6	1685.60	6, 49	Memory	0.11	21	0.10	130
	0.8	1839.06	6, 56	Memory	0.06	16	0.16	170
3	0.2	1245.84	4, 26, 58	Memory	1.33	85	3.70	519
	0.4	1416.00	3, 12, 81	Memory	1.20	103	2.55	489
	0.6	1572.90	12, 20, 41	Memory	0.43	58	0.88	315
	0.8	1778.22	3, 34, 49	Memory	0.33	62	0.56	326
4	0.2	1110.22	4, 20, 41, 58	Memory	8.80	128	29.59	755
	0.4	1274.94	3, 49, 59, 60	Memory	5.23	127	10.96	658
	0.6	1464.18	3, 25, 59, 73	Memory	1.33	65	1.90	330
	0.8	1721.90	25, 34, 64, 73	Memory	1.01	98	0.73	412
5	0.2	1016.08	13, 29, 38, 41, 64	Memory	113.55	295	216.43	1210
	0.4	1184.40	3, 25, 40, 56, 59	Memory	22.76	149	50.04	1247
	0.6	1417.32	3, 12, 24, 59, 73	Memory	5.62	122	8.75	651
	0.8	1690.12	13, 24, 30, 59, 64	Memory	2.21	120	1.03	344
Average				–	10.27		20.49	

Table 3
Results for the Stratified p-hub center problem with the AP data set (small instances).

$\|N\|$	p	α	Opt	Hubs	MILP	BD1		BD2	
					CPU(s)	CPU(s)	# cuts	CPU(s)	# cuts
25	2	0.2	45030.60	4, 16	246.42	0.07	20	0.07	118
		0.4	46956.24	4, 16	222.65	0.03	14	0.04	109
		0.6	47940.87	5, 12	122.52	0.03	14	0.04	80
		0.8	50594.43	5, 12	70.82	0.03	12	0.05	70
	3	0.2	40090.01	5, 8, 22	326.93	0.08	41	0.11	236
		0.4	40970.84	5, 8, 11	258.74	0.05	29	0.08	196
		0.6	42504.02	5, 7, 16	183.90	0.04	22	0.05	139
		0.8	46269.52	5, 8, 21	126.35	0.03	18	0.03	65
	4	0.2	32561.90	2, 5, 9, 16	321.60	0.08	46	0.13	215
		0.4	35292.80	2, 5, 13, 21	262.67	0.06	34	0.07	180
		0.6	38414.38	5, 7, 13, 21	218.48	0.04	35	0.06	194
		0.8	45168.19	5, 7, 17, 21	196.31	0.03	30	0.04	105
	5	0.2	29452.84	2, 5, 12, 14, 21	314.87	0.14	70	0.18	363
		0.4	32165.65	2, 5, 13, 16, 21	224.79	0.08	51	0.13	315
		0.6	36455.32	5, 7, 13, 17, 21	209.25	0.06	43	0.09	217
		0.8	44337.62	2, 5, 7, 13, 21	150.78	0.04	34	0.03	73
50	2	0.2	55372.11	7, 32	5h (17.28%)	0.06	23	0.07	163
		0.4	56953.47	7, 22	5h (10.20%)	0.05	26	0.05	168
		0.6	58610.88	8, 12	5h (8.02%)	0.04	27	0.05	159
		0.8	60600.83	10, 13	11949.55	0.03	22	0.03	144
	3	0.2	45240.18	4, 8, 44	5h (20.01%)	0.11	45	0.20	325
		0.4	50425.81	1, 8, 31	5h (16.77%)	0.14	65	0.23	300
		0.6	53998.03	3, 8, 31	5h (8.39%)	0.15	70	0.18	387
		0.8	55631.54	10, 14, 42	15793.94	0.08	24	0.03	118
	4	0.2	39119.77	4, 9, 28, 31	5h (21.13%)	0.31	63	0.23	260
		0.4	44593.45	4, 9, 27, 31	5h (18.50%)	0.26	57	0.59	333
		0.6	48203.68	1, 10, 15, 42	5h (7.67%)	0.15	63	0.11	200
		0.8	54440.88	3, 10, 16, 42	5h (2.57%)	0.11	58	0.09	257
	5	0.2	32591.50	1, 4, 9, 28, 31	5h (31.68%)	0.23	47	0.25	257
		0.4	39729.18	1, 4, 9, 25, 42	5h (14.22%)	0.34	63	0.51	407
		0.6	45657.90	1, 4, 10, 18, 42	5h (9.07%)	0.23	75	0.22	364
		0.8	52677.18	1, 10, 16, 41, 42	10963.22	0.08	31	0.06	175
Average					> 8630.12	0.10		0.13	

are reported by the solver. From a solution time perspective we can observe that both Benders algorithms solved the instances in a fraction of a second.

Table 4 gives the solution results for the stratified p-hub center problem with the medium size AP instances. The MILP model could not be solved for any of these instances due to excessive memory requirements. Nevertheless, the instances were solved within

Table 4
Results for the Stratified p-hub center problem with the AP data set (medium instances).

$\|N\|$	p	α	Opt	Hubs	MILP	BD1		BD2	
					CPU(s)	CPU(s)	# cuts	CPU(s)	# cuts
75	2	0.2	56500.54	9, 65	Memory	0.07	18	0.07	127
		0.4	58494.80	9, 32	Memory	0.06	24	0.11	170
		0.6	59571.38	10, 32	Memory	0.06	16	0.08	133
		0.8	62688.18	10, 32	Memory	0.05	17	0.07	159
	3	0.2	46423.53	7, 11, 64	Memory	0.18	44	0.33	337
		0.4	51604.40	7, 28, 46	Memory	0.25	67	0.29	280
		0.6	55390.18	5, 11, 46	Memory	0.29	57	0.34	295
		0.8	58032.33	13, 21, 62	Memory	0.10	38	0.15	298
	4	0.2	40222.47	7, 12, 38, 46	Memory	0.43	59	0.69	400
		0.4	46847.82	7, 12, 39, 46	Memory	0.68	82	1.47	442
		0.6	52105.90	4, 13, 39, 46	Memory	1.07	125	3.16	620
		0.8	56045.86	5, 11, 15, 62	Memory	0.19	52	0.23	247
	5	0.2	37845.95	2, 7, 12, 41, 64	Memory	2.38	109	5.17	605
		0.4	42981.91	1, 7, 12, 39, 46	Memory	2.43	136	3.28	662
		0.6	47384.35	1, 7, 13, 36, 62	Memory	0.79	114	1.10	513
		0.8	55474.73	4, 12, 15, 22, 62	Memory	0.65	116	0.45	336
100	2	0.2	57204.35	13, 44	Memory	0.14	27	0.26	242
		0.4	58614.65	14, 23	Memory	0.13	27	0.20	187
		0.6	60707.28	14, 22	Memory	0.12	26	0.18	226
		0.8	63769.69	17, 23	Memory	0.09	24	0.17	251
	3	0.2	46214.74	5, 16, 62	Memory	0.41	50	0.69	279
		0.4	50688.61	5, 16, 62	Memory	0.65	64	0.82	350
		0.6	55418.50	6, 16, 61	Memory	0.55	82	0.97	471
		0.8	58519.51	11, 18, 83	Memory	0.27	56	0.20	213
	4	0.2	38064.22	5, 17, 61, 77	Memory	0.73	72	1.04	353
		0.4	44098.90	5, 17, 61, 77	Memory	0.86	81	0.74	365
		0.6	50177.12	5, 19, 32, 61	Memory	0.85	97	1.18	263
		0.8	56688.72	5, 18, 31, 83	Memory	0.64	106	0.53	336
	5	0.2	35538.00	5, 17, 37, 61, 69	Memory	3.19	112	6.70	428
		0.4	41111.43	5, 18, 37, 61, 69	Memory	2.86	104	3.97	602
		0.6	47936.05	2, 12, 18, 46, 83	Memory	4.91	121	3.82	476
		0.8	55633.68	4, 18, 20, 36, 82	Memory	1.09	108	0.80	319
Average					–	0.85		1.23	

quite short computation times using the proposed Benders algorithms. The solution times for BD1 are much smaller than those of BD2. Note that the maximum solution times for the instances with 100 nodes are less than 5 and 7 s by using BD1 and BD2, respectively. This indicates the high efficiency of the proposed algorithms.

Table 5 presents the solution results for the stratified p-hub center problem with the large size AP instances. As can be seen from these results, the solution times of BD1 and BD2 are very small for solving such large-scale instances of the problem, with average computational times of around 7 and 14 s, respectively. Although both the algorithms are highly efficient, the results indicate the superiority of BD1 over BD2.

5.2.2. Results for SpHMCP

Table 6 presents the results for solving the stratified p-hub maximal covering problem with the CAB data set. Here, the objective function value shows the average percentage of the covered demand over all strata. We present the results for both the mathematical formulations MILP1 and MILP2 and also the proposed Benders algorithms BD1 and BD2. According to the reformulations presented in Section 4.2, the first Benders algorithm (BD1) adds a single cut at a time to the master problem as in **MP1-SpHMCP**, while the second algorithm (BD2) adds $|N|$ cuts at a time to the MP as in **MP2-SpHMCP**.

Table 6 shows that the proposed solution algorithms solve all the instances of the CAB data set in a small fraction of a second, while the average time taken by CPLEX to solve the MILP1 models is around 6 s and 0.7 s, respectively. The superiority of the MILP2 formulation over the MILP1 model stems from the fact that MILP2 is a more compact model in terms of the number of variables compared to MILP1. As expected, for a given number of the opened hubs p, as the value of the discount factor (α) increases (i.e., as the inter-hub speed decreases), the optimal objective value decreases. This is because larger values of α result in increased travel time between the O/D pairs and hence, decrease the percentage of the covered traffic. On the other hand, for a fixed value of the discount factor (inter-hub speed), the total covered traffic increases as the number of opened hubs (p) gets larger.

Solution results for the stratified p-hub maximal covering problem with the TR data set are given in Table 7. Note that the MILP1 model could not be solved using the standard solver due to excessive memory requirements for the instances in the TR data set. However, the MILP2 model has solved all the instances of the TR data set within an average computational time of around 1000 s. The solution times for the proposed Benders algorithms, on the other hand, are quite short for such a realistic data set. Results also

Table 5
Results for the Stratified p-hub center problem with the AP data set (large instances).

