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Modeling dynamic VaR and CVaR of cryptocurrency returns with alpha-stable innovations

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ABSTRACT

We employ alpha-stable distribution to dynamically compute risk exposure measures for the five most traded cryptocurrencies. Returns are jointly modeled with an ARMA-GARCH approach for their conditional mean and variance processes with alpha-stable innovations. We use the MLE method to estimate the parameters of this distribution, along with those of conditional mean and variance. Our results show that the dynamic approach is superior to the static method. We also find out that these risk measures of five cryptocurrencies do not offer the same pattern of behavior across subperiods (i.e., pre-, during- and post-COVID pandemic).

1. Introduction

As their number and market capitalization grow at an unimaginable rate, cryptocurrencies have become unignorable financial investment instruments for individual and institutional investors by serving as attractive alternative investment opportunities. However, cryptocurrency investments appear to be very risky. For example, holding Bitcoin can lead to a gain of more than 20% a day but also a loss of similar magnitude in the last four years, as demonstrated in Table 1. Investing in cryptocurrencies has a much more volatile outcome than investing in traditional financial assets. Accordingly, it begs the question of how to deal with this type of risk so that one can use these high-gain opportunities and how to obtain reliable risk exposure measures for risk management when dealing with this type of investment instrument.

In risk management, Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) are widely used to evaluate an investment or portfolio's possible downside risk exposure. Computing VaR and CVaR would not be a challenging problem if returns were to behave according to the normal distribution. However, the returns of cryptocurrencies, like returns of many financial assets, immensely differ from the normal distribution. This leptokurtic property problem is even more severe for cryptocurrencies, as seen in Table 1. Drozdz et al. (2018) have documented some specific patterns of returns of cryptocurrencies. Aware of this phenomenon, Fung et al. (2021), Guo (2021), Feng et al. (2017), and Conlon and McGee (2020) have used alternative distributions with heavy tails like General Error Distribution, t-distribution, or Normal Inverse Gaussian distribution to model returns of cryptocurrencies.

In this regard, the alpha-stable distribution can be a viable alternative given its desirable features, which fit cryptocurrency returns' distributional characteristics. Researchers and practitioners have used this type of distribution for modeling returns of

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Table 1
Descriptive statistics of daily log-returns of sample cryptocurrencies: 23 November 2018–16 June 2022.

Characteristic	BTC	LTC	BCH	ETH	XRP
Mean	0.0012	2.61e−04	−4.71e−4	0.0017	−1.98e−4
Median	0.0015	6.94e−04	−3.40e−04	0.0021	−7.78e−4
1th quartile	−0.0156	−0.0266	−0.0253	−0.0210	−0.0219
3rd quartile	0.0189	0.0272	0.0258	0.0276	0.0220
Maximum	0.1718	0.2687	0.4208	0.2307	0.4448
Minimum	−0.4647	−0.4491	−0.5613	−0.5507	−0.5505
St. dev.	0.0394	0.0541	0.0602	0.0503	0.0590
Skewness	−1.2316	−0.8456	−0.2283	−1.2414	0.1029
Kurtosis	19.790	11.814	17.090	16.663	18 645
Num. of obs	1301	1301	1301	1301	1301

financial assets (e.g., Nolan (2003)) as well as for computing their VaR and CVaR (e.g., Stoyanov et al. (2013) and Mohammadi (2017)). It seems to be a good candidate for cryptocurrencies' risk modeling as it allows for capturing heavier tails that might reflect the extremely volatile nature of their returns. Moreover, like returns of financial assets generally, the volatility of cryptocurrencies is not constant, with a high degree of persistence over time. This stylized fact has been documented and modeled with various ARMA-GARCH equations in which distributions with the above-mentioned heavy tails have been employed (see, e.g., Dyhrberg (2016), Thies and Molnar (2018), Takaishi (2020), Shen et al. (2020) and Zhang and Wang (2021)). However, the alpha-stable distribution has not yet been fully utilized in such a framework.

Another stylized fact is that the volatility in returns of cryptocurrencies increased considerably during the COVID-19 pandemic. To the extent that the coronavirus outbreak can act as one source of external shocks that may substantially change the dynamics of returns of cryptocurrencies (e.g., Salisu and Ogbonna (2021), Jiang et al. (2021) and Shahzad et al. (2021)), it is worth knowing whether there is a change in the price dynamics profile of cryptocurrencies during this period and how the risk exposure of investing in cryptocurrencies under the pandemic shock has progressed. Suppose the price dynamics and the return profile of cryptocurrencies change. In that case, the risk exposure of investment in these assets will follow accordingly, and risk managers must adapt to the new reality. So far, this problem has not been thoroughly investigated. Furthermore, at the end of February 2022, the war in Ukraine started and can also work as a new impulse affecting the behavior of cryptocurrency investors.

The above discussions led us to investigate the risk exposures of cryptocurrency returns over the recent period from November 2018 to June 2022 using GARCH-based volatility models with alpha-stable distributed innovations. Our approach is mainly motivated by how high uncertainties caused by external factors, among others, affect the need to re-balance the portfolio structure, which may exacerbate the already volatile nature of cryptocurrency returns. More concretely, we first develop ARMA-GARCH and ARMA-GJR-GARCH models with various ARMA specifications to model the returns and variances of the top five most frequently traded cryptocurrencies using their daily closing price data. Next, the corresponding VaR and CVaR series are generated, and their statistical properties are examined based on the estimation results of the best-suited specification. Finally, we analyze whether there are any substantial changes in the dynamics of these risk measures during three subperiods: pre-, during, and post-COVID pandemic. The post-COVID period thus incorporates the impact of the war in Ukraine.

