EXTENSIONS OF STRONGLY $\pi$-REGULAR RINGS

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Dedicated to Professor Abdullah Harmanci on his 70th birthday

Abstract. An ideal $I$ of a ring $R$ is strongly $\pi$-regular if for any $x \in I$ there exist $n \in \mathbb{N}$ and $y \in I$ such that $x^n = x^{n+1}y$. We prove that every strongly $\pi$-regular ideal of a ring is a $B$-ideal. An ideal $I$ is periodic provided that for any $x \in I$ there exist two distinct $m, n \in \mathbb{N}$ such that $x^m = x^n$. Furthermore, we prove that an ideal $I$ of a ring $R$ is periodic if and only if $I$ is strongly $\pi$-regular and for any $u \in U(I)$, $u^{-1} \in \mathbb{Z}[u]$.

1. Introduction

A ring $R$ is strongly $\pi$-regular if for any $x \in R$ there exist $n \in \mathbb{N}, y \in R$ such that $x^n = x^{n+1}y$. For instance, all artinian rings and all algebraic algebra over a filed. Such rings are extensively studied by many authors from very different view points (cf. [1, 3, 4, 7, 9, 10, 11, 12, 13, 14]). We say that an ideal $I$ of a ring $R$ is strongly $\pi$-regular provided that for any $x \in I$ there exist $n \in \mathbb{N}, y \in I$ such that $x^n = x^{n+1}y$. Many properties of strongly $\pi$-regular rings were extended to strongly $\pi$-regular ideals in [5].

Recall that a ring $R$ has stable range one provided that $aR + bR = R$ with $a, b \in R$ implies that there exists $y \in R$ such that $a + by \in R$ is invertible. The stable range one condition is especially interesting because of Evans' Theorem, which states that a module cancels from direct sums whenever has stable range one. For general theory of stable range conditions, we refer the reader to [5]. An ideal $I$ of a ring $R$ is a $B$-ideal provided that $aR + bR = R$ with $a \in 1 + I$, $b \in R$ implies that there exists $y \in R$ such that $a + by \in R$ is invertible. An ideal $I$ is a ring $R$ is stable provided that $aR + bR = R$ with $a \in I, b \in R$ implies that there exists $y \in R$ such that $a + by \in R$ is invertible. As is well known, every $B$-ideal of a ring is stable, but the converse is not true.

In [1, Theorem 4], Ara proved that every strongly $\pi$-regular ring has stable range one. This was extended to ideals, i.e., every strongly $\pi$-regular ideal of a
Theorem 2.1. We define $I$ of a ring $R$ and show that every strongly $\pi$-regular ring is stable (cf. [6]). The main purpose of this note is to extend these results, and show that every strongly $\pi$-regular ideal of a ring is a $B$-ideal. An ideal $I$ of a ring $R$ is periodic provided that for any $x \in I$ there exist two distinct $m, n \in \mathbb{N}$ such that $x^m = x^n$. Furthermore, we show that an ideal $I$ of a ring $R$ is periodic if and only if $I$ is strongly $\pi$-regular and for any $u \in U(I)$, $u^{-1} \in \mathbb{Z}[u]$. Several new properties of such ideals are also obtained.

Throughout, all rings are associative with an identity and all modules are unitary modules. $U(R)$ denotes the set of all invertible elements in the ring $R$ and $U(I) = (1 + I) \cap U(R)$.

2. Strongly $\pi$-regular ideals

The aim of this section is to investigate more elementary properties of strongly $\pi$-regular ideals and construct more related examples. For any $x \in R$, we define $\sigma_x : R \to R$ given by $\sigma_x(r) = xr$ for all $r \in R$.

**Theorem 2.1.** Let $I$ be an ideal of a ring $R$. Then the following are equivalent:

1. $I$ is strongly $\pi$-regular.
2. For any $x \in I$, there exists $n \geq 1$ such that $R = \ker(\sigma_x^n) \oplus \text{im}(\sigma_x^n)$.

**Proof.** (1) $\Rightarrow$ (2) Let $x \in I$. In view of [5, Proposition 13.1.15], there exist $n \in \mathbb{N}, y \in I$ such that $x^n = x^{n+1}y$ and $xy = yx$. It is easy to check that $\sigma_x^n = \sigma_x^{n+1} \sigma_y$. If $a \in \ker(\sigma_x^n) \cap \text{im}(\sigma_x^n)$, then $a = \sigma_x^n(r)$ and $\sigma_x^n(a) = 0$. This implies that $x^{2n}r = \sigma_x^{2n}(r) = 0$, and so $a = x^n r = x^{n+1} y r = y x^{n+1} r = y^n x^{2n} r = 0$. Hence, $\ker(\sigma_x^n) \cap \text{im}(\sigma_x^n) = 0$. For any $r \in R$, we see that $r = (r - \sigma_x^n(y r)) + \sigma_x^n(y r)$, and then $R = \ker(\sigma_x^n) + \text{im}(\sigma_x^n)$, as required.