$\|N\|$	p	α	Opt	Hubs	MILP	BD1		BD2	
					CPU(s)	CPU(s)	# cuts	CPU(s)	# cuts
150	2	0.2	59743.84	15, 97	Memory	0.40	38	0.44	250
		0.4	61399.95	17, 64	Memory	0.28	37	0.40	256
		0.6	63211.06	17, 62	Memory	0.31	41	0.20	200
		0.8	66647.16	18, 122	Memory	0.23	36	0.19	230
	3	0.2	48548.05	5, 23, 97	Memory	1.03	56	2.82	323
		0.4	53722.77	5, 24, 92	Memory	0.93	53	2.13	413
		0.6	58526.66	7, 25, 92	Memory	1.16	88	2.22	486
		0.8	62052.84	27, 37, 122	Memory	0.38	48	0.25	227
	4	0.2	39572.61	5, 25, 79, 91	Memory	1.44	82	3.56	389
		0.4	46183.92	5, 26, 78, 91	Memory	1.49	52	2.07	352
		0.6	52432.26	5, 27, 47, 91	Memory	1.62	97	1.61	436
		0.8	60273.13	25, 29, 38, 122	Memory	0.60	48	0.49	232
	5	0.2	37626.47	5, 30, 56, 91, 113	Memory	18.20	147	32.04	800
		0.4	42710.96	5, 26, 68, 84, 91	Memory	4.57	99	11.49	854
		0.6	50415.59	5, 27, 36, 56, 123	Memory	4.17	130	3.29	482
		0.8	59346.63	5, 21, 29, 91, 121	Memory	1.87	111	1.19	310
200	2	0.2	60598.53	25, 130	Memory	0.73	50	1.04	345
		0.4	62743.37	25, 89	Memory	0.59	46	0.70	315
		0.6	64371.97	25, 122	Memory	0.55	48	0.54	288
		0.8	67288.71	26, 166	Memory	0.39	40	0.38	256
	3	0.2	49262.34	7, 34, 127	Memory	2.70	49	5.61	409
		0.4	54542.59	10, 34, 87	Memory	4.23	112	6.49	642
		0.6	59851.02	7, 32, 124	Memory	4.60	98	6.87	520
		0.8	62894.27	37, 53, 165	Memory	0.93	63	0.72	294
	4	0.2	39977.96	7, 35, 105, 122	Memory	6.06	92	11.36	593
		0.4	47422.09	7, 40, 111, 122	Memory	5.29	103	14.36	561
		0.6	54021.80	7, 36, 104, 121	Memory	4.94	77	5.30	629
		0.8	61181.20	7, 31, 39, 165	Memory	1.80	98	1.23	343
	5	0.2	38157.49	7, 40, 73, 122, 199	Memory	69.21	180	204.78	880
		0.4	44534.99	7, 36, 96, 114, 121	Memory	24.10	174	41.73	579
		0.6	51930.78	7, 36, 88, 111, 165	Memory	52.51	212	85.89	1005
		0.8	60120.64	4, 33, 39, 55, 163	Memory	5.53	162	1.77	325
Average					–	6.96		14.16	

Table 6
Results for the Stratified p-hub maximal covering problem with the CAB data set.

p	α	Opt	Hubs	MILP1	MILP2	BD1		BD2	
				CPU(s)	CPU(s)	CPU(s)	# cuts	CPU(s)	# cuts
2	0.2	56.70%	2, 21	21.05	0.38	0.60	42	0.43	271
	0.4	54.15%	18, 21	6.41	0.28	0.16	37	0.11	270
	0.6	53.12%	2, 4	2.41	0.09	0.08	20	0.08	165
	0.8	52.15%	2, 4	2.78	0.04	0.07	23	0.06	120
3	0.2	73.85%	12, 17, 21	5.54	1.28	0.16	42	0.15	253
	0.4	65.71%	2, 4, 12	5.01	1.66	0.16	48	0.08	245
	0.6	60.49%	4, 12, 17	6.68	0.33	0.14	45	0.08	250
	0.8	58.26%	2, 4, 12	2.62	0.39	0.10	31	0.04	135
4	0.2	84.36%	12, 18, 21, 24	9.44	1.45	0.21	49	0.14	336
	0.4	74.96%	2, 4, 12, 14	5.58	0.86	0.22	55	0.15	308
	0.6	67.13%	4, 12, 17, 24	5.70	0.60	0.41	78	0.10	296
	0.8	63.76%	1, 4, 12, 17	2.26	0.13	0.13	36	0.03	140
5	0.2	89.75%	4, 7, 12, 17, 24	7.05	2.02	0.27	63	0.13	347
	0.4	81.91%	2, 4, 7, 12, 14	2.94	0.74	0.19	46	0.05	176
	0.6	72.98%	4, 7, 12, 14, 18	4.74	0.38	0.23	56	0.06	170
	0.8	67.39%	4, 7, 12, 14, 17	2.08	0.13	0.16	44	0.04	142
Average				5.77	0.67	0.20		0.11	

show that the average time spent by the multi-cut algorithm (BD2) is less than the average time for the single-cut algorithm (BD1).

Table 8 reports the results for solving the stratified p-hub maximal covering problem with the small instances of the AP data set. The MILP1 and MILP2 models are solved by CPLEX within the average CPU time of around 82 and 6 s, respectively. However, the

Table 7
Results for the Stratified p-hub maximal covering problem with the TR data set.

p	α	Opt	Hubs	MILP1	MILP2	BD1		BD2	
				CPU(s)	CPU(s)	CPU(s)	# cuts	CPU(s)	# cuts
2	0.2	53.23%	38, 41	Memory	709.23	29.60	117	12.75	3608
	0.4	46.78%	38, 41	Memory	139.71	21.24	98	6.69	2838
	0.6	43.15%	41, 68	Memory	49.54	12.85	67	3.12	1449
	0.8	41.20%	41, 68	Memory	11.13	6.68	38	1.90	775
3	0.2	67.20%	12, 41, 68	Memory	1682.60	133.52	645	96.12	13720
	0.4	57.43%	41, 46, 71	Memory	876.58	94.18	579	43.17	9294
	0.6	52.07%	6, 41, 46	Memory	119.04	41.11	289	11.98	4537
	0.8	48.22%	6, 41, 46	Memory	24.79	18.70	143	2.96	1328
4	0.2	79.16%	3, 27, 41, 60	Memory	2895.89	281.89	1143	355.64	15695
	0.4	67.71%	3, 41, 60, 80	Memory	1393.43	166.24	886	106.94	9391
	0.6	58.48%	1, 3, 41, 60	Memory	349.34	126.43	819	40.88	7202
	0.8	53.75%	1, 3, 41, 60	Memory	37.90	37.82	273	5.48	1841
5	0.2	88.50%	12, 19, 41, 64, 80	Memory	5819.73	514.01	1203	541.20	9702
	0.4	75.19%	3, 12, 19, 41, 80	Memory	1604.32	591.53	1804	349.46	11794
	0.6	63.72%	1, 3, 19, 21, 41	Memory	520.54	426.50	1725	190.37	10677
	0.8	57.25%	1, 3, 21, 41, 60	Memory	39.86	105.29	572	11.74	2575
Average				–	1017.10	162.97		111.28	

Table 8
Results for the Stratified p-hub maximal covering problem with the AP data set (small instances).

$\|N\|$	p	α	Opt	Hubs	MILP1	MILP2	BD1		BD2	
					CPU(s)	CPU(s)	CPU(s)	# cuts	CPU(s)	# cuts
25	2	0.2	37.37%	8, 18	2.75	1.34	0.37	18	0.17	127
		0.4	34.38%	8, 18	2.48	0.09	0.13	22	0.10	104
		0.6	32.53%	8, 18	2.10	0.04	0.07	21	0.14	147
		0.8	31.80%	8, 18	1.86	0.03	0.05	15	0.08	90
	3	0.2	44.51%	7, 18, 20	15.15	1.45	0.24	84	0.27	300
		0.4	40.43%	8, 18, 20	4.24	0.36	0.12	57	0.10	211
		0.6	37.60%	8, 16, 18	2.29	0.10	0.12	57	0.08	160
		0.8	36.14%	8, 16, 18	1.87	0.03	0.20	32	0.05	94
	4	0.2	51.68%	2, 7, 14, 18	10.32	1.82	0.42	157	0.21	337
		0.4	46.52%	8, 16, 18, 20	2.45	0.29	0.18	80	0.06	168
		0.6	41.52%	8, 16, 18, 20	2.79	0.23	0.21	97	0.06	187
		0.8	38.60%	8, 14, 16, 18	2.03	0.16	0.32	95	0.06	149
	5	0.2	58.75%	2, 8, 15, 16, 18	8.32	0.88	0.51	151	0.11	306
		0.4	51.57%	2, 8, 16, 18, 20	2.47	0.23	0.25	92	0.07	125
		0.6	44.54%	2, 7, 16, 18, 20	5.92	0.32	0.58	183	0.06	222
		0.8	41.00%	2, 7, 14, 16, 18	2.02	0.15	0.43	118	0.04	137
50	2	0.2	37.59%	14, 35	185.26	6.85	0.87	37	0.35	463
		0.4	34.75%	16, 35	112.24	1.97	0.74	34	0.26	356
		0.6	32.98%	16, 35	85.58	0.71	0.76	37	0.27	309
		0.8	32.34%	15, 35	56.58	0.33	0.40	20	0.22	247
	3	0.2	44.48%	14, 29, 35	223.09	19.36	4.09	178	0.73	869
		0.4	39.85%	16, 33, 35	206.44	4.13	3.78	181	0.59	779
		0.6	37.53%	16, 33, 35	132.24	1.59	1.88	97	0.34	457
		0.8	36.08%	15, 32, 35	102.77	0.54	1.23	67	0.23	318
	4	0.2	50.92%	14, 29, 32, 35	183.44	18.77	8.16	314	0.82	748
		0.4	44.75%	14, 27, 33, 35	155.79	9.95	7.15	320	0.74	789
		0.6	41.12%	14, 28, 33, 35	73.52	1.66	4.01	199	0.43	524
		0.8	38.96%	14, 27, 32, 35	43.95	0.59	2.67	142	0.25	343
	5	0.2	54.54%	4, 15, 29, 32, 35	312.69	72.38	43.39	1170	1.90	1104
		0.4	47.37%	14, 25, 29, 33, 35	402.36	29.97	54.15	1669	2.08	1229
		0.6	43.26%	14, 18, 33, 35, 38	143.76	9.84	16.62	735	1.10	998
		0.8	40.41%	14, 18, 33, 35, 38	149.17	4.63	10.78	503	0.58	552
Average					82.43	5.96	5.15		0.39	

proposed solution algorithms solve the corresponding instances in much smaller time. The average solution time for BD1 is around 5 s, while BD2 solves the problems in less than half of a second on average.

The results for the stratified p-hub maximal covering problem with the medium instances of the AP data set are shown in Table 9. The MILP1 model was not solved by CPLEX due to high memory requirements. However, the MILP2 model was able to solve all the

Table 9
Results for the Stratified p-hub maximal covering problem with the AP data set (medium instances).

