Decision Support
Equilibrium in an ambiguity-averse mean–variance investors market

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A R T I C L E  I N F O

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A B S T R A C T

In a financial market composed of \( n \) risky assets and a riskless asset, where short sales are allowed and mean–variance investors can be ambiguity averse, i.e., diffident about mean return estimates where confidence is represented using ellipsoidal uncertainty sets, we derive a closed form portfolio rule based on a worst case max–min criterion. Then, in a market where all investors are ambiguity-averse mean–variance investors with access to given mean return and variance–covariance estimates, we investigate conditions regarding the existence of an equilibrium price system and give an explicit formula for the equilibrium prices. In addition to the usual equilibrium properties that continue to hold in our case, we show that the diffidence of investors in a homogeneously diffident (with bounded diffidence) mean–variance investors’ market has a deflationary effect on equilibrium prices with respect to a pure mean–variance investors’ market in equilibrium. Deflationary pressure on prices may also occur if one of the investors (in an ambiguity-neutral market) with no initial short position decides to adopt an ambiguity-averse attitude. We also establish a CAPM-like property that reduces to the classical CAPM in case all investors are ambiguity-neutral.

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1. Introduction and background

A major theme in mathematical finance is the study of investors’ portfolio decisions using the well-established theory of mean–variance that began with the seminal work of Markowitz (1952). The mean–variance portfolio theory then formed the basis of the celebrated Capital Asset Pricing Model (CAPM) (Sharpe, 1964), the most commonly used equilibrium and pricing model in the financial literature. However, it is a well-known fact that the investors’ portfolio holdings in the mean–variance portfolio theory are very sensitive to the estimated mean returns of the risky assets; see e.g., Best and Grauer (1991a), Best and Grauer (1991b), Black and Litterman (1992). The purpose of the present paper is to investigate equilibrium relations in a financial market composed of \( n \) risky assets and a riskless asset using an approach that takes into account the imprecision in the mean return estimates. In our model, investors act as mean–variance investors with a degree of diffidence (or confidence) towards the mean return estimates of risky assets. We refer to this attitude of diffidence as ambiguity aversion to distinguish it from risk aversion quantified by a mean–variance objective function. Decision making under ambiguity aversion is an active research area in decision theory and economics; see e.g., Klibanoff, Marinacci, and Mukerji (2005, 2009). Our study follows the earlier work of Konno and Shirakawa (1994, 1995), and is in particular inspired by the previous work of Deng, Li, and Wang (2005) where the authors study a similar problem allowing the mean returns of risky assets to vary over a hyper-rectangle, i.e., an interval is specified for each mean return estimate and a max–min approach is used in the portfolio choice as in the present paper. We adopt an ellipsoidal uncertainty set instead of a hyper-rectangle, and obtain a closed-form portfolio rule using a worst-case max–min approach as in the robust optimization framework of Ben-Tal and Nemirovski (1999, 1998). In contrast, in Deng et al. (2005) a closed-form portfolio rule is not possible due to the polyhedral nature of their ambiguity representation. The ellipsoidal model controls the diffidence of investors using a single positive parameter \( \epsilon \) while the interval model of Deng et al. (2005) requires the specification of an interval for each risky asset, and has to resort to numerical solution of a linear programming problem to find a worst-case interval of return vector in the hyper-rectangle. The linear programming nature of the procedure may cause several components of the rate of return vector in question to assume the lower or upper end values of the interval as a by-product of the simplex method (i.e., an extreme point of the hyper-rectangle will be found). Since the worst case return occurs at an extreme point of the hyper-rectangle, it corresponds to an extreme scenario where most (or all) risky assets assume their worst possible return values, which may translate into an unnecessarily
conservative portfolio. Such extreme behavior does not occur with an ellipsoidal representation of the uncertainty set due to the non-linear geometry of the ellipsoid. Besides, the ellipsoidal representation is also motivated by statistical considerations alluded to in Section 2. As in Deng et al. (2005), in the contributions of Konno and Shirakawa (1994, 1995) where short sales are not allowed, the formula for the equilibrium price vector requires the solution of an optimization problem as input to the formula whereas we have an explicit formula for the equilibrium price.

To the best of our knowledge, the present paper is one the few studies next to Deng et al. (2005), Wu, Song, Xu, and Liu (2009) to incorporate ambiguity aversion in asset returns in an equilibrium framework. However, unlike the present paper, in neither Deng et al. (2005) nor Wu et al. (2009) there is a truly closed-form result, and furthermore they do not study the impact of ambiguity aversion on equilibrium prices.