(2) $\Rightarrow$ (1) Write $1 = a + b$ with $a \in \ker(\sigma_x^n)$ and $b \in \text{im}(\sigma_x^n)$. For any $x \in I$, $\sigma_x^n(1) = \sigma_x^n(b)$, and so $x^n \in x^{2n} R$. Thus, $I$ is strongly $\pi$-regular. $\square$

**Corollary 2.2.** Let $I$ be a strongly $\pi$-regular ideal of a ring $R$, and let $x \in I$. Then the following are equivalent:

1. $\sigma_x$ is a monomorphism.
2. $\sigma_x$ is an epimorphism.
3. $\sigma_x$ is an isomorphism.

**Proof.** (1) $\Rightarrow$ (2) In view of Theorem 2.1, there exists $n \geq 1$ such that $R = \ker(\sigma_x^n) \oplus \text{im}(\sigma_x^n)$. Since $\sigma_x$ is a monomorphism, so is $\sigma_x^n$. Hence, $\ker(\sigma_x^n) = 0$, and then $R = \text{im}(\sigma_x^n)$. This implies that $\sigma_x$ is an epimorphism.

(2) $\Rightarrow$ (3) Since $R = \ker(\sigma_x^n) \oplus \text{im}(\sigma_x^n)$, it follows from $R = \text{im}(\sigma_x^n)$ that $\ker(\sigma_x^n) = 0$. Hence, $\sigma_x$ is a monomorphism. Therefore $\sigma_x$ is an isomorphism.

(3) $\Rightarrow$ (1) is trivial. $\square$

**Proposition 2.3.** Let $I$ be an ideal of a ring $R$. Then the following are equivalent:

1. $I$ is strongly $\pi$-regular.
2. For any $x \in I$, $RxR$ is strongly $\pi$-regular.
Proof. (1)⇒(2) Let $x \in I$. For any $a \in RxR$, there exists an element $b \in I$ such that $a^n = a^{n+1}b$ for some $n \in \mathbb{N}$. Hence, $a^n = a^{n+1}(ab^2)$. As $ab^2 \in RxR$, we see that $RxR$ is strongly $\pi$-regular.

(2)⇒(1) For any $x \in I$, $RxR$ is strongly $\pi$-regular, and so there exists $y \in RxR$ such that $x^n = x^{n+1}y$. Clearly, $y \in I$, and therefore $I$ is strongly $\pi$-regular. □

The index of a nilpotent element in a ring is the least positive integer $n$ such that $x^n = 0$. The index $i(I)$ of an ideal $I$ of a ring $R$ is the supremum of the indices of all nilpotent elements of $I$. An ideal $I$ of a ring $R$ is of bounded index if $i(I) < \infty$. It is well known that $i(I) \leq n$ if and only if $I$ contains no direct sums of $n + 1$ nonzero pairwise isomorphic right ideals (cf. [9, Theorem 7.2]).

**Theorem 2.4.** Let $R$ be a ring, and let

$$I = \{a \in R \mid i(RaR) < \infty\}.$$

Then $I$ is a strongly $\pi$-regular ideal of $R$.

**Proof.** Let $x, y \in I$ and $z \in R$. Then $RxzR, RzxR \subseteq RxR$. This implies that $RxzR$ and $RzxR$ are strongly $\pi$-regular of bounded index. Hence, $xz, zx \in I$.

Obviously, $R(x - y)R \subseteq RxR + RyR$. For any $a \in R(x - y)R, a = c + d$ where $c \in RxR$ and $d \in RyR$. Since $RxR$ is strongly $\pi$-regular, there exists some $n \in \mathbb{N}$ such that $c^n = c^{n+1}r$ for a $r \in R$. Let $RyR$ is of bounded index $m$. Then $c^n = c^{nm+1}s$ for a $s \in R$. Hence, $a^{nm+1}s - a^n \in RyR$. As $RyR$ is strongly $\pi$-regular, we can find $k \in \mathbb{N}$ and $d \in RyR$ such that

$$(a^{nm+1}s - a^n)^k = (a^{nm+1}s - a^n)^{k+1}d,$$

$$d = d(a^{nm+1}s - a^n)d,$$

$$d(a^{nm+1}s - a^n) = (a^{nm+1}s - a^n)d.$$

Hence,

$$\left((a^{nm+1}s - a^n) - (a^{nm+1}s - a^n)^2d\right)^k = (a^{nm+1}s - a^n)^k(1 - (a^{nm+1}s - a^n)d)^k,$$

$$= (a^{nm+1}s - a^n)^k(1 - (a^{nm+1}s - a^n)d),$$

$$= 0.$$

Therefore $a^{nm+1}s - a^n)^m = (a^{nm+1}s - a^n)^{m+1}$. As a result, $a^{nm} \in a^{nm+1}R$. Hence, we can find $r \in R$ such that $a^{nm} = a^{nm+1}(ar)$. Therefore $I$ is a strongly $\pi$-regular ideal of $R$. □

**Corollary 2.5.** Let $R$ be a ring of bounded index. Then

$$I = \{a \in R \mid RaR \text{ is strongly } \pi\text{-regular}\}$$

is the maximal strongly $\pi$-regular ideal of $R$. 