In sum, our study makes two main contributions to the related literature. First, we show the relevance of alpha-stable distributions in modeling the fundamental risk measures in finance. The empirical method we propose is effective and advantageous in that it allows us to capture the most important characteristics of cryptocurrency returns (Sathe and Upadhye, 2021), including heavy tails, leptokurtic behavior, and possibly skewness. Our finding indicates that, among the non-Gaussian distribution innovations, the GARCH model with alpha-stable innovations performs the best. Furthermore, the inclusion of the ARMA part into the model enables us to deal with the instability of the mean process in returns which has rarely been the focus of the existing research strand. Second, new empirical insights from our study regarding the better evaluation of the risk exposure measures by alpha-stable distributions will help risk managers make better decisions with cryptocurrency investments.

The rest of this paper is organized as follows. Section 2 describes our methodology. Section 3 presents the data used. Section 4 reports and discusses the empirical results. Section 5 provides some concluding remarks.

2. Methodology

2.1. Value-at-Risk and conditional Value-at-Risk

VaR measures the risk of loss of investments. It tells us the potential loss of a risky investment or a portfolio for a given probability and over a given period under certain market conditions. Firms and regulators widely use VaR to measure the risk exposure of firms due to unfavorable market movements. It helps firms create sufficient capital reserves to ensure their financial stability with respect to those potential losses. Let a random variable X be the potential loss of an investment and $F_X(.)$ be the cumulative distribution function of X. Let $Y = -X$ and $\alpha \in (0,1)$. Then, VaR at level α (VaR_α) is the minimal value y such that the probability is at least $1-\alpha$ that Y does not exceed y. Formally, $\mathrm{VaR}_\alpha(X)$ is the $(1-\alpha)$-quantile of Y (see Artzner et al. (1999) for more details), i.e.,

$$\mathrm{VaR}_\alpha(X) = -\min\{x : F_X(x) \geq \alpha\} = F_Y^{-1}(1-\alpha). \tag{1}$$

One of the disadvantages of VaR is that it is not coherent.[1] To overcome this problem, conditional value at risk (also known as expected shortfall (ES)) has been introduced as an alternative to VaR. CVaR can inform us about on the expected loss over a period of time beyond a given VaR boundary. If X is the potential loss of an investment at some point in the future and $\alpha \in (0, 1)$, then the CVaR is defined as

$$\mathrm{CVaR}_\alpha(X) = \frac{1}{1-\alpha}\int_0^{1-\alpha} \mathrm{VaR}_\gamma(X)\mathrm{d}\gamma. \tag{2}$$

For more details about CVaR concept, see Acerbi and Tasche (2002).

2.2. ARMA-GARCH model

An ARMA-GARCH model of a random variable y_t is a two-part econometric model in which its conditional mean follows an ARMA process

$$y_t = c + \sum_{i=1}^{m} \phi_i y_{t-i} + \sum_{j=1}^{n} \theta_j \epsilon_{t-j} + \epsilon_t, \tag{3}$$

where $\phi_i, i = 1, \ldots, m$ and $\theta_j, j = 1, \ldots, n$ are coefficients of the AR and MA terms, and m and n are their length of lags, respectively. The error term ϵ_t has these properties

$$\epsilon_t = z_t \sigma_t, \tag{4}$$

where z_t are the standardized residuals with zero mean and unit variance and the conditional variance term σ_t^2 follows a GARCH process proposed by Bollerslev (1986) as

$$\sigma_t^2 = \eta + \sum_{i=1}^{p} \mu_i \sigma_{t-i}^2 + \sum_{j=1}^{q} \lambda_j \epsilon_{t-j}^2, \tag{5}$$

where $\mu_i, i = 1, \ldots, p$ and $\lambda_j, j = 1, \ldots, q$ are coefficients of the ARCH and GARCH terms, and p and q are their length of lags, respectively. Parameters of an ARMA-GARCH model are estimated jointly using the maximum likelihood estimation method. The shape of the likelihood function depends on type of distribution according to which the error term z_t is distributed.

2.3. Alpha-stable distribution

The alpha-stable distribution is from a family of distributions which can capture both asymmetry and heavy tails of random variables. It was first studied in 1920s by Levy, hence, sometimes it is also called Levy alpha-stable distribution (Feller, 1970). In general, the probability density function (PDF) and cumulative distribution function (CDF) of the alpha stable in a closed form is unknown. It can only be characterized by its characteristic function $\phi(t)$,

$$\log \phi(t) = \begin{cases} -\sigma^\alpha |t|^\alpha \{1 + i\beta \mathrm{sign}(t) \tan \frac{\alpha\pi}{2} [(\sigma|t|)^{(1-\alpha)} - 1]\} + i\mu t, & \alpha \neq 1 \\ -\sigma|t|\{1 + i\beta \mathrm{sign}(t) \frac{2}{\pi} \log(\sigma|t|)\} + i\mu t, & \alpha = 1 \end{cases}$$

for $\forall t \in \mathbb{R}$. There are four parameters $\alpha, \beta, \mu, \sigma$ included in this characteristic function. α, β are shape parameters where $\alpha \in (0, 2]$ is the tail power as it captures the tail thickness of the distribution and $\beta \in [-1, 1]$ is skewness parameter. The two remaining parameters are location parameter μ and scale parameter σ. The normal distribution is a special case of the alpha-stable distribution when $\alpha = 2$ as well as Cauchy and Levy distributions when $\alpha = 1$ and $\alpha = 0.5$, respectively. An interesting feature of the alpha-stable distribution is that all p-th moments, $p > \alpha$ do not exist. The lack of closed forms for PDF and CDF of the alpha-stable distribution is impractical for computation purposes. Therefore, Zolotarev (1986) proposes integration formulas by which its PDF and CDF can be directly computed. They are included in Appendix A. As ARMA-GARCH model can generate series of conditional volatility, a dynamic series of VaR and CVaR can be adequately computed using the series of conditional variance.