The seminal results on equilibrium in capital markets were established in the early works of Lintner (1965), Mossin (1966) and Sharpe (1964), which resulted in the celebrated CAPM; see Elton and Gruber (1991), Markowitz (1987) for textbook treatments of the subject. The theory of equilibrium in capital asset markets were later extended in several directions in e.g., Black (1972), Jensen (1987, 1989, 1980, 1989). In a recent study, Rockafellar, Uryasev, and Zabarankin (2007) use the so-called diversions measures (an example is Conditional Value at Risk, CVaR) to investigate equilibrium in capital markets. Balbás, Balbás, and Balbás (2010) use coherent risk measures, expectation bounded risk measures and general deviations in optimal portfolio problems, and study CAPM-like relations. Grechuk and Zabarankin (2012) consider an optimal risk sharing problem among agents with utility functionals depending only on the expected value and a deviation measure of an uncertain payoff. They characterize Pareto optimal solutions and study the existance of an equilibrium. Kalinchenko, Uryasev, and Rockafellar (2012) use the generalized CAPM based on mixed Conditional Value at Risk deviation for calibrating the risk preferences of investors. Hasuikè (2010) use fuzzy numbers to represent investors’ preferences in an extension of the CAPM. Zabarankin, Pavlikov, and Uryasev (2014) use the Conditional Drawdown-at-Risk (CDaR) measure to study optimal portfolio selection and CAPM-like equilibrium models. Won and Yannelis (2011) examine equilibrium with an application to financial markets without a riskless asset where uncertainty makes preferences incomplete. They assume a normal distribution for the mean return with an uncertainty set due to the non-linear geometry of the ellipsoid. Besides, the ellipsoidal representation is also motivated by statistical considerations alluded to in Section 2. As in Deng et al. (2005), in the contributions of Konno and Shirakawa (1994, 1995) where short sales are not allowed, the formula for the equilibrium price vector requires the solution of an optimization problem as input to the formula whereas we have an explicit formula for the equilibrium price.

The paper is organized as follows. In Section 2 we examine the problem of portfolio choice of an ambiguity-averse investor using an ellipsoidal ambiguity set and worst case max–min criterion. We derive an explicit optimal portfolio rule. In Section 3, we study conditions under which an equilibrium system of prices exist in different capital markets characterized by the presence of ambiguity-averse or neutral investors, and give an explicit formula for equilibrium prices. We illustrate the results with a numerical example. Section 4 gives some properties of equilibrium. In particular, separation and proportion properties are shown, as well as a CAPM-like result which reduces to the classical CAPM when investors have full confidence in the estimates of mean rate of return. We conclude in Section 5 with a summary and future research directions.

2. Ambiguity-averse mean–variance investor’s portfolio rule

Let the price per share of asset $j$ be denoted $p_j$, $j = 1, 2, \ldots, n$ for the first $n$ risky assets in the market, we assume the price of the $(n+1)$th riskless asset to be equal to one. We denote by $x^0_j$ the number of shares of asset $j$ held initially by the investor while we use $x_j$ to denote the number of shares of asset $j$ held by the investor after the transaction, for all $j = 1, 2, \ldots, n+1$. Unlimited short positions are allowed, i.e., there is no sign restriction on $x_j$.

The $n$ risky assets have random rate of return vector $\hat{r} = (r_1, r_2, \ldots, r_n)$ and estimate of mean rate of return vector $\hat{r} = (\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_n)$ (that we shall also refer to as the nominal rate of return) with variance–covariance matrix estimate $\hat{\Sigma}$ which is assumed positive definite. The $(n+1)$th position is reserved for the riskless asset with deterministic rate of return equal to $k$. The investor has a risk aversion coefficient $\omega \in (0, 1)$ and an initial endowment $W_0$ assumed positive such that

$$W_0 = \sum_{j=1}^{n} p_j x_j^0 + x_n^0.$$

Since there are no withdrawals from and injections to the portfolio, we still have, after the transaction,

$$W_0 = \sum_{j=1}^{n} p_j x_j + x_n.$$

Dividing the last equation by $W_0$ and defining the proportions

$$y_j \equiv \frac{x_j}{W_0}$$

for $j = 1, \ldots, n+1$ we have that

$$\sum_{j=1}^{n+1} y_j = 1.$$

If we denote the true (unknown) mean rates of return by $\bar{r}_j$ for $j = 1, \ldots, n$ the mean rate of return of portfolio $x$ (with proportions $y_j$) is equal to

$$\sum_{j=1}^{n} r_j y_j + R_{n+1},$$

with variance equal to $\sum_{j=1}^{n} \sum_{k=1}^{n} \Gamma_k y_k y_j = \mathbf{y}^T \Sigma \mathbf{y}$ where $\mathbf{y}$ denotes the vector with components $(y_1, \ldots, y_n)$. Note that the random end-of-period wealth $W_1$ is given as
The investor is also ambiguity averse with ambiguity aversion coefficient $\epsilon$ such that his/her confidence in the mean rate of return vector estimate is expressed as a belief that the true mean rate of return lies in the ellipsoidal set

$$U_r = \{ r \in \mathbb{R}^{1/2} (r - \hat{r}) \leq \epsilon \},$$

that is, an $n$-dimensional ellipsoid centered at $\hat{r}$ (the estimated mean return vector) with radius $\epsilon$. The idea is that the decisions of an ambiguity averse investor are made by considering the worst case occurrences of the true mean rate of return $r$ within the set $U_r$. Therefore, more conservative portfolio choices are made when the volume of the ellipsoid is larger, i.e., for greater values of $\epsilon$, while an ambiguity-neutral investor with no doubt about errors in the estimated values sets $\epsilon$ equal to zero. The differences between the true mean rate of return $r$ and its forecast $\hat{r}$ depend on the variance of the returns, hence they are scaled by the inverse of the covariance matrix. To quote Fabozzi, Kolm, Pachamanova, and Focardi (2007): “The parameter $\epsilon$ corresponds to the overall amount of scaled deviations of the realized returns from the forecasts against which the investor would like to be protected”.