Proof. Since $R$ is of bounded index, so is $RaR$ for any $a \in R$. In view of Theorem 2.4, $I = \{a \in R \mid RaR$ is strongly $\pi$-regular$\}$ is a strongly $\pi$-regular ideal of $R$. Thus we complete the proof by Proposition 2.3.  \hfill \Box

**Example 2.6.** Let $V$ be an infinite-dimensional vector space over a field $F$, let $R = \text{End}_F(V)$, and let $I = \{\sigma \in R \mid \dim F \sigma(V) < \infty\}$. Then $I$ is strongly $\pi$-regular, while $R$ is not strongly $\pi$-regular.

**Proof.** Clearly, $I$ is an ideal of the ring $R$. We have the descending chain $\sigma(V) \supseteq \sigma^2(V) \supseteq \cdots$. As $\dim F \sigma(V) < \infty$, we can find some $n \in \mathbb{N}$ such that $\sigma^n(V) = \sigma^{n+1}(V)$. Since $V$ is a projective right $F$-module, we can find some $\tau \in R$ such that the following diagram

$$
\begin{array}{c}
V \\
\downarrow \sigma^n \\
\sigma^{n+1}(V)
\end{array}
$$

commutes, i.e., $\sigma^{n+1} \tau = \sigma^n$. Hence, $\sigma^n = \sigma^{n+1}(\tau^2)$. Therefore $I$ is a strongly $\pi$-regular ideal of $R$. Let $\varepsilon$ be an element of $R$ such that $\varepsilon(x_i) = x_i+1$ where $\{x_1, x_2, \ldots\}$ is the basis of $V$. If $R$ is strongly $\pi$-regular, there exists some $m \in \mathbb{N}$ such that $\varepsilon^mR = \varepsilon^{m+1}R$, and so $\varepsilon^m(V) = \varepsilon^{m+1}(V)$. As $\varepsilon^m(x_i) = x_i+m$ for all $i$, we see that $\varepsilon^m(V) = \sum_{i \geq m} x_iF \neq \sum_{i \geq m+1} x_iF = \varepsilon^{m+1}(V)$. This gives a contradiction. Therefore $R$ is not a strongly $\pi$-regular ring. \hfill \Box

**Example 2.7.** Let $V$ be an infinite-dimensional vector space over a field $F$, let $R = \text{End}_F(V)$, and let $S = (\begin{smallmatrix} 0 & R \\ R & 0 \end{smallmatrix})$. Then $I = \{\sigma \mid \sigma = (\begin{smallmatrix} 0 & R \\ R & 0 \end{smallmatrix})\}$ is a strongly $\pi$-regular ideal of $R$, while $S$ is not a strongly $\pi$-regular ring.

**Proof.** By the discussion in Example 2.6, $R$ is not strongly $\pi$-regular. Hence, $S$ is not strongly $\pi$-regular. As $I^2 = 0$, one easily checks that $I$ is a strongly $\pi$-regular ideal of the ring $S$. \hfill \Box

An ideal $I$ of a ring $R$ is called a gs-$\pi$-ideal if for any $a \in I$ there exists some integer $n \geq 2$ such that $a^nRa^n$. For instance, every ideal of strongly regular rings is a gs-$\pi$-ideal.