$\|N\|$	p	α	Opt	Hubs	MILP1	MILP2	BD1		BD2	
					CPU(s)	CPU(s)	CPU(s)	# cuts	CPU(s)	# cuts
75	2	0.2	35.97%	22, 52	Memory	18.12	5.78	55	1.45	711
		0.4	33.23%	22, 52	Memory	3.53	4.25	43	1.21	695
		0.6	31.84%	22, 52	Memory	1.98	3.38	36	0.88	524
		0.8	31.17%	22, 52	Memory	0.82	2.48	25	0.60	289
	3	0.2	42.59%	21, 41, 52	Memory	282.80	31.61	311	4.05	1741
		0.4	38.29%	22, 48, 52	Memory	88.73	24.84	273	4.35	2141
		0.6	36.15%	22, 48, 52	Memory	9.24	14.58	168	2.40	1154
		0.8	34.98%	22, 48, 52	Memory	3.33	7.53	92	1.02	514
	4	0.2	49.01%	21, 41, 48, 52	Memory	460.93	74.37	701	6.86	2374
		0.4	43.49%	21, 41, 48, 52	Memory	68.36	44.52	472	3.56	1604
		0.6	39.81%	21, 39, 48, 52	Memory	24.53	35.49	405	3.24	1416
		0.8	38.10%	21, 39, 48, 52	Memory	3.63	14.10	169	1.45	621
	5	0.2	53.27%	5, 22, 42, 48, 52	Memory	751.48	278.37	1869	17.07	3686
		0.4	46.39%	5, 22, 41, 48, 52	Memory	232.33	215.72	1794	11.23	2752
		0.6	41.93%	5, 23, 48, 52, 57	Memory	132.42	189.46	1794	10.71	2704
		0.8	39.66%	21, 39, 48, 52, 55	Memory	13.80	66.07	736	2.75	1187
100	2	0.2	37.89%	30, 70	Memory	386.41	19.81	63	5.53	1268
		0.4	35.38%	28, 70	Memory	25.01	13.69	45	4.68	1055
		0.6	33.71%	28, 70	Memory	4.22	8.33	30	3.10	669
		0.8	32.94%	30, 70	Memory	8.08	6.90	22	2.28	413
	3	0.2	45.45%	28, 56, 70	Memory	1068.33	91.81	299	13.98	2820
		0.4	41.09%	28, 56, 70	Memory	384.19	75.62	268	11.13	2444
		0.6	37.95%	28, 63, 70	Memory	103.24	79.88	295	6.98	1760
		0.8	36.52%	28, 63, 70	Memory	12.66	39.55	150	3.93	884
	4	0.2	51.88%	28, 56, 63, 70	Memory	2022.99	211.91	667	22.52	3821
		0.4	46.43%	28, 56, 63, 70	Memory	285.32	119.79	423	11.41	2306
		0.6	42.22%	28, 56, 63, 70	Memory	67.10	104.85	396	7.52	1542
		0.8	39.80%	28, 56, 63, 70	Memory	10.70	75.44	295	4.24	932
	5	0.2	54.66%	7, 30, 57, 63, 70	Memory	9898.00	4662.63	6329	152.47	8877
		0.4	48.76%	28, 56, 63, 70, 75	Memory	1969.23	1308.14	3223	43.45	4352
		0.6	44.21%	28, 50, 63, 70, 78	Memory	567.10	758.86	2323	31.48	4454
		0.8	41.55%	28, 54, 63, 70, 75	Memory	35.18	269.58	980	9.02	1737
Average					–	591.99	276.85		12.70	

instances within an average CPU time of less than 10 min. The solution time for the proposed Benders algorithms are quite small. The average solution time for BD1 is less than 5 min, while BD2 has solved the medium instances in less than 13 s on average.

Finally, Table 10 presents the results obtained by solving the stratified p-hub maximal covering problem with the large instances of the AP data set. MILP2 was able to solve only 15 out of 32 instances within the allowed solution time of 5 h. For these instances, the gap percentages between the upper and lower bounds are reported. In one instance ($|N| = 200$, $p = 2$, and $\alpha = 0.2$) CPLEX could not even find an integer feasible solution within 5 h. BD1 was not able to solve 4 out of 32 instances within the allowed CPU time of 5 h. BD2, in contrast, was able to solve all the large instances within an average solution time of around 10 min.

5.2.3. Results with larger values of p

In order to test the efficiency of the proposed BD algorithms, we solve instances with larger numbers of opened hubs ($p = 6$, 8, and 10) from the CAB, TR and AP ($|N| = 100$) data sets. Table 11 presents the results with these larger values of p. In these experiments, we have removed the limit of 5 h on the solution times.

The results in Table 11 show that the solution times for instances with large values of p are reasonable (less than 5000 s on average), but are significantly larger than the corresponding solution times for the instances with smaller values of p. This is mainly due to the fact that by increasing the value of p, the number of feasible solutions to the problems increases exponentially. For instance, the number of feasible solutions for the instances of the AP data set ($|N| = 100$) with $p = 5$ is $\binom{100}{5}$, where this number increases to $\binom{100}{10}$ when p is set to 10, which is around (23×10^4) times the earlier value (i.e., $\binom{100}{10} \simeq 23 \times 10^4 \times \binom{100}{5}$).

5.2.4. Comparison to hierarchical hub solutions

This section provides a comparison of stratified hub center and stratified maximal covering solutions to hierarchical hub solutions. The hierarchical hub networks (e.g., Lin and Chen (2004), Yaman (2009), Korani and Sahraeian (2013), Ghaffarinasab (2020b)) use several layers of hubs that are used to connect the network for the O/D demands. The hierarchical hub location problem has been introduced in Yaman (2009) with two levels of hubs to capture this characteristic. Here, we present the results for solving the hierarchical p-hub center and hierarchical p-hub maximal covering problems, which have been extended from the models proposed

Table 10
Results for the Stratified p-hub maximal covering problem with the AP data set (large instances).

$\|N\|$	p	α	Opt	Hubs	MILP1	MILP2	BD1		BD2	
					CPU(s)	CPU(s)	CPU(s)	# cuts	CPU(s)	# cuts
150	2	0.2	38.02%	38, 100	Memory	10178.69	105.92	71	32.05	2095
		0.4	35.40%	38, 100	Memory	1210.95	68.42	49	22.76	1634
		0.6	33.81%	38, 100	Memory	420.95	37.57	27	13.05	962
		0.8	32.83%	38, 100	Memory	82.93	23.18	17	19.35	854
	3	0.2	44.92%	38, 82, 100	Memory	5h (751.40%)	617.10	424	85.25	5412
		0.4	40.70%	38, 93, 100	Memory	4506.51	468.15	350	54.86	4290
		0.6	38.15%	38, 93, 101	Memory	1046.63	187.09	149	29.25	2193
		0.8	36.40%	38, 93, 100	Memory	128.60	143.60	115	19.46	1294
	4	0.2	51.17%	38, 82, 94, 101	Memory	5h (568.57%)	1596.57	1065	149.36	8431
		0.4	45.91%	38, 81, 93, 101	Memory	10344.22	842.20	630	57.16	4131
		0.6	41.97%	38, 81, 93, 101	Memory	5312.82	507.49	402	48.17	3452
		0.8	39.49%	38, 79, 93, 100	Memory	160.70	331.69	270	29.38	1930
	5	0.2	54.67%	7, 41, 83, 94, 100	Memory	5h (495.64%)	12870.98	5010	347.62	9372
		0.4	48.33%	38, 69, 83, 93, 100	Memory	5h (1.26%)	7886.78	4352	323.85	10703
		0.6	43.76%	38, 81, 93, 100, 109	Memory	13373.28	5308.92	3654	266.28	11870
		0.8	40.96%	38, 49, 93, 100, 110	Memory	516.45	2401.09	1826	75.08	4272
200	2	0.2	36.95%	57, 140	Memory	5h (-)	774.16	175	198.57	5494
		0.4	34.55%	57, 140	Memory	5h (696.65%)	563.68	137	172.19	4665
		0.6	33.14%	61, 140	Memory	5212.38	350.69	87	84.89	2584
		0.8	32.52%	61, 140	Memory	579.02	159.89	38	53.10	1186
	3	0.2	44.24%	57, 111, 140	Memory	5h (769.84%)	4091.42	945	432.70	11271
		0.4	39.96%	57, 111, 140	Memory	5h (652.61%)	3246.15	817	304.92	9343
		0.6	37.24%	56, 129, 140	Memory	5h (246.61%)	2479.47	656	233.55	6896
		0.8	35.87%	57, 128, 140	Memory	1987.53	1412.99	384	124.32	3420
	4	0.2	50.30%	57, 111, 128, 140	Memory	5h (587.79%)	14305.86	2968	1032.79	18001
		0.4	45.15%	57, 111, 128, 140	Memory	5h (498.38%)	6524.56	1528	337.04	8117
		0.6	41.32%	57, 109, 130, 141	Memory	5h (400.04%)	4308.15	1075	195.36	5886
		0.8	39.12%	56, 109, 128, 140	Memory	1251.14	2626.21	684	89.56	2607
	5	0.2	53.58%	56, 90, 111, 132, 141	Memory	5h (519.99%)	5h (8.76%)	–	10645.59	29723
		0.4	47.61%	56, 102, 113, 128, 140	Memory	5h (440.17%)	5h (6.29%)	–	3016.43	18721
		0.6	43.32%	57, 99, 130, 141, 155	Memory	5h (352.09%)	5h (4.09%)	–	1014.88	14298
		0.8	40.61%	57, 69, 128, 140, 148	Memory	5h (0.34%)	5h (1.71%)	–	482.01	8052
Average					–	> 10759.77	> 5132.50		624.71	

Table 11
Results for the Stratified p-hub center and maximal covering problems with large values of p.

p	α	SpHCP						SpHMCP					
		CAB		TR		AP ($\|N\|$ = 100)		CAB		TR		AP ($\|N\|$ = 100)	
		Opt	CPU(s)	Opt	CPU(s)	Opt	CPU(s)	Opt	CPU(s)	Opt	CPU(s)	Opt	CPU(s)
6	0.2	1210.68	0.27	921.14	400.93	33437.01	41.39	93.22%	0.169	93.07%	656.76	57.48%	966.36
	0.4	1449.33	0.18	1134.88	109.33	38163.14	15.56	84.50%	0.135	80.81%	485.13	50.93%	172.52
	0.6	1812.65	0.10	1369.96	14.34	45323.10	5.15	75.16%	0.117	68.23%	134.88	45.90%	47.82
	0.8	2204.10	0.06	1660.22	1.75	54898.89	1.15	69.55%	0.063	59.41%	46.12	42.78%	9.05
8	0.2	1029.77	0.32	805.82	3749.46	27928.61	652.26	96.59%	0.248	96.64%	9189.30	62.47%	16301.98
	0.4	1273.97	0.13	1052.72	1096.41	34170.08	158.96	88.30%	0.147	85.96%	3578.18	54.89%	1508.91
	0.6	1682.17	0.07	1311.62	109.86	42806.35	19.05	78.43%	0.238	72.26%	2981.14	48.90%	492.45
	0.8	2156.39	0.04	1621.94	2.29	54170.03	2.21	71.87%	0.073	62.33%	252.22	44.84%	70.55
10	0.2	810.55	0.31	750.94	28864.26	24413.65	2109.47	98.43%	0.75	98.18%	16146.70	66.96%	15439.24
	0.4	1191.39	0.19	1021.40	16524.22	31325.87	106.88	90.65%	0.20	88.97%	14514.67	57.89%	12478.15
	0.6	1620.66	0.08	1282.68	2095.50	41458.65	41.89	80.46%	0.27	75.25%	11054.39	51.11%	5026.38
	0.8	2141.15	0.02	1613.98	3.28	54040.00	0.27	73.18%	0.12	64.31%	583.62	46.24%	275.91
Average			0.15		4414.30		262.85		0.21		4968.59		4399.11

in Yaman (2009), and we compare these results with those of the corresponding stratified models. The hierarchical network can be viewed as having three levels of nodes, with demand points at the lowest level, regional hubs in a middle level, and central hubs at the top level. The hierarchical hub network operates as in Yaman (2009), where each demand point is assigned to a single regional or central hub, each regional hub is assigned to a single central hub, and the central hubs are connected by a complete network; and we assume each O/D flow must pass through at least one central hub on its path from the corresponding origin to destination. We install a total of p, hubs which consists of p_0 central hubs and $(p - p_0)$ regional hubs. Transportation time between regional and