Garlappi, Uppal, and Wang (2007) show that the ellipsoidal representation of the ambiguity of estimates may also lead to more stable portfolio strategies, delivering a higher out-of-sample Sharpe ratio compared to the classical Markowitz portfolios. It is also well-known (see Johnson & Wichern (1997, p. 212)), that the random variable 

$$(r - \hat{r})^T \Gamma^{-1} (r - \hat{r}),$$

has a known distribution (F-distribution under standard assumptions on the time series of returns), and this fact can be exploited using a quantile framework to set meaningful values for $\epsilon$ in practical computation with return data, c.f. Garlappi et al. (2007).

The ambiguity-averse mean–variance investor is interested in choosing his/her optimal portfolio according to the solution of the following problem

$$\max_{y} \min_y (1 - \omega)(r^T y + (1 - \epsilon^T y) R - \omega y^T \Gamma y)$$

where $\omega$ represents an $n$-vector of ones and the scalar $\omega \in (0, 1)$ represents the degree of risk aversion of the investor. The larger the value of $\omega$, the more risk averse (in the sense of aversion to variance of portfolio return) the investor. Processing the inner min we obtain as usual the problem:

$$\max_{y} (1 - \omega)(r^T y + (1 - \epsilon^T y) R - \epsilon \|r^T y\|_2) - \omega y^T \Gamma y$$

that is referred to as AAMVP (abbreviation of Ambiguity Averse Mean–Variance Portfolio). Let $\hat{\mu} = \hat{r} - Re$. Hence we can re-write AAMVP as

$$\max_{y} (1 - \omega)(\hat{\mu}^T y + R - \epsilon \|r^T y\|_2) - \omega y^T \Gamma y$$

Let us define the market optimal Sharpe ratio as $H^2 = \hat{\mu}^T \Gamma^{-1} \hat{\mu}$.

**Proposition 1.** If $\epsilon < H$ then AAMVP admits the unique optimal solution

$$y^* = \left( \frac{1 - \omega}{2\omega} \right) \left( \frac{H - \epsilon}{H} \right) \Gamma^{-1} \hat{\mu}, \quad y_{n+1}^*$$

$$= 1 - \sum_{j=1}^{n} \left( \frac{1 - \omega}{2\omega} \right) \left( \frac{H - \epsilon}{H} \right) (\Gamma^{-1})_j$$

i.e., an ambiguity-averse mean–variance investor with limited diffidence ($\epsilon < H$) makes the optimal portfolio choice in the risky assets

$$X_j = \frac{W_0}{\hat{\mu}} \left( \frac{1 - \omega}{2\omega} \right) \left( \frac{H - \epsilon}{H} \right) (\Gamma^{-1})_j, \quad j = 1, \ldots, n.$$
We denote the price system by the vector \( (p_1, p_2, \ldots, p_m) \) for the \( n \) risky assets. The price of the riskless asset is assumed to be equal to one. We make the following assumptions:

1. The total number of shares of asset \( j \) is \( x_0^j, j = 1, 2, \ldots, n + 1 \).
2. Investors \( i = 1, \ldots, m \) make their static portfolio choices according to the ambiguity-averse mean–variance portfolio model AAMPV of the previous section; they all agree on the nominal excess return vector \( \mu \) (i.e., they all agree on the same nominal rate of return vector \( \bar{r} \) and the same riskless rate \( R \)) and positive-definite variance–covariance matrix \( \Sigma \).
3. Investor \( i \) invests an initial wealth \( W_i^0 \) in an initial portfolio \( (x_1^i, x_2^i, \ldots, x_m^i) \).
4. Investor \( i \) has risk aversion coefficient \( \omega_i \) and ambiguity aversion coefficient (diffidence level) \( \epsilon_i \).

We have

\[ \sum_{j=1}^{n} x_0^j = x_0^i, j = 1, 2, \ldots, n + 1, \]  
\[ n \sum_{j=1}^{n} p_j x_0^j + x_0^{n+1} = W_i^0. \]  

3.1. Uniform markets

Using the result from the previous section we have that each investor \( i \) holds the percentage portfolio

\[ y_j^i = \frac{1 - \omega_i H - \epsilon_i(I^{-1} \mu)}{2 \omega_i H} x_j^i, j = 1, 2, \ldots, n, \]  
\[ y_{n+1}^i = 1 - \sum_{j=1}^{n} y_j^i = 1 - \frac{1 - \omega_i H - \epsilon_i}{2 \omega_i H} \sum_{j=1}^{n} (I^{-1} \mu) \]  

under the assumption that each investor \( i \) operates under limited diffidence, i.e., \( \epsilon_i < H, i = 1, \ldots, m \). Passing to the corresponding asset portfolio holdings (shares) \( x_j^i \) we have

\[ x_j^i = \frac{W_i^0 x_j^i}{p_j^i} = \frac{W_i^0 (1 - \omega_i H - \epsilon_i(I^{-1} \mu))}{2 \omega_i H} x_j^i, j = 1, 2, \ldots, n. \]  
\[ x_{n+1}^i = W_i^0 y_{n+1}^i = W_i^0 \left( 1 - \sum_{j=1}^{n} y_j^i \right) = W_i^0 \left( 1 - \frac{1 - \omega_i H - \epsilon_i}{2 \omega_i H} \sum_{j=1}^{n} (I^{-1} \mu) \right). \]