**Example 2.8.** Every gs-$\pi$-ideal of a ring is strongly $\pi$-regular.

**Proof.** Let $I$ be a gs-$\pi$-ideal of a ring $R$. Given $\pi^2 = \sigma$ in $I/(I \cap J(R))$, then $x^2 \in I \cap J(R)$. As $I$ is a gs-$\pi$-ideal, we see that $Rx = x^2Rx^2 \subseteq J(R)$, i.e., $(RxR)^2 \subseteq J(R)$. As $J(R)$ is semiprime, it follows that $RxR \subseteq J(R)$, and so $x \in J(R)$. That is, $\pi = 0$. This implies that $I/(I \cap J(R))$ is reduced. For any idempotent $e \in I/(I \cap J(R))$ and any $a \in R/J(R)$, it follows from $(ea(\pi - e))^2 = 0$ that $ea(\pi - e) = 0$, thus $ea = eae$. Likewise, $ae = eae$. This implies that $ea = ae$. As a result, every idempotent in $I/(I \cap J(R))$ is central. For any $x \in I \cap J(R)$, there exists some $y \in R$ such that $x^2 = x^2yx^2$, and then $x^2(1 - yx^2) = 0$. This implies that $x^2 = 0$. Conversely, we let $x^2 = 0$. As $I$ is a gs-$\pi$-ideal, we see that $Rx = x^2Rx^2 = 0$. That is, $(RxR)^2 \subseteq J(R)$, and
so \( x \in J(R) \). Therefore \( I \cap J(R) = \{ x \in I \mid x^2 = 0 \} \). Let \( x \in I \). Then there exists some \( n \geq 2 \) such that \( xRx = x^nRx^n \). Hence, \( x^2 = x^2yx^2 \). As \( x^2y \in I \) is an idempotent, we see that \( x^2 - x^6y^2 \in I \cap J(R) \). By the preceding discussion, we get \( (x^2 - x^6y^2)^2 = 0 \). This implies that \( x^4 = x^5r \) for some \( r \in I \). Thus \( I \) is strongly \( \pi \)-regular.

3. Stable range condition

For any \( x, y \in R \), write \( x \circ y = x + y + xy \). We use \( x^{[n]} \) to stand for \( x \circ \cdots \circ x \) \( (n \geq 1) \) and \( x^{[0]} = 0 \). The following result was firstly observed in \( [8, \text{Lemma 1}] \), we include a simple proof to make the paper self-contained.

**Lemma 3.1.** Let \( x_i, y_j \in R \), and let \( p_i, q_j \in \mathbb{Z} \) \((1 \leq i \leq m, 1 \leq j \leq n) \). If \( \sum_i p_i = \sum_j q_j = 1 \), then \( (\sum_i p_i x_i) \circ (\sum_j q_j y_j) = \sum_{i,j} (p_i q_j)(x_i \circ y_j) \). If \( \sum_i p_i = \sum_j q_j = 0 \), then \( (\sum_i p_i x_i)(\sum_j q_j y_j) = \sum_{i,j} (p_i q_j)(x_i \circ y_j) \).

**Proof.** For any \( p_i, q_j \in \mathbb{Z} \), one easily checks that

\[
\sum_{i,j} (p_i q_j)(x_i \circ y_j) = (\sum_i p_i x_i)(\sum_j q_j y_j) + (\sum_j q_j y_j)(\sum_i p_i x_i) + (\sum_i p_i)(\sum_j q_j y_j).
\]

Therefore the result follows. \( \square \)

**Lemma 3.2.** Let \( I \) be a strongly \( \pi \)-regular ideal of a ring \( R \). Then for any \( x \in I \), there exists some \( n \in \mathbb{N} \) such that \( x^{[n]} = x^{[n+1]} \circ y = z \circ x^{[n+1]} \) for \( y, z \in I \).

**Proof.** Let \( x \in I \). Then \( -x - x^2 \in I \). Since \( I \) is a strongly \( \pi \)-regular ideal, there exists some \( n \in \mathbb{N} \) such that \( (-x - x^2)^n = (-x - x^2)^{n+1} s = s(-x - x^2)^{n+1} \). Clearly, \( x - x^{[2]} = -x - x^2 \). Thus,

\[
(x - x^{[2]})^n = (x - x^{[2]})^{n+1} s = (x - x^{[2]})^{2n} t,
\]

where \( t = s^n \). Since \( \sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0 \), it follows from Lemma 3.1 that

\[
\sum_{i=0}^{n} (-1)^i \binom{n}{i} (x^{[n-i]} \circ (x^{[2]})^i) = (x - x^{[2]})^n.
\]

Thus,

\[
(x - x^{[2]})^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x^{[n+i]}
\]

\[
= x^{[n]} + \sum_{i=1}^{n} (-1)^i \binom{n}{i} x^{[n+i]}.
\]
Let \( u = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} x[i] \). Then \( u \circ x^n = x^n \circ u \). Since \( \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} \) = 1, by using Lemma 2.1 again, \( (x - x^2)^n = x^n - x^n \circ u \). Thus, we get

\[
x^n - x^n \circ u = (x^n - x^n \circ u)^2 t = (x^n - x^n \circ u)(x^n - x^n \circ u)(t - 0) = (x^{2n} - x^{2n} \circ u - x^{2n} \circ u + x^{2n} \circ u^2)(t - 0) = x^{2n} \circ (t - u \circ t - u \circ t + u^2 \circ t + u + u - u^2) - x^{2n} = x^{2n} \circ (u^2 t) - x^{2n}.
\]