3. Data

As of early 2022, there exist more than 8000 cryptocurrencies. We select the five most well-established currencies from this immense collection of cryptocurrencies for our analysis, namely Bitcoin (BTC), Ethereum (ETH), Ripple (XRP), and two of Bitcoin's branched-offs, i.e., Litecoin (LTC) and Bitcoin Cash. This choice allows us to examine a potential similarity between Bitcoin's returns and its branched-offs. The daily sample data used for the econometric analysis are from the Yahoo Finance database. The daily data covers the closing price for almost four years, from 23 November 2018 to 16 June 2022, sufficiently long for the maximum likelihood estimation procedure.

[1] A coherent risk measure should have these properties: monotonicity, sub-additivity, homogeneity, and translational invariance. See Francis and Kim (2013) for details.

Table 2
The results of the unit root testing of five return series.

Parameter	BTC	LTC	BCH	ETH	XRP
Estimated γ	−1.0301	−1.0267	−1.0191	−1.0301	−1.0928
std. deviation	0.0243	0.0242	0.0241	0.0244	0.0254
t-statistic	−42.225	−42.048	−42.244	−42.225	−43.024
p-value	0.0001***	0.0001***	0.0001***	0.0001***	0.0001***

We transform the daily closing prices of cryptocurrencies into logarithmic returns. Table 1 displays several descriptive statistics of daily returns. A glance at those numbers tells us that the extreme values of these five cryptocurrencies' returns are much higher than those of traditional financial instruments, as there is no daily price limitation for the former. In contrast, their returns have higher kurtosis, similar to those of other financial assets. This feature justifies the use of a distribution with heavy tail property for modeling the returns. Finally, the skewness values are not close to zero, indicating a certain degree of asymmetry in the returns. Figs. 1–5 show the evolution of returns of the five cryptocurrencies during 2018–2022.

4. Empirical results

In this empirical part, we model the returns of five cryptocurrencies by an ARMA-GARCH model with an error term following an alpha-stable distribution, i.e., $r_t \sim S(\alpha, \beta, \delta_t, \gamma_t)$, where α, β are shape parameters and δ_t and γ_t are conditional scale and location processes, respectively. Then the estimated parameters of this model are used to compute VaR and CVaR. To correctly assess the ARMA-GARCH model, we must identify the most suitable specification for the given dataset. For the ARMA part of the model, the Box–Jenkins methodology is used (see Box et al. (2015)). First, we use the ADF test to test the stationarity of all five series of returns of cryptocurrencies. Table 2 reports the results. The low p-value of all five cases rejects the null hypothesis that a unit root is present in these series at the 1% significance level.[2] Furthermore, the values of estimated γ shown in the table indicate that all series are stationary and random, with some persistence present in the return series of Ripple. Then, the autocorrelation function (ACF) and partial autocorrelation function (PACF) of all five series are estimated. Based on the PACF and ACF analysis results, we identify that ARMA(1,1) is a potentially good model for conditional mean for XRP returns. Otherwise, there are no substantial autocorrelations in the remaining four series. Therefore, we assume that there is no need to model these series' mean by an ARMA model.

For the GARCH part of the model, we use the ARCH test proposed by Engle (1982) to detect the presence of heteroscedasticity in the conditional variance of cryptocurrency returns. The test strongly rejects the null hypothesis of homoscedastic variances in all five cases. Regarding the length of lags of the ARCH and GARCH terms, the results of Hansen and Linde (2005) suggest that we do not have to extend the specification of the GARCH part further than GARCH(1,1). We also consider including the leverage effect accounting for asymmetric responses in the volatility documented by Glosten et al. (1993) into the GARCH part. As all specifications are nested, we verify their presence by the likelihood ratio (LR) test. We applied the same approach to various alternative specifications for the ARMA part, where various lag lengths of the ARMA part were tested.

The parameters of our ARMA-GARCH model are estimated jointly using the maximum likelihood technique. As there is no econometric package providing the estimation of an ARMA-GARCH model with alpha-stable innovations, all computational works have been done in MATLAB. The numerical integration formula for PDF in Eq. (A.1) is used to construct the likelihood function. The numerical algorithm for parameter estimation is very stable and converges to optimal solutions from various random initial values. However, it is a bit time-consuming due to the complexity of the numerical evaluation of the PDF function of the alpha-stable distribution. Based on the likelihood ratio (LR) test results, the presence of the leverage effect is not confirmed, and our original specification for the ARMA part is fully justified. In Tables B.8 to B.12 in the Appendix B, due to the limited space of the article, we report the estimation results where the values of estimates and their corresponding asymptotic standard errors, their asymptotic z-score as well as the p-values are shown. For values of α, the null hypothesis is $\alpha = 2$. Otherwise, the null hypothesis is that the value of the estimated parameter equals zero.

Accordingly, the estimated values of the ARMA-GARCH model's parameters in Tables B.8–B.12 show that in all five cases, the estimated value of tail parameter α is above 1.5 and statistically significantly less than 2 (at the significant level of 5%, and for now on always so) indicating that the daily returns of these cryptocurrencies have relatively strong heavy tails. The skewness parameter β does not significantly differ from 0, which is in line with the results of the descriptive statistics. It also justifies the absence of the leverage effect in the GARCH part. The values of the location parameter are not significantly different from 0, indicating the symmetry around zero of returns of the five cryptocurrencies. All GARCH(1,1) model parameters are statistically significant, justifying its usage for modeling conditional variance of returns in our analysis.