The market clearing condition requires the following equation to hold:

\[ \sum_{j=1}^{n} x_j^i = x_0^i, j = 1, 2, \ldots, n + 1, \]  
i.e., we have

\[ \sum_{i=1}^{m} W_i^0 \frac{1 - \omega_i H - \epsilon_i(I^{-1} \mu)}{H} x_j^i = x_0^j, j = 1, 2, \ldots, n + 1, \]  

Re-arranging this equation and recalling (4) we have the equation system with \( n \) equations and \( n \) unknowns:

\[ p_j x_0^j = (I^{-1} \mu) \sum_{i=1}^{m} \left( \frac{1 - \omega_i}{2 \omega_i} \right) \left( \frac{H - \epsilon_i}{H} \right) \left( \sum_{k=1}^{n} p_k x_k^0 + x_{n+1}^0 \right), \]  
\[ j = 1, 2, \ldots, n. \]  

Now, define for convenience \( \zeta_j = (I^{-1} \mu) \) for \( j = 1, 2, \ldots, n \) and

\[ \zeta = \sum_{i=1}^{m} \sum_{j=1}^{n} \left( \frac{1 - \omega_i}{2 \omega_i} \right) \left( \frac{H - \epsilon_i}{H} \right) x_0^j \zeta_j. \]

Proposition 2. In an ambiguity-averse mean–variance investors’ market where every investor has limited diffidence (i.e., \( \epsilon_i < H \) for all \( i = 1, \ldots, m \)) if \( \epsilon \neq 1 \), then there exists a unique solution \( p^* \) to the equilibrium system (11) given by

\[ p_j^* = \frac{1}{1 - \alpha} \sum_{i=1}^{m} \left( \frac{1 - \omega_i}{2 \omega_i} \right) \left( \frac{H - \epsilon_i}{H} \right) x_0^i, j = 1, \ldots, n. \]  

...
the ratio term \( x_j^0 \) (which represents the investor \( i \)'s initial fraction of shares of asset \( j \)) would give the total of fraction portfolio holdings \( (y_j) \) in the market, summed over all investors and all risky assets. Thus, the scalar \( x \) gives a measure of the weighted total of fraction portfolio holdings where each \( y_j \) is weighted by the corresponding ratio \( x_j^0 \). If this weighted total is strictly less than 1, an equilibrium price exists as is shown in the proposition above. The condition is also necessary. The condition
\[
\sum_{i=1}^{m} \left( 1 - \alpha_i \right) \left( \frac{H - \xi_i}{\beta_i} \right) x_j^0 \geq 0, \quad j = 1, 2, \ldots, n + 1
\]
also represents a weighted total of initial portfolio holdings over all investors in the market. The weight \( \left( \frac{1 - \alpha_i}{\beta_i} \right) \left( \frac{H - \xi_i}{\beta_i} \right) \) encodes the risk aversion and ambiguity aversion attitudes of the investor.

The existence of strictly positive prices is a harder question that is rarely addressed (with the exception of Rockafellar et al. (2007)) although zero prices would hardly make economic sense in practice. Interestingly, we can also prove the following negative result on the existence of a strictly positive system of equilibrium prices.

If the condition of Proposition 2 \( \sum_{i=1}^{m} \left( 1 - \alpha_i \right) \left( \frac{1}{\beta_i} \right) x_j^0 \geq 0 \), \( j = 1, 2, \ldots, n \) partially holds (only for the risky assets), i.e., a weighted total of initial portfolio holdings of risky assets over all investors in the market is non-negative, while this total is negative for the riskless asset then it is not possible to have positive equilibrium prices in the market.

**Proposition 3.** If \( \sum_{i=1}^{m} \left( 1 - \alpha_i \right) \left( \frac{1}{\beta_i} \right) x_j^0 \geq 0, j = 1, 2, \ldots, n, c_{n+1} < 0 \), no investor is short on risky assets, i.e., \( \zeta_j > 0 \) for all \( j = 1, 2, \ldots, n \), and \( \alpha \in (0, 1) \) then a strictly positive equilibrium price system does not exist in an ambiguity-averse mean–variance investors’ market where every investor has limited diffidence (i.e., \( \epsilon_i < H \) for all \( i = 1, \ldots, m \)).

**Proof.** We shall invoke the non-homogeneous Stiemek theorem (Stiemek, 1915) for the system:

\[
(I - d^T)p = c_{n+1}d, \quad p > 0.
\]
The alternative of the above system according to Stiemek’s theorem is the system
\[
\begin{align*}
(I - c^T)x &\geq 0, \\
-\xi &\geq 0,
\end{align*}
\]
If \( x = c \) then
\[
\begin{align*}
(I - c^T)x &= c(1 - \alpha), \\
-\xi &\geq \xi(c - \alpha).
\end{align*}
\]
Since by assumption we have
\[
\sum_{i=1}^{m} \left( 1 - \alpha_i \right) \left( \frac{1}{\beta_i} \right) x_j^0 \geq 0, \quad j = 1, 2, \ldots, n,
\]
we have \( c \geq 0 \). Due to the hypotheses that \( \alpha \in (0, 1) \) and \( c_{n+1} < 0 \) we have \( x = c \) that satisfies the alternative system.