Let \( v = x^{2n} \circ (u^2 t) - x^{2n} \). Then

\[
x^n = x^n \circ u + v = (x^n \circ u + v - 0) \circ u + v = x^n \circ u^2 + (v \circ u - u) + v = \cdots = x^n \circ u^{n+1} + \sum_{i=0}^{n} (v \circ u^i - u^i).
\]

Further,

\[
v \circ u^i - u^i = (x^{2n} \circ (u^2 t) - x^{2n} \circ u^i) \circ u^i - u^i = (x^{2n} \circ (u^2 t) - x^{2n} + 0) \circ u^i - u^i = x^{2n} \circ (u^2 t) \circ u^i - x^{2n} \circ u^i.
\]

Hence

\[
x^n = x^n \circ u^{n+1} + \sum_{i=0}^{n} (x^{2n} \circ (u^2 t) \circ u^i - x^{2n} \circ u^i).
\]

Further, we see that

\[
\sum_{i=0}^{n} (x^{2n} \circ (u^2 t) \circ u^i - x^{2n} \circ u^i) = x^{2n} \circ (\sum_{i=0}^{n} (u^2 t) \circ u^i - u^i) + 0) - x^{2n}.
\]

As \( \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} \) = 1, we see that

\[
u^{n+1} = (\sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} x[i])^{n+1} = \sum_{i_1 + \cdots + i_n = n+1} C_{i_1, \ldots, i_n} x^{i_1 + i_2 + \cdots + i_n} = \sum_{i_1 + \cdots + i_n = n+1} C_{i_1, \ldots, i_n} x^{[1+i_2+i_3+\cdots+(n-1)i_n]}.
\]
It is easy to check that \( \sum_{i=1}^{n} u \) and so \( u^{[n+1]} = x^{[n]} \circ v, \) where \( v = \sum_{i=1}^{n} C_{i1, \ldots, in} x^{[1+i+\cdots+(n-1)i]} \). Therefore

\[
x^{[n]} = x^{[2n]} \circ v + x^{[2n]} \circ (\sum_{i=0}^{n} ((u^2t) \circ u^{[i]} - u^{[i]})) - x^{[2n]}
\]

\[
= x^{[2n]} \circ (v + \sum_{i=0}^{n} ((a^2t) \circ u^{[i]} - u^{[i]})) - 0
\]

\[
= x^{[2n]} \circ (v + \sum_{i=0}^{n} ((u^2t) \circ u^{[i]} - u^{[i]})).
\]

Let \( y = x^{[n-1]} \circ (v + \sum_{i=0}^{n} ((u^2t) \circ u^{[i]} - u^{[i]})). \) Then \( x^{[n]} = x^{[n+1]} \circ y \) with \( y \in I. \) Likewise, \( x^{[n]} = z \circ x^{[n+1]} \) for \( z \in I, \) as required. \( \square \)

**Theorem 3.3.** Every strongly \( \pi \)-regular ideal of a ring is a \( B \)-ideal.

**Proof.** Let \( I \) be a strongly \( \pi \)-regular ideal of a ring \( R. \) Let \( a \in 1 + I. \) Then \( a-1 \in I. \) In view of Lemma 2.2, we can find some \( n \in \mathbb{N}, b, c \in 1 + I \) such that \( (a-1)^{[n]} = (a-1)^{[n+1]} \circ (b-1) = (c-1) \circ (a-1)^{[n+1]} \). One easily checks that \( (a-1)^{[n]} = a^n - 1 \) and \( (a-1)^{[n+1]} = a^{n+1} - 1. \) Therefore \( a^n = a^{n+1} \circ b = ca^{n+1}, \) and so \( a^n \in a^{n+1} R \cap Ra^{n+1}. \) According to [5, Proposition 13.1.2], \( a \in 1 + I \) is strongly \( \pi \)-regular. According to [5, Theorem 13.1.7], \( I \) is a \( B \)-ideal. \( \square \)

**Corollary 3.4.** Let \( I \) be a strongly \( \pi \)-regular ideal of a ring \( R, \) and let \( A \) be a finitely generated projective right \( R \)-module. If \( A = AI, \) then for any right \( R \)-modules \( B \) and \( C, \) \( A \oplus B \cong A \oplus C \) implies that \( B \cong C. \)

**Proof.** For any \( x \in I, \) we have \( n \in \mathbb{N} \) and \( y \in R \) such that \( x^n = x^{n+1} y \) and \( xy = yx. \) Hence \( x^n = x^n z x^n, \) where \( z = y^n. \) Let \( g = zx^n \) and \( e = g + (1-g)x^n g. \) Then \( e \in Rx \) is an idempotent. In addition, we have \( 1 - e = (1-g)(1-x^n g) = (1-g)(1-x^n) \in Rx. \) Set \( f = 1-e. \) Then there exists an idempotent \( f \in I \) such that \( f \in Rx \) and \( 1 - f \in Rx. \) Therefore \( I \) is an exchange ideal of \( R. \) In view of Theorem 3.3, \( I \) is a \( B \)-ideal. Therefore we complete the proof by [5, Lemma 13.1.9]. \( \square \)