We use the estimation results in the previous part to compute VaR at $\alpha = 95\%$ (VaR95), VaR at $\alpha = 99\%$ (VaR99), and CVaR at $\alpha = 97.5\%$ (CVaR975) with the use of formula for CDF in Eq. (A.4).[3] First, we use the estimated values of the ARMA-GARCH model to

[2] The asterisks at the upper right hand side of p-values indicate the level of significance of 1% (***), 5% (**), and 10% (*), respectively. These symbols hold further throughout the text.

[3] Although the second moment of a random variable with alpha stable distribution does not exist, according to Paolella (2016), this non-existence is a theoretical concept. However, the simulation results show that simulated stable data with $\alpha = 1.7$ are fit with Student t-distribution with the number of degrees of freedom $\nu \approx 4$ instead of $\nu < 2$. Hence, the (non)-existence is not quite relevant for risk forecasting, but distribution properties like leptokurtic, asymmetric, and bell-shaped are so.

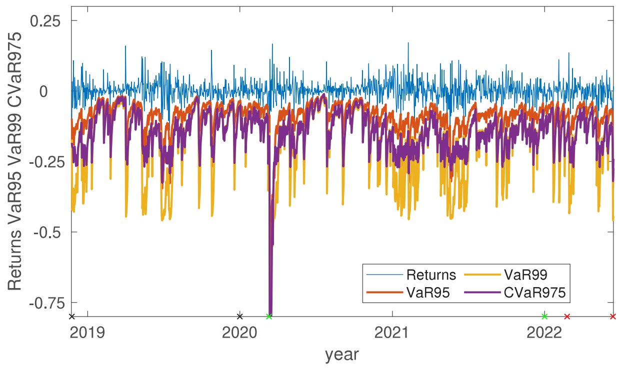

Fig. 1. Bitcoin returns and the corresponding VaR and CVaR.

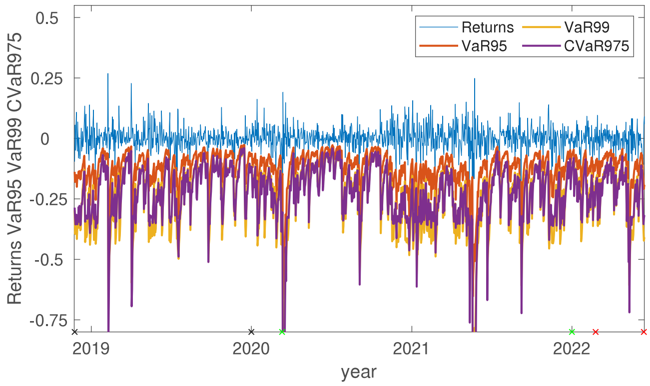

Fig. 2. Litecoin returns and the corresponding VaR and CVaR.

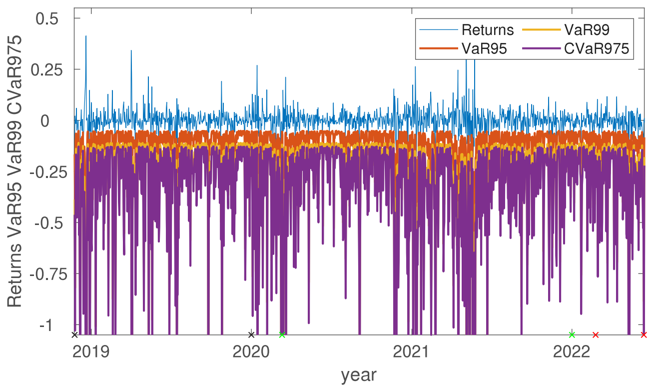

Fig. 3. Bitcoin Cash returns and the corresponding VaR and CVaR.

generate conditional variance series for all five cryptocurrencies and the conditional mean series for Ripple's returns. Then, we use them to calculate VaR95 and VaR99 according to Eq. (1) and CVaR975 according to Eq. (2). In the case of CVaR975, the integration is numerically evaluated with the trapezoidal rule. As the alpha-stable distribution has a strong heavy tail, we stop generating VaR when α is too small, and the algorithm cannot calculate the corresponding value of VaR. The results of VaR95 and VaR99 are shown in Figs. 1–5. We also calculate selected statistics of VaR95, VaR99, and CVaR975, which we display in Tables 3–5. The estimation follows the procedure suggested by Nolan (1997).

In order to make the results of dynamic VaR and CVaR comparable with those of these quantities in the static case, we compute the same quantities in the corresponding static mode. The procedure is as follows. We use the whole dataset to estimate the parameters of the alpha-stable distribution for the entire period, assuming that its parameters are unchanged during this period. Then, using the estimated values of parameters of the invariant distribution, we compute VaR95, VaR99, and CVaR975 for all five

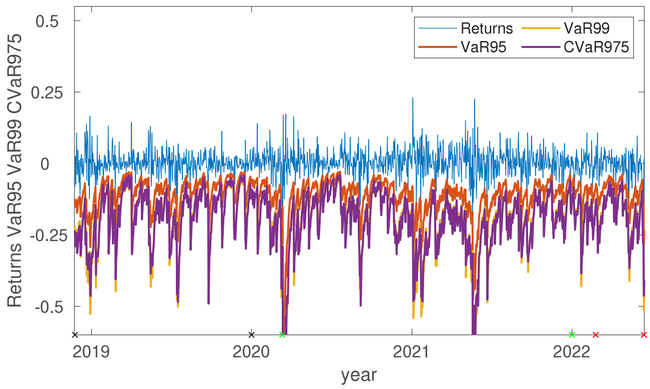

Fig. 4. Ethereum returns and the corresponding VaR and CVaR.