If the market consists of fully confident (in the mean rate of return estimates) investors (i.e., ambiguity-neutral), we have the following equilibrium result in a mean–variance capital market. Let us define for convenience
\[
\gamma = \sum_{i=1}^{m} \sum_{j=1}^{n} \left( 1 - \alpha_i \right) \left( \frac{1}{\beta_i} \right) x_j^0 x_j^0.
\]

**Corollary 1.** In a mean–variance investors’ market (with no ambiguity aversion) if \( \gamma > 1 \), then there exists a unique solution \( x^* \) to the equilibrium system (11) given by
\[
\sum_{j=1}^{n+1} \left( 1 - \alpha_j \right) \left( \frac{H - \xi_j}{\beta_j} \right) x_j^0 = 0, \quad j = 1, 2, \ldots, n.
\]

If \( \sum_{i=1}^{m} \left( 1 - \alpha_i \right) \left( \frac{1}{\beta_i} \right) x_j^0 > 0, j = 1, 2, \ldots, n + 1, \) and \( \zeta_j > 0 \) for all \( j = 1, \ldots, n \), then the market admits a unique non-negative equilibrium price vector \( p^* \) if and only if \( \gamma < 1 \).

As in Proposition 2 the scalar \( \gamma \) gives a measure of the weighted total of fraction portfolio holdings where each \( y_j \) is weighted by the corresponding ratio \( x_j^0 \).

An interesting case is when all ambiguity-averse investors agree on the same level of limited diffidence, i.e., \( \epsilon_i < H \) for all \( i = 1, \ldots, m \). In that case, the equilibrium price vector \( p^* \) has a simplified expression:
\[
p_j^* = \frac{H - \epsilon}{H(1 - \alpha)} \sum_{i=1}^{m} \left( 1 - \alpha_i \right) \left( \frac{1}{\beta_i} \right) x_j^0, j = 1, \ldots, n.
\]

Obviously, the above expression implies \( p_j^* = \frac{H - \epsilon}{H(1 - \alpha)} x_j^0 \). Now, since we have
\[
0 < \frac{(H - \epsilon)(1 - \gamma)}{H - (H - \gamma)^2} < \frac{H - H' + \epsilon_j - \epsilon}{H - H' + \epsilon_j} < 1
\]
as \( \gamma < 1 \) in equilibrium, and \( H > \epsilon > 0 \). Therefore, in a homogeneously and mildly diffident ambiguity-averse mean–variance investors’ market (where diffidence is bounded above by the slope of the Capital Market Line), equilibrium prices are under downward pressure with respect to a purely confident mean–variance investors’ market. We summarize these observations below.

**Proposition 4.** In a homogeneously and mildly diffident (where all investors have the same \( \epsilon < H \)) ambiguity-averse mean–variance investors’ market in equilibrium prices are smaller than the equilibrium prices in a pure mean–variance investors’ market.

Another interesting observation is the following. Assume no investor has an initial liability, i.e., \( x_j^0 > 0 \) for all \( i = 1, \ldots, m \) and \( \zeta_j > 0 \) for all \( j = 1, \ldots, n \). Then we have the immediate consequence that \( x < \gamma \). This implies straightforwardly that \( p_j^* < p_j^H \), for all \( j = 1, \ldots, n \). In other words, in an ambiguity-averse mean–variance investors market with bounded diffidence, if all investors have long initial positions, then equilibrium leads to smaller prices compared to the equilibrium prices of purely mean–variance investors’ market, everything else being equal. Hence, the introduction of ambiguity aversion or diffidence in rate of return estimates into a market with all positive initial positions creates a deflationary pressure on equilibrium prices.

**A Numerical Example.** For illustration we consider an example with three investors and three assets (two risky assets and one riskless asset). The relevant data for the risky assets are specified as follows
\[
\mu = \begin{bmatrix} 0.1287 \\ 0.1096 \end{bmatrix},
\]
\[
\Gamma = \begin{bmatrix} 0.4218 & 0.0530 \\ 0.0530 & 0.2230 \end{bmatrix}.
\]
We assume \( x_j^0 = 10 \) for all three assets \( j = 1, 2, 3 \), and the initial portfolio holdings
\[
\begin{bmatrix} 4 & 3 & 3 \\ 6 & 2 & 2 \\ 3 & 3 & 4 \end{bmatrix}^T
\]
for each asset respectively, e.g., investor 1 holds initially 4 shares of asset 1, 6 shares of asset 2 and 3 units of the riskless asset. We have \( H = 0.2822 \) and \( \zeta = H^\gamma \hat{\mu} = |0.2509 0.4319| \). In Fig. 1 we plot the
Fig. 1. Effect of increasing risk aversion coefficient $\omega$ when all investors are equally ambiguity averse with $\epsilon_i = 0.01$ for $i = 1, 2, 3$.

evolution of the prices of the two risky assets in a uniformly ambiguity-averse investors’ market with $\epsilon_i = 0.01$ for $i = 1, 2, 3$. Increasing $\omega$, i.e., increasing the risk aversion of investors (expressed as an increasing emphasis on a smaller variance of portfolio return) equally for all investors while ambiguity aversion remains fixed across the board has a sharp deflationary effect on asset prices. In Figs. 2 and 3 we show the impact of increasing ambiguity aversion equally across the board has a sharp deflationary effect on asset prices. In Figs. 2 and 3 we show the impact of increasing ambiguity aversion equally across the board has a sharp deflationary effect on asset prices. In Figs. 2 and 3 we show the impact of increasing ambiguity aversion equally across the board has a sharp deflationary effect on asset prices. In Figs. 2 and 3 we show the impact of increasing ambiguity aversion equally across the board has a sharp deflationary effect on asset prices.