**Corollary 3.5.** Let \( I \) be a strongly \( \pi \)-regular ideal of a ring \( R, \) and let \( a, b \in 1 + I. \) If \( aR = bR, \) then \( a = bu \) for some \( u \in U(R). \)

**Proof.** Write \( ax = b \) and \( a = by. \) As \( a, b \in 1 + I, \) we see that \( x, y \in 1 + I. \) In view of Theorem 3.3, \( I \) is a \( B \)-ideal. Since \( yx + (1-yx) = 1, \) there exists an element \( z \in R \) such that \( u := y + (1-yx)z \in U(R). \) Therefore \( bu = b(y + (1-yx)z) = by = a, \) as required. \( \square \)

**Corollary 3.6.** Let \( I \) be a strongly \( \pi \)-regular ideal of a ring \( R, \) and let \( A \in M_n(\mathbb{I}) \) be regular. Then \( A \) is the product of an idempotent matrix and an invertible matrix.
Proof. By virtue of Theorem 3.3, \( I \) is a \( B \)-ideal. As \( A \in M_n(I) \) is regular, we have a \( B \in M_n(I) \) such that \( A = ABA \). Since \( AB + (I_n - AB) = I_n \), we get \( (A + (I_n - AB))B + (I_n - AB)(I_n - B) = I_n \) where \( A + (I_n - AB) \in I_n + M_n(I) \). Thus, we can find a \( Y \in M_n(R) \) such that \( U := A + (I_n - AB) + (I_n - AB)(I_n - B)Y \in GL_n(R) \). Therefore \( A = ABA = AB(A + (I_n - AB) + (I_n - AB)(I_n - B)Y) = ABU \), as required.

Let \( A \) be an algebra over a field \( F \). An element \( a \) of an algebra \( A \) over a field \( F \) is said to be algebraic over \( F \) if \( a \) is the root of some non-constant polynomial in \( F[x] \). An ideal \( I \) of \( A \) is said to be an algebraic ideal of \( A \) if every element in \( I \) is algebraic over \( F \).

Proposition 3.7. Let \( A \) be an algebra over a field \( F \), and let \( I \) be an algebraic ideal of \( A \). Then \( I \) is a \( B \)-ideal.

Proof. For any \( a \in I \), \( a \) is the root of some non-constant polynomial in \( F[x] \). So we can find \( a_m, \ldots, a_n \in F \) such that \( a_n a^n + a_{n-1} a^{n-1} + \cdots + a_0 a^0 = 0 \), where \( a_m \neq 0 \). Hence, \( a^m = -(a_n a^n + \cdots + a_{m+1} a^{m+1})a^{-1} = -a^{m+1}(a_n a^{n-1} + \cdots + a_{m+1})a^{-1} \). Set \( b = -(a_n a^{n-1} + \cdots + a_{m+1})a^{-1} \). Then \( a^m = a^{m+1}b \). Therefore \( I \) is strongly \( \pi \)-regular, and so we complete the proof by Theorem 3.3.

In the proof of Theorem 3.3, we show that for any \( a \in 1 + I \), there exists some \( n \in \mathbb{N} \) such that \( a^n = a^{n+1}b \) for a \( b \in 1 + I \) if \( I \) is a strongly \( \pi \)-regular ideal. A natural problem asks that if the converse of the preceding assertion is true. The answer is negative from the following counterexample. Let \( p \in \mathbb{Z} \) be a prime and set \( \mathbb{Z}_{(p)} = \{ a/b \mid b \not\in \mathbb{Z}p \text{ (in lowest terms)} \} \). Then \( \mathbb{Z}_{(p)} \) is a local ring with maximal \( p\mathbb{Z}_{(p)} \). Thus, the Jacobson radical \( p\mathbb{Z}_{(p)} \) satisfies the condition above. Choose \( p/(p+1) \in p\mathbb{Z}_{(p)} \). Then \( p/(p+1) \in J(\mathbb{Z}_{(p)}) \) is not nilpotent. This shows that \( p\mathbb{Z}_{(p)} \) is not strongly \( \pi \)-regular.

4. Periodic ideals

An ideal \( I \) of a ring \( R \) is periodic provided that for any \( x \in I \) there exist distinct \( m, n \in \mathbb{N} \) such that \( x^m = x^n \). We note that an ideal \( I \) of a ring \( R \) is periodic if and only if for any \( a \in I \), there exists a potent element \( p \in I \) such that \( a - p \) is nilpotent and \( ap = pa \).