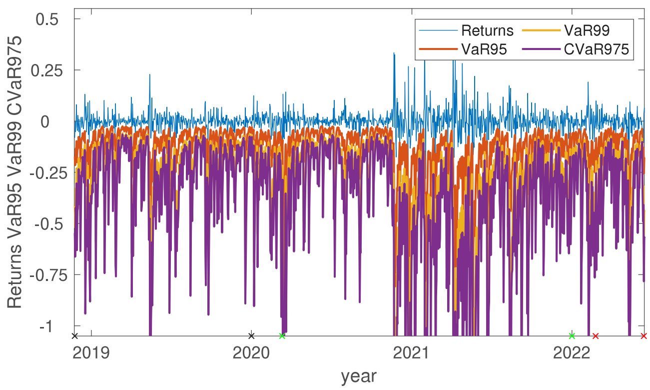

Fig. 5. Ripple returns and the corresponding VaR and CVaR.

Table 3
Selected statistics on computed VaR95 of returns of sample cryptocurrencies.

Characteristic	BTC	LTC	BCH	ETH	XRP
Mean	−0.0872	−0.1215	−0.1285	−0.1087	−0.1288
Median	−0.0766	−0.1058	−0.0902	−0.0957	−0.0949
1th quartile	−0.1035	−0.1462	−0.1438	−0.1302	−0.1551
3rd quartile	−0.0545	−0.0735	−0.0631	−0.0703	−0.0575
Maximum	−0.0107	−0.02736	−0.0511	−0.0287	−0.0241
Minimum	−0.6114	−0.6634	−1.5293	−0.7232	−1.2773
St. dev.	0.0565	0.0736	0.1204	0.0628	0.1195
Num. of crosses	38	46	55	37	49

Table 4
Selected statistics on computed VaR99 of returns of sample cryptocurrencies.

Characteristic	BTC	LTC	BCH	ETH	XRP
Mean	−0.1732	−0.2330	−0.2788	−0.3410	−0.3048
Median	−0.1355	−0.1909	−0.1944	−0.2410	−0.2237
1th quartile	−0.2074	−0.3254	−0.3318	−0.3955	−0.3739
3rd quartile	−0.0961	−0.1329	−0.1358	−0.1624	−0.1345
Maximum	−0.0198	−0.0494	−0.1098	−0.1275	−0.0553
Minimum	−1.0657	−1.1960	−3.3053	−3.8046	−3.0255
St. dev.	0.1174	0.1371	0.2579	0.3175	0.2812
Num. of crosses	6	10	8	5	6

cryptocurrencies. We also calculate the number of cases when the actual returns fall below these values for each cryptocurrency. Table 6 shows the obtained results.

Table 3 shows that the average daily VaR95 for Bitcoin, Litecoin, Bitcoin Cash, and Ethereum are similar, ranging from −13 to −9%, with a relatively high standard deviation in the range of 5.5 to 13%. Their dynamics can be observed in Figs. 1–5. They

Table 5
Selected statistics on computed CVaR975 of returns of sample cryptocurrencies.

Characteristic	BTC	LTC	BCH	ETH	XRP
Mean	−0.1468	−0.2313	−0.3410	−0.1967	−0.3929
Median	−0.1401	−0.2135	−0.2410	−0.1756	−0.2946
1th quartile	−0.1949	−0.3054	−0.3955	−0.2476	−0.4867
3rd quartile	−0.0934	−0.1408	−0.1624	−0.1228	−0.1708
Maximum	−0.0108	−0.0403	−0.1275	−0.0401	−0.0607
Minimum	−1.2779	−1.4427	−3.8046	−1.4701	−3.5542
St. dev.	0.0851	0.1369	0.3175	0.1159	0.3570
Num. of crosses	8	8	5	11	3

Table 6
Computed VaR95, VaR99 and CVaR975 on static basis.

Quantity	BTC	LTC	BCH	ETH	XRP
VaR95	−0.0569	−0.0779	−0.0954	−0.0714	−0.0810
Num. of crosses	79	81	56	76	69
VaR99	−0.1449	−0.1709	−0.1545	−0.2198	−0.2330
Num. of crosses	4	10	5	11	5
CVaR975	−0.1924	−0.2109	−0.1857	−0.5147	−0.3459
Num. of crosses	1	4	2	3	3

Table 7
The results of equal mean testing and the computed values of relative changes.

	Tested periods	BTC	LTC	BCH	ETH	XRP
VaR95	Pre- vs. During	3.47e−5***	0.0085***	**0.2333**	3.05e−5***	1.35e−10***
	Difference	−11.6946	−13.8130	−4.3489	−19.7862	−63.5519
	Pre- vs. Post-	0.0179**	**0.5059**	**0.2333**	**0.3845**	1.27e−5***
	Difference	−7.1279	−0.8452	2.5278	−0.2750	−22.7631
	During vs. Post-	**0.6362**	0.0130**	**0.3439**	0.0073***	0.0069***
	Difference	4.0885	11.3940	6.5901	16.2884	24.9394
VaR99	Pre- vs. During	3.47e−5***	0.0085***	**0.2333**	3.05e−5	1.35e−10
	Difference	−11.2783	−12.3099	−4.5676	−18.7795	−63.3268
	Pre- vs. Post-	0.0179**	**0.5059**	**0.2333**	**0.3845**	1.27e−5***
	Difference	−6.0085	1.9049	2.5836	0.1559	−23.2049
	During vs. Post-	**0.6362**	0.0130	**0.3439**	0.0073***	0.0069***
	Difference	4.7357	12.6568	6.8388	15.9417	24.5654
CVaR975	Pre- vs. During	5.85e−5***	0.0010***	**0.2332**	3.08e−7***	1.44e−10***
	Difference	−18.3438	−13.8491	−4.3715	−21.0630	−63.2423
	Pre- vs. Post-	5.85e−5***	**0.4212**	**0.1332**	**0.2342**	1.44e−10***
	Difference	−12.5446	−0.8483	2.2407	−1.3715	−23.7280
	During vs. Post-	**0.1190**	0.0127**	**0.3439**	0.0072***	0.0069***
	Difference	4.9003	11.4193	6.3352	16.2655	24.2059