3.2. Mixed markets

Consider now a uniform market with ambiguity-neutral investors where an investor decides to adopt an ambiguity-averse position. For simplicity we shall examine the case where we have two investors. Investor indexed 1 is ambiguity-neutral with risk aversion coefficient $\omega_1$, investor indexed 2 is ambiguity-averse with coefficient $\epsilon < H$ and risk aversion coefficient $\omega_2$. All other assumptions about the assets traded in the market are still valid.

The ambiguity-neutral investor makes the portfolio choice

$$x_{i1} = \frac{W_0^i}{p_j} \frac{1 - \omega_1}{2\omega_1} \zeta_{ij}, j = 1, \ldots, n, x_{n+1} = W_0^1 \left(1 - \frac{1 - \omega_1}{2\omega_1} \sum_{j=1}^n \zeta_{ij}\right).$$

while the ambiguity-averse investor makes the choice

$$x_{i2} = \frac{W_0^i}{p_j} \left(1 - \omega_2 - \frac{(H - \epsilon)}{H} \right) \zeta_{ij}, j = 1, \ldots, n,$$

$$x_{n+1} = W_0^2 \left(1 - \frac{(1 - \omega_2)(H - \epsilon)}{2H\omega_2} \sum_{j=1}^n \zeta_{ij}\right).$$

As in the proof of Proposition 2 we define

$$c_j^1 = \frac{1 - \omega_1}{2\omega_1} x_j^0, j = 1, \ldots, n + 1;$$

for investor 1, and

$$c_j^2 = \frac{1 - \omega_2}{2\omega_2} \frac{H - \epsilon}{H} x_j^0, j = 1, \ldots, n + 1;$$

for investor 2, and $d_j = \zeta_j x_j^0$ for $j = 1, \ldots, n$. Then we can express the equilibrium price system for the mixed market as

$$p^n = \frac{c_{n+1}^1 + c_{n+1}^2}{1 - 2\rho},$$

where $\rho = d^T(c^1 + c^2)$, and we assume that the conditions guaranteeing the non-negative of $p^n$ as expressed in Proposition 2 hold.

Now, we compare the equilibrium price system $p^n$ to the equilibrium price system of a uniform ambiguity-neutral investors market. i.e., if investor 2 were to be ambiguity-neutral as well, we would have the following price system $\tilde{p}$:

$$\tilde{p} = \frac{c_{n+1}^1 + c_{n+1}^2}{1 - 2\rho},$$

where $\rho = d^T(c^1 + c^2)$ with

$$c_j^i = \frac{1 - \omega_i}{2\omega_i} x_j^0, j = 1, \ldots, n + 1.$$

We assume again the conditions guaranteeing the non-negative of $p^n$ expressed in Corollary 1 hold. Now, it is a simple exercise to see that
\[ c_j^2 = \frac{H - \epsilon}{H} c_j^2, \quad j = 1, \ldots, n + 1 \]
and
\[ \alpha^m = \alpha^0 + \left( \frac{H - \epsilon}{H} - 1 \right) d^t c^2. \]

These observations imply that
\[ p^m = \frac{c_j^2 + \frac{1 - \epsilon}{\epsilon} c_j^2}{1 - \alpha^m} d. \]

Therefore, if \( c_j^2 > 0 \) and \( d^t c^2 > 0 \), we have \( p^m < p^i \), i.e., if an investor with positive initial holdings moves from ambiguity-neutral position to (bounded) ambiguity-averse position, this change has a deflationary effect on equilibrium prices. We summarize this result below. We define
\[ c_j^i = \frac{1 - \epsilon}{2\epsilon} \alpha^0_j, \quad j = 1, \ldots, n + 1, \quad i = 1, \ldots, m \]
for every investor \( i \), and refer to the \( n \)-vector with components \( (c_1^i, \ldots, c_n^i) \) as \( c^i \).

**Proposition 5.** In a uniform market of \( m \) ambiguity-neutral mean-variance investors in equilibrium, assume investor \( m \) adopts an ambiguity-averse attitude with coefficient \( \epsilon < H \). Then the following statements hold:

1. A non-negative equilibrium price system
\[ p^m = \frac{\sum_{i=1}^{n+1} c_i^m + \alpha^m}{1 - \alpha^m} d \]
exists, if and only if \( \alpha^m < 1 \) where \( \alpha^m \) is defined as
\[ \alpha^m = d^t \left( \sum_{i=1}^{n+1} c_i^m + \epsilon \right). \]
2. If the initial holdings \( \alpha_{1m}^j \) for all \( j = 1, \ldots, n + 1 \) of investor \( m \) are positive, the equilibrium prices of the mixed market are smaller than the equilibrium prices of the uniform market.

The above result is not surprising if one bears in mind that an ambiguity-averse investor holds smaller long positions in risky assets compared to an ambiguity-neutral investor, which leads to a decreased demand for risky assets, and hence exerts a downward pressure on equilibrium prices.

A similar analysis can be made when the ambiguity aversion of one investor is not classified as mildly diffident, but rather, significantly diffident, i.e., with \( \epsilon > H \), in which case this investor would put all his/her initial wealth into the riskless asset. It can be shown again that such behavior leads to a drop in equilibrium prices. This is left as an exercise.