Lemma 4.1. Let \( I \) be an ideal of a ring \( R \). If \( I \) is periodic, then for any \( x \in 1 + I \) there exist \( m \in \mathbb{N} \) such that \( x^m = x^{m+1}f(x) \).

Proof. For any \( a \in I \), there exists some \( n \in \mathbb{N} \) such that \( a^n = a^{n+1}(a^{m-n-1}) \) where \( m \geq n + 1 \). For any \( x \in 1 + I \), we see that \( x - 1 \in I \). As in the proof in Lemma 3.2, we can find \( f(t) \in R[t] \) such that \( (x-1)^{[n]} = (x-1)^{[n+1]} \circ f(x-1) \). One easily checks that \( (x-1)^{[n]} = x^n - 1 \) and \( (x-1)^{[n+1]} = x^{n+1} - 1 \). Therefore \( x^n = x^{n+1}f(x) \), as required.
Lemma 4.2. Let $R$ be a ring, and let $c \in R$. If there exist a monic $f(t) \in \mathbb{Z}[t]$ and some $m \in \mathbb{N}$ such that $mc = 0$ and $f(c) = 0$, then there exist $s, t \in \mathbb{N}(s \neq t)$ such that $c^s = c^t$.

Proof. Clearly, $\mathbb{Z}c \subseteq \{0, c, \ldots, (m - 1)c\}$. Write $f(t) = t^k + b_1t^{k-1} + \ldots + b_{k-1}t + b_k \in \mathbb{Z}[t]$. Then $c^{k+1} = -b_1c^k - \ldots - b_{k-1}c^2 - b_k c$. This implies that $\{c, c^2, c^3, \ldots\} \subseteq \{c, c^2, c^3, \ldots, c^k, 0, c, \ldots, (m - 1)c, c^2, \ldots, (m - 1)c^k\}$. That is, $\{c, c^2, c^3, \ldots\}$ is a finite set. Hence, we can find some $s, t \in \mathbb{N}, s \neq t$ such that $c^s = c^t$, as desired. \hfill \Box

As is well known, a ring $R$ is periodic if and only if for any $x \in R$, there exist $n \in \mathbb{N}$ and $f(t) \in \mathbb{Z}[t]$ such that $x^n = x^{n+1}f(x)$. We extend this result to periodic ideals.

Lemma 4.3. Let $I$ be an ideal of a ring $R$. If for any $x \in I$, there exist $n \in \mathbb{N}$ and $f(t) \in \mathbb{Z}[t]$ such that $x^n = x^{n+1}f(x)$, then $I$ is periodic.

Proof. Let $x \in I$. If $x$ is nilpotent, then we can find some $n \in \mathbb{N}$ such that $x^n = x^{n+1} = 0$. Thus, we may assume that $x \in I$ is not nilpotent. By hypothesis, there exist $n \in \mathbb{N}$ and $g(t) \in \mathbb{Z}[t]$ such that $x^n = x^{n+1}g(x)$. Thus, $x^n = x^{n+1}f(x)$, where $f(x) = x(g(x))^2 \in \mathbb{Z}[t]$. In addition, $f(0) = 0$. Let $e = x^n(f(x))^2$. Then $0 \neq e = c^2 \in R$ and $x^n = xe$. Set $S = cRe$ and $\alpha = ex = xe$. Then $f(\alpha) = ef(x)$. Further,

$$\alpha^n(f(\alpha))^n = e, \alpha^n = x^n, \alpha^n = \alpha^{n+1}f(\alpha).$$

Thus, $e = \alpha^n(f(\alpha))^n = \alpha^n(f(\alpha))^{n+1} = \alpha^n(f(\alpha))^n\alpha f(\alpha) = c\alpha f(\alpha) = \alpha f(\alpha)$. Write $f(t) = at^n + \ldots + a_1t + a_0$. Then $\alpha(a_1\alpha + \cdots + a_na^n) = e$. This implies that $(\alpha^{-1})^{n+1} - a_1(\alpha^{-1})^{n-1} - \cdots - a_n \in \mathbb{Z}[t]$. Then $g(t)$ is a monic polynomial such that $g(\alpha^{-1}) = 0$.

Let $T = \{me \in S \mid m \in \mathbb{Z}\}$. Then $T$ is a subring of $S$. For any $me \in I$, by hypothesis, there exists $g(t) \in \mathbb{Z}[t]$ such that $(me)^p = (me)^{p+1}g(me) \in (me)^{p+1}T$. This implies that $T$ is strongly $\pi$-regular. Construct a map $\varphi : Z \to T, m \mapsto me$. Then $Z/\text{Ker}\varphi \cong T$. As $Z$ is not strongly $\pi$-regular, we see that $\text{Ker}\varphi \neq 0$. Hence, $T \cong \mathbb{Z}_q$ for some $q \in \mathbb{N}$. Thus, $qe = 0$. As a result, $q\alpha^{-1} = 0$. In view of Lemma 4.2, we can find some $s, t \in \mathbb{N}(s \neq t)$ such that $(\alpha^{-1})^s = (\alpha^{-1})^t$. This implies that $\alpha^s = \alpha^t$. Hence, $x^{ns} = x^{nt}$, as asserted. \hfill \Box

Theorem 4.4. Let $I$ be an ideal of a ring $R$. Then $I$ is periodic if and only if

(1) $I$ is strongly $\pi$-regular,
(2) For any $u \in U(I)$, $u^{-1} \in \mathbb{Z}[u]$.