are high, but they are in line with the descriptive statistics. The average daily VaR95 of Litecoin, Ethereum, and Ripple are over 12%, with a standard deviation of similar magnitude, meaning that investing in those cryptocurrencies is even much riskier than investing in Bitcoin. Another interesting feature is the number of cases when actual returns are under the VaR95 threshold in the last row of the tables. This number is much less than 5% in all cases. Let us compare this result with the result of the static case when the number of cases should always be roughly 5%. We can see that the dynamic approach can identify more risky events related to investing in cryptocurrencies. This makes the dynamic approach superior to the static one because, in the dynamic case, one distribution for the whole dataset is replaced by a set of distributions specific to each observation.

Regarding the results of computing VaR99, the average daily values for Bitcoin, Litecoin, Bitcoin Cash, and Ethereum range from −18 to −15% with a standard deviation of the same size, as shown in Table 4. Thus, one must consider a potential daily loss of at least 17% with a 1% probability. The number for Ethereum is much higher, which is around 34%. Like in the case of VaR95, the number of cases falling below the VaR99 threshold is much less than 1%. Hence, in both VaR95 and VaR99 cases, the dynamic approach identifies less risky cases than the static approach.

As far as the results of computed values of daily CVaR975 are concerned, they are also very high, ranging from −40% to −15%, with standard deviation varying from 9% to over 30%. The highest values belong, again, to the returns of Ripple. Their dynamics is rather rich and can be observed in Figs. 1–5. The reason why CVaR975 is so high comes from the fact that the alpha-stable distribution has heavier tails. The probability of extreme loss occurrence does not die out fast enough. Hence, these extreme values can still affect the CVaR when they are considered. Unlike the previous two cases, the dynamic approach identifies more possibilities of falling under the threshold than the number obtained from the static one. We can notice that VaR99 and CVaR975, which should be similar when the normal distribution is used, differ substantially with the use of alpha-stable distribution.

Next, we use the computed series of VaR95, VaR99, and CVaR975 to examine how these measures developed in the pre-pandemic, during-pandemic, and post-pandemic periods with the war in Ukraine. The subsample for the first period is from 24 November 2018 to 31 December 2019. The second subsample is the period from 11 March 2020 to 31 December 2021.[4] The last one is from 24 February 2022 to the sample end date. In Figs. 1–5, each subperiod is marked by a small cross on the horizontal axis (pre-COVID in black, during the COVID pandemic in green, and the last one in red). If there is no change in these quantities, their means should be equal. For this purpose, the non-parametric Wilcoxon rank sum test is used instead of a two-sample t-test for equal means, as returns of cryptocurrencies are not normally distributed. The null hypothesis of the test is the two populations are equal. The test statistic is derived from the number of time when observations in one sample precede observation in the other sample in an ordered arrangement of two independent samples and under the null hypothesis, it has an asymptotic normal distribution (for more information on Wilcoxon rank sum test, see Gibbons and Chakraborti (2014)).

We compare three possible pairs of cases within each returns series: Pre-pandemic vs. During pandemic, Pre-pandemic vs. Post-pandemic, and During pandemic vs. Post-pandemic. To do so, we use function *ranksum* in Matlab to perform the test for series VaR95, VaR99, and CVaR975. Besides the statistical testing, we also compute the relative change of the mean of one subperiod of computed risk measures to its counterpart in the second subperiod to provide a quantitative picture of the differences. The relative change d is computed as follows

$$d = \frac{\bar{\mu}_{first} - \bar{\mu}_{second}}{\bar{\mu}_{first}} * 100, \quad (6)$$

where $\bar{\mu}_{.}$ is the computed mean of the subperiod, and the order "first" or "second" corresponds to the earlier or later subperiod, respectively. We calculate the relative distance for VaR95, VaR99, and CVaR975 according to Eq. (6) within each returns series. We display the test results and the computed relative changes in Table 7, where the p-value of the null hypothesis is shown in the first line. The numbers in bold are the cases when the null hypothesis is not rejected.

Table 7 shows that the risk measures of all five cryptocurrencies do not behave similarly. While those of Bitcoin Cash do not significantly differ over time, these measures of Ripple change all the time according to the p-values of the Wilcoxon rank sum test. In the pre-pandemic period, they are lowest on average. Then, they attain the highest values during the pandemic period and fall in the post-COVID period. The drops in values are statistically significant on average (at the significant level of 5% and it will always be so further). However, they do not return to similar levels in the pre-COVID subperiod. With the three remaining cryptocurrencies, there is a significant growth in values of three computed risk measures during the pandemic subperiod compared to the pre-COVID subperiod, whose size is above 10% up to around 20% in the case of Ethereum. Then, the decrease is observed in the post-COVID subperiod compared to the previous subperiod. However, this decrease of VaR95 and VaR99 is statistically insignificant in the case of Bitcoin as the hypothesis of two equal means cannot be rejected, while the change of these metrics in the case of Litecoin and Ethereum is significant. Comparing these measures in the pre-COVID subperiod with those in the post-COVID subperiod, the changes in VaR95, VaR99 and CVaR975 are insignificant in the case of Litecoin and Ethereum and significant in the case of Bitcoin. Though the price dynamics of all cryptocurrencies is always volatile, the relatively more stable behavior of riskiness measures for Bitcoin Cash may come from the fact that it acts more like a mean of payment rather than as an investment anchor.