Table 12
Results for the hierarchical p-hub center problem with the CAB data set.

p	α	$p_0 = 1$		$p_0 = 2$		$p_0 = 3$		$p_0 = 4$		$p_0 = 5$	
		Opt	Hubs	Opt	Hubs	Opt	Hubs	Opt	Hubs	Opt	Hubs
2	0.2	2131.20	**21**, 22	2079.92	**5, 8**	–	–	–	–	–	–
	0.4	2501.92	8, **11**	2131.20	**21, 22**	–	–	–	–	–	–
	0.6	2711.09	8, **11**	2281.32	**21, 22**	–	–	–	–	–	–
	0.8	2931.74	8, **11**	2402.55	**8, 21**	–	–	–	–	–	–
3	0.2	1951.60	**5**, 19, 23	1760.15	9, **13, 22**	1760.15	**2, 13, 22**	–	–	–	–
	0.4	2131.20	12, **21**, 23	1923.12	**5, 19**, 23	1923.12	**5, 19, 23**	–	–	–	–
	0.6	2438.41	**8**, 18, 24	2049.13	**1, 8**, 17	2049.13	**8, 20, 24**	–	–	–	–
	0.8	2716.43	**11**, 22, 23	2231.95	**5, 8**, 14	2100.47	**1, 8, 20**	–	–	–	–
4	0.2	1760.15	2, 12, **13**, 23	1520.06	**11, 22**, 24, 25	1467.81	**9, 16, 19**, 23	1467.81	**9, 16, 19, 23**	–	–
	0.4	2009.54	3, 12, **13**, 23	1691.14	**11**, 14, 18, **22**	1619.48	**9, 16, 19**, 23	1619.48	**9, 16, 19, 23**	–	–
	0.6	2222.79	**7**, 12, 23, 25	1915.52	**13**, 17, **19**, 23	1760.15	3, **12, 13, 23**	1760.15	**3, 12, 13, 23**	–	–
	0.8	2501.92	**11**, 12, 22, 23	2163.19	**5, 8**, 14, 23	1959.58	**12, 13**, 17, **23**	1884.84	**5, 12, 13, 23**	–	–
5	0.2	1348.96	**11**, 12, 18, 23, 24	1284.42	**11**, 12, 23, **24**, 25	1284.42	**11**, 12, 23, **24, 25**	1284.42	**11**, 12, **18, 23, 24**	1284.42	**6, 11, 12, 23, 24**
	0.4	1740.92	**7**, 12, 14, 23, 25	1389.83	**11, 12**, 18, 23, 24	1355.41	**11, 12, 20**, 23, 24	1346.36	**11, 12**, 23, **20, 24**	1346.36	**6, 11, 12, 23, 24**
	0.6	2131.20	12, 19, **21**, 22, 23	1777.64	17, **19, 21**, 23, 24	1617.20	**12**, 18, **21, 23**, 24	1577.96	8, **9, 12, 16, 23**	1443.48	**2, 11, 12, 23, 24**
	0.8	2472.38	**11**, 12, 21, 22, 23	2081.91	**5, 8**, 14, 22, 23	1875.03	3, **12**, 14, **21, 23**	1824.42	**7, 12**, 14, **23, 25**	1599.74	**11, 12, 18, 23, 24**

Table 13
Results for the hierarchical p-hub maximal covering problem with the CAB data set.

p	α	$p_0 = 1$		$p_0 = 2$		$p_0 = 3$		$p_0 = 4$		$p_0 = 5$	
		Opt	Hubs	Opt	Hubs	Opt	Hubs	Opt	Hubs	Opt	Hubs
2	0.2	56.16%	**6**, 24	61.48%	**6, 12**	–	–	–	–	–	–
	0.4	54.02%	**6**, 24	56.70%	**6, 24**	–	–	–	–	–	–
	0.6	48.48%	14, **20**	55.20%	**6, 14**	–	–	–	–	–	–
	0.8	44.86%	14, **25**	54.65%	**6, 24**	–	–	–	–	–	–
3	0.2	71.53%	12, 18, **21**	75.49%	**6, 12**, 24	78.59%	**12, 13, 17**	–	–	–	–
	0.4	60.56%	**5**, 18, 24	67.67%	**6, 12**, 24	73.05%	**12, 18, 21**	–	–	–	–
	0.6	56.15%	**5**, 14, 18	60.19%	**5**, 17, **24**	66.20%	**6, 13, 14**	–	–	–	–
	0.8	45.81%	14, 24, **25**	56.10%	14, **21, 25**	63.92%	**6, 13, 24**	–	–	–	–
4	0.2	85.44%	12, 14, 18, **21**	88.76%	**12**, 14, 17, **21**	90.50%	**6, 12, 13**, 14	92.31%	**11, 12, 24, 25**	–	–
	0.4	67.14%	**5**, 7, 14, 17	78.10%	**5, 12**, 14, 17	85.18%	**12, 18, 21**, 24	86.96%	**12, 18, 21, 24**	–	–
	0.6	59.80%	**5**, 13, 14, 17	68.12%	**5, 12**, 14, 18	72.72%	**12, 17, 21**, 24	78.57%	**12, 17, 21, 24**	–	–
	0.8	46.93%	**6**, 14, 17, 21	57.53%	**6, 14**, 17, 21	66.40%	**6, 13, 14**, 17	70.99%	**12, 14, 17, 21**	–	–
5	0.2	89.75%	8, 12, 14, 18, **21**	93.30%	10, **12**, 14, 17, **21**	94.90%	**7, 12**, 14, 18, **21**	95.96%	**11, 12**, 14, **16, 25**	95.96%	**11, 12, 14, 16, 25**
	0.4	70.62%	**5**, 7, 12, 14, 17	85.83%	**5**, 10, **12**, 14, 17	89.28%	**5, 7, 12**, 14, 17	90.93%	8, **12, 18, 21, 24**	91.60%	**4, 7, 12, 18, 24**
	0.6	62.27%	1, **5**, 7, 14, 17	71.77%	**5, 12**, 13, 14, 17	78.33%	**5, 10, 12**, 14, 18	83.85%	**4, 12**, 14, **16, 17**	86.02%	**4, 12, 13, 17, 24**
	0.8	47.69%	14, 15, 17, 21, **25**	58.70%	8, 14, 17, **21, 25**	67.85%	**6**, 10, **13, 14**, 17	73.44%	**6, 12, 13, 14**, 17	79.52%	**4, 12, 13, 14, 17**

central hubs is discounted by the discount factor α, whereas the time decrease between two central hubs is discounted by factor α_0, where we set $\alpha_0 = \frac{1}{2}\alpha$. Thus, the vehicles traveling between two central hubs have average speed $2/(\alpha v)$, vehicles traveling between a central hub and regional hub have speed $1/(\alpha v)$, and vehicles traveling between a demand point and regional hub have speed $1/v$. To obtain results for the hierarchical p-hub center and hierarchical p-hub maximal covering problems we modified the formulation from Yaman (2009) (to reflect these service-oriented objectives) and solved problems for the CAB data set using CPLEX. The results including the optimal objective function values and opened hubs are presented in Tables 12 and 13. Note that these results are new to the literature and are based on the model presented in Yaman (2009), but modified to obtain center and maximal covering versions. The central hubs are shown in boldface in Tables 12 and 13.

Pairwise comparison of solutions for the same values of p and α in Table 1 for the SpHCP and in Table 12 for the hierarchical p-hub center problem shows that while the objective function value is larger in the hierarchical solutions with one central hub (as expected due to the extra travel with the 2 layers of hubs), the optimal hub locations can be quite similar. The average pairwise similarity of hub locations is 48.6%. Pairwise similarity is the fraction of hubs in Table 12 that are also a hub for the corresponding instance in Table 1. The central hubs in Table 12 are also often used as hubs in the non-hierarchical problem (Table 1), as the average similarity for central hubs is 37.6% (here, similarity is the fraction of the central hubs in Table 12 that are also a hub for the corresponding instance in Table 1). The similarity for the hubs decreases with α from 54.6% for $\alpha = 0.2$ to 38% for $\alpha = 0.8$. The similarity for the central hubs also decreases with α from 47.3% for $\alpha = 0.2$ to 29.9% for $\alpha = 0.8$. Results also show that as the number of central hubs increases, the travel times decrease because of reduced travel time between central hubs; and having two or more central hubs is enough to provide a lower objective value in the hierarchical hub problems, except when $\alpha = 0.2$.

Pairwise comparison of solutions for the same values of p and α in Table 6 for the SpMHCP and in Table 13 for the hierarchical p-hub maximal covering problem shows the same pattern of results as the comparison of results in Tables 1 and 12, although the similarities are about one-third smaller. For example, the average pairwise similarity of hub locations for the maximal covering problems is 31.4% vs. 48.6% for the hub center problems. In summary, there is considerable similarity of the optimal hubs (about 30%–50%) between the stratified and hierarchical hub problems. This is not surprising given that remote, isolated hubs are often optimal locations for center and covering problems. For example, with the CAB data set, nodes 23 (Seattle) is optimal in 94% of the hierarchical hub center solutions and in 100% of the SpHCP solutions for $p \geq 4$.

5.3. Discussion of results

The results presented in Tables 1–10 show that the proposed solution algorithms are able to efficiently solve large scale instances of the stratified p-hub center and the stratified p-hub maximal covering problems in short computational times. As expected, the solution time for both problems increases with the size of the problems (i.e., the number of nodes $|N|$ and/or the number of installed hubs p). Moreover, for the maximal covering problem, the solution times decrease with an increase in the discount factor. One reason for this behavior is that by increasing the value of the discount factor, the travel times over paths using inter-hub arcs get larger and hence there are fewer paths that do not exceed the time threshold (i.e., the covering radius), which reduces the number of feasible paths to be explored by the algorithm. Since both stratified problems belong to the class of NP-hard optimization problems, developing exact solution algorithms that are capable of solving large-scale instances of problems in short computational times is important both from the theoretical and practical standpoints.