### 4. Properties of the equilibrium price system

We devote this section to the study of some interesting properties of portfolios in equilibrium. More precisely, we follow the references Deng et al. (2005), Konno and Shirakawa (1994), Konno and Shirakawa (1995) to examine the properties of the portfolios in equilibrium in a market of mildly diffident mean-variance investors. Define the master fund \( z_j^* = \gamma_j^*/\epsilon \) for all \( j = 1, \ldots, n \). We begin with the following two-fund separation property. Let us define \( A = e^t \Gamma^{-1} e \) and \( B = e^t \Gamma^{-1} F \).

**Proposition 6.** Let the price system in the mildly diffident mean-variance investors’ market be as defined in (12). Then, after the transaction, each investor \( i \) holds

(i) a portfolio composed of the riskless asset and a non-negative multiple \( \lambda_i \) of the initial total holdings \( x^0 = (x_1^0, x_2^0, \ldots, x_n^0) \) of risky assets, where \( \sum_{i=1}^n \lambda_i = 1 \) and
\[ \lambda_i = \frac{(1 - \epsilon)w^0_i(1 - \epsilon)w^1_i(H - \epsilon_i)}{\epsilon w^i(\sum_{i=1}^n \epsilon w^i (H - \epsilon_i))} \]
for \( i = 1, \ldots, m; \)
(ii) a percentage portfolio which is a linear \( (\zeta_j, 1 - \zeta_j) \) combination of the percentage riskless portfolio \( (0, 0, \ldots, 0, 1) \) and the (augmented) master fund \( (z_1^*, z_2^*, \ldots, z_n^*, 0) \) consisting only of risky assets where \( \zeta_j = \frac{1 - \epsilon}{\epsilon} w_{j1}^0 (B - RA) \).

**Proof.** Recall that in equilibrium each investor \( i \) holds the optimal portfolio
\[ x_{ij}^* = \frac{w_{i1}^0(1 - \epsilon w_{i1}^1(H - \epsilon_i))}{2\epsilon w_{ij} (1 - \epsilon)w_{ij}^1} \sum_{k=1}^m \frac{1 - \epsilon}{\epsilon} w_{ik}^1 (H - \epsilon_i) x_{kn}^i \]
for \( j = 1, \ldots, n \).

Since we have \( \sum_{j=1}^n x_{ij}^* = x_{0j}^i \), we infer immediately that \( \sum_{j=1}^n \lambda_j = 1 \). This proves part (i).

For part (ii), we have \( y_{ij}^* = \frac{(1 - \epsilon)w_{ij}^1(H - \epsilon_i)}{2\epsilon w_{ij}^0} \).

Hence, the result follows. \( \square \)

We note that the weight \( \gamma_j^* \) in part (ii) of the previous result is smaller than the corresponding weight that would result if \( \epsilon \) were taken equal to zero, i.e., the investor were ambiguity-neutral. This observation implies that ambiguity aversion leads to giving less weight to master fund \( z^* \).

Let the vector \( y^m \) and \( z^m \) be defined with components
\[ y_{ij}^m = \frac{x_{ij}^m}{\sum_{j=1}^m x_{ij}^m}, \quad j = 1, 2, \ldots, n + 1 \]
and
\[ z_{ij}^m = \frac{x_{ij}^m}{\sum_{j=1}^m x_{ij}^m}, \quad j = 1, 2, \ldots, n, \]
called, respectively, the market portfolio of all assets and the market portfolio of risky assets in Deng et al. (2005). We also have the following proportion property.

**Proposition 7.** Let the capital market be in equilibrium. Then the following hold:

(i) the market portfolio \( y^m \) is proportional to the market portfolio \( z^m \) of risky assets;
(ii) the market portfolio \( z^m \) of risky assets is identical to \( z^* \).

**Proof.** Using the definition of \( y^m \) we have
\[ y_{ij}^m = \frac{\sum_{j=1}^m x_{ij}^m}{\sum_{j=1}^m x_{ij}^m} \cdot \frac{\sum_{j=1}^m x_{ij}^m}{\sum_{j=1}^m x_{ij}^m} \cdot \frac{\sum_{j=1}^m x_{ij}^m}{\sum_{j=1}^m x_{ij}^m} \]
\[ = \frac{\sum_{j=1}^m \frac{(1 - \epsilon)w_{ij}^1(H - \epsilon_i)}{2\epsilon w_{ij}^0} \sum_{j=1}^m \frac{(1 - \epsilon)w_{ij}^1(H - \epsilon_i)}{2\epsilon w_{ij}^0}} {\sum_{j=1}^m \frac{(1 - \epsilon)w_{ij}^1(H - \epsilon_i)}{2\epsilon w_{ij}^0} \sum_{j=1}^m \frac{(1 - \epsilon)w_{ij}^1(H - \epsilon_i)}{2\epsilon w_{ij}^0}} \]
\[ = \frac{(B - RA) \sum_{j=1}^m \frac{(1 - \epsilon)w_{ij}^1(H - \epsilon_i)}{2\epsilon w_{ij}^0}} {\sum_{j=1}^m \frac{(1 - \epsilon)w_{ij}^1(H - \epsilon_i)}{2\epsilon w_{ij}^0} \sum_{j=1}^m \frac{(1 - \epsilon)w_{ij}^1(H - \epsilon_i)}{2\epsilon w_{ij}^0}} \]
\[ = z_{ij}^*. \]
For the second part we have
\[
z^*_M = \sum_{j=1}^{n} \frac{x_j^* P_j}{\sum_{j=1}^{n} (1 - \alpha_j)(1 - e_j)} \sum_{j=1}^{n} \frac{(1 - \alpha_j)(1 - e_j)}{\alpha_j e_j} W^j_{\beta_j} s_j = z^* . \]