Proof. Suppose that $I$ is periodic. Then $I$ is strongly $\pi$-regular. For any $u \in U(I)$, it follows by Lemma 4.1 that there exist $m \in \mathbb{N}, f(t) \in \mathbb{Z}[t]$ such that $u^n = u^{n+1}f(u)$. Hence, $uf(u) = 1$, and so $u^{-1} \in \mathbb{Z}[u]$.

Suppose that (1) and (2) hold. For any $x \in I$, there exist $m \in \mathbb{N}$ and $y \in I$ such that $x^n = x^ny^m, y = yx^m$ and $xy = yx$ from [5, Proposition
13.1.15]. Set \( u = 1 - x^m y + x^m \). Then \( u^{-1} = 1 - x^m y + y \). Hence, \( u \in U(I) \).
By hypothesis, there exists \( g(t) \in \mathbb{Z}[t] \) such that \( ud(u) = 1 \). Further, \( x^m = x^m y (1 - x^m y + x^m) = x^m y u \). Hence, \( x^m u^{-1} = x^m y \), and so \( x^m = x^m y x^m = x^{2m} g(u) = x^{2m} x^m (g(u))^2 \). Write \( (g(u))^2 = b_0 + b_1 u + \cdots + b_n u^n \in \mathbb{Z}[u] \). For any \( n \geq 0 \), it is easy to check that \( x^m u^i = x^m (1 - x^m y + x^m)^i \in \mathbb{Z}[x] \). This implies that \( x^m (g(u))^2 \in \mathbb{Z}[x] \). According to Lemma 4.3, \( I \) is periodic. \( \Box \)

It follows by Theorem 4.4 and Theorem 3.3 that every periodic ideal of a ring is a \( B \)-ideal.

**Corollary 4.5.** Let \( I \) be a strongly \( \pi \)-regular ideal of a ring \( R \). If \( U(I) \) is torsion, then \( I \) is periodic.

**Proof.** For any \( u \in U(I) \), there exists some \( m \in \mathbb{N} \) such that \( u^m = 1 \). Hence, \( u^{-1} = u^{m-1} \in \mathbb{Z}[u] \). According to Theorem 4.4, we complete the proof. \( \Box \)

**Example 4.6.** Let \( R = \left( \begin{array}{cc} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & 0 \end{array} \right) \) and \( I = \left( \begin{array}{cc} 0 & \mathbb{Z} \\ \mathbb{Z} & 0 \end{array} \right) \). Then \( I \) is a nilpotent ideal of \( R \); hence, \( I \) is strongly \( \pi \)-regular. Clearly, \( \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \in U(I) \), but \( \left( \begin{array}{c} 1 \\ 1 \end{array} \right)^m \neq 0 \) for any \( m \in \mathbb{N} \). Thus, \( U(I) \) is torsion.

The example above shows that the converse of Corollary 4.6 is not true. But we can derive the following.

**Proposition 4.7.** Let \( I \) be an ideal of a ring \( R \). If \( \text{char}(R) \neq 0 \), then \( I \) is periodic if and only if

1. \( I \) is strongly \( \pi \)-regular,
2. \( U(I) \) is torsion.

**Proof.** Suppose that \( I \) is periodic. Then \( I \) is strongly \( \pi \)-regular. Let \( x \in U(I) \). Then \( x \) is not nilpotent. By virtue of Lemma 4.1, there exist \( m \in \mathbb{N} \), \( f(t) \in \mathbb{Z}[t] \) such that \( x^m = x^{m+1} f(x) \). As in the proof of Lemma 4.3, we have a monic polynomial \( g(t) \in \mathbb{Z}[t] \) such that \( g(a^{-1}) = 0 \). As \( \text{char}(R) \neq 0 \), we assume that \( \text{char}(R) = q \neq 0 \). Then \( qa^{-1} = 0 \). According to Lemma 4.2, we can find two distinct \( s, t \in \mathbb{N} \) such that \( (a^{-1})^s = (a^{-1})^t \). Similarly to Lemma 4.3, \( x^{ts} = x^{st} \), and so \( x \) is torsion. Therefore \( U(I) \) is torsion. The converse is true by Corollary 4.5. \( \Box \)

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