Examination of the results for SpHCP and SpHMCP suggests some implications for managerial practice. First, increasing the number of opened hubs, can increase the quality of service provided to the customers considerably in terms of the weighted maximum travel time and the percentage of covered demands across different strata. However, these benefits come with the added fixed costs for opening new facilities, added complexity as more routes are available through the network, and a reduction in average traffic on arcs (since more arcs are available). Note that with the increase in open hubs, many O/D demands will travel a longer part of their path between two hub facilities on the faster vehicles, which may have additional benefits (e.g., comforts of passenger plane travel vs. bus travel). Furthermore, the results indicate that by decreasing the discount factor value (i.e., employing faster modes of transportation on inter-hub connections), the service provider can offer better service to the customers. These benefits likely come with the added costs for the faster transportation mode (e.g., planes vs. trucks for freight transport), but do not affect the network complexity as does adding new hubs. Further, faster transport modes (a smaller discount factor) will tend to increase traffic on the inter-hub arcs (to take advantage of the faster transportation) and may thereby allow a higher frequency of service ion those arcs as well. Thus, the results clearly show the impacts of the two ways to improve service: by adding hubs (increasing p) or by using faster vehicles (reducing α). Each option has associated costs, and each option brings benefits from the improved service, which ultimately may translate into increased demand and/or being able to charge higher prices for the better service level. Thus, the number of opened hubs and type of vehicles are two main levers that transportation firms can use to gain a competitive advantage over their rivals in the increasingly competitive business environment.

Examination and comparison of the optimal solutions for SpHCP and SpHMCP reveals some interesting behaviors. The stratified hub center problem solutions in aggregate use a larger selection of optimal hubs, though none very intensively across all problems, while the stratified maximal covering problems use fewer different hubs more intensively. For example, comparing the optimal hubs in Tables 1 and 6 for the CAB data set shows that SpHCP uses 17 of the 25 nodes as hubs in at least one optimal solution, while SpHMCP uses only nine of the 25 nodes as hubs in at least one optimal solution. For SpHCP, the nodes most often used as hubs are node 23 in nine of the 16 problems, node 12 in seven of the 16 problems and node 8 in six of the 16 problems; further, one of the western nodes 8, 12 or 22 is a hub in all of the 16 problems. In contrast, for SpHMCP, the nodes most often used as hubs are node 4 in 12 of the 16 problems (and node 4 is never a hub for SpHCP), node 12 in 12 of the 16 problems, and node 2 in seven of the 16 problems (and node 2 is never a hub for SpHCP). There is also a geographic component of hub location evident in the solutions, as for nearby nodes 17 and 18, or 14 and 24, one of which is a hub in nine, and seven respectively, of the 16 problems. We also note that the more peripheral nodes 3, 10 and 15 are never optimal hubs in Tables 1 and 6 for SpHCP and SpHMCP, respectively. The behavior described above for the CAB data set is also reflected in the solutions with the TR data set. Comparing the optimal hubs in Tables 2 and 7 shows that SpHCP uses 24 different nodes as hubs in at least one optimal solution, while SpHMCP uses only 15 different nodes as hubs, eight of which are not hubs for SpHCP. With the TR data set, for SpHCP, the nodes most often used as hubs are node 3 in six of the 16 problems, and node 59 in five of the 16 problems (with node 59 never a hub for SpHMCP). In contrast, for SpHMCP, the nodes most often used as hubs are node 41 in all 16 problems (and a hub in only 3 of the 16 problems for SpHCP), node 3 in six of the 16 problems (note that node 3 is the most common hub for SpHCP), and node 60 in five of the 16 problems. These results reflect a quite different usage of hubs in the two problems SpHCP and SpHMCP, where with CAB only six of the 22 nodes used as hubs are hubs for both problems, and with TR only seven of the 32 nodes used as hubs are hubs for both problems.

6. Generalized models

This section considers three generalizations of the stratified hub location models presented earlier.

6.1. Partial coverage in SpHMCP

In the models presented for the SpHMCP, we assumed a binary coverage mechanism, i.e., if the travel time for the O/D pair $(i,j) \in A^s$ is smaller than the corresponding coverage radius (β_{ij}^s), then the demand for that O/D pair is fully covered. Otherwise, the corresponding demand is entirely uncovered. However, earlier research (see e.g., Peker and Kara (2015)) has shown that in some real-life cases binary coverage may be unrealistic and partial coverage may be more appropriate. In this section, we generalize our earlier models to accommodate partial coverage where the coverage level gradually decreases as the travel time increases. To this end, we need to alter the definition of the covering parameter, a_{ijkm}^s, which previously was defined as a binary parameter in (10). Thus, we define the following five-step function in our mathematical model to provide three levels of partial coverage:

$$a_{ijkm}^s = \begin{cases} 1, & \text{if } c_{ijkm} \le 0.7\beta_{ij}^s \\ 0.75, & \text{if } 0.7\beta_{ij}^s < c_{ijkm} \le 0.9\beta_{ij}^s \\ 0.5, & \text{if } 0.9\beta_{ij}^s < c_{ijkm} \le 1.1\beta_{ij}^s \\ 0.25, & \text{if } 1.1\beta_{ij}^s < c_{ijkm} \le 1.3\beta_{ij}^s \\ 0, & \text{if } 1.3\beta_{ij}^s < c_{ijkm} \end{cases} \qquad s \in S, (i,j) \in A^s, k,m \in N \tag{90}$$

Table 14
Results for the Stratified p-hub maximal covering problem with gradual coverage.

p	α	CAB		TR		AP ($\lvert N\rvert = 100$)	
		Opt	Hubs	Opt	Hubs	Opt	Hubs
2	0.2	55.25%	18, 21	51.37%	38, 41	37.28%	28, 70
	0.4	53.22%	4, 18	45.89%	38, 41	34.84%	28, 70
	0.6	52.17%	4, 18	42.37%	38, 54	33.45%	28, 71
	0.8	51.14%	4, 18	40.69%	38, 54	32.72%	28, 71
3	0.2	71.25%	12, 18, 21	63.39%	21, 41, 68	44.03%	28, 56, 70
	0.4	64.33%	4, 12, 18	55.35%	41, 46, 71	39.83%	28, 65, 70
	0.6	59.90%	4, 12, 18	50.42%	6, 41, 46	37.64%	28, 65, 70
	0.8	57.48%	4, 12, 18	47.31%	6, 41, 46	36.19%	28, 65, 70
4	0.2	81.34%	4, 12, 18, 24	74.14%	3, 27, 34, 60	50.16%	28, 55, 63, 70
	0.4	72.86%	4, 12, 18, 24	63.58%	3, 34, 60, 80	44.82%	28, 55, 65, 70
	0.6	66.11%	4, 12, 14, 18	55.94%	1, 3, 41, 60	41.19%	28, 55, 65, 70
	0.8	61.70%	1, 4, 12, 18	51.28%	1, 3, 41, 60	38.99%	28, 53, 65, 70
5	0.2	87.75%	4, 7, 12, 18, 24	82.62%	12, 34, 64, 71, 80	52.87%	28, 45, 55, 66, 70
	0.4	79.02%	4, 7, 12, 14, 18	70.37%	12, 34, 64, 71, 80	47.14%	28, 55, 65, 70, 75
	0.6	71.14%	4, 7, 12, 14, 18	61.00%	1, 6, 23, 34, 64	43.13%	28, 55, 65, 70, 75
	0.8	65.12%	4, 7, 12, 14, 18	54.46%	1, 3, 21, 41, 60	40.54%	28, 54, 65, 70, 75

We then solve SpHMCP with (90) instead of (10) for the CAB, the TR, and the AP (with $|N| = 100$) data sets, and the results are presented in Table 14. Results show that in most of the cases, the optimal hub set under partial coverage differs from the corresponding optimal set under the binary coverage. For the CAB data set, 13 of 16 solutions in Table 14 differed in one hub with solutions that used binary coverage (from Table 6) and two other solutions differed in two hubs. For the larger TR data set, eight of 16 solutions in Table 14 differed at least in one hub with solutions that used binary coverage (from Table 7), and with the largest AP ($|N| = 100$) data set only two solutions in Table 14 had identical sets of hubs with solutions that used binary coverage (from Table 9). As expected, the coverage percentage with the partial coverage under the defined coverage (90) is smaller than the corresponding value with the binary coverage for all the instances.

6.2. Inclusion of operational times at hubs

As the proposed models deal with enhancing the customer service level, it is important to study the effect of the operational times at hubs on the optimal solutions as these times directly affect the total in-transit time of the O/D traffic. To this end, we modify the definition of the total travel time for the O/D traffic $(i, j) \in A$ that is routed via hubs $k \in N$ and $m \in N$, originally defined by (1), as follows:

$$c_{ijkm} = \begin{cases} \tau_{ik} + ot_k + \alpha\tau_{km} + ot_m + \tau_{mj}, & \text{if } k \neq m, \\ \tau_{ik} + ot_k + \tau_{kj}, & \text{if } k = m \end{cases} \tag{91}$$

where, ot_k represents the operational time value at hub $k \in N$. Using (91) in the proposed models for the SpHCP and the SpHMCP, we solved the instances from the CAB, TR and AP ($|N| = 100$) data sets by setting the values for the operational times as $ot_k = \frac{\bar{\tau}}{20}$, where $\bar{\tau}$ denotes the average travel time on the network arcs, i.e., $\bar{\tau} = \sum_{(i,j)\in A} \frac{\tau_{ij}}{|A|}$. The results for solving the SpHCP and the SpHMCP considering operational times at hubs are presented in Tables 15 and 16, respectively.

These results show that including operational time at hubs worsens the value of the objective function in both the stratified p-hub center and maximal covering problems. This is quite expected as adding operational time at hubs increases the total travel time for the O/D traffic. An interesting observation is that the inclusion of the operational time at hubs often alters the optimal set of hub locations. For instance, in 6 out of 16 instances, the optimal hub sets for the SpHCP with operational times under the CAB data set are different from the optimal hub set for the corresponding instances without operational times. For the TR and AP ($|N| = 100$) data set, the number of nonidentical hubs sets are 7 and 9, respectively. For the SpHMCP, the number of instances with different optimal hub sets when considering operational times are 12, 6, and 9 for the CAB, TR, and AP ($|N| = 100$) data sets, respectively.