Let the random (uncertain) rate of return of the market portfolio be denoted by
\[
\tilde{r}_M = \sum_{j=1}^{n} \beta_j^* z^*_j ,
\]
with the worst-case value
\[
\tilde{r}_M = E[\tilde{r}_M] = \sum_{j=1}^{n} \beta_j^* z^*_j .
\]

where \( r^* \) is as defined in (2). It is the rate of return where the maximum in the min–max portfolio selection model AAMVP of Section 2 is attained. Then, we have the following CAPM-like property which expresses the nominal excess rate of return of risky asset \( j \) as proportional to the worst-case excess rate of return of the market portfolio of risky assets. In addition to the terms that are encountered in the classical CAPM, the proportionality also depends on the square root of the market optimal Sharpe ratio \( H^2 \) and the ambiguity aversion coefficient \( \epsilon \).

**Proposition 8.** Let a capital market of homogeneously different investors with common \( \epsilon < H \) be in equilibrium. Then the excess nominal rate of return on each risky asset is proportional to the excess worst-case rate of return on the market portfolio of risky assets; i.e., the following holds
\[
\tilde{r}_j - R = \frac{H \text{cov}[\tilde{r}_j, \tilde{r}_M]}{(H - \epsilon) \text{Var}[\tilde{r}_M]} (\tilde{r}_M - R) , \quad j = 1, 2, \ldots, n .
\]

**Proof.** Let us re-write \( z^M = z^* = \frac{H}{(H - \epsilon)(1 - \alpha)(1 - e)} \Gamma^{-1}(r^* - Re) \) where \( r^* \) is defined in (2) of Section 2. Then, we have
\[
\text{Var}[, \tilde{r}_M] = (z^M)^T \Gamma z^M = \frac{H (\tilde{r}_M - R)}{(H - \epsilon)(1 - \alpha)(1 - e)}
\]
and
\[
\text{cov}[\tilde{r}_j, \tilde{r}_M] = \epsilon^j (z^M)^T \Gamma z^M = \frac{H (\tilde{r}_j - R)}{(H - \epsilon)(1 - \alpha)(1 - e)}
\]
where \( \epsilon^j \) is the \( n \)-vector with all components equal to zero except the \( j \)th component which is equal to one. Then, the result follows by taking the ratio \( \frac{\text{cov}[\tilde{r}_j, \tilde{r}_M]}{\text{Var}[\tilde{r}_M]} = \frac{\tilde{r}_j - R}{\tilde{r}_M - R} \) and recalling the definition (2) of \( r^* \). □

Note that this result reduces to the classical CAPM when \( \epsilon = 0 \), i.e., there is no ambiguity aversion, \( r^* \) reduces to \( \tilde{r} \) (which we can take as the true mean rate of return when no ambiguity aversion is present), and the coefficient \( \frac{\tilde{r}_j - R}{\tilde{r}_M - R} \) is equal to one. A possible interpretation of the previous result in terms of the classical CAPM is as follows. Recall that in classical CAPM, the factor of proportionality is called the beta of asset \( j \) (written \( \beta_j \)) and tells us how the nominal risk of this asset is correlated with the nominal risk of the whole market. If \( \beta_j \) is positive, then the risk of the asset is positively related to the market, and the investor holding that asset is partaking to the risk of the market and gets a premium for taking this position. If \( \beta_j \) is negative, the risk of the asset is inversely related with the risk of the market, i.e., if the market pays well, the asset pays poorly and vice versa. In our version of the CAPM like result, we have the beta that is scaled by the ratio \( \frac{\tilde{r}_j - R}{\tilde{r}_M - R} \) which is a number larger than one when we have \( 0 < \epsilon < H \). Therefore, the constant of proportionality and hence the new beta which relates in our case the nominal excess return to the total worst case return of the market is larger than the beta of the classical CAPM.

5. Conclusions

In this paper, we analyzed existence of equilibrium in a financial market composed of risky assets and a riskless asset, where mean–variance investors can display aversion to ambiguity, i.e., aversion to imprecision in the estimated mean rates of return of risky assets. We first derived a closed-form optimal portfolio rule for a mean–variance investor with aversion to ambiguity modeled using an ellipsoidal uncertainty set, borrowing the concept from robust optimization. The optimal portfolio rule reduces to the portfolio choice of a mean–variance investor when the investor is ambiguity-neutral. We examined conditions under which an equilibrium exists in a market of ambiguity-averse investors as well as conditions that lead to deflationary pressure on equilibrium prices with respect to a pure mean–variance investors’ (i.e., ambiguity-neutral) market. A CAPM-like result is derived, which reduces to the usual CAPM in the absence of ambiguity aversion.

Future research can address equilibrium in the absence of the riskless asset, limitations or exclusion of short sales, equilibrium with other risk measures such as robust CVaR or expected shortfall under mean return ambiguity, and equilibrium under transaction costs.

**References**