On the other hand, the most volatile fluctuation of Ripple among the five cryptocurrencies in terms of VaR95, VaR99, and CVaR975 may result from its extremely high volume in circulation and overreaction of a certain portion of the Ripple holders may induce when extraordinary events in the world occur. The similar but not identical pattern of riskiness measures of Bitcoin and Litecoin may come from their similarity. Ethereum is the second most popular cryptocurrency after Bitcoin, which may be why it is more volatile than Bitcoin. Since the leading cryptocurrencies are often viewed as a safer means of wealth stock, events like the pandemic or the war in Ukraine can prompt investors to rethink their investment strategy, which in turn may cause some changes in their investment behavior and give rise to the change in the riskiness profile of investment into these assets. The econometric analysis identifies a change in the dynamics of the risk exposure metrics in returns of cryptocurrencies (except in the case of Bitcoin Cash). While the COVID pandemic may have triggered investor anxiety regarding actual economic activities, which induced the need for asset re-balancing, the war in Ukraine itself may not be the only cause of the change. However, it can be one factor that contributed to the beginning of the end of the COVID pandemic period and initiated the gradual reverse dynamics of the post-COVID pandemic period. However, a more thorough investigation is required to confirm this trend's existence.

5. Conclusion

The results of previous works suggest that alpha-stable distribution is a worthy alternative that has yet to gain appropriate attention. In this study, we examined the use of this distribution to model cryptocurrency returns in a dynamic setting. Given their excessively volatile nature, we used the ARMA-GARCH model with alpha-stable distributed innovations in computing dynamic VaR and CVaR measures for the five most commonly traded cryptocurrencies (Bitcoin, Litecoin, Ethereum, Ripple, and Bitcoin Cash). While Ripple's conditional mean process of returns differs from those of the remaining four cryptocurrencies, their conditional variance processes behave similarly without the leverage effect, as it is probably already captured by the skew parameter of the

[4] Though it is called the COVID-19 pandemic, the pandemic was declared on 11 March 2020. Although no official end of the COVID pandemic has been declared up to now, the practical end of this subperiod was chosen at the end of 2021 as, at that moment, the impact of the pandemic on the investment decision of investors is considered to be negligible. Also, we want to leave a gap between subperiods to avoid the overlapping effect since the pandemic may still be present.

stable alpha distribution. As a general case of other alternatives, the alpha-stable distribution is an applicable candidate for modeling extremely volatile returns of cryptocurrencies in the ARMA-GARCH framework. Also, VaR and CVaR estimates in a dynamic setting are more accurate than those obtained from a static approach since they can more precisely identify the risk exposure of investment in cryptocurrencies. Moreover, our results confirm that the theoretical non-existence of the second moment of an alpha-stable distribution is not an obstacle to using this distribution to model the returns of assets with extreme volatility. Therefore, using the computed values of risk measures like VaR and CVaR, we can recognize and quantify their significant changes before, during, and after the COVID-19 pandemic. For example, these measures did not show the same pattern of behavior in these three subperiods. The use of stable alpha distributions can also be extended to the multivariate case in the fashion as in Arellano-Valle and Genton (2010) under the framework of the ARMA-GARCH model to compute VaR and CVaR as a topic of our future research. Also, as the dynamics of the risk metrics are unstable over time, it would be interesting to identify potential economic and financial factors and quantify their potential impacts on the behavior of risk metrics.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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Appendix A. PDF and CDF of alpha stable distribution

The density function of the stable distribution is not generally known in the explicit form. However, the integral expression proposed by Zolotarev (1986) is often used as an alternative. Using substitution $\zeta = -\beta \tan \frac{\pi\alpha}{2}$, the PDF of standard alpha stable random variable $S(\alpha, \beta, 0, 1)$, can be expressed as:

- if $\alpha \neq 1$ and $x \neq \zeta$

$$f(x;\alpha,\beta) = \begin{cases} \frac{\alpha(x-\zeta)^{\frac{1}{1-\alpha}}}{\pi|\alpha-1|} \int_{-\theta_0}^{\frac{\pi}{2}} V \exp\left(-(x-\zeta)^{\frac{\alpha}{\alpha-1}}\right) V \mathrm{d}\theta, & x > \zeta \\ f(-x;\alpha,-\beta), & x < \zeta \end{cases} \tag{A.1}$$

- if $\alpha \neq 1$ and $x = \zeta$

$$f(x;\alpha,\beta) = \frac{\Gamma\left(1+\frac{1}{\alpha}\right)\cos(\xi)}{\pi(1+\zeta^2)^{\frac{1}{2\alpha}}}, \tag{A.2}$$

- if $\alpha = 1$

$$f(x;1,\beta) = \begin{cases} \frac{1}{2|\beta|} \exp\left(\frac{x\pi}{2\beta}\right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} V \exp\left(-\exp\left(\frac{x\pi}{2\beta}\right) V\right) \mathrm{d}\theta, & \beta \neq 0 \\ \frac{1}{\pi(1+x^2)}, & \beta = 0 \end{cases} \tag{A.3}$$

where

$$\xi = \begin{cases} \frac{\arctan(-\zeta)}{\alpha}, & \alpha \neq 1 \\ \frac{\pi}{2}, & \alpha = 1 \end{cases}$$

$$V = \begin{cases} (\cos(\alpha\xi))^{\frac{1}{\alpha-1}} \left(\frac{\cos(\theta)}{\sin(\alpha(\xi+\theta))}\right)^{\frac{\alpha}{\alpha-1}} \frac{\cos(\alpha\xi+(\alpha-1)\theta)}{\cos(\theta)}, & \alpha \neq 1 \\ \frac{2}{\pi}\left(\frac{0.5\pi+\beta\theta}{\cos(\theta)}\right) \exp\left(\frac{1}{\beta}(0.5\pi+\beta\theta)\tan(\theta)\right), & \alpha = 1, \beta \neq 0, \end{cases}$$

and $\Gamma(.)$ is the so called gamma function. It is clear that V is a function of θ for given α and β.