6.3. Multi-modal problems with fixed costs

In this section we propose MILP formulations for the multi-modal stratified hub center and maximal covering problems with fixed set-up costs for hubs and hub edges. To explicitly model different modes, let Q be the set of available modes of transport that can be chosen for inter-hub arcs. Accordingly, we define f_k^q and $\hat{f}_{km}^q$ as the fixed costs associated with installing a hub facility for mode $q \in Q$ at node $k \in N$ and setting up a hub edge with mode $q \in Q$ between hubs $k, m \in N, (k < m)$, respectively. The discount factor to be applied to the transportation cost over inter-hub arcs using transport mode $q \in Q$ is denoted by α^q. We assume that an installed hub edge between hubs k and m activates both arcs (k, m) and (m, k) on the hub level network. Moreover, we assume that

Table 15
Results for the Stratified p-hub center problem considering operational times at hubs.

p	α	CAB		TR		AP ($\|N\| = 100$)	
		Opt	Hubs	Opt	Hubs	Opt	Hubs
2	0.2	2114.28	5, 22	1454.13	23, 26	58362.42	13, 44
	0.4	2434.32	5, 8	1609.66	12, 26	59746.29	13, 43
	0.6	2634.01	8, 21	1761.06	6, 49	62148.04	15, 22
	0.8	2720.84	8, 9	1907.62	6, 56	65340.74	18, 25
3	0.2	1846.20	9, 16, 22	1321.30	4, 26, 58	47522.23	5, 16, 62
	0.4	2109.66	5, 12, 23	1469.08	3, 12, 41	52778.55	5, 16, 62
	0.6	2338.27	8, 17, 24	1640.65	3, 12, 59	56711.71	5, 16, 61
	0.8	2573.52	8, 9, 24	1848.02	3, 34, 49	60062.25	18, 28, 82
4	0.2	1669.22	3, 12, 13, 23	1181.32	4, 20, 41, 58	40144.33	5, 17, 61, 77
	0.4	1856.99	9, 12, 16, 23	1348.76	3, 49, 59, 60	46188.84	5, 17, 61, 77
	0.6	2204.92	12, 16, 20, 23	1535.58	3, 25, 59, 73	52267.06	5, 18, 32, 61
	0.8	2519.75	3, 8, 9, 24	1792.90	3, 4, 34, 73	58557.35	7, 20, 38, 82
5	0.2	1381.80	11, 12, 23, 24, 25	1087.95	38, 41, 56, 64, 69	37528.61	5, 17, 37, 61, 69
	0.4	1699.19	11, 12, 14, 18, 23	1259.22	3, 25, 40, 56, 59	42760.48	2, 12, 18, 54, 61
	0.6	2011.10	1, 17, 19, 22, 23	1484.51	3, 25, 30, 59, 72	49654.42	2, 12, 18, 46, 83
	0.8	2355.39	6, 12, 22, 23, 24	1764.34	3, 4, 21, 30, 34	57446.80	5, 16, 20, 61, 81

Table 16
Results for the Stratified p-hub maximal covering problem considering operational times at hubs.

p	α	CAB		TR		AP ($\|N\| = 100$)	
		Opt	Hubs	Opt	Hubs	Opt	Hubs
2	0.2	53.59%	18, 21	46.47%	38, 41	30.99%	29, 70
	0.4	51.61%	4, 18	41.24%	38, 41	28.64%	28, 70
	0.6	50.72%	4, 18	38.69%	41, 68	27.36%	28, 70
	0.8	49.10%	21, 25	37.10%	41, 68	26.74%	28, 70
3	0.2	69.04%	12, 17, 21	59.46%	21, 41, 68	37.45%	28, 56, 70
	0.4	61.57%	4, 12, 18	51.27%	6, 41, 46	33.48%	28, 63, 70
	0.6	57.18%	4, 12, 18	46.43%	6, 41, 46	31.36%	28, 63, 70
	0.8	54.78%	12, 21, 25	43.74%	6, 41, 46	30.02%	28, 63, 70
4	0.2	79.97%	12, 18, 21, 24	71.60%	3, 27, 34, 60	43.28%	28, 56, 63, 70
	0.4	70.75%	4, 12, 14, 18	59.82%	1, 41, 60, 64	38.24%	28, 55, 63, 70
	0.6	63.25%	1, 4, 12, 18	52.58%	1, 3, 41, 60	34.59%	28, 56, 63, 70
	0.8	59.64%	1, 4, 12, 17	48.25%	1, 3, 41, 58	32.51%	28, 53, 63, 70
5	0.2	86.39%	4, 7, 12, 18, 24	81.47%	12, 19, 41, 64, 80	46.09%	28, 45, 56, 66, 70
	0.4	77.12%	4, 7, 12, 14, 18	67.07%	1, 21, 41, 60, 64	40.33%	28, 50, 57, 63, 70
	0.6	68.79%	4, 7, 12, 14, 18	56.56%	1, 3, 21, 41, 60	36.39%	28, 56, 63, 70, 75
	0.8	62.75%	4, 7, 12, 14, 17	51.35%	1, 3, 21, 41, 60	33.97%	28, 54, 65, 70, 75

the available budget for paying the upfront costs of installing hubs and arcs is limited and denoted by B. We introduce the binary variables y_k^q and e_{km}^q as follows:

$$y_k^q = \begin{cases} 1, & \text{if a hub with transport mode } q \in Q \text{ is installed at node } k \in N, \\ 0, & \text{otherwise} \end{cases}$$

$$e_{km}^q = \begin{cases} 1, & \text{if an edge with transport mode } q \in Q \text{ is installed between hubs } k, m \in N, (k < m), \\ 0, & \text{otherwise.} \end{cases}$$

We define flow variable $\bar{x}_{ijk}^s$ as the fraction of traffic associated with the O/D pair $(i,j) \in A^s$ that is routed through the access arc (i,k) in stratum $s \in S$. Similarly, flow variable $\bar{\bar{x}}_{ijm}^s$ is defined as the fraction of demand associated with the O/D pair $(i,j) \in A^s$ that is routed through the access arc (m,j). Finally, flow variable x_{ijkm}^{sq} represents the fraction of traffic for the O/D pair $(i,j) \in A^s$ that is routed through the hub arc (k,m) using transportation mode $q \in Q$. The following MILP model solves the multi-modal stratified hub center problem with fixed costs (MSHCP-FC):

$$\min \sum_{s \in S} w^s z^s \tag{92}$$

$$\text{s.t.:} \sum_{k \in N} \bar{x}_{ijk}^s = 1 \qquad s \in S, (i,j) \in A^s \tag{93}$$

$$\sum_{m \in N} \bar{\bar{x}}_{ijm}^s = 1 \qquad s \in S, (i,j) \in A^s \tag{94}$$

$$\sum_{m\in N}\sum_{q\in Q} x^{sq}_{ijkm} + \bar{\bar{x}}^{s}_{ijk} - \sum_{m\in N}\sum_{q\in Q} x^{sq}_{ijmk} - \bar{x}^{s}_{ijk} = 0 \quad s\in S,(i,j)\in A^s,k\in N \tag{95}$$

$$\bar{x}^{s}_{ijk} \le \sum_{q\in Q} y^q_k \quad s\in S,(i,j)\in A^s,k\in N \tag{96}$$

$$\bar{\bar{x}}^{s}_{ijm} \le \sum_{q\in Q} y^q_m \quad s\in S,(i,j)\in A^s,m\in N \tag{97}$$

$$x^{sq}_{ijkm} + x^{sq}_{ijmk} \le e^q_{km} \quad s\in S,(i,j)\in A^s,k,m\in N,(k<m),q\in Q \tag{98}$$

$$2e^q_{km} \le y^q_k + y^q_m \quad k,m\in N,(k<m),q\in Q \tag{99}$$

$$\bar{x}^{s}_{ijk} \le 2 - y^q_i - y^{q'}_k \quad s\in S,(i,j)\in A^s,k\in N,(i\neq k),q,q'\in Q \tag{100}$$

$$\bar{\bar{x}}^{s}_{ijm} \le 2 - y^q_j - y^{q'}_m \quad s\in S,(i,j)\in A^s,m\in N,(j\neq m),q,q'\in Q \tag{101}$$

$$\sum_{k\in N}\sum_{q\in Q} f^q_k y^q_k + \sum_{k\in N}\sum_{\substack{m\in N\\(m>k)}}\sum_{q\in Q} \hat{f}^q_{km} e^q_{km} \le B \tag{102}$$

$$z^s \ge \sum_{k\in N} \tau_{ik}\bar{x}^s_{ijk} + \sum_{k\in N}\sum_{m\in N}\sum_{q\in Q} \alpha^q \tau_{km} x^{sq}_{ijkm} + \sum_{m\in N} \tau_{mj}\bar{\bar{x}}^s_{ijm} \quad s\in S,(i,j)\in A^s \tag{103}$$

$$z^s \in \mathbb{R} \quad s\in S \tag{104}$$

$$\bar{x}^s_{ijk}, \bar{\bar{x}}^s_{ijm}, x^{sq}_{ijkm} \in \{0,1\} \quad s\in S,(i,j)\in A^s,k,m\in N,q\in Q \tag{105}$$

$$y^q_k \in \{0,1\} \quad k\in N,q\in Q \tag{106}$$

$$e^q_{km} \in \{0,1\} \quad k,m\in N,(k<m),q\in Q \tag{107}$$

The objective function (92) minimizes the weighted sum of the maximum travel times between O/D pairs within all strata. Constraints (93)–(95) are flow conservation constraints which ensure that for each pair of nodes $(i,j)\in A^s$, the corresponding traffic leaves node i and arrives at node j, and is being properly accounted for whenever a hub k is used. Constraints (96) and (97) assure that O/D traffic can only access or leave the inter-hub network through installed hubs. Constraints (98) ensure that flows can travel via a hub arc with a specific mode of transport only if the corresponding hub edge is installed. Constraints (99) guarantee that a hub edge can only be activated only if the respective hubs are installed. Constraints (100) and (101) prevent formation of bridge arcs (i.e., arcs connecting pairs of hubs but without the reduced unit flow cost of a hub arc). The budget limitation for setting up the hubs and the hub arcs is reflected by (102). Constraints (103) calculate the maximum travel time between the O/D pairs within each stratum. Finally, (104)–(107) are the standard domain constraints.

To model the multi-modal stratified hub maximal covering problem with fixed costs, we use the additional binary variable r^s_{ij} which takes the value of 1 if the traffic from origin node i to destination node j in stratum s is covered; and 0, otherwise. The MILP formulation for the multi-modal stratified hub maximal covering problem with fixed costs (MSHMCP-FC) can be written as follows:

$$\max \sum_{s\in S} w^s \left(\sum_{(i,j)\in A^s} t^s_{ij} r^s_{ij} \right) \tag{108}$$

s.t.: (93)–(102), (105)–(107) (109)

$$\sum_{k\in N} \tau_{ik}\bar{x}^s_{ijk} + \sum_{k\in N}\sum_{m\in N}\sum_{q\in Q} \alpha^q \tau_{km} x^{sq}_{ijkm} + \sum_{m\in N} \tau_{mj}\bar{\bar{x}}^s_{ijm} \le \beta^s_{ij} + (1-r^s_{ij})M \quad s\in S,(i,j)\in A^s \tag{110}$$

$$r^s_{ij} \in \{0,1\} \quad s\in S,(i,j)\in A^s \tag{111}$$

The objective function (108) maximizes the weighted sum of the covered traffic over all the strata. Constraints (110) determine whether the traffic from node i to node j in stratum s is covered or not in which M is a sufficiently large number. More specifically, if the left-hand side of (110) exceeds the covering radius β^s_{ij}, then the variable r^s_{ij} will be forced to take the value of zero in order to satisfy the inequality. On the other hand, if the left-hand side of (110) does not exceed the covering radius β^s_{ij}, it means that the corresponding flow is covered and the variable r^s_{ij} will take the value of one due to maximization sense in the objective function. A tight value for M can be calculated as $(2 + max^a \times min^\alpha)max^\tau$ in which max^τ is the maximum travel time on network arcs, max^a is the maximum number of hub edges that can be installed in the network, and min^α is the smallest discount factor corresponding to different modes of transport. This number can be calculated a priori based on the fixed costs for hubs and hub edges, discount factor values, and the available budget.

6.3.1. Fixed cost results

In order to focus on the role of fixed costs, we first solve the MSHCP-FC and MSHMCP-FC with only one mode of transport (i.e., $|Q|$=1), and compare the results to those for the corresponding solutions of the SpHCP and SpHMCP. For this purpose, we use the CAB data set with 15 and 25 nodes (i.e., $|N|$=15, 25). Three strata are used as presented in our numerical example in Section 5.1. We use the fixed setup cost values for hubs as $f_k = 100$ for all $k\in N$ and the fixed activation cost values for hub edges

Table 17
Results for the MSHCP-FC with $|Q| = 1$ for the CAB data set.

$\lvert N \rvert$	MSHCP-FC						SpHCP		
	B	α	Opt	Hubs	Activated Hub Edges	Consumed Budget	Opt	Hubs	% Improve
15	300	0.2	1717.98	12, 13	(12,13)	280.40	1717.98	12, 13	0.00%
		0.4	1722.82	12, 13	(12,13)	280.40	1722.82	12, 13	0.00%
		0.6	1827.78	12, 13	(12,13)	280.40	1827.78	12, 13	0.00%
		0.8	1941.68	5, 12	(5,12)	294.48	1941.68	5, 12	0.00%
	600	0.2	1205.95	1, 3, 11, 12	(1,3), (1,11), (11,12)	549.27	1205.95	1, 3, 11, 12	0.00%
		0.4	1316.69	1, 3, 11, 12	(1,3), (1,11), (11,12)	549.27	1304.41	1, 3, 11, 12	0.94%
		0.6	1545.63	1, 3, 11, 12	(1,3), (1,11), (11,12)	549.27	1521.07	1, 3, 11, 12	1.61%
		0.8	1794.07	2, 11, 12, 13	(2,11), (2,13), (11,12), (11,13)	302.35	1751.73	1, 3, 11, 12	2.42%
25	300	0.2	1994.29	21, 22	(21,22)	286.80	1988.50	5, 22	0.29%
		0.4	2219.29	21, 22	(21,22)	286.80	2219.29	21, 22	0.00%
		0.6	2380.06	8, 18	(8,18)	278.74	2380.06	8, 18	0.00%
		0.8	2531.50	21, 22	(21,22)	286.80	2531.50	21, 22	0.00%
	600	0.2	1511.20	8, 16, 20, 22	(8,16), (8,20), (8,22)	566.19	1492.49	6, 8, 16, 22	1.25%
		0.4	1745.80	12, 13, 17, 23	(12,13), (12,23), (13,17)	577.53	1742.48	12, 13, 17, 23	0.19%
		0.6	2026.02	6, 8, 16, 22	(6,8), (8,16), (8,22)	561.96	2026.02	6, 8, 16, 22	0.00%
		0.8	2346.69	3, 8, 9, 22	(3,8), (8,9), (8,22)	593.22	2312.14	3, 4, 7, 22	1.49%

as $\hat{f}_{km} = \frac{d_{km}}{20}$ for all $k, m \in N, (k < m)$. We consider five values of $\alpha \in \{0.2, 0.4, 0.6, 0.8\}$ and two values for the budget $B \in \{300, 600\}$. The results for solving the MSHCP-FC with $|Q| = 1$ are shown in Table 17. The first two columns show the number of nodes and the available budget in the problem instances, respectively. The third column presents the discount factor applied to the inter-hub connections. The next four columns show the solution results for the MSHCP-FC, including the optimal objective function value, the set of opened hubs and activated hub arcs and the consumed budget, respectively. In order to compare the optimal network configurations obtained by the MSHCP-FC with those of the S*p*HCP, we provide the optimal solution results for S*p*HCP in the last three columns, including the optimal objective function value, the opened hubs, and the percentage improvement in the objective function values of S*p*HCP over those of the MSHCP-FC. Note that S*p*HCP is solved by setting the parameter p as the number of opened hubs in the solution of the MSHCP-FC.

It can be seen from Table 17 that in all instances with $B = 300$, two hubs are opened and a hub arc connects them resulting in a complete inter-hub network. Therefore, in such cases the optimal solution obtained by the MSHCP-FC is identical to the optimal solution of S*p*HCP unless the fixed cost needed to establish the hubs and the hub arc exceeds the available budget. For example, when $|N| = 25$, $B = 300$, and $\alpha = 0.2$, the optimal solution for the MSHCP-FC is to open hubs at nodes 21 and 22. However, the optimal hubs for S*p*HCP are at nodes 5 and 22 with a slightly better objective value. The reason is that the fixed cost required to establish the hub arc (5,22) is larger (due to longer distance) than the fixed cost needed to activate the hub arc (21,22). This makes the total cost of the S*p*HCP solution exceed the available budget of 300 units. The largest budget $B = 600$ allows more opened hubs and activated hub arcs which provides better service and shorter travel times (compared to $B = 300$). Note that the inter-hub network is incomplete in these cases but the hubs are connected in all the cases. It is interesting to see that even though the inter-hub network is incomplete in the solutions of the MSHCP-FC, in some instances with identical sets of hubs, the provided service is the same with that of S*p*HCP solution which has a complete inter-hub network. As an example, see the instance with $|N| = 25$, $B = 600$, and $\alpha = 0.6$ where only three hub arcs (6,8), (8,16), and (8,22) are activated by the MSHCP-FC but the objective value is identical with that of S*p*HCP. The reason is that the worst-case travel times in all the strata pass through these three hub arcs in the optimal solution of S*p*HCP and hence the objective values of the two problems coincide at the value of 2026.02.

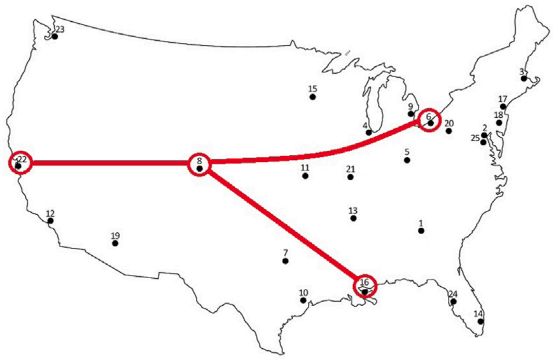
Table 18 presents the results for solving the MSHMCP-FC with $|Q| = 1$. It can be observed that for $B = 300$, the optimal solutions for the MSHMCP-FC are identical to those of the S*p*HMCP. In contrast, for three out of four instances with $|N| = 15$ and $B = 600$, the optimal hub sets for the MSHMCP-FC and S*p*HMCP are different. The instances with $|N| = 25$ and $B = 600$ could not be solved within the time limit of 20 h using CPLEX. Therefore, we used the optimal set of hubs for S*p*HMCP with $p = 4$ and fixed these hubs as opened hubs in the MSHMCP-FC model and then allowed the set of activated hub arcs be determined by the model (108)–(111). The obtained values are shown in boldface with an asterisk in Table 18. Note that for the instance with α=0.6, the objective function values of the solutions attained for the MSHMCP-FC and S*p*HMCP are the same (53.06%) regardless of the fact that the inter-hub network in the former solution is incomplete.

The optimal networks for MSHCP-FC and MSHMCP-FC with $|Q| = 1$, $B = 600$, and $\alpha = 0.6$ are depicted in Fig. 4. Note that in both cases, the inter-hub network is incomplete but the provided service level is the same as the corresponding problems with complete inter-hub networks (see Figs. 2-d and 3-d). This can be deemed as an important advantage of the incomplete networks as they can obtain solutions with the same (or slightly inferior) objective value but with significantly lower upfront cost of constructing the network. Nevertheless, incomplete hub networks make some O/D traffic travel through more circuitous paths which can be regarded as a disadvantage in applications where fast delivery is required by the clients. For instance, in Fig. 4-a the traffic corresponding to the O/D pair (6,16) travels through the path 6 → 8 → 16 rather than going directly from 6 to 16 as it is the case when the inter-hub network is complete.

From a managerial perspective, being able to add or remove service between hub nodes can be a valuable tool to respond to changes in the market. For example, the tremendous drop in air travel and freight transport with COVID-19 beginning in 2019 could

Table 18
Results for the MSHMCP-FC with $|Q| = 1$ for the CAB data set.

$\lvert N\rvert$	MSHMCP-FC						SpHMCP		
	B	α	Opt	Hubs	Activated Hub Edges	Consumed Budget	Opt	Hubs	% Improve
15	300	0.2	39.03%	4, 7	(4,7)	239.51	39.03%	4, 7	0.00%
		0.4	37.62%	4, 7	(4,7)	239.51	37.62%	4, 7	0.00%
		0.6	37.31%	4, 7	(4,7)	239.51	37.31%	4, 7	0.00%
		0.8	37.25%	4, 7	(4,7)	239.51	37.25%	4, 7	0.00%
	600	0.2	65.99%	4, 7, 12, 14	(4,7), (7,12), (7,14)	556.87	66.06%	4, 7, 12, 14	0.10%
		0.4	58.45%	4, 7, 8, 14	(4,7), (4,8), (4,14), (7,14)	599.10	60.27%	4, 7, 12, 14	3.11%
		0.6	51.96%	4, 6, 7, 8, 11	(4,6), (4,11), (7,11), (8,11)	585.85	53.68%	4, 7, 8, 9, 15	3.32%
		0.8	45.62%	4, 6, 7, 8, 11	(4,6), (4,11), (7,11), (8,11)	585.85	46.41%	4, 7, 8, 9, 15	1.73%
25	300	0.2	49.22%	18, 24	(18,24)	246.33	49.22%	18, 24	0.00%
		0.4	47.52%	2, 14	(2,14)	247.48	47.52%	2, 14	0.00%
		0.6	41.97%	4, 18	(4,18)	233.77	41.97%	4, 18	0.00%
		0.8	41.62%	2, 4	(2,4)	230.65	41.62%	2, 4	0.00%
	600	0.2	**70.92%***	4, 18, 22, 24	(4,18), (4,22), (18,24)	572.78	71.12%	4, 18, 22, 24	0.29%
		0.4	**63.46%***	2, 4, 12, 14	(2,14), (4,12), (4,14)	593.91	64.55%	2, 4, 12, 14	1.71%
		0.6	**53.06%***	1, 4, 18, 22	(1,4), (1,18), (4,18), (4,22)	589.96	53.06%	1, 4, 18, 22	0.00%
		0.8	**51.43%***	2, 4, 7, 22	(2,4), (4,7), (2,22)	592.97	51.46%	2, 4, 7, 22	0.07%