The CDF $F(x;\alpha,\beta)$ of a standard stable random variable is

Table B.8
Estimation results of parameters of GARCH model for Bitcoin.

Parameter	Coefficient	Asym. Std. error	z-score	p-value
α	1.802	0.0586	−3.3788	0.0007***
β	−0.0234	0.0989	−0.2366	0.8130
c	0.0024	0.0101	0.2376	0.8122
η	7.39e−6	1.83e−5	0.4038	0.6863
μ	0.7355	0.0617	11.9206	0***
λ	0.2645	0.062	4.2661	1.99e−05***

Table B.9
Estimation results of parameters of GARCH model for Litecoin.

Parameter	Coefficient	Asym. Std. error	z-score	p-value
α	1.7693	0.0424	−5.4410	5.29e−08***
β	−0.0549	0.0691	−0.7945	0.4269
c	0.0003	0.0012	0.2725	0.7852
η	1.13e−07	2.07e−5	0.0055	0.9956
μ	0.6768	0.0705	9.6000	0***
λ	0.3231	0.0607	5.3229	1.02e−07***

- if $\alpha \neq 1$ and $x \neq \zeta$

$$F(x;\alpha,\beta) = \begin{cases} c_1(\alpha,\beta) + \frac{\text{sign}(1-\alpha)}{\pi} \int_{-\zeta}^{\frac{\pi}{2}} \exp\left(-(x-\zeta)^{\frac{\alpha}{\alpha-1}} V\right) \mathrm{d}\theta, & x > \zeta \\ 1 - F(-x;\alpha,-\beta), & x < \zeta, \end{cases} \tag{A.4}$$

where

$$c_1(\alpha,\beta) = \begin{cases} \frac{1}{\pi}\left(\frac{\pi}{2} - \xi\right), & \alpha < 1 \\ 1, & \text{otherwise} \end{cases}$$

- if $\alpha \neq 1$ and $x = \zeta$

$$F(x;\alpha,\beta) = \frac{1}{\pi}\left(\frac{\pi}{2} - \xi\right), \tag{A.5}$$

- if $\alpha = 1$

$$F(x;1,\beta) = \begin{cases} \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\left(-\exp\left(-\frac{x\pi}{2\beta}\right) V(\theta;1,\beta)\right) \mathrm{d}\theta, & \beta > 0 \\ \frac{1}{2} + \frac{1}{\pi}\arctan(x), & \beta = 0 \\ 1 - F(x;1,-\beta), & \beta < 0 \end{cases} \tag{A.6}$$

Appendix B. Parameter estimation results of model (ARMA)-GARCH

The log-likelihood function for MLE estimation is

$$\ln L = \sum_{i=1}^{T} \ln f(\theta; r_i), \tag{B.1}$$

where f is the PDF defined id (A.1), $\theta = [\alpha, \beta, \delta_t, \gamma_t]$ is the vector of four parameters of alpha stable distribution we want to estimate, $r_i, i = 1, 2, \dots, T$, is series of daily returns of a cryptocurrency and T is the number of observation. The conditional mean and conditional variance are $\delta_t = c + \phi r_{t-1} + \theta \epsilon_{t-1}$ and $\gamma_t^2 = \eta + \mu \gamma_{t-1}^2 + \lambda \epsilon_{t-1}^2$, respectively. The estimate of θ is the vector of parameters that maximizes the loglikelihood value defined in (B.1). The estimation procedure is performed with Matlab function $fmincon$ and the results are obtained together with the estimated Hessian which is used to computed the asymptotic standard errors of the estimates. We estimate both two specifications: ARMA-GARCH and ARMA-GJR-GARCH. Using the LR test, we exclude the ARMA-GJR-GARCH specification as a better alternative since it does not significantly increase the value of loglikelihood function of an ARMA-GARCH specification. The ARMA-GARCH model is the suitable specification and we will report only the estimation results of this model. The estimation results with the corresponding significance testing for each cryptocurrency are displayed in tables below.

Table B.10
Estimation results of parameters of GARCH model for Bitcoin Cash.

Parameter	Coefficient	Asym. Std. error	z-score	p-value
α	1.642	0.056	−6.3929	1.63e−10***
β	−0.0756	0.037	−2.0432	0.0410**
c	−0.0005	2.69e−6	−186.2454	0***
η	0.0003	9.98e−5	3.2365	0.0012***
μ	0.0303	0.0772	0.3925	0.6947
λ	0.9694	0.0746	12.9946	0***

Table B.11
Estimation results of parameters of GARCH model for Ethereum.

Parameter	Coefficient	Asym. Std. error	z-score	p-value
α	1.802	0.0586	−3.3788	0.000***
β	−0.0234	0.0989	−0.2366	0.8130
c	0.0024	0.0101	0.2376	0.8122
η	7.39e−6	1.83e−5	0.4038	0.6863
μ	0.7355	0.0617	11.9206	0***
λ	0.2645	0.062	4.2661	1.99e−05***

Table B.12
Estimation results of parameters of ARMA-GARCH model for Ripple.

Parameter	Coefficient	Asym. Std. error	z-score	p-value
α	1.5459	0.0499	−9.1002	0***
β	−0.0237	0.0262	−0.9046	0.3657
c	−0.0008	0.0002	−5.2026	1.97e−07***
ϕ	0.2283	0.0093	24.4957	0***
θ	−0.4380	2.31e−05	−18 961.04	0***
η	1.13e−7	5.29e−8	2.1366	0.0326**
μ	0.3643	0.0583	6.2487	4.14e−10***
λ	0.6356	0.0822	7.7324	1.07e−14***

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