(a) MSHCP-FC: Obj = 2026.02

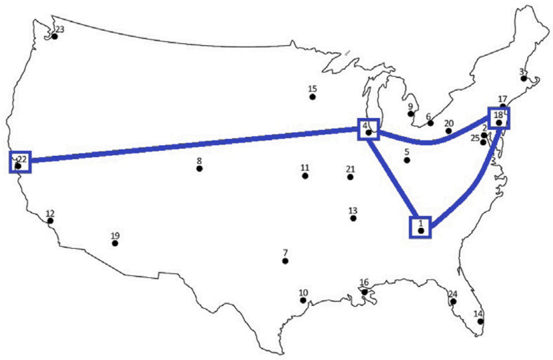

(b) MSHMCP-FC: Obj = 53.06%

Fig. 4. Solutions for the MSHCP-FC and MSHMCP-FC for the CAB data with $\alpha^1 = 0.6$, $B = 600$, $|N|$=25, and $|Q| = 1$.

lead to reduced services for certain arcs O/D pairs and the elimination of some inter-hub connections. Fixed cost models that do not assume full connectivity of the hub nodes can thus give valuable strategic insights for flexible network designs. Also, note that considering fixed costs for installing hubs and activating hub arcs may add realism in some cases as constructing transportation infrastructure (e.g., road, rail, port, etc.) or offering new services is highly capital-intensive. Therefore, it is not viable in many practical situations to have a complete transportation network between the installed hubs. Results with the fixed cost (FC) models show how a range of hub network topologies can be generated when there are incomplete networks. For example, with MSHCP-FC for $|N| = 25$, $B = 600$ and $\alpha^1 = 0.4$ (see Table 17), the hub network is a "line" of 3 connected hub arcs 23-12-13-17; with $|N| = 25$, $B = 600$ and $\alpha = 0.6$, the hub network is a "star" with three hub arcs from central node 8 to nodes 6, 16 and 22 (see Fig. 4-a), and for MSHMCP-FC with $|N| = 15$, $B = 600$ and $\alpha = 0.6$ (see Table 18), the network is a tree with 4 edges. Other incomplete topologies as in Fig. 4-b are seen as well, and with greater numbers of hubs more variety is possible.

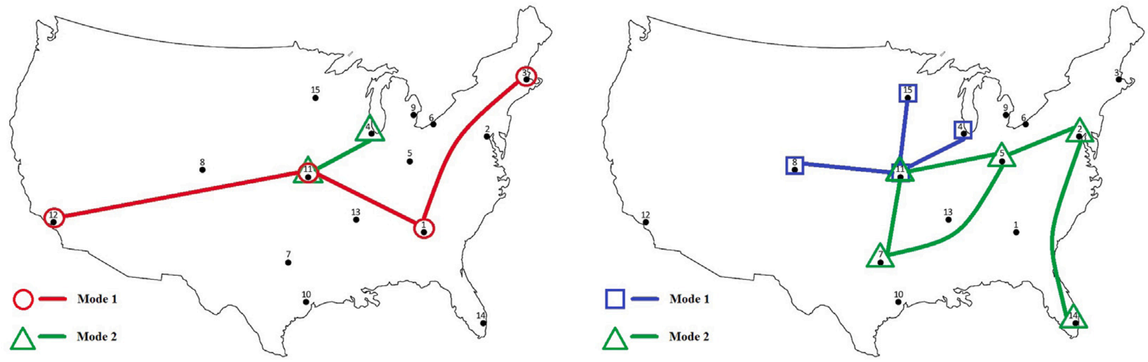
6.3.2. Multi-modal solutions

Now we consider the multi-mode case and assume we have two available modes of transport that can be used between hubs (i.e., $|Q| = 2$). Let the discount factor values for these two modes of transport be set as $\alpha^1 = 0.6$ and $\alpha^2 = 1$ (i.e., the first mode is faster). We use the fixed setup cost values for the two types of hubs as $f_k^1 = 100$ and $f_k^2 = 20$ for all $k \in N$ and the fixed activation cost values for the corresponding hub edges as $\hat{f}_{km}^1 = \frac{d_{km}}{20}$ and $\hat{f}_{km}^2 = \frac{d_{km}}{100}$ for all $k, m \in N, (k < m)$. The models MSHCP-FC and MSHMCP-FC with $|Q| = 2$ are solved for the CAB data with $|N| = 15$ and 25 and the results are presented in Table 19. The mode type of the installed hubs and edges are shown with the superscript in fifth and sixth columns of the table.

Table 19 shows that four of the eight instances use only one mode of transport (mode 1) and the other four instances (in rows 2, 5, 6 and 7) use both modes. Fig. 5 shows two of the optimal two-mode networks (from rows 2 and 6) for MSHCP-FC and MSHMCP-FC with the CAB data when $|N|$=15 and $B = 600$. In the solution for the MSHCP-FC (Fig. 5-a), three hubs serve only transport mode 1 (hubs 1, 3 and 12), one hub serves only mode 2 (hub 4) and one hub serves both modes 1 and 2 (hub 11). Three arcs of transport mode 1 connect hub 3 to hub 1 to hub 11 to hub 12. One arc of transport mode 2 connects hub 4 to hub 11. Note that there are also access arcs with a different travel time rate (representing a third transport mode) connecting the non-hub nodes to hubs. The

Table 19
Results for the MSHCP-FC and MSHMCP-FC with $|Q| = 2$ for the CAB data set.

Model	$\lvert N \rvert$	B	Opt	Hubs	Activated Hub Edges	Consumed Budget
MSHCP-FC	15	300	1827.78	12^1, 13^1	$(12,13)^1$	280.40
	15	600	1484.23	1^1, 3^1, 11^1, 12^1, 4^2, 11^2	$(1,3)^1$, $(1,11)^1$, $(11,12)^1$, $(4,11)^2$	593.33
	25	300	2380.06	8^1, 18^1	$(8,18)^1$	278.74
	25	600	**2026.02***	6^1, 8^1, 16^1, 22^1	$(6,8)^1$, $(8,16)^1$, $(8,22)^1$	561.96
MSHMCP-FC	15	300	45.31%	4^1, 8^1, 4^2, 7^2	$(4,8)^1$, $(4,7)^2$	293.27
	15	600	54.16%	4^1, 8^1, 11^1, 15^1, 2^2, 5^2, 7^2, 11^2, 14^2	$(4,11)^1$, $(8,11)^1$, $(11,15)^1$, $(2,5)^2$, $(2,14)^2$, $(5,7)^2$, $(5,11)^2$, $(7,11)^2$	599.69
	25	300	**42.97%***	1^1, 2^1, 1^2, 5^2, 14^2	$(1,2)^1$, $(1,5)^2$, $(1,14)^2$	298.51
	25	600	**53.06%***	1^1, 4^1, 18^1, 22^1	$(1,4)^1$, $(1,18)^1$, $(4,18)^1$, $(4,22)^1$	589.96



(a) MSHCP-FC: Obj = 1484.23

(b) MSHMCP-FC: Obj = 54.16%

Fig. 5. Solutions for the MSHCP-FC and MSHMCP-FC for the CAB data with $|N|$=15, $B = 600$, and $|Q| = 2$.

optimal network for MSHMCP-FC (Fig. 5-b) is more complex and includes three hubs that serve mode 1 only, 4 hubs that serve mode 2 only, and one hub that serves both modes 1 and 2. There are five mode 2 arcs that connect the five hubs that serve mode 2, and three mode 1 arcs that connect the four hubs that serve mode 1. Note that in both solutions, node 11 accommodates two hubs with different modes of transport. These more complex networks allow interesting O/D paths that use multiple modes. For example, with MSHCP-FC the path from node 12 to node 4 uses mode 1 for 12–11, then mode 2 for 11–4. With MSHMCP-FC, the path from node 12 to node 4 uses an access arc 12–8, then mode 1 for 8-11-4.

7. Conclusions

Hub networks are the backbone of many transportation systems and they often support service to several different classes of demand (i.e., traffic receiving different levels of service). We introduced the stratified multiple allocation p-hub center and p-hub maximal covering problems where the commodities to be transported between each O/D pair were divided into different strata with varying service level requirements. MILP formulations were proposed for the problems and efficient Benders decomposition algorithms were developed to solve them. Extensive computational experiments were conducted to analyze the efficiency of the proposed solution algorithms and the proposed MILP formulations using three well-known data sets from the HLP literature with up to 200 nodes (39,800 O/D pairs) and 10 strata. Results indicate that the optimal set of hub locations when considering different strata can be quite dissimilar to that of the traditional p-hub center or p-hub maximal covering problem. Furthermore, by altering the values of different input parameters, we studied the resulting changes on the optimal solutions of the problems. Results showed how the stratified center and maximal covering problems can have quite different optimal hub sets, and thus quite different traffic flow patterns. Results also demonstrate that large-scale instances of the problems can be solved by the developed BD algorithms in quite short computational times. Finally, we generalized the basic MILP formulations to include multiple modes of transport and also fixed set-up costs for hubs and hub edges. These features allow incomplete inter-hub networks. We compared the results obtained by solving the generalized models to those of the classical models and observed that the optimal solutions for the fixed cost models tend to be incomplete networks with lower network construction costs but not necessarily inferior service levels compared to the classical models.

The present work can be expanded in a number of ways. An interesting extension would assign different weights to the O/D traffic within each stratum. For example, one may want to weight a high flow O/D pair in one strata different than a low flow O/D pair in the same strata. Alternatively, some of the assumptions made in this work can be relaxed in order to reflect more realistic situations. To this end, one can adopt more detailed modeling of different modes of transportation between hub facilities, or allow direct connections between O/D pairs. Furthermore, looking at the stratified HLP from a cost perspective and addressing the stratified p-hub median problem would be a worthwhile contribution. Ideally, researchers would develop stratified demand models that address both cost and service considerations. As another line for future research one can propose heuristic solution algorithms for the generalized models in order to solve large-scale instances in reasonable time. It is also valuable to study the same problems under the single allocation or r-allocation settings. A final extension can be the incorporation of routing decisions into the proposed models which is important in postal services and LTL transportation.

CRediT authorship contribution statement

Nader Ghaffarinasab: Conceptualization, Methodology, Software, Investigation, Computing resources, Writing – original draft. **Bahar Y. Kara:** Conceptualization, Supervision, Validation, Writing – review & editing. **James F. Campbell:** Conceptualization, Supervision, Validation, Writing – review & editing.